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Periodic solutions and bifurcations of delay-differential equations

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Abstract

In this Letter a simple but effective iteration method is proposed to search for limit cycles or bifurcation curves of delay-differential equations. An example is given to illustrate its convenience and effectiveness.

Keywords: Delay-differential equation; Periodic solution; Bifurcation

1. Introduction

This Letter considers a non-linear delay-differential system in the form

$$\dot{x} = -\mu \frac{x(t-1)}{1+[x(t-1)]^4}. \tag{1}$$

This system was studied by Peng and Ucar [1] by numerical approach. The numerical result shows that the system, Eq. (1), has multiple bifurcations, stable limit cycles (periodic or quasiperiodic solutions), and chaotic behavior.

Recently many methods were suggested to deal with non-linear equations, for example, homotopy perturbation method [2–4], variational iteration method [5,6], various Lindstedt–Poincaré methods [7–9], variational method [10–15], extended tanh method [16,17], Adomian Pade approximation [18]. There also exist many approaches to bifurcation of various non-linear problems [19–24].

This Letter suggests a simple but effective iteration approach to Eq. (1) to find its periodic solutions and bifurcations.

2. An iteration method

Delay systems are widely found in engineering, see Ref. [25] and references cited thereby. In this Letter we suggest an iteration method for the discussed problem.

We rewrite Eq. (1) in the form

$$\dot{x} = -\mu x(t-1) - \dot{x}\left[x(t-1)\right]^4, \tag{2}$$

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and construct an iteration formulation for (2) as follows

$$\dot{x}_{n+1} = -\mu x_n(t-1) - \dot{x}_n\big[x_n(t-1)\big]^4. \tag{3}$$

We feel interest in its periodic solutions (or limit cycles) and its bifurcations. If we know that the discussed system has limit cycles, then the energy method (or variational method) suggested by He [26] can be powerfully applied to the search for its solution. Homotopy perturbation method was first introduced to find the bifurcation curves [3]. Generally speaking, limit cycles can be determined approximately in the form [3,26]

$$x = b + a\cos\omega t + \sum_{n=1}^{m}(C_n \cos n\omega t + D_n \sin n\omega t), \tag{4}$$

where a, b, C_n and D_n are constants.

For simplicity we begin with

$$x_0 = A\cos\omega t, \tag{5}$$

where A is the amplitude and ω its frequency. Substituting (5) into (3), we obtain

$$\begin{aligned}\dot{x}_1 &= -\mu A\cos\omega(t-1) + A^5\omega\sin\omega t\cos^4\omega(t-1)\\ &= -\mu A(\cos\omega t\cos\omega + \sin\omega t\sin\omega)\\ &\quad + \frac{1}{8}A^5\omega\sin\omega t\big[3 + 4\cos 2\omega(t-1)\\ &\quad + \cos 4\omega(t-1)\big].\end{aligned} \tag{6}$$

If $\dot{x}_1 = \dot{x}_0$, then x_0 happens to be the exact solution. Generally such case will not arise for non-linear problems, but we can minimize

$$J = \int_0^T (\dot{x}_{n+1} - \dot{x}_n)^2\,dt, \quad T = \frac{2\pi}{\omega}, \tag{7}$$

to identify ω in trial function, Eq. (5). The second approach to the identification of the unknown constants in (4) is the Galerkin method, which requires

$$\int_0^T (\dot{x}_{n+1} - \dot{x}_n)\cos i\omega t\,dt = 0, \tag{8}$$

$$\int_0^T (\dot{x}_{n+1} - \dot{x}_n)\sin i\omega t\,dt = 0,$$

$$i = 0, 1, 2, 3, \ldots. \tag{9}$$

In this Letter we will apply the Galerkin method to the determination of the frequency, ω, in Eq. (5). Setting

$$\int_0^T (\dot{x}_1 - \dot{x}_0)\sin\omega t\,dt = 0, \tag{10}$$

we have

$$-\mu A\sin\omega + \frac{3}{8}A^5\omega - \frac{1}{4}A^5\omega\sin 2\omega + A\omega = 0. \tag{11}$$

We rewrite (11) in the form

$$A^4 = \frac{\mu\frac{\sin\omega}{\omega} - 1}{\frac{3}{8} - \frac{1}{4}\sin 2\omega}. \tag{12}$$

Since $A^4 \geqslant 0$, the above equation has no solution when $\mu\sin\omega/\omega - 1 < 0$, so Eq. (1) has no periodic solution. However, when $\mu\sin\omega/\omega - 1 > 0$, Eq. (12) has real solutions, so Eq. (1) exists periodic solution, its frequency can be solved from (12). The so-called simple bifurcation occurs when

$$\mu\sin\omega - \omega = 0, \tag{13}$$

Eq. (11) has multiple roots depending upon the value of μ.

When $\mu = \pi/2$, we have $\omega = \pi/2$. And $\mu = \pi/2$ is the Hopf bifurcation point, which agrees exactly with the numerical result [1]. Other bifurcation points occur at $\mu = 2n\pi + \pi/2$, where n is nature number.

3. Conclusion

The preceding analysis has the virtue of utter simplicity. We conclude from the result obtained that the method developed here is extremely simple in its principle, quite easy to use, and gives a very good accuracy in the whole solution domain, even with the simplest trial functions. Theoretically any accuracy can be achieved by a suitable choice of the trial functions. With the help of some mathematical software, such as MATHEMATICA, MATLAB, the method provides a powerful mathematical tool to the determination of limit cycles and bifurcation curves for complex non-linear systems.

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