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A prototype model for chaos studies

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Abstract

In this paper, a simple model is proposed for chaos studies. The model consists of a state, time delay and a nonlinear element. It can be described as an autonomous continuous-time difference–differential equation with only one variable. The rich behaviour of the system is numerically illustrated. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Time delay; Chaotic model; Bifurcation diagram; Nonlinear feedback systems

1. Introduction

Traditionally, it was implicitly assumed that random behaviour was due to extreme complexity of the dynamical systems with higher number of independent degrees of freedom. However, recent introduction of chaotic dynamics tells us that randomness in the dynamical systems does not necessarily involve an enormous number of independent degrees of freedom [1]. In the presence of a nonlinearity only a few independent variables are sufficient to generate chaotic motion. Consider the following deterministic ordinary differential equations:

$$\dot{x} = f(x, t), \quad x(0) = x_0, \tag{1}$$

in which $x \in \mathscr{R}^n$ is the state vector dependent on t and $\dot{x}$ denotes the derivative. The nonlinear function is $f(x, t) : \mathscr{R}_+ \times \mathscr{R}^n \rightarrow \mathscr{R}^n$ and may depend on t and x. The initial-state vector at $t = 0$ is $x(0)$. The nonlinear function $f(\cdot)$ may include continuous or discontinuous nonlinearities. The system defined in (1) must contain nonlinearity and has at least three degrees of freedom, $n = 3$, in order to exhibit chaotic behaviour [2]. However this two conditions are not only necessary

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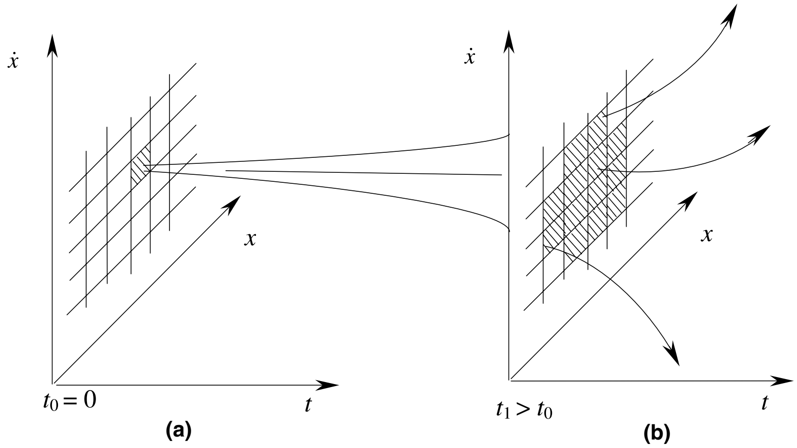

Fig. 1. An illustration of uncertainty growth in chaotic systems.

properties for a chaotic system, but also its trajectory needs to be sensitive to the initial conditions [2]. Suppose the system output, $x(t)$, and its velocity, $\dot{x}(t)$, are measured accurately. Then in $x(t) - \dot{x}(t)$ plane (known as the phase plane) one can divide up the space into areas of size $\Delta x(t)$, $\Delta\dot{x}(t)$ as shown in Fig. 1(a). If a nonchaotic system is released from two nearby initial conditions, its trajectories start close to one another in phase space and their future values can be predicted somewhere in the shaded box, $N_0(t)$, illustrated in Fig. 1(a).

However, if the system is chaotic, its trajectories will move exponentially away from each other for small times on the average and resulting loss of information about initial conditions. The uncertainty grows in time to $N(t)$ boxes as shown in Fig. 1(b). The growth in the uncertainity given by

$$N = N_0 e^{ht}, \tag{2}$$

where h is related to concept called the Lyapunov exponent, which the rate at which nearby trajectories of the system phase space diverge [2]. This complexity of the behaviour is due to the internal, rather then external dynamics.

For autonomous continuous time nonlinear system it has been indicated in [2] that chaos cannot occur when $n = 1, 2$. Confirming this result, the systems given in [3] all have at least three-state variables and models for chaos studies; Chua's circuits [4], Lorenz equations [5] and pressure transducer model [6] are all in third-order forms. This seems not the case for systems with delay time, a two-cell nonautonomous neural networks with delay (which is a third-order continuous time autonomous system) may appear chaotic attractor [7]. Here further it points out that a simple continuous-time nonlinear system modelled with delay differential equation can exhibit chaotic behaviour without taken delay time too large and can be used as a prototype model in chaos studies.

2. The system description

Consider the following nonlinear continuous system in dimensionless form with one-state variable

$$\dot{x}(t) = \delta x(t-\tau) - \varepsilon[x(t-\tau)]^3 \quad (t \geqslant t_0), \tag{3}$$

where δ and ε are positive system parameters. Here t_0 is the initial interval and $\tau > 0$ corresponds to the delay time in which represents the time interval between the start of an event at one point and its resulting action at another point in the system. The solution of the system equation defined in (3) is determined uniquely when an initial function $\Psi(t)$ defined on an initial interval is prescribed as

$$x(t) = \Psi(t) \quad \text{for } t \in [t_0 - \tau, t_0].$$

The time delay has been approximated or its effects are generally ignored by modellers. There are several researches have comments on the dangers that modellers risk if they ignore delays which they think are small [8]. In the case of approximation used for the delay elements, its effects on the system behaviour can only partly be observed. The delay element arises naturally in various diverse fields: infectious diseases, chemical kinetics, and the navigational control of ships and aircraft [8]. Therefore the subject of delay differential equations is now a rapidly growing one and numerical solution of system modelled by delay differential equations is both theoretical and practical interest. The bibliography prepared by Baker et al. [9] shows recent developments and interests on this field.

3. From regular to complex behaviours

The system given in (3) has been used as a prototype model to observe self-oscillations in the shipbuilding industry [10]. However, the complexity of the system has not been mentioned or discussed yet. Since the system behaviour is generally studied with the small signal analysis, the local behaviours around the equilibrium points are only obtained and discussed. Consider the system in (3) with zero delay time, $\tau = 0$, has three equilibrium points namely origin and $\pm\sqrt{\delta/\varepsilon}$. The vector field of the system is depicted in Fig. 2. The arrows on the x-axis indicate the corresponding velocity vectors at each interval of equilibrium points. The arrows point to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. There is no flow at the equilibrium points.

In Fig. 2 solid black dots at $\pm\sqrt{\delta/\varepsilon}$ represent stable equilibrium points where open circle at origin represents unstable one. Fig. 2 shows that the system trajectory moves and approaches the stable fixed point at $\sqrt{\delta/\varepsilon}$ for any initial condition satisfies $x_0 > 0$. Similarly, if negative initial

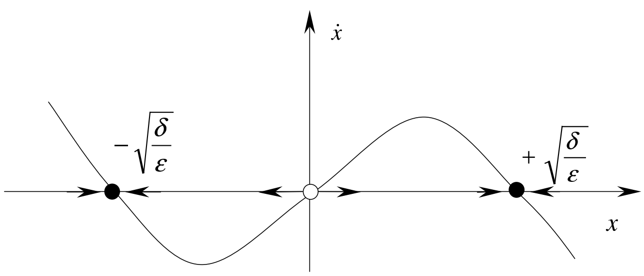

Fig. 2. Vector field of system given in (3) for $\tau = 0$.

condition, $x_0 < 0$, is chosen then the system trajectory approaches to the negative equilibrium point at $-\sqrt{\delta/\varepsilon}$.

The effects of delay on the system dynamics are discussed and investigated in the feedback nonlinear systems [11]. It has been shown that the time delay occurred due to the sensors dynamics in feedback path leads to undesirable phenomena such as oscillations and instability.

4. Numerical results

The dynamical behaviour of the system given in (3) without any restrictions and approximations may only be examined numerically. Here the system is modelled in Matlab/SIMULINK® environment and is numerically solved by the use of fifth-order Runge–Kutta ordinary differential solver, embedded in Matlab toolboxes. Since the system has not contain discontinuous elements, ode45 solver is found to be suitable in this case [12]. The following numerical results are obtained for 0.001 integration step size and 10^{-6} absolute and relative tolerances. The delay time $\tau = 0$ and the nonlinearity gain ε in (3) are fixed at unity. Here $\Psi(t) = 0.1$ is chosen on the initial interval as initial function. The system response is examined by changing the gain δ of the linear part of the feedback signal. The delay value, $\tau = 1$, is not taken large and can be occurred in many engineering systems especially due to the measurement devises in process control systems. The system trajectory is first observed from time response and its a typical time response is shown in Fig. 3, depicted for $\delta = 1.6$.

In order to monitoring the system response, its trajectory is depicted in phase plane by plotting the integrator input versus its output as illustrated in Fig. 1. It can be seen from (3) that the system without linear feedback part is asymptotically stable. However the linear part is destabilise the system since positively fed back to the system input.

The numerical results are obtained for $\delta = 0.5$, 0.9, 1.51 and 1.7, and the system trajectories are depicted in phase plane, Figs. 4(a), (b), (c) and (d), respectively. Since positive initial function is

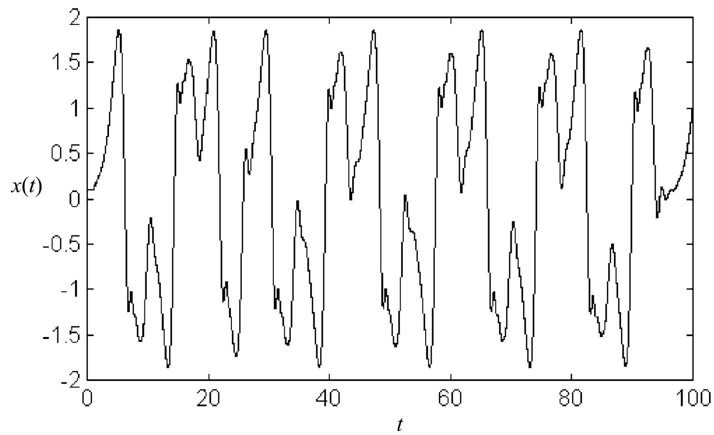

Fig. 3. Time response of the system for $\tau = \varepsilon = 1$ and $\delta = 1.6$.

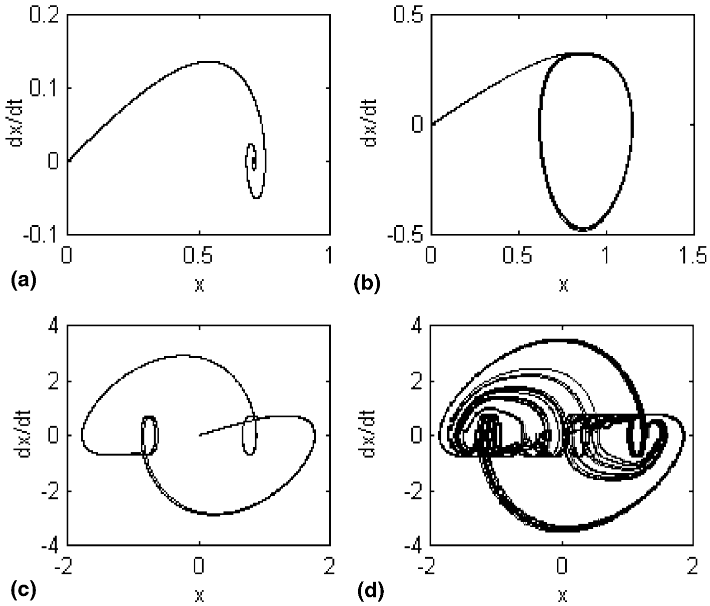

Fig. 4. Phase portrait of the system with $\tau = \varepsilon = 1$: (a) for $\delta = 0.5$ system trajectory converges to the equilibrium point, (b) self-sustained oscillation for $\delta = 0.9$, (c) oscillations for $\delta = 1.51$, (d) chaotic behaviour for $\delta = 1.7$.

chosen, Fig. 4(a) shows the system trajectory is asymptotically decaying at positive equilibrium point for $\delta = 0.5$. Conversely, one can show that the trajectory will decay onto the negative equilibrium point at $-\sqrt{\delta/\varepsilon}$ if a negative initial function is chosen. Increasing the value of δ to 0.9 leads to a self-oscillation as depicted in Fig. 4(b) and encloses the stable equilibrium point located at $\sqrt{\delta/\varepsilon}$. One can shows that this self-sustained oscillation is stable for a positive range of the state initial conditions. Choosing the parameter $\delta = 1.5$ yields to the behaviour depicted in Fig. 3(c) which indicates several oscillations with different amplitude and frequencies. Note that the simulation interval time for Fig. 4 is chosen large enough in order to observe both the transient and study-state trajectories of the system. The trajectories depicted in Figs. 4(a)–(c) are usually considered as regular behaviours since their nature are known and can be analysed analytically. However, increasing the values of δ leads to complex behaviours: for example, choosing the parameter $\delta = 1.7$ results to a strange behaviour depicted in Fig. 4(d). Note that the phase portrait of the system depicted in Fig. 2(d) shows the last-half part of the system trajectory in order to monitoring the steady-state behaviour of the system. The phase portrait depicted in Fig. 4(d) is a clear indication of double-scroll chaotic-type behaviour, observed in the study of chaotic systems [1] with three-state variables.

The unique character of chaotic dynamics may be seen most clearly by examining the system to be started twice, but from slightly different initial functions. For nonchaotic system, the uncertainty between two nearby initial functions leads to an error in prediction that groves linearly with time. For chaotic systems, on the other hand, the error between two trajectories grows exponentially in time (see Fig. 1), so that the state of the system is essentially unpredictable after a very short time. These phenomena can be observed from error function depicted in Fig. 5 for two

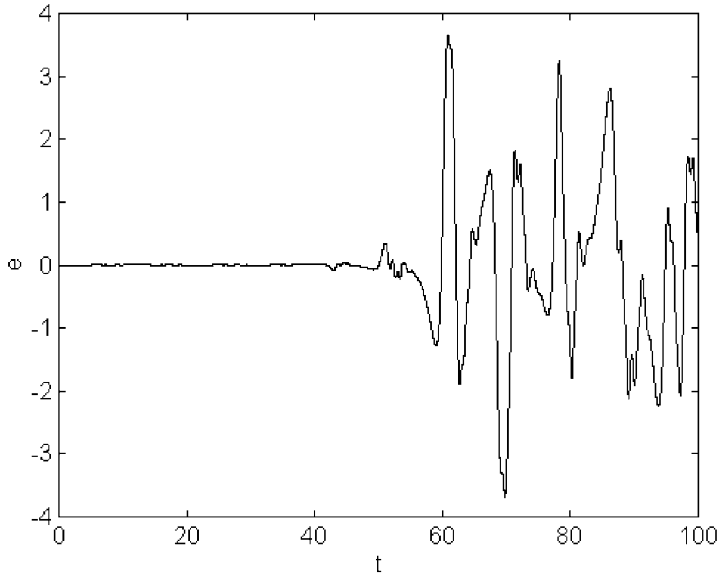

Fig. 5. The error, $e = x(t)_{0.1} - x(t)_{0.10001}$, between two trajectories obtained for nearby initial conditions of the system state.

nearby the state initial values, $x_0 = 0.1$ and $x_0 = 0.10001$. Fig. 5 shows simultaneous difference taken between them, $e = x(t)_{0.1} - x(t)_{0.10001}$, for $\delta = 1.7$.

Fig. 5 indicates the error between two trajectories initially almost unchanged within the time interval of $0 < t < 40$, but it increases and gives raise to large error in later times resulting in a loss of final-state predictability. This phenomenon is known as sensitivity of the system trajectory to the value of the initial conditions. This is a clear evidence of chaotic behaviour of the proposed simple structure system given in (3) with a realistic delay time.

5. Bifurcation diagram

The error between two system trajectories depicted in Fig. 5 only illustrates the results for fixed system parameters. However, the details of the system responses to a range of a system parameter is considerably important and chaotic regions are generally needed to produce chaos or operate the system in nonchaotic regions. There are two methods namely the bifurcation diagrams and Lyapunov exponents are usually used and obtained for monitoring the overall system behaviour [2].

The system behaviour qualitatively can be observed as a function of the system parameters δ or ε for the fixed delay time. It has been shown in [11] that changing the system parameter ε actually stabilises the system and does not effect the system behaviour, since it fed to the system negatively. Here the bifurcation diagram, which is a widely used technique for examining the pre-chaotic or post-chaotic changes in the system under parameter variations, is obtained. A Matlab programme is developed in order to obtain bifurcation diagram based on the procedure given in [2]. Again the system parameter ε is fixed at unity and its behaviour is observed by changing δ within the range of $0.5 \leqslant \delta \leqslant 1.78$ with 0.2 step size. The system trajectories are obtained for [0 20000] simulation times and considering only its last-half part to eliminate the possible transient responses. In order to illustrate the algorithm developed for bifurcation diagram, consider Fig. 6 that illustrates a system trajectory in phase plane for fixed system parameters. Let S be an one-dimensional surface of section. S is required to be transverse to the flow, i.e., all trajectories starting on S flow through it, not parallel to it.

The mapping from S to itself, obtained by following trajectories from one interaction with S, solid black dots, to the next, is known Poincare' section [2]. Note Fig. 4(b) shows a solid black dot only in Poincare' section for $\delta = 0.9$. If two self-sustained oscillations exist, then two dots will be appearing on Poincare section. The bifurcation diagram shows the black dots, obtained from each Poincare' sections, versus a system parameter.

The system bifurcation diagram is depicted in Fig. 7, in which the system output x stereoscopically observed from Poincare' section and depicted versus the selected range of positive feedback gain δ. Fig. 7 shows the trajectory of the system has initially settled down at a fixed point

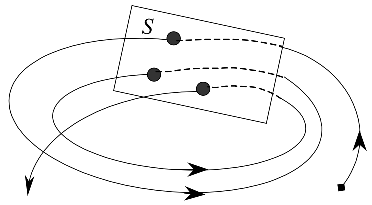

Fig. 6. A typical Poincare' section diagram.

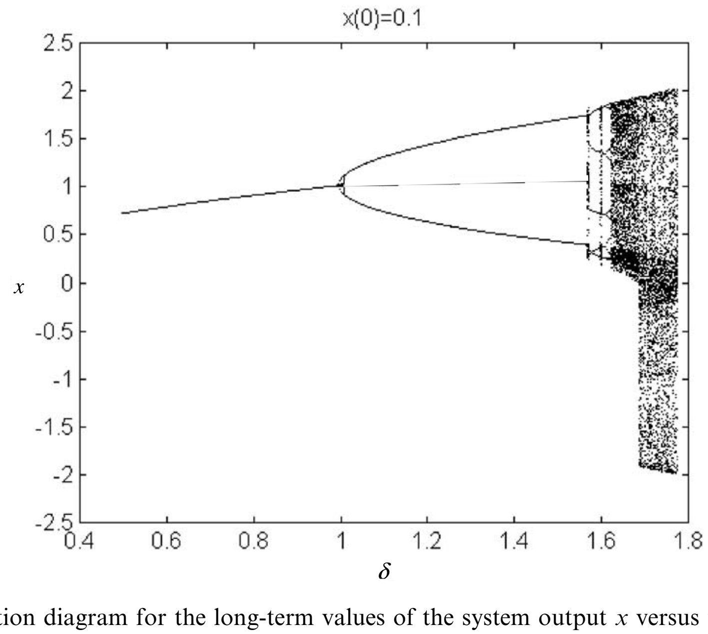

Fig. 7. Bifurcation diagram for the long-term values of the system output x versus the parameter δ.

for $\delta < 1$. Increasing $\delta > 1$ leads to period two until the value of δ reach to 1.56 where a small chaotic region sets. For the range of $1.64 < \delta < 1.8$ the bifurcation diagram clearly indicates that the system exhibits chaotic behaviour. Further increasing δ leads to unbounded solution namely unstable behaviour.

6. Conclusions

The dynamic behaviour of a simple nonlinear system with delay element is studied in this paper. It has been shown that such system may exhibit complex behaviour which cannot be observed from approximation on the nonlinearity and time delay elements. The effective results presented here may be considered as:

1. a simple dynamical system with a time delay can exhibit very complex behaviour include chaos,
2. the system presented in this paper can be used as a prototype model for studying chaotic behaviours in engineering science.

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