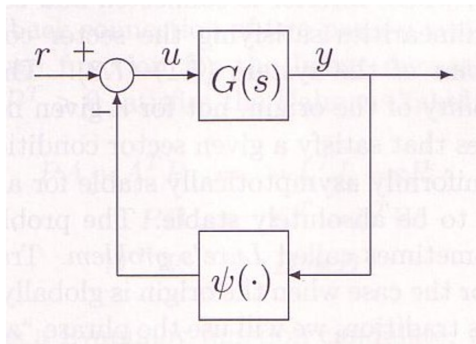
1 Popov and Circle Criterion

Many nonlinear physical systems can be represented as a feedback connection of a linear dynamical system and a nonlinear element.



The process for representing a system in this form depends on the particular system involved. For instance, in the case in which a control system's only nonlinearity is in the form of a relay or actuator/sensor nonlinearity, there is no difficulty in representing the system in this feedback form. In other cases, the representation may be less obvious. We assume that the extern input $r = 0$ and study the behavior of the unforced system. What is unique about this chapter is the use of the frequency response of the linear system, which builds on classical control tools like Nyquist plot and Nyquist criterion.
The system is said to be absolutely stable if it has a globally uniformly asymptotically stable equilibrium point at the origin for all nonlinearities in a given sector. The circle and Popov criteria give frequency-domain sufficient conditions for absolute stability in the form of strict positive realness of certain transfer functions. In the single-input-single-output case, both criteria can be applied graphically.

We assume the external input $r = 0$ and study the behavior of the unforced system represented by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$u = -\psi(y)$$

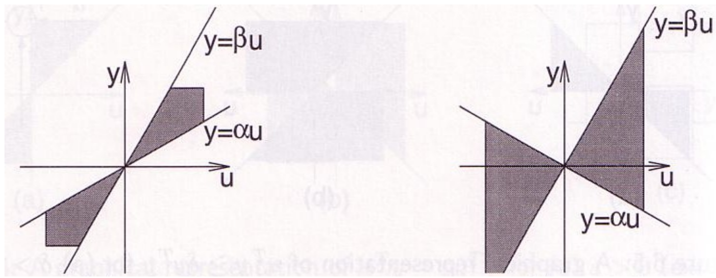
where $x \in \Re^n, u, y \in \Re^p, (A, B)$ is controllable, (A, C) is observable, and $\psi : [0, \infty) \times \Re^p \to \Re^p$ is a memoryless, possibly time-varying, nonlinearity, which is piecewise continuous in t and locally Lipschitz in y.

Definition 1.1: A memoryless function $h : [0, \infty) \to \Re$ is said to be to belong to the sector

1) $[0, \infty]$ if $u \cdot h(t, u) \geq 0$

2) $[\alpha,\infty]$ if $u\left[h(t,u)-\alpha u\right]\geq 0$

3) $[0,\beta]$ with $\beta>0$ if $h(t,u)\left[h(t,u)-\beta u\right]\leq 0$



Definition 1.2: The closed loop system is called absolutely stable in the sector $[0,\beta]$ if the origin is globally uniformly asymptotically stable for any nonlinearity in the given sector. It is absolutely stable in a finite domain if the origin is uniformly asymptotically stable.

1.1 Popov Criterion

We can apply the Popov criterion if the following conditions are satisfied:

1) The time-invariant nonlinearity $\psi:\Re\to\Re$ satisfies the sector condition $[0,\beta]$
2) The time-invariant nonlinearity $\psi:\Re\to\Re$ satisfies $\psi(0)=0$
3) $G(s)=\frac{1}{s^n}\frac{p(s)}{q(s)}$ with $\deg(p(s))<\deg(q(s))$
4) The poles of $G(s)$ are in the LHP or on the imaginary axis
5) The system is marginally stable in the singular case

Theorem 1.1: The closed loop system is absolutely stable if, $\psi\in[0,k], 0<k<\infty$, and there exists a constant q such that the following equation is satisfied

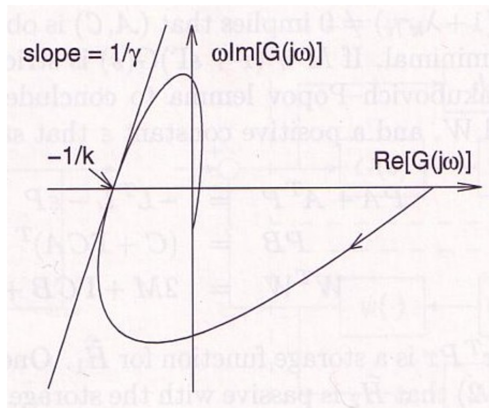
$$\operatorname{Re}\left[G(j\omega)\right]-qj\omega\operatorname{Im}\left[G(j\omega)\right]>-\frac{1}{k}\ ,\ \forall\omega\in[-\infty,\infty]$$

Graphical Interpretation

We define the Popov plot P

$$P(j\omega)=\left\{z=\operatorname{Re}\left[G(j\omega)\right]+j\omega\operatorname{Im}\left[G(j\omega)\right]\mid\omega>0\right\}$$

Then the closed loop system is absolutely stable if P lies to the right of the line that intercepts the point $-(1/k)+j0$ with a slope $1/q$.



Example 1.1: Consider the second-order system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - h(y)$$
$$y = x_1$$

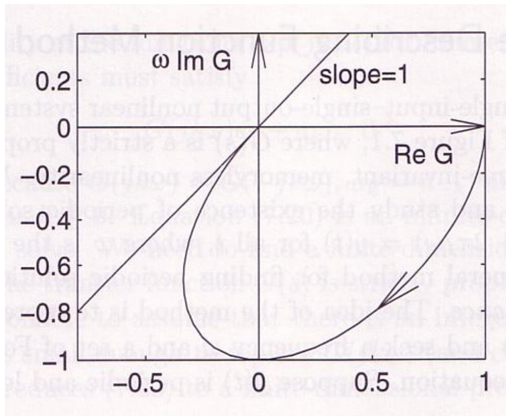
This system would fit the required form for using the Popov criterion if we took $\psi = h$, but the matrix A would not be Hurwitz. Adding and subtracting the term αy to the right-hand side of the second state equation, where $\alpha > 0$, and defining $\psi(y) = h(y) - \alpha y$, the system takes the required form, with

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \text{and}\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Assume that h belongs to a sector $[\alpha, \beta]$, where $\beta > \alpha$. Then ψ belongs to the sector $[0, k]$, where $k = \beta - \alpha$. The Popov condition takes the form

$$\frac{1}{k} + \frac{\alpha - \omega^2 + q\omega^2}{\left(\alpha - \omega^2\right)^2 + \omega^2} > 0\ ,\ \forall \omega \in [-\infty, \infty]$$

For all finite positive values of α and k, this inequality is satisfied by choosing $q > 1$. Even at $k = \infty$, the foregoing inequality is satisfied for all $\omega \in (-\infty, \infty)$. Hence, the system is absolutely stable for all nonlinearities h in the sector $[\alpha, \infty]$, where α can be arbitrarily small.



The figure shows the Popov plot of $G(j\omega)$ for $\alpha = 1$. The plot is drawn only for $\omega \geq 0$, since $\mathrm{Re}\left[G(j\omega)\right]$ and $\omega\,\mathrm{Im}\left[G(j\omega)\right]$ are even functions of ω. The Popov plot asymptotically approaches the line through the origin of unity slope from the right side. Therefore, it lies to the right of any line of slope less than one that intersects the real axis at the origin and approaches it asymptotically as ω tends to ∞.

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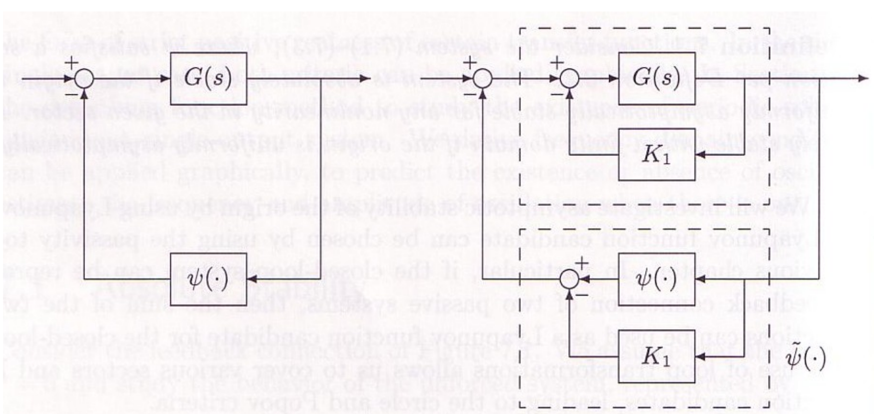
1.1.1 Loop transformations for more general sector conditions

As we have seen in the previous example, if the nonlinearity ψ belongs to a sector $\left[K_1, K_2\right]$, it is still possible to use the Popov criterion. The idea is to transform the loop such that all the conditions for using the Popov criterion are still satisfied.



If we apply this transformation, $\psi \in \left[K_1, K_2\right]$ is transformed to $\tilde{\psi} \in \left[0, K_2 - K_1\right]$.

We can the use the Popov criterion with $\tilde{\psi} \in [0, K_2 - K_1]$ and the new linear system

$$\tilde{G}(s) = {G(s)} \Big/ {(1 + K_1 G(s))}.$$

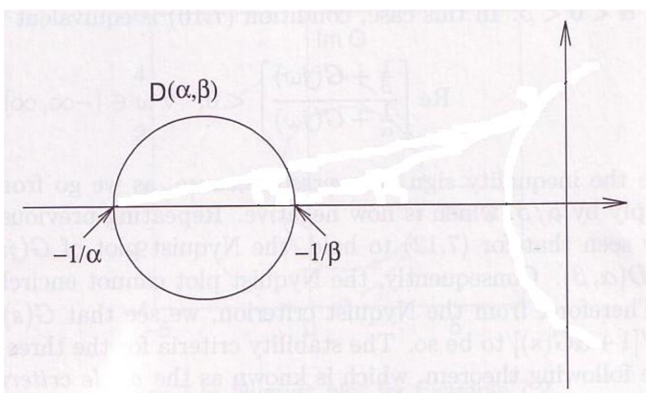
Remark 1.1: Consider $G(s) = e^{\tau s} G_1(s)$ with $G_1(s)$ stable. Then the control loop is absolute stable in the sector $[0, K]$ if it exists a $q \geq 0$ that fulfills the Popov inequality .

Remark 1.2: The Popov criterion only gives a sufficient condition for absolute stability.

1.2 Circle Criterion

When we allow the nonlinearity to become time varying, the Popov Criterion is no longer applicable. The Circle Criterion gives us a tool to analyse absolute stability for a time varying nonlinearity.

Definition 1.3: We define $D(\alpha, \beta)$ to be the closed disk in the complex plane whose diameter is the line segment connecting the points $-(1/\alpha) + j0$ and $-(1/\beta) + j0$.



Theorem 1.2: Consider a scalar system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ u &= -\psi(y, t) \end{aligned}$$

where $\{A, B, C, D\}$ is a minimal realization of $G(s)$ and $\psi \in [\alpha, \beta]$. Then, the system is absolutely stable if one of the following conditions is satisfied, as appropriate:

1) If $0 < \alpha < \beta$, the Nyquist plot of $G(j\omega)$ does not enter the disk $D(\alpha,\beta)$ and encircles it m times in the counterclockwise direction, where m is the number of poles of $G(s)$ with positives real parts.

2) If $0 = \alpha < \beta$, $G(s)$ is Hurwitz and the Nyquist plot of $G(j\omega)$ lies to the right of a vertical line defined by $\text{Re}[s] = -1/\beta$.

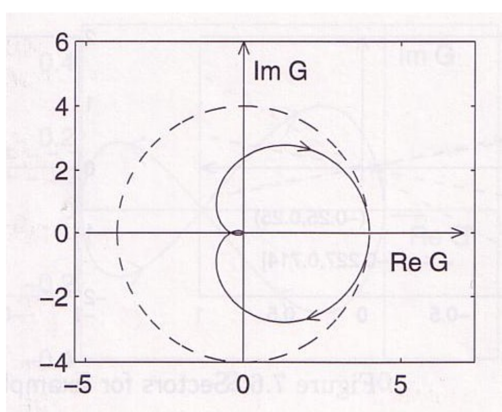
3) If $\alpha < 0 < \beta$, $G(s)$ is Hurwitz and the Nyquist plot of $G(j\omega)$ lies in the interior of the disk $D(\alpha,\beta)$.

If the sector condition is satisfied only on an interval $[a,b]$, then the foregoing conditions ensure that the system is absolutely stable with a finite domain.

Example 1.2: Let

$$G(s) = \frac{4}{(s+1)(1/2\,s+1)(1/3\,s+1)}$$

The Nyquist plot of $G(j\omega)$ is shown in the next figure. Since $G(s)$ is Hurwitz, we can allow α to be negative and apply the third case of the circle criterion. So, we need to determine a disk $D(\alpha,\beta)$ that encloses the Nyquist plot. Clearly, the choice of the disk is not unique. Suppose we decide to locate the center of the disk at the origin of the complex plane. This means that we will work with a disk $D(-\gamma_2,\gamma_2)$, where the radius $(1/\gamma_2) > 0$ is to be chosen. The Nyquist plot will be inside this disk if $|G(j\omega)| < 1/\gamma_2$. In particular, if we set $\gamma_1 = \sup_{\omega\in\Re} |G(j\omega)|$, then γ_2 must be chosen to satisfy $\gamma_1\gamma_2 < 1$.



It is not hard to see that $|G(j\omega)|$ is maximum at $\omega = 0$ and $\gamma_1 = 4$. Thus, γ_2 must be lass than 0.25. Hence, we can conclude that the system is absolutely stable for all nonlinearities in the sector $[-0.25+\varepsilon, 0.25-\varepsilon]$, where $\varepsilon > 0$ can be arbitrarily small.

Inspection of the Nyquist plot suggests that the choice to locate the center at the origin may not be the best one. By locating the center at another point, we might be able to obtain a disk that encloses the Nyquist plot more tightly.

Another direction we can pursue in applying the circle criterion is to restrict α to zero and apply the second case of the circle criterion. The Nyquist plot lies to the right of the vertical line $\mathrm{Re}[s] = -0.857$. Combining these results allows us to conclude that the system is absolutely stable for all nonlinearities in the sector (-0.25, 1.166].

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