Hamiltonicity of maximal planar graphs and planar triangulations

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Preface

One of the typical topics in graph theory is the study of planar graphs. Planar graphs arise quite naturally in real-world applications, such as road or railway maps and chemical molecules. The cartographers of the past were aware of the fact that any map on the plane could be colored with four or fewer colors so that no adjacent countries were colored alike. It was the Four Color Problem which stimulated interest in hamiltonian planar graphs. A graph is said to be hamiltonian if it has a cycle that contains all vertices exactly once. If a planar graph is hamiltonian, then it is easy to color its faces with four or fewer colors so that no two adjacent faces are colored alike. Moreover, planar graphs play an important role as some practical problems can be efficiently solved for planar graphs even if they are intractable for general graphs.

The study of planar graphs was initiated by L. Euler in the 18th century. It was subsequently neglected for 180 years and only came to life again with Kuratowski's theorem, which provides a criterion for a graph to be planar. At the same time, one of the most outstanding results for maximal planar graphs was proved by H. Whitney. A maximal planar graph is a planar graph having the property that no additional edges can be added so that the resulting graph is still planar. In 1931 H. Whitney [55] proved that each maximal planar graph without separating triangles is hamiltonian; a separating triangle is a triangle whose removal separates the graph. In particular, all 4–connected maximal planar graphs are hamiltonian.

Another connection of maximal planar graphs and the Four Color Problem is the following. In 1884 P.G. Tait [49] conjectured that every cubic, 3–connected and planar graph is hamiltonian. Note that the dual graph of a cubic, 3–connected and planar graph is a maximal planar graph. The Conjecture of Tait became more important, in the sense that its correctness would imply the positive correctness of the Four Color Problem. Unfortunately, the latter conjecture is wrong, as shown by W.T. Tutte in 1946. Therefore, the Four Color Problem is true if and only if every hamiltonian planar graph is 4–colorable.

In 1956 W.T. Tutte [53] generalized Whitney's result and proved that each 4–connected planar graph is hamiltonian. In 1990 M.B. Dillencourt [24] generalized Whitney's result in a different direction, namely to graphs called NST–triangulations. M.B. Dillencourt

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defined NST–triangulations to be those planar graphs that are triangulations in the sense that every face, perhaps except for the exterior face, is a triangle but that there is no separating triangle. In the class of the 3–connected maximal planar graphs numerous examples which are non-hamiltonian were found. In 2003 C. Chen [17] obtained a stronger version of Whitney's result and he proved that any maximal planar graph with only one separating triangle is hamiltonian. An interesting problem with respect to the class of the 3–connected maximal planar graphs is the following: What is the maximal number α, so that every maximal planar graph with at most α separating triangles is hamiltonian?

In this thesis we will extend the results of C. Chen and M.B. Dillencourt in various ways. We will mainly study the existence of hamiltonian cycles and hamiltonian paths in maximal planar graphs and planar triangulations. We will prove that each maximal planar graph with at most five separating triangles is hamiltonian. In the case of more separating triangles, we will introduce a special structure of the position of the separating triangles to each other. The latter structure will also generate hamiltonicity.

Chapter 1 will be devoted to an introduction of the terminology and the basic concepts of planar graphs and maximal planar graphs. We will introduce one of the most powerful tools in the theory of planar graphs, namely Euler's formula. This concept will be the corner-stone of many results of us.

In Chapter 2 we will study some fundamentals of maximal planar graphs. The first section will be concerned with the generation of some classes of maximal planar graphs. There are two main reasons why this generation will be of interest: On the one hand, this will provide a basis for inductive proof; and on the other hand, this can be used to develop efficient algorithms for the constructive enumeration of the structures. In the second section we will focus on the structures of some classes of maximal planar graphs. Moreover, we will study the subgraphs of a maximal planar graph G induced by vertices of G with the same degree. In the last section we will turn towards the connectivity of maximal planar graphs.

The most interesting problem regarding hamiltonian maximal planar graphs will be investigated in Chapter 3. In the first section we will extend the result of Chen [17]. We will prove that each maximal planar graph with at most five separating triangles is hamiltonian. We will give a counterexample for the case of six separating triangles. In the case of more separating triangles, we will establish a special structure of the position of the separating triangles to each other, which will also generate hamiltonicity. In the second section we will deal with hamiltonian paths. Since maximal planar graphs with more than five separating triangles need not be hamiltonian, we will show that maximal planar graphs with exactly six separating triangles have at least one hamiltonian path. It would be nice to have a theorem that says: "A graph G is non-hamiltonian if G has

property 'Q', where 'Q' can be checked in polynomial time". This motivates the concept of 1–toughness, which is introduced in the fourth section. In the fifth section we will construct specific maximal planar graphs. We will show that for all $n \geq 13$ and $k \geq 6$, there exist a hamiltonian as well as a non-hamiltonian maximal planar graph with n vertices and k separating triangles. In the sixth section we will study the number of cycles of certain length that a maximal planar graph G on n vertices may have. In the seventh section we will consider pancyclicity rather than hamiltonicity. In fact, we will determine whether a maximal planar graph G contains a cycle of each possible length l with $3 \leq l \leq n$. In the eighth section we will deal with the question: How many vertices of a hamiltonian maximal planar graph can be deleted, so that the remaining graph is still hamiltonian?

In Chapter 4 we will turn to general planar triangulations. Clearly, each maximal planar graph is a planar triangulation. In the first section we will consider some properties of planar triangulations. The second section of this chapter deals with general results on 2–connected planar triangulation. In the third section we will see that the deletion of vertices of a hamiltonian maximal planar graph leads to some interesting results in the class of 3–connected planar triangulations. In the last section we will concentrate on 4–connected planar triangulation.

In Chapter 5 we will consider some applications of hamiltonian maximal planar graphs and planar triangulations. In the first section of this chapter we will turn towards algorithms, which will be useful for applications. In the second section we will show that the dual graph of a maximal planar graph is of interest for applications. In the third section we will deal with some applications of hamiltonian maximal planar graphs and planar triangulations in computer graphics. In the last section we will focus on applications in chemistry.

Contents

Chapter 1

Introduction

In this chapter we present the basic notation and terminology of graph theory which will be used throughout this thesis. Before discussing results related to maximal planar graphs in more detail in the second section, we will briefly explain the basic definitions and concepts in the first section. The notation mainly follows that of L. Volkmann [54] as well as G. Chartrand and L. Lesniak [16] and we refer the reader to these books for any information not provided here. Special notation and definitions will be defined where needed.

1.1 Terminology and notation

General concepts

If not stated otherwise, the term *graph* will be used throughout the thesis to represent a finite, simple and undirected graph. We denote the *vertex set* of a graph G with $V(G)$ and the *edge set* with $E(G)$. The cardinalities of these sets will be the *order $n(G)$* and the *size $m(G)$* of G, respectively. If two vertices $u, v \in V(G)$ are connected with an edge, we simply write $uv \in E(G)$ and say that u and v are *adjacent*. An edge $e = uv$ is called *incident* with both *end-vertices* u and v. Two distinct edges with a common end-vertex are called *adjacent edges*.

A cycle with the vertices $x_1, x_2, ..., x_k$ and the edges $x_1x_2, x_2x_3, ..., x_kx_1$ is called a *k-cycle* and is denoted by $x_1x_2...x_kx_1$. A path from x to y is called a $P(x, y)$ *path*.

The *neighborhood $N_G(v)$* of a vertex v is defined as $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. The *degree $d_G(v)$* of a vertex v is defined as the number of edges incident with v. If $d_G(v) = 0$, we call v an *isolated vertex*. The *minimum degree $\delta(G)$* and the *maximum degree $\Delta(G)$* of a graph G are the minimum and maximum over all vertex-degrees in G, respectively. $\tau_i(G)$ of a graph G denotes the number of all vertices with degree i. If $\delta(G) = \Delta(G) = d$, we call the graph G *d–regular*. A 3–regular graph G is also called *cubic*.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph H is called *induced* if $E(H) = \{uv \in E(G) \mid u, v \in V(H)\}$. For a vertex subset

$S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S. The deletion of a vertex x in G, in symbols $G - x$, denotes the induced subgraph $G[V(G) \setminus \{x\}]$. The deletion of a set of vertices S is defined as $G - S = G[V(G) \setminus S]$. Let u and v be two vertices of G. If $uv \in E(G)$, then $G + uv = G$; otherwise $G + uv$ denotes the graph obtained from G by adding the edge uv.

A graph G is *homomorphic* to a graph H if a mapping $g : V(G) \to V(H)$ exists, which is called *homomorphism*, so that g preserves adjacency. That is, $xy \in E(G)$ if and only if $g(x)g(y) \in E(H)$. If g is a one-to-one mapping, g is called *isomorphism* and we call the two graphs G and H *isomorphic*, in symbols $G \cong H$.

Connectivity

A vertex x is said to be *connected* to a vertex y in a graph G if there exists a path between these two vertices in G. A graph G is called *connected* if it contains a path between any two vertices. All vertices which are connected to each other induce a *component*, and $\kappa(G)$ denotes the number of components of G. Note that if $\kappa(G) = 1$, then G is connected. A *k–cut* of a graph G is a set S of k vertices of a graph G so that $G - S$ is not connected. Note that a k–cut is also called a separating set of k vertices. The *connectivity* $\sigma(G)$ of a graph G is the size of a smallest k–cut of G if G is not complete, and $\sigma(G) = n - 1$ if $G = K_n$ for any positive integer n. Hence $\sigma(G)$ is the minimum number of vertices whose removal results in a disconnected or a trivial graph. A graph G is said to be *k–connected*, $k \geq 1$ if $\sigma(G) \geq k$. In general, G is k–connected if and only if the removal of fewer than k vertices results in neither a disconnected nor a trivial graph.

Planar graphs

A graph G is said to be *embeddable* on a surface S if it can be drawn on S in a way that no two edges intersect geometrically except at a vertex, at which they are both incident. In this thesis we are concerned exclusively with the case in which S is a plane or sphere. A graph G is called *planar* if it can be embedded in the plane. Note embedding a graph in the plane is equivalent to embedding it on the sphere. A planar graph divides the plane into regions, which are called *faces*. The cardinality of the set of all faces is denoted by $l(G)$. The unbounded region is called *exterior face*, the other faces are called *interior faces*. In a 2–connected planar graph G, the *exterior cycle* is the cycle bounding the exterior face and will be denoted by X_G. Cycles bounding an interior face are called *facial cycles*. An edge is called a *boundary edge* of a graph G if it is on the exterior cycle; otherwise it is an *interior edge*. A vertex is called a *boundary vertex* of a graph G if it is on the exterior cycle. A vertex that is not a boundary vertex is called an *interior vertex*. A *chord* is an interior edge whose end vertices are both boundary vertices.

A planar graph G is called *maximal planar* if for every pair u, v of non-adjacent vertices of G the graph $G + uv$ is non-planar. Thus in any embedding of a maximal planar graph G with $|V(G)| \geq 3$ vertices, the boundary of every face is a triangle.

A planar graph is called *planar triangulation* if the boundary of every face is a triangle, except possibly the exterior face. A graph G is called *triangulated* if every cycle of G with length greater than three has a chord.

A *separating k–cycle* is a k–cycle so that both the interior as well as the exterior contain one or more vertices. A separating 3–cycle is called a *separating triangle*. Therefore, a separating triangle does not form the boundary of a face.

For a given connected planar graph G we define the *dual graph G^** as follows: All faces of G constitute the vertex set of G^*. Two distinct vertices of G^* are joined by an edge if the corresponding faces in G have a common edge as a border. Each edge of G^* is drawn so that it crosses its associated edge of G but no other edge of G or G^*.

Hamiltonicity

A cycle of a graph G is called a *hamiltonian cycle* if it contains all vertices of G. A graph G is defined to be *hamiltonian* if it has a hamiltonian cycle. Similarly a *hamiltonian path* is a path of a graph G which contains all vertices of G.

An embedding of a maximal planar graph G is called *hamiltonian for any two boundary edges of G* if for any two boundary edges of G, the graph G has a hamiltonian cycle passing through these two edges. A maximal planar graph G is called *hamiltonian for any two boundary edges of G* if for each embedding of G, the graph G is hamiltonian for any two boundary edges of G.

1.2   Subject

One of the main topics in graph theory is the concept of planarity. In the 18th century, Euler made one of the first discoveries in this field, the well known "Polyederformel".

Theorem 1.1 (Euler [27]) *Let G be a planar graph on n vertices and m edges. Then*

$$l(G) = 1 + \mu(G) = 1 + m(G) - n(G) + \kappa(G).$$

Since planar triangulations are always connected, we can use the following observation.

Observation 1.2 *Let G be a connected planar graph on n vertices and m edges. Then*

$$l(G) = 2 + m(G) - n(G).$$

According to Theorem 1.1 of Euler, each embedding of a planar graph has the same number of faces.

It was Kuratowski [38] who gave a characterization of planar graphs in terms of forbidden homeomorphic subgraphs. We say a graph H is homeomorphic from G if either H is isomorphic to G or H is isomorphic to a graph obtained by subdividing any sequence

of edges of G. Note that by subdivision of an edge $e = xy$ of a non-empty graph G, we mean that the edge xy is removed from the graph and a new vertex w is inserted instead, along with the edges wx and wy. We say G is homeomorphic with H if both G and H are homeomorphic from a graph F.

Theorem 1.3 (Kuratowski [38]) *A graph is planar if and only if it possesses no subgraph homeomorphic with either K_5 or $K_{3,3}$.*

Now we will study some elementary properties of maximal planar graphs.

Theorem 1.4 (Chartrand and Lesniak [16]) *Let G be a maximal planar graph on $n \geq 3$ vertices and m edges. Then*

$$l(G) = \frac{2}{3}m(G) \quad \text{and} \quad m(G) = 3n(G) - 6.$$

This result provides the next corollary for planar triangulations, in order to answer the following question: How many edges has a planar triangulation?

Corollary 1.5 *Let G be a planar triangulation on $n \geq 3$ vertices and m edges. Then*

$$2n(G) - 3 \leq m(G) \leq 3n(G) - 6.$$

In the inequality of the left side the equality holds if every vertex of G is on the exterior face. Moreover, Theorem 1.4 yields the following observation.

Observation 1.6 *Let G be a maximal planar graph on $n \geq 3$ vertices. Then*

$$l(G) = 2n(G) - 4.$$

Thus we obtain a necessary condition for a planar graph to be maximal planar. If a planar graph G has an odd number of faces, then G is not maximal planar. Another important consequence of Theorem 1.1 of Euler is the following corollary.

Corollary 1.7 *Let G be a maximal planar graph on $n \geq 4$ vertices. Then*

$$3\tau_3 + 2\tau_4 + \tau_5 = (7 - 6)\tau_7 + (8 - 6)\tau_8 + ... + (\Delta(G) - 6)\tau_{\Delta(G)} + 12.$$

The latter corollary leads to an immediate but important consequence.

Corollary 1.8 *Every planar graph G contains a vertex of degree at most 5, that is, $\delta(G) \leq 5$.*

An interesting feature of maximal planar graphs on $n \geq 4$ vertices is the fact that the minimum degree is at least 3. Together with the previous corollary we obtain $\delta(G) \in \{3, 4, 5\}$ in a maximal planar graph G of order at least four. This leads to a very interesting observation.

Observation 1.9 *Let G be a planar graph. Then G is at most 5–connected.*

While considering the stereographic projection, Chiba and Nishizeki showed the following lemma, which will be used in some proofs in Chapter 3.

Lemma 1.10 (Chiba and Nishizeki [18]) *Let G be a planar graph. Then G can always be embedded in the plane so that a given face of G becomes the exterior face.*

Consider the stereographic projection and rotate the sphere so that the north pole is in that face.

Chapter 2

Fundamentals of maximal planar graphs

2.1   Generating maximal planar graphs

In this section we deal with the generation of some classes of maximal planar graphs, which will provide a basis for inductive proofs.

An inductive class I is defined by giving:

- initial specifications which define the class A of initial elements – the basis of I.

- generating specifications which define the class R of rules (modes) of combination, any such rule applied to an appropriate sequence of elements, already in A, produces an element of A.

The inductive class $I = Cn(A; R)$ consists exactly of the elements which can be obtained (constructed) from the basis by a finite number of applications of the generating rules.

There are two main reasons why methods to construct an infinite class from a finite set of starting graphs are of interest. On the one hand, this provides a basis for inductive proof; and on the other hand, this can be used to develop efficient algorithms for the constructive enumeration of the structures.

In the following inductive definitions, the starting graphs and rules should be understood as embedded on a surface. The small black triangles attached to the vertices in the description of the rule denote any number (zero or more) of edges. The condition ≤ 2 means that there can be no more than two edges on that side. The condition $1, 2$ means there should be one or two edges on that side, see Figure 2.2. In the following we shall use the half-edges to indicate that there must be an edge.

Definition 2.1 *The set of all maximal planar graphs with minimum degree i is called* MPG-*i; and we denote the subset of* MPG-*i with n vertices by* MPG_n-*i.*

In 1967 Bowen and Fisk [12] presented an inductive definition of the class of all maximal planar graphs.

Theorem 2.2 (Bowen and Fisk [12]) *The inductive class $Cn(S; \mathcal{A})$, see Figure 2.1, is equal to the class of all maximal planar graphs.*
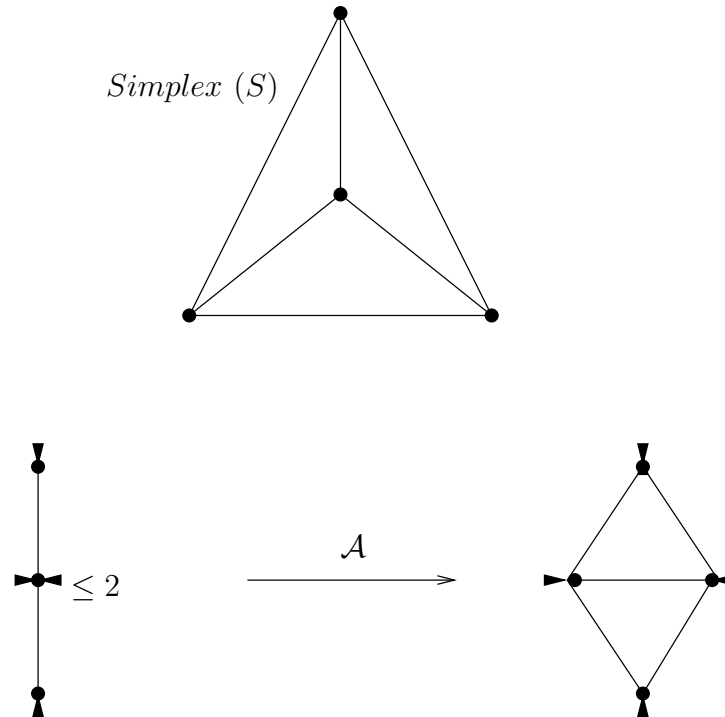
Figure 2.1: Bowen and Fisk's operations.

In 1989 Batagelj [8] implemented a similar inductive procedure to generate the class of all maximal planar graphs without vertices of degree three.

Theorem 2.3 (Batagelj [8]) *The inductive class $Cn(O; \mathcal{B}, \mathcal{C})$, see Figure 2.2, is equal to the class of all maximal planar graphs without vertices of degree three.*

We need the following definition, in order to get a better formulation of the following result.

Definition 2.4 *A graph is called cyclically k-edge-connected if at least k edges must be deleted to disconnect a component so that every remaining component contains a cycle.*

Barnette [9] and Butler [14] independently described a method for constructing all planar cyclically 5-edge-connected cubic graphs. In the language of the dual graph this class is the set of all 5–connected maximal planar graphs. We call such maximal planar graphs 5–*connected MPG*-5 graphs.
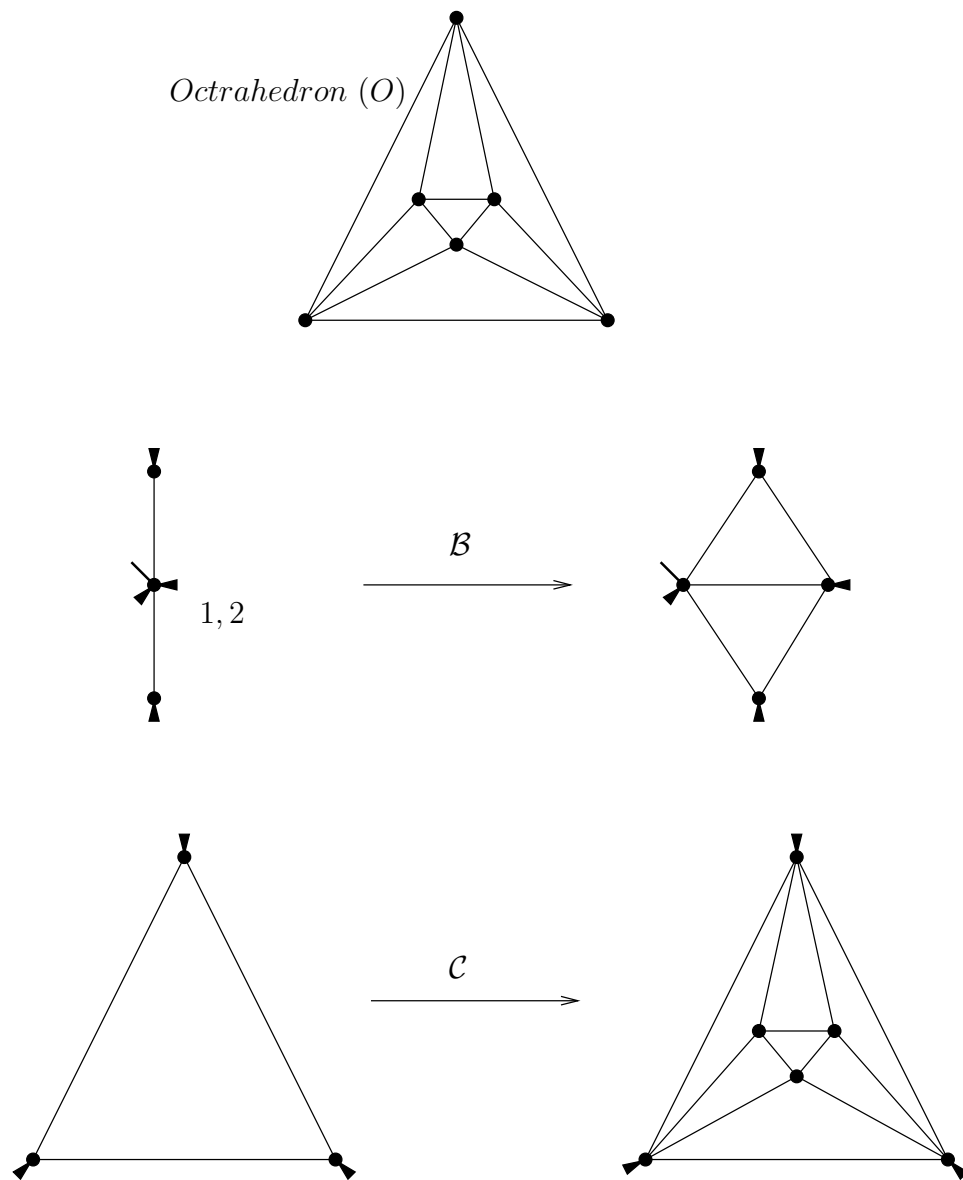
Figure 2.2: Batagelj's operations.

Note that a maximal planar graph with minimum degree five is always a 3–connected MPG-5 graph, a 4–connected MPG-5 graph if there are no separating triangles and a 5–connected MPG-5 graph if there are no separating triangles or separating 4–cycles.

Barnette and Butler's method starts with the icosahedron graph and uses the operations called $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ given in Figure 2.3. Since both proved an inductive definition of planar cyclically 5-edge-connected cubic graphs, the operations depictured here are just the dual forms of the operations used by Barnette and Butler.

Theorem 2.5 (Barnette [9] and Butler [14]) *The inductive class $Cn(I; \mathcal{D}, \mathcal{E}, \mathcal{F})$, see Figure 2.3, is equal to the class of all 5–connected maximal planar graphs with minimum degree five.*

Batagelj [6] described a method for constructing all 3–connected MPG-5 graphs. He uses the operations $\mathcal{D}$ and $\mathcal{E}$ also used by Barnette and Butler, and in addition, a switching operation $\mathcal{G}$. This operation assumes that the top and bottom vertices do not share an edge.

Theorem 2.6 (Batagelj [6]) *The inductive class $Cn(I; \mathcal{D}, \mathcal{E}, \mathcal{G})$, see Figure 2.4, is equal to the class of all 3–connected maximal planar graphs with minimum degree five.*

We denote an operation $\mathcal{G}$ such that the central edge does not belong to a separating triangle after the operation by an operation $\mathcal{G}_3$. Brinkmann and McKay [13] described a method for constructing all 4–connected MPG-5 graphs.

Theorem 2.7 (Brinkmann and McKay [13]) *The inductive class $Cn(I; \mathcal{D}, \mathcal{E}, \mathcal{G}_3)$ is equal to the class of all 4–connected maximal planar graphs with minimum degree five.*

An important subclass of all 5–connected MPG-5 graphs, with many practical applications, are those with maximum degree 6, best known via their dual graphs, the fullerene graphs, see Chapter 5.

2.2   Properties of maximal planar graphs

In this section we analyse the structure of some classes of maximal planar graphs. Moreover, we study the subgraphs of G induced by vertices of G with the same degree.

Definition 2.8 *Let $G = (V, E)$ be a graph.*

$$Di = \{x \in V | d(x) = i\}$$

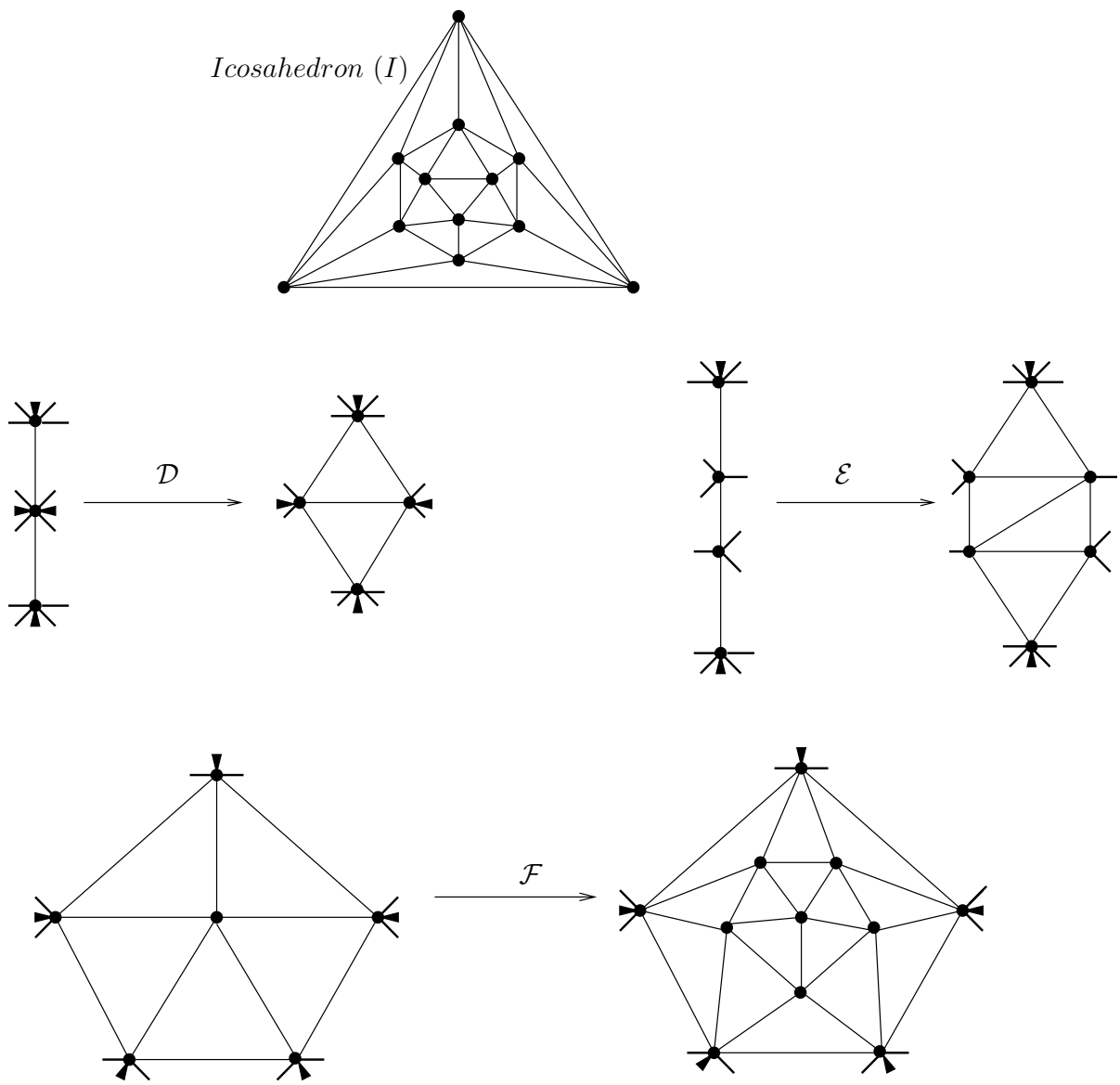*denotes the vertex set of G with degree i.*

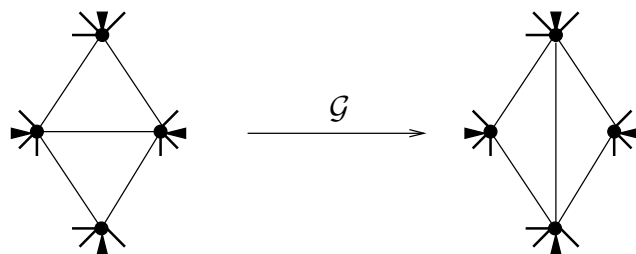Figure 2.3: Barnette and Buttler's operations.

Figure 2.4: Switching operation.

2.2.1   Properties of MPG-3 graphs

We start with the following observation.

Observation 2.9 *If $G = (V, E)$ is an arbitrary maximal planar graph with $\delta(G) = 3$, then G has the K_4 as a subgraph.*

Proof. Each MPG-3 graph has a vertex e of degree 3. Let the vertices a, b and c be the neighbors of e, see Figure 2.5. Then a, b and c are pairwise adjacent, since G is maximal planar. If not, then e has a further neighbor, which yields a contradiction to $d(e) = 3$.
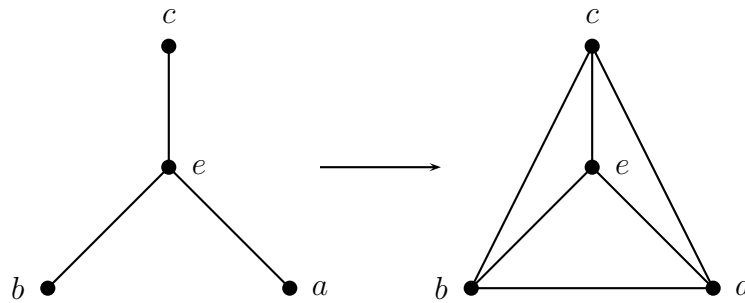
Figure 2.5: Each MPG-3 graph has a subgraph isomorphic to K_4.

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This observation has an easy corollary.

Corollary 2.10 *If $G = (V, E)$ is a maximal planar graph with $\delta(G) = 3$ and $n \geq 5$, then G has at least one separating triangle.*

Now we study the subgraph of G induced by vertices of G with degree three.

Theorem 2.11 *Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 3$. All components of the subgraph $G[D3]$ belong to one of the following kind of graphs:*

- *K_4 if $n = 4$,*

- *trivial graph if $n \geq 5$.*

Proof. Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 3$. Assume x and y are two adjacent vertices with degree 3. Let the vertices a, b be the neighbors of x, and let the vertices c, d be the neighbors of y, see Figure 2.6. If the vertices a and c are distinct, then either x or y is adjacent to another vertex, since all faces of G are triangles, then the degree of this vertex is greater than three. Therefore, the vertices a and c are equal. We call this vertex e. With the same argumentation as above we show that the vertices b and d are equal. We call this vertex f.
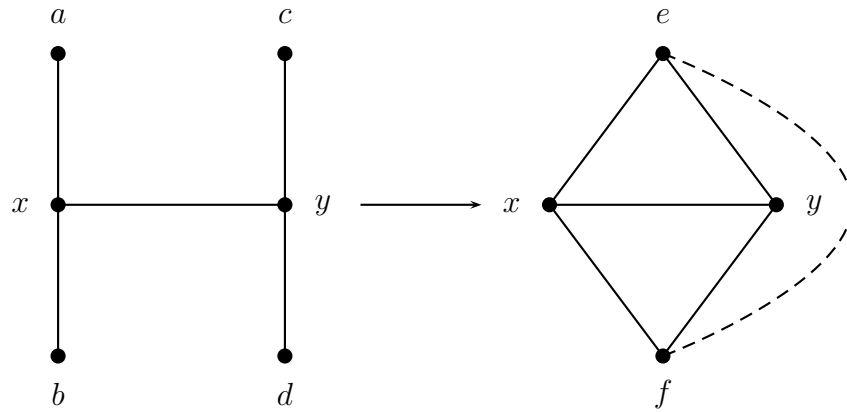
Figure 2.6: The subgraph $D3$.

Case 1: Let $n = 4$. Since G is maximal planar, the vertices e and f are adjacent. Therefore, $G[D3] = K_4$.

Case 2: Let $n \geq 5$. Since G is maximal planar, there exists a vertex which is adjacent to x or y. This leads to a contradiction to the assumption. Thus all components of the subgraph $G[D3]$ are isolated vertices. ∎

This theorem and the following trivial observation lead to a very useful corollary.

Observation 2.12 *If $G = (V, E)$ is a maximal planar graph with $\delta(G) = 3$ and $n = 5$, then $\tau_3 = 2$ and G has one separating triangle.*

This is the only case in which there are more vertices of degree three than separating triangles. Therefore, the following result holds.

Corollary 2.13 *If $G = (V, E)$ is a maximal planar graph with $\delta(G) = 3$ and $n \geq 6$, then the neighbors of every vertex of degree 3 induce a separating triangle.*

Remark. For this reason the number of vertices with degree three, τ_3, is a lower bound of the number of separating triangles if $n \geq 6$.

The number of vertices with degree three has still another important property. It supplies a necessary condition of whether a graph is hamiltonian.

Theorem 2.14 *Let $G = (V, E)$ be an arbitrary maximal planar graph with $\delta(G) = 3$ and $n \geq 13$. If $2\tau_3 > n$, then G is non-hamiltonian.*

Proof. Assume G is hamiltonian. Let C be a hamiltonian cycle of G and x be a vertex of degree 3. Let the vertices a and b be the neighbors of x on the hamiltonian cycle C. If we replace the edges ax and xb by the edge ab, then we obtain a hamiltonian cycle C' in the maximal planar graph $G - x$. If we delete all vertices of degree 3 in this way, we
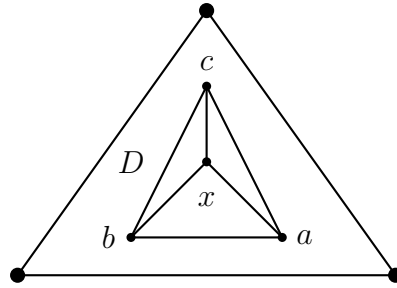
Figure 2.7: $2\tau_3 \geq n$.

obtain a maximal planar graph $G - D3$. For two distinct vertices of degree 3 one obtains different edges. After construction the hamiltonian cycle of $G - D3$ has at least τ_3 edges. $G - D3$ consists of $n - \tau_3$ vertices. Therefore, $\tau_3 \leq n - \tau_3$, i.e. $2\tau_3 \leq n$. Thus we get a contradiction to the hypothesis.   ∎

2.2.2   Properties of MPG-4 graphs

While considering MPG-4 graphs we must distinguish two different classes of MPG-4 graphs, those which are 3–connected and those which are 4–connected, see Section 2.3. Inspired by Observation 2.9, we now present a degree condition for maximal planar graphs with $\delta(G) = 4$.

Theorem 2.15 (Helden [35]) *If $G = (V, E)$ is an arbitrary maximal planar graph with $\delta(G) = 4$, then there is no vertex of degree four with $d_{G[D4]}(x) = 3$.*

Proof. Assume that there is a vertex a with degree four and $d_{G[D4]}(a) = 3$. This vertex is adjacent to three vertices with degree four, we call them b, c and d, as well as to one vertex with degree ≥ 5. We call this vertex z. Since G is maximal planar, all faces of the graph G are triangles. Because of that, the edge which is incident to z and a belongs to two triangles, to which the vertices a and z belong as well. The vertex a has degree four, thus it can be connected with no further edge. From this it follows that the vertex z is adjacent to b and d. The same argument holds for the edges ad and ab. This means the vertex d is adjacent to the vertex c and the vertex b is adjacent to the vertex c. Moreover, a vertex y with $d_G(y) \geq 4$ exists and y is adjacent to the vertex d. The vertex d can be connected with no further vertex, because $d_G(d) = 4$. Thus y is adjacent to z, as well as y is adjacent to c. Since $d_G(c) = 4$ and y is adjacent to the vertex c, it follows that y is adjacent to the vertex b, because the graph G is maximal planar. Since $d_G(z) \geq 5$, a vertex x exists with $d_G(x) \geq 4$. The vertex x is adjacent to the vertex b because the graph G is maximal planar. Thus we get a contradiction to $d_G(b) = 4$ and the proof is complete. ∎

Studying the subgraph $G[D4]$, we obtained the following result.

Theorem 2.16 (Helden [35]) *Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 4$. If there exists a vertex x in the subgraph $G[D4]$ with $d_{G[D4]}(x) = 4$, then $\Delta(G) = 4$ and $n(G) = 6$.*

Proof. We prove this theorem by using Theorem 2.15 and using the attribute that the graph G is maximal planar. We build the graph around the vertex x. Since $d_{G[D4]}(x) = 4$, the vertex x is adjacent to four vertices with degree four. We call these four vertices a, b, c and d. As the graph G is maximal planar we obtain that the vertex a is adjacent to the vertices b and d. With the same argument we get that the vertex c is adjacent to the vertices d and b. Assume the vertex a is adjacent to the vertex c, then the vertex d cannot be adjacent to b. Thus we have $d_{G[D4]}(a) = d_{G[D4]}(b) = d_{G[D4]}(c) = d_{G[D4]}(d) = 3$. This is not possible because of Theorem 2.15. Therefore, all vertices must have degree four in the subgraph $G[D4]$. Thus there must be another vertex e with degree four. Now we have obtained all six vertices with degree four. An extension is not possible. ∎

The graph constructed in the proof of Theorem 2.16 is denoted by MPG_6-4, see Figure 2.8.

Figure 2.8: An MPG_6-4 graph.

Next we will determine the structure of components of the subgraph $G[D4]$.

Theorem 2.17 (Helden [35]) *Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 4$. All components of the subgraph $G[D4]$ belong to the following types of graphs:*

- *trivial graph,*

- *cycle C_p with $p \geq 5$ with $p \in I\!N$,*

- *path of length q with $q \in I\!N$,*

- MPG_6-4 *graph.*

Proof. The vertices in the subgraph $G[D4]$ have maximal degree four. Because of Theorem 2.15 we know there is no vertex with degree three in the subgraph $G[D4]$. Therefore, the vertices in the subgraph $G[D4]$ only have the degrees $0, 1, 2,$ or 4. If $d_{G[D4]}(v) = 4$ for

one vertex v, Theorem 2.16 supplies an MPG_6-4 graph as a component and $\kappa(G[D4]) = 1$. If the vertices in the subgraph $G[D4]$ have the degrees $0, 1$ or 2 only the trivial graph, a path of length q with $q \in I\!N$ or a cycle C_p with $p \geq 5$ can occur as a component. ∎

The next lemma provides properties which will be needed later.

Lemma 2.18 (Helden [35]) *Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 4$. If one component K of the subgraph $G[D4]$ consists of a path of length q with $q \in I\!N$, then $|N_G(K)| = 4$ and for all $x \in N_G(K)$ is $d(x) \geq 5$.*

Proof. A path of length q with $q \in I\!N$ consists of $q - 1$ vertices with degree two in the subgraph $G[D4]$ and two vertices with degree one in the subgraph $G[D4]$. The vertex x_1 with $d_{G[D4]}(x_1) = 1$ is adjacent to three vertices with degree at least five, we call this three vertices a, b and c, and x_1 is adjacent to one vertex with degree four, we call this vertex x_2. As the graph G is maximal planar, we get that the vertex c is adjacent to the vertices a and b and with the same argument we obtain that the vertex x_2 is adjacent to the vertices a and b. If $q \geq 2$ there exist $q - 1$ more vertices x_i with degree four which are adjacent to x_{i-1}, a and b, see Figure 2.9. The last vertex with degree four, we will call x_{q+1}, satisfies $d_{G[D4]}(x_{q+1}) = 1$. The vertex x_{q+1} is also adjacent to a and b. Now we have to prove that $cx_{q+1} \notin E(G)$.
Assume $cx_{q+1} \in E(G)$.

Case 1: Let $q \leq 3$. This yields a contradiction to $d_G(a) \geq 5$, $d_G(b) \geq 5$ or $d_G(c) \geq 5$.

Case 2: Let $q \geq 4$. This yields a contradiction to $c \notin G[D4]$. This completes the proof of the lemma. ∎
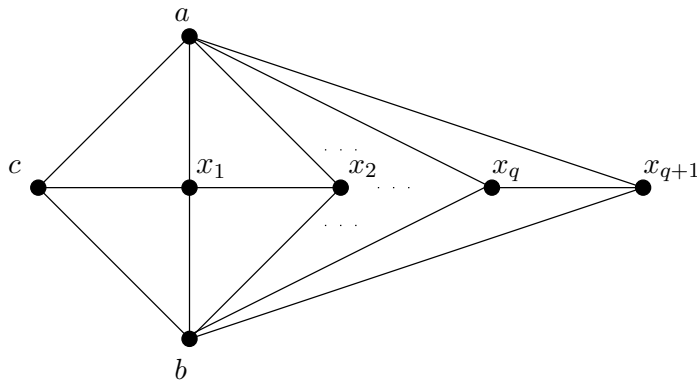

Figure 2.9: Proof of Lemma 2.18

In analogy to Lemma 2.18 we can prove the next corollary.

Corollary 2.19 *Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 4$. If one component K of the subgraph $G[D4]$ consists of one cycle C_p with $p \geq 5$ with $p \in I\!N$, then $|N_G(K)| = 2$ and $d(x) \geq 5$ for all $x \in N_G(K)$.*

Since the neighborhood of a cycle component consists of only two vertices, adding one more vertex to the graph provides that the degree of at least one vertex of the cycle component increases to at least five. Hence, we get a path component. This means if one component of the subgraph $G[D4]$ consists of one cycle, then no other component can exist. This yields the next easy corollary.

Corollary 2.20 (Helden [35]) *Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 4$. If one component of the subgraph $G[D4]$ consists of one cycle C_p with $p \geq 5$ with $p \in I\!N$, then $\kappa(G[D4]) = 1$.*

The next lemma provides properties which will be needed later.

Lemma 2.21 (Helden [35]) *Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 4$. If the subgraph $G[D4]$ consists of one cycle C_p with $p \geq 5$ with $p \in I\!N$, then G contains exactly two vertices with degree at least five.*

Proof. Let $G[D4]$ consists of one cycle C_p with $p \geq 5$ with $p \in I\!N$. We call the vertices of the cycle $x_1, x_2, ..., x_p$. Since $d_G(x_i) = 4$ for all $i = 1, 2, ..., p$ there exist at least two vertices a and b. Because the graph G is maximal planar, one vertex lies inside the cycle and one vertex lies outside the cycle. Furthermore both vertices a and b are adjacent to all vertices $x_1, x_2, ..., x_p$. Because $d_G(x_i) = 4$ for all $i = 1, 2, ..., p$ and the graph G is maximal planar, there exists no further vertex. ∎

Now we consider the trivial components.

Corollary 2.22 (Helden [35]) *Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 4$. If one component K_1 of the subgraph $G[D4]$ consists of one vertex x, then $|N_G(x)| = 4$ and $d(y) \geq 5$ for all $y \in N_G(K_1)$.*

Now we consider 3–connected MPG-4 graphs. This means there exists at least one separating triangle. In 1978 Hakimi and Schmeichel showed the following result for the vertices with degree four in the interior of a separating triangle.

Theorem 2.23 (Hakimi and Schmeichel [30]) *Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 4$. Suppose G contains a separating triangle induced by the set of vertices $X = \{x_1, x_2, x_3\}$. Consider any planar embedding of G, and let $D_{in}i$ denote the vertices of degree i occurring in the interior of the triangle X in this embedding.*

a) If $|D_{in}4| = 0$, then $|D_{in}5| \geq 7$.

b) If $|D_{in}4| = 1$, then $|D_{in}5| \geq 5$.

c) If $|D_{in}4| = 2$, then $|D_{in}5| \geq 3$.

The octahedron, see Figure 2.2, yields the following observation.

Observation 2.24 *Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 4$. Suppose G contains a separating triangle induced by the set of vertices $X = \{x_1, x_2, x_3\}$. Consider any planar embedding of G, and let $D_{in}i$ denote the vertices of degree i occurring in the interior of the triangle X in this embedding.*

If $|D_{in}4| = 3$, then $|D_{in}5| \geq 0$.

Thus the previous results show that the number of vertices with degree 5 occurring in the interior of a separating triangle X depends on the number of vertices with degree 4 occurring in the interior of the same triangle X.

2.2.3   Properties of MPG-5 graphs

In 1983 Batagelj showed that there is only one 5–regular maximal planar graph.

Theorem 2.25 (Batagelj [6]) *Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 5$. If $G = G[D5]$ with $\Delta(G) = 5$, then this graph is unique up to isomorphism. We call this graph icosahedron or MPG_{12}-5, see Figure 2.3.*

Since maximal planar graphs with $\delta(G) = 5$ and $\Delta(G) = 6$ play a decisive role in their applications, we will show the next result. Rolland-Balzon [44] proved this result.

Theorem 2.26 (Rolland-Balzon [44]) *Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 5$ and with 14 vertices. If $\Delta(G) = 6$, then this graph is unique up to isomorphism. We call this graph MPG_{14}-5, see Figure 2.10.*

Note that a maximal planar graph with minimum degree five is always a 3–connected MPG-5 graph, a 4–connected MPG-5 graph if there are no separating triangles and a 5–connected MPG-5 graph if there are no separating triangles or separating 4–cycles. Inspired by Theorem 2.23 we present a related condition for maximal planar graphs with $\delta(G) = 5$.

Theorem 2.27 *Let $G = (V, E)$ be a maximal planar graph with $\delta(G) = 5$. Suppose G contains a separating 4–cycle induced by the set of vertices $X = \{x_1, x_2, x_3, x_4\}$. Then in any planar embedding of G, the interior and exterior of X must each contain at least seven vertices of degree at least five.*

Proof. Let $I(X)$ denote the set of vertices inside of the 4-cycle X. Let $d_{I(X)}(x_i)$ denote the number of vertices in $I(X)$ to which x_i is adjacent, for $i = 1, 2, 3, 4$.

Clearly $d_{I(X)}(x_i) > 0$ for each i, since G is maximal planar and $I(X) \neq \emptyset$. We can show that $d_{I(X)}(x_i) \geq 2$ for every i, with $d_{I(X)}(x_i) \geq 3$ for some i. That is
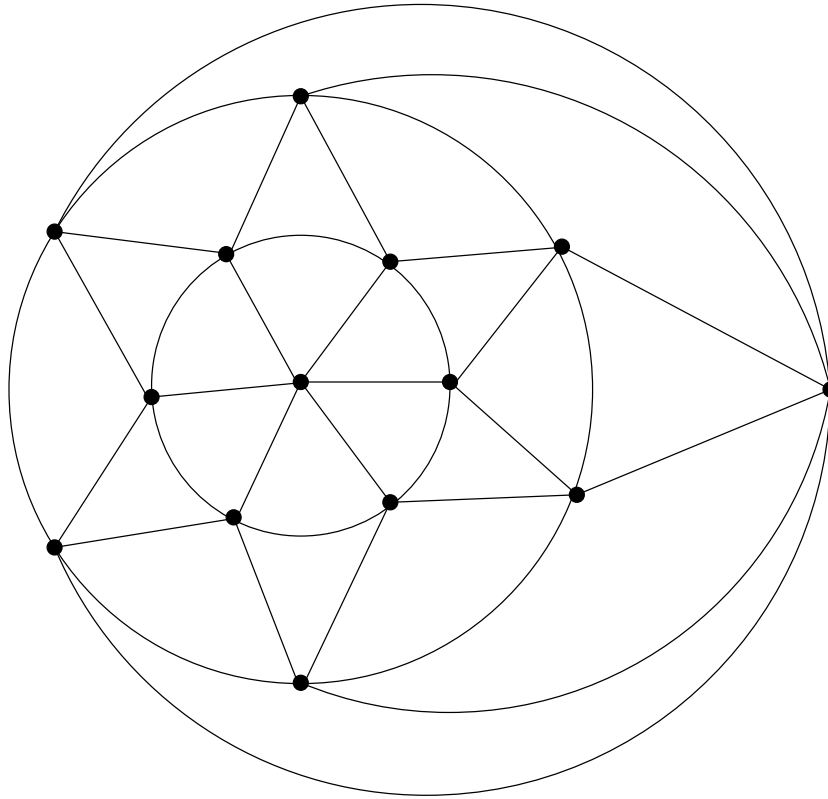
$$\sum_{i=1}^{4} d_{I(X)}(x_i) \geq 9. \tag{2.1}$$

Figure 2.10: A maximal planar graph with $\delta(G) = 5$ and $\Delta(G) = 6$.

Let G' denote the planar graph induced by $X \cup I(X)$. Since one more edge can be added to G' to obtain a maximal planar graph, it follows from Theorem 1.1 of Euler and Equation 2.1 that

$$\sum_{v \in I(X)} d(v) \leq 6|I(X)| - 7.$$

The desired result for the interior follows directly from this last inequality. As X is the vertex set of an arbitrary separating 4–cycle, the result holds also for the exterior of X. ∎

2.3 Connectivity of maximal planar graphs

In this section we turn towards the connectivity of maximal planar graphs. Note that there is only one 2–connected maximal planar graph on three vertices, see Figure 2.11. As from now we consider only maximal planar graphs on $n \geq 4$ vertices. First we will consider the connectivity of maximal planar graphs with minimum degree three.

Theorem 2.28 *Let G be a maximal planar graph with minimum degree three. Then G is 3–connected.*

Proof. Since G is maximal planar the boundary of every face is a 3–cycle. Therefore, G is at least 2–connected. G is at most 3–connected, since the minimum degree is three.
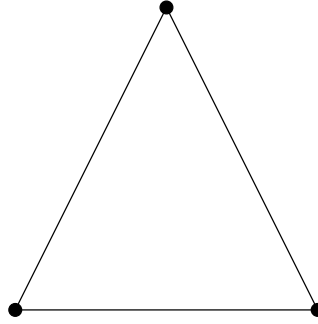
Figure 2.11: A 2–connected maximal planar graph with three vertices.

Assume G is not 3–connected. There exists a separating 2–cut with vertices $u, v \in V(G)$. If $uv \in E(G)$, then $N(u) \cap N(v) \neq \emptyset$. Hence, $G - \{u, v\}$ is connected. If $uv \notin E(G)$, then $N(u) \cap N(v) \neq \emptyset$ or $N(u) \cap N(v) = \emptyset$ but in both cases we get $G - \{u, v\}$ is connected. ∎

We proceed in proving the connectivity of maximal planar graphs with minimum degree four.

Theorem 2.29 *Let G be a maximal planar graph with minimum degree four. Then G is either 4–connected or G has a separating triangle.*

Proof. Since G is a maximal planar graph the boundary of every face is a 3–cycle. Therefore, G is at least 2–connected. G is at most 4–connected, since the minimum degree is four. Assume there exists a separating 2–cut with vertices $u, v \in V(G)$. If $uv \in E(G)$, then $N(u) \cap N(v) \neq \emptyset$. Hence, $G - \{u, v\}$ is connected. If $uv \notin E(G)$, then $N(u) \cap N(v) \neq \emptyset$ or $N(u) \cap N(v) = \emptyset$ but in both cases we get $G - \{u, v\}$ is connected. Therefore, if G has no separating triangle and no separating 4–cycle then G is 5–connected. If G is not 5–connected, then there exists a separating triangle or a separating 4–cycle. ∎

This theorem has an easy corollary.

Corollary 2.30 *Let G be a maximal planar graph with minimum degree four and without a separating triangle. Then G is 4–connected.*

Note that for a maximal planar graph, 3–cuts correspond to separating triangles, while 4–cuts correspond to separating 4–cycles.

Theorem 2.31 *Let G be a maximal planar graph with minimum degree five. Then G is either 5–connected or G has a separating triangle or a separating 4–cycle.*

Proof. Since G is a maximal planar graph the boundary of every face is a 3–cycle. Therefore, G is at least 2–connected. G is at most 5–connected, since the minimum degree is five. Assume there exists a separating 2–cut with vertices $u, v \in V(G)$. If $uv \in E(G)$, then $N(u) \cap N(v) \neq \emptyset$. Hence, $G - \{u, v\}$ is connected. If $uv \notin E(G)$, then $N(u) \cap N(v) \neq \emptyset$ or $N(u) \cap N(v) = \emptyset$ but in both cases we get $G - \{u, v\}$ is connected. Therefore, if G has no separating triangle then G is 4–connected. If G is not 4–connected, then there exists a separating triangle. ∎

This theorem yields the following two corollaries.

Corollary 2.32 *Let G be a maximal planar graph with minimum degree five and without a separating triangle. Then G is 4–connected.*

Corollary 2.33 *Let G be a maximal planar graph with minimum degree five and without a separating triangle and without a separating 4–cycle. Then G is 5–connected.*

Now consider a maximal planar graph with vertex degrees $d_1 \geq d_2 \geq ... \geq d_n$. Remember $\tau_i(G)$ denotes the number of all vertices with degree i. Our purpose is to give sufficient conditions on $d_1 \geq d_2 \geq ... \geq d_n$ for G to be d_n–connected. If $d_n = 3$, G will always be d_n–connected. Hence, we assume that $4 \leq d_n \leq 5$. In 1978 Hakimi and Schmeichel proved the following result.

Theorem 2.34 (Hakimi and Schmeichel [30]) *Let G be a maximal planar graph with vertex degrees $d_1 \geq d_2 \geq ... \geq d_n$, with $d_n \geq 4$. If*

$$\left\lfloor \frac{7}{4}\tau_4(G) \right\rfloor + \tau_5(G) < 14,$$

then G is d_n–connected.

One of the most elementary properties that a graph can have is that of being connected. This is a global property in the sense that it is defined in terms of all pairs of vertices in the graph. Now we will present a result which uses the property of being connected in a localized sense.

Definition 2.35 *A graph G is called locally k–connected with $k \in I\!N$ if for each vertex v, the subgraph induced by $N(v)$ is k–connected.*

We obtain the following lemma.

Lemma 2.36 *Each maximal planar graph $G = (V, E)$ with $|V| \geq 4$ is locally 2–connected.*

Proof. Let G be a maximal planar graph and $v \in V(G)$ a vertex of G. The subgraph induced by $N(v)$ is a cycle with or without chords and thus 2–connected. ∎

Thus we get the next corollary.

Corollary 2.37 *If v is a vertex of a locally 2–connected planar graph, then v belongs to exactly $d(v)$ triangles.*

It is well known that every 6–connected graph is non-planar, while there are 5–connected graphs which are planar. For local connectedness, however, the situation is quite different.

Lemma 2.38 *Every locally 3–connected graph G is non-planar.*

Proof. Let G be locally 3–connected and v be a vertex of G. We consider two cases.

Case 1: $N(v)$ is complete. Since $N(v)$ is 3–connected for each $v \in V(G)$, $N(v)$ contains the complete graph K_4 as a subgraph. In G, the vertex v is adjacent to all vertices of K_4 so that G contains the K_5 as a subgraph, and by Kuratowski's Theorem 1.3, G is non-planar.

Case 2: $N(v)$ is not complete. Therefore, there exist two non-adjacent vertices u and w and three disjoint $P(u, w)$ paths, each of which has length at least two. In G, the vertex v is adjacent to the interior vertices of these three paths. Thus G contains a subgraph homeomorphic with $K_{3,3}$ and by Kuratowski's Theorem 1.3, G is non-planar. ∎

Chapter 3

Hamiltonicity of maximal planar graphs

The hamiltonian problem; determining when a graph G contains a cycle, which contains all vertices of G exactly once, has been fundamental in graph theory. Named after Sir William Rowan Hamilton, this problem has its origins in the 1850s. It was the Four Color Problem which stimulated interest in hamiltonian planar graphs; if a planar cubic graph is hamiltonian then it is easy to color its faces by four colors in such a way that adjacent faces always receive distinct colors.

3.1 Hamiltonian cycle

It is well known that the problem of recognizing hamiltonian graphs is NP-complete. The problem does not get any easier when the input is restricted to 3–connected planar graphs, as proved by M. R. Garey, D. S. Johanson and R. E. Tarjan [29]. In 1980 T. Akiyama, T. Nishizeki and T. Saito [1] modified the argument to show that the problem remains NP-complete even if the input is restricted to cubic bipartite planar graphs. In particular, V. Chvátal and A. Wigderson showed independently that the problem of determining whether a maximal planar graph is hamiltonian is NP-complete, see V. Chvátal [20]. However, H. Whitney proved that any maximal planar graph without separating triangles is hamiltonian. Since Whitney's result plays a decisive role, we state this result first. For Whitney's Theorem, we need the following definition and the next lemma. H. Whitney introduced the following condition.

Definition 3.1 *Let G be a planar triangulation, let R be the exterior face of G, and let a and b be two vertices on R. We say that (G, R, a, b) satisfies* Whitney's Condition *(Condition (W) for short) if (G, R, a, b) satisfies Condition (W1) and (W2) described below.*
*We say that (G, R, a, b) satisfies* Condition (W1) *if G has no separating triangles.*
We say that (G, R, a, b) satisfies Condition (W2) *if either (G, R, a, b) satisfies*
Condition (W2a) *R is divided into two paths, $a_0 a_1 ... a_m$ is the path from a to b and $b_0 b_1 ... b_n$ is the path from b to a ($a_0 = b_n = a$, $b_0 = a_m = b$), and there is no chord of the form $a_i a_j$ or $b_i b_j$, see Figure 3.1, or*

Condition (W2b) *R is divided into three paths, $a_0a_1...a_m$ is the path from a to b, $b_0b_1...b_n$ is the path from b to c and $c_0c_1...c_k$ is the path from c to a ($a_0 = c_k = a$, $b_0 = a_m = b$, $c_0 = b_n = c$) for some vertex c on R distinct from a and b and there is no chord of the form a_ia_j, b_ib_j or c_ic_j, see Figure 3.1.*

In 1931 H. Whitney proved the following.

Lemma 3.2 (Whitney [55]) *Let G be a planar triangulation, let R be the exterior face of G, and let a and b be two vertices on R. If (G, R, a, b) satisfies Condition (W), then G has a hamiltonian path from a to b.*
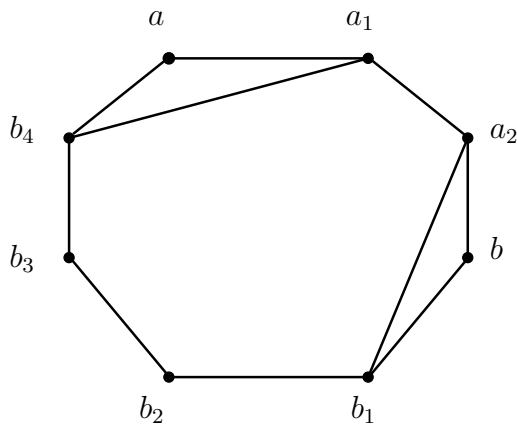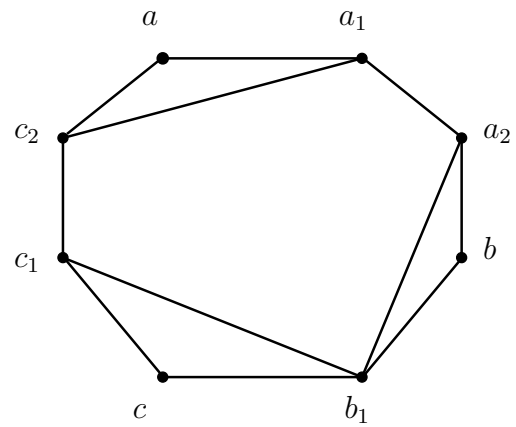
(W2a)   (W2b)

Figure 3.1: An illustration of conditions (W2a) and (W2b).

Theorem 3.3 (Whitney [55]) *Let G be a maximal planar graph without separating triangles. Then G is hamiltonian.*

Tutte [53] generalized Whitney's result.

Theorem 3.4 (Tutte [53]) *Let G be a 4–connected planar graph. Then G is hamiltonian.*

He proved this by showing that every planar graph has a special kind of path. This will be called a Tutte path, and is a generalization of a hamiltonian path. It is convenient to consider only 2–connected graphs.

Definition 3.5 *For a subgraph H of a graph G, a* trivial bridge *and a* non-trivial bridge *of H in G are defined as follows: A* trivial bridge *of H in G is an edge in $E(G) \setminus E(H)$ with both ends in $V(H)$. A* non-trivial bridge *of H in G is a component K of $G - V(H)$ together with all vertices of H adjacent to vertices of K and all edges with one end in H and the other in K. The* vertices of attachment *of a bridge B of H in G are $V(B) \cap V(H)$. A bridge is attached to its vertices of attachment. A path (cycle) P, as a subgraph of a planar graph G, is a* Tutte path (cycle) *if and only if each bridge of P has at most three vertices of attachment and each bridge containing an edge of X_G has at most two vertices of attachment.*

Lemma 3.6 (Tutte [53]) *Let G be a 2–connected planar graph. Let x, y and α be two vertices and an edge, respectively, of X_G. Then G has a Tutte path P from x to y containing α.*

Tutte's Theorem follows by choosing x and y to be adjacent. In 1983, C. Thomassen [51] improved the result of W. T. Tutte by removing the restriction on the location of y.

Lemma 3.7 (Thomassen [51]) *Let G be a 2–connected planar graph. Let x and α be a vertex and an edge, respectively, of X_G and let y be any vertex of G distinct from x. Then G has a Tutte path P from x to y containing α.*

For a proof of Lemma 3.7 see [18]. Note that a Tutte path in a 4–connected graph is also a hamiltonian path. This provides the next corollary.

Corollary 3.8 (Thomassen [51]) *Every 4–connected planar graph has a hamiltonian cycle through any edge of G.*

Moreover, Lemma 3.7 allows a more generalized result. Therefore, we need the next definition.

Definition 3.9 *A graph G is called* hamiltonian–connected *if it contains a hamiltonian path between any two prescribed vertices.*

It is well known that hamiltonian–connectedness implies hamiltonicity. Thus the following corollary that follows from Lemma 3.7 is a generalization of Tutte's result.

Corollary 3.10 (Thomassen [51]) *Let G be a 4–connected planar graph. Then G is hamiltonian–connected.*

D. P. Sanders [45] improved the result of C. Thomassen by removing the restriction on the location of x. He proved the following variation of Tutte's Lemma 3.6.

Theorem 3.11 (Sanders [45]) *Let G be a 2–connected planar graph. Let α be an edge of X_G, and let x and y be arbitrary distinct vertices of G. Then G has a Tutte path P from x to y containing α.*

The previous theorem leads to the next corollary that improves the result of C. Thomassen.

Corollary 3.12 (Sanders [45]) *Let G be a 4–connected planar graph. Let α be an edge of X_G, and let x and y be arbitrary distinct vertices of G. Then G has a hamiltonian path P from x to y containing α.*

Note that Tutte's Theorem shows that a 4–connected planar graph has a hamiltonian cycle and while Thomassen's result shows that a 4–connected planar graph has a hamiltonian cycle through any edge, this corollary shows that a 4–connected planar graph has a hamiltonian cycle through any two edges.

Corollary 3.13 (Sanders [45]) *Every 4–connected planar graph has a hamiltonian cycle through any two of its edges.*

In 2003 C. Chen [17] generalized Whitney's result in a different direction, he proved the following variation of Whitney's lemma.

Lemma 3.14 (Chen [17]) *Let G be a maximal planar graph without separating triangles, let R be the exterior face of G, and let a, b and c be the three vertices on R. Then G has a hamiltonian path from a to b starting from the edge ac.*

From the Lemma above, C. Chen obtained a stronger version of Whitney's Theorem.

Theorem 3.15 (Chen [17]) *Let G be a maximal planar graph with only one separating triangle. Then G is hamiltonian.*

While trying to answer the question of what makes a maximal planar graph with separating triangles non-hamiltonian, C. Chen introduced the definition of hamiltonian for any two boundary edges. From Lemma 3.14 above, C. Chen obtained a stronger version of Whitney's Theorem.

Theorem 3.16 (Chen [17]) *Let G be a maximal planar graph without separating triangles. Then G is hamiltonian for any two boundary edges.*

With the following theorem we extend Theorem 3.16 to exactly one separating triangle.

Theorem 3.17 (Helden [33]) *Let G be a maximal planar graph with only one separating triangle. Then G is hamiltonian for any two boundary edges.*

Proof. Let x, y and z be the vertices of the separating triangle T and let xy be no boundary edge. Let G_{in} (G_{out}, respectively) be the subgraph of G derived by deleting all the vertices outside (inside, respectively) the separating triangle T. G_{out} and G_{in} are both maximal planar graphs without separating triangles. By Theorem 3.16 the graph G_{in} is hamiltonian for any two boundary edges. Consider G_{out}, let a, b and c be the vertices which form the exterior face of G_{out}. Let a be a vertex which is distinct from the vertices x and y. Furthermore, let G'_{out} be the subgraph of G_{out} derived by deleting a. By Theorem 3.11 G'_{out} has a Tutte path P from x to y containing bc. Thus we obtain a Tutte path P in G_{out} from x to y containing ba and ac. Since G_{out} is a maximal planar graph without separating triangles, G_{out} is 4–connected. Note that a Tutte path in a 4–connected graph is also a hamiltonian path. Since G_{in} is hamiltonian for any two boundary edges, we can always find a path $P_{in}(x, y)$. Then $acb...xP_{in}(x, y)y...a$ is a hamiltonian cycle in G passing through two boundary edges. Since a is arbitrary G is hamiltonian for any two boundary edges. ∎

With Theorem 3.16 and Theorem 3.17 we can prove the following result.

Theorem 3.18 (Helden [33]) *Let G be a maximal planar graph with exactly two separating triangles. Then G is hamiltonian.*

Proof. Let a, b and c be the vertices of one separating triangle T. Let G_{in} (G_{out}, respectively) be the subgraph of G derived by deleting all the vertices outside (inside, respectively) the separating triangle T. We distinguish the two cases in which the two separating triangles are either nested or not.

Case 1: The two separating triangles are not nested. The two separating triangles could be disjoint or not. In both cases G_{in} is a graph without separating triangles and G_{out} is a graph with only one separating triangle. By Theorem 3.16 the graph G_{in} is hamiltonian for any two boundary edges. Note that a planar graph can always be embedded in the plane such that a given face of the graph becomes the exterior face, see Lemma 1.10. Therefore, we can embed G_{out} in the plane such that $abca$ becomes the exterior cycle of G_{out}. By Theorem 3.17 G_{out} is also hamiltonian for any two boundary edges. Then $acP_{in}(c,b)bP_{out}(b,a)a$ is a hamiltonian cycle in G. Since G_{in} and G_{out} are hamiltonian for any two boundary edges, we can always find one common edge of both paths P_{in} and P_{out}.

Case 2: The two separating triangles are nested. The two separating triangles could be disjoint or not. In both cases we have to distinguish the following two cases.

Subcase 2.1: G_{in} is a graph without separating triangles and G_{out} is a graph with only one separating triangle.

Subcase 2.2: G_{in} is a graph with only one separating triangle and G_{out} is a graph without separating triangles.

In both Subcases we obtain that G_{in} is hamiltonian for any two boundary edges and G_{out} is hamiltonian for any two boundary edges. Then $acP_{in}(c,b)bP_{out}(b,a)a$ is a hamiltonian cycle in G. ∎

Now we describe how to decompose a maximal planar graph G with at least one separating triangle into at least two maximal planar graphs with fewer separating triangles than G.

Definition 3.19 *Let G be a maximal planar graph and let $D_1, D_2, ..., D_k$ be k separating triangles. Let G_{in,D_1} ($G_{in,D_2}, ..., G_{in,D_k}$, respectively) be the subgraph of G derived by deleting all the vertices outside the separating triangle D_1 ($D_2, ..., D_k$, respectively). Let $G_{out,D_1,D_2,...,D_k}$ be the subgraph of G derived by deleting all the vertices inside the separating triangles $D_1, D_2, ..., D_k$.*

Note that the graphs G_{in,D_1} ($G_{in,D_2}, ..., G_{in,D_k}$, respectively) and $G_{out,D_1,D_2,...,D_k}$ are maximal planar graphs with fewer separating triangles than G. The following lemma shows how to obtain a hamiltonian cycle in G from two known hamiltonian cycles in these subgraphs.

Lemma 3.20 (Helden and Vieten [34]) *Let G be a maximal planar graph with at least one separating triangle D. If $G_{in,D}$ is hamiltonian for any two boundary edges of $G_{in,D}$ and $G_{out,D}$ is hamiltonian for any two boundary edges of $G_{out,D}$, then G is hamiltonian.*

Proof. Let a, b and c be the vertices of the separating triangle D. Consider $G_{in,D}$. a, b and c are the vertices of the exterior cycle of $G_{in,D}$. Without loss of generality, $G_{in,D}$ has a hamiltonian cycle $C_{in} = abcP_{in}(c, a)a$ where $P_{in}(c, a)$ is a hamiltonian path of $G_{in,D} - b$ between c and a. Next consider $G_{out,D}$. a, b and c are the vertices of a facial cycle of $G_{out,D}$. Note that a planar graph can always be embedded in a surface such that a given face of the graph becomes the exterior face, see Chiba and Nishizeki [19]. Without loss of generality, $G_{out,D}$ has a hamiltonian cycle $C_{out} = bcaP_{out}(a, b)b$ where $P_{out}(a, b)$ is a hamiltonian path of $G_{out,D} - c$ between a and b. Then $C = aP_{out}(a, b)bcP_{in}(c, a)a$ is a hamiltonian cycle of G. ∎

In 2002 B. Jackson and X. Yu improved Corollary 3.13 of D.P. Sanders in a different direction. They showed the following.

Theorem 3.21 (Jackson and Yu [37]) *Let G be a planar triangulation without separating triangles. Let T, T_1 and T_2 be distinct triangles in G. Let $V(T) = \{u, v, w\}$. Then there exists a hamiltonian cycle C of G and edges $e_1 \in E(T_1)$ and $e_2 \in E(T_2)$ such that uv, uw, e_1, e_2 are distinct and contained in $E(C)$.*

This implies the following two easy corollaries.

Corollary 3.22 (Helden and Vieten [34]) *Let G be a maximal planar graph without separating triangles. Let T_1 and T_2 be distinct triangles in G. Let $V(X_G) = \{u, v, w\}$. Then there exists a hamiltonian cycle C of G and edges $e_1 \in E(T_1)$ and $e_2 \in E(T_2)$ such that uv, uw, e_1, e_2 are distinct and contained in $E(C)$.*

Corollary 3.23 (Helden and Vieten [34]) *Let G be a maximal planar graph without separating triangles. Let T_1 and T_2 be distinct, arbitrary and facial triangles in G. Then there exist edges $e_1 \in E(T_1)$ and $e_2 \in E(T_2)$ such that G is hamiltonian for any two boundary edges of G and the hamiltonian cycle contains the edges e_1 and e_2.*

With Corollary 3.22 we obtain a stronger version of Theorem 3.16 of C. Chen.

Corollary 3.24 (Helden and Vieten [34]) *Let G be a maximal planar graph without separating triangles. Let T be an arbitrary facial triangle in G. Then there exists an edge $e \in E(T)$ such that G is hamiltonian for any two boundary edges of G and the hamiltonian cycle contains the edge e.*

Let G be a maximal planar graph. Suppose that T is a separating triangle in G. Then we may separate G into two graphs $G_{in,T}$ and $G_{out,T}$. Then $G_{in,T}$ and $G_{out,T}$ are maximal planar graphs and T is a facial cycle of $G_{in,T}$ and $G_{out,T}$. We shall refer to T as a marker triangle in $G_{in,T}$ and $G_{out,T}$. We now iterate this procedure for both $G_{in,T}$ and $G_{out,T}$. We continue until we obtain a collection S of maximal planar graphs each of which has no separating triangles. We shall refer to S as pieces of G. Note that each separating triangle will occur as a marker triangle in exactly two pieces of G. We define a new graph B whose vertices are the pieces in S, and in which two pieces are joined by an edge if they have a marker triangle in common. It follows from the decomposition theory

developed by W. H. Cunningham and J. Edmonds [22] that B is a tree and also that the set S and the tree B are uniquely defined by G. We shall refer to B as the decomposition tree of G. For a given embedding of G with k separating triangles $D_1, D_2, ..., D_k$ we define the piece $G_{out,D_1,D_2,...,D_k}$ as the root of the decomposition tree B. Thus we get a rooted decomposition tree B. For the proof of Theorem 3.27, we need the following two lemmata.

Lemma 3.25 (Helden and Vieten [34]) *Let G be a maximal planar graph with at least one separating triangle D_1 and let B be the rooted decomposition tree of the given embedding of G. Let each vertex of the tree B have at most two children and the root $G_{out,D_1,D_2,...,D_k}$ has exactly one child G_{in,D_1}. If G_{in,D_1} is hamiltonian for any two boundary edges of G_{in,D_1}, then G is hamiltonian for any two boundary edges of G for the given embedding of G.*

Proof. Let a, b and c be the vertices of the separating triangle D_1. Consider G_{in,D_1}. Let a, b and c be the vertices of the exterior cycle of G_{in,D_1}. Without loss of generality, G_{in,D_1} has a hamiltonian cycle $C_{in} = abcP_{in}(c,a)a$ where $P_{in}(c,a)$ is a hamiltonian path of $G_{in,D_1} - b$ between c and a.

Next consider G_{out,D_1}. G_{out,D_1} is a maximal planar graph without separating triangles. a, b and c are the vertices of the facial cycle G_{out,D_1}. By Corollary 3.24 G_{out,D_1} is hamiltonian for any two boundary edges of G_{out,D_1} and contains without loss of generality the edge ac. We replace the edge ac by the path $P_{in}(c,a)$ in the graph G. Then G is hamiltonian for any two boundary edges of G for the given embedding of G. ∎

Lemma 3.26 (Helden and Vieten [34]) *Let G be a maximal planar graph with at least two separating triangles D_1 and D_2 and let B be the rooted decomposition tree of the given embedding of G. Let each vertex of the tree B have at most two children and the root $G_{out,D_1,D_2,...,D_k}$ has exactly two children G_{in,D_1} and G_{in,D_2}. If G_{in,D_1} is hamiltonian for any two boundary edges of G_{in,D_1} and G_{in,D_2} is hamiltonian for any two boundary edges of G_{in,D_2}, then G is hamiltonian for any two boundary edges of G for the given embedding of G.*

Proof. Let a_1, b_1 and c_1 be the vertices of the separating triangle D_1. Consider G_{in,D_1}. a_1, b_1 and c_1 are the vertices of the exterior cycle of G_{in,D_1}. Without loss of generality, G_{in,D_1} has a hamiltonian cycle $C_{in,D_1} = a_1b_1c_1P_{in,D_1}(c_1,a_1)a_1$ where $P_{in,D_1}(c_1,a_1)$ is a hamiltonian path of $G_{in,D_1} - b_1$ between c_1 and a_1.

Let a_2, b_2 and c_2 be the vertices of the separating triangle D_2. Consider G_{in,D_2}. a_2, b_2 and c_2 are the vertices of the exterior cycle of G_{in,D_2}. Without loss of generality, G_{in,D_2} has a hamiltonian cycle $C_{in,D_2} = a_2b_2c_2P_{in,D_2}(c_2,a_2)a_2$ where $P_{in,D_2}(c_2,a_2)$ is a hamiltonian path of $G_{in,D_2} - b_2$ between c_2 and a_2.

Next consider G_{out,D_1,D_2}. G_{out,D_1,D_2} is a maximal planar graph without separating triangles. a_1, b_1, c_1 and a_2, b_2, c_2 are the vertices of distinct facial cycles of G_{out,D_1,D_2}. By Corollary 3.23 G_{out,D_1,D_2} is hamiltonian for any two boundary edges of G_{out,D_1,D_2} and contains without loss of generality the edges a_1c_1 and a_2c_2. We replace the edge a_1c_1 by the path $P_{in,D_1}(c_1,a_1)$ and the edge a_2c_2 by the path $P_{in,D_2}(c_2,a_2)$ in the graph G. Then G is hamiltonian for any two boundary edges of G for the given embedding of G. ∎

From Lemma 3.25 and Lemma 3.26 above we obtain a stronger version of Theorem 3.16 and Theorem 3.17.

Theorem 3.27 (Helden and Vieten [34]) *If G is a maximal planar graph with exactly two separating triangles, then G is hamiltonian for any two boundary edges of G.*

Proof. Let D_1 and D_2 be the separating triangles of G. Let B be the rooted decomposition tree of an arbitrary embedding of G. Each vertex of the tree B has at most two children and the root G_{out,D_1,D_2} has at least one child. We distinguish the following two cases.

Case 1: The root G_{out,D_1,D_2} has exactly one child G_{in,D_1}. Then the piece G_{in,D_2} is a child of the piece G_{in,D_1}. The graph G_{in,D_1} has one separating triangle D_2. By Theorem 3.17 G_{in,D_1} is hamiltonian for any two boundary edges of G_{in,D_1}. By Lemma 3.25 G is hamiltonian for any two boundary edges of G.

Case 2: The root G_{out,D_1,D_2} has exactly two children G_{in,D_1} and G_{in,D_2}. The graph G_{in,D_1} has no separating triangles and the graph G_{in,D_2} has no separating triangles. Then by Theorem 3.16 G_{in,D_1} is hamiltonian for any two boundary edges of G_{in,D_1} and G_{in,D_2} is hamiltonian for any two boundary edges of G_{in,D_2}. By Lemma 3.26 G is hamiltonian for any two boundary edges of G. ■

The hypothesis of Theorem 3.27 that G has at most two separating triangles cannot be weakened to at most three separating triangles. To show this we consider the graph G obtained from K_4 by inserting a new vertex into each interior face of K_4 then joining the new vertex to every vertex incident with the face. Then G is not immediately hamiltonian for any two boundary edges of G.

The following theorem presents one of our main results, because we do not claim an integer which is an upper bound on the number of separating triangles, but we assume a special structure of the position of the separating triangles to each other.

Theorem 3.28 (Helden and Vieten [34]) *Let G be a maximal planar graph and let B be the rooted decomposition tree of the given embedding of G. If each vertex of the tree B has at most two children, then G is hamiltonian for any two boundary edges of G for the given embedding of G.*

Proof. The proof is by induction on the number s of separating triangles. Clearly, the theorem is true for $s = 0$, $s = 1$ and $s = 2$. The claim follows from Theorem 3.16, Theorem 3.17 and Theorem 3.27. We proceed with the induction step. Assume that G has $s+1$ separating triangles. The root $G_{out,D_1,D_2,...,D_{s+1}}$ has at least one child. We distinguish the following two cases.

Case 1: The root $G_{out,D_1,D_2,...,D_{s+1}}$ has exactly one child G_{in,D_1}. The graph G_{in,D_1} has s separating triangles. Since the rooted decomposition tree T of G_{in,D_1} is a subtree of the tree B, each vertex of the tree T has at most two children. Then by induction G_{in,D_1} is

hamiltonian for any two boundary edges of G_{in,D_1}. By Lemma 3.25 G is hamiltonian for any two boundary edges of G for the given embedding of G.

Case 2: The root $G_{out,D_1,D_2,...,D_{s+1}}$ has exactly two children G_{in,D_1} and G_{in,D_2}. The graph G_{in,D_1} has t separating triangles and the graph G_{in,D_2} has r separating triangles. Since $r + t = s - 1$ it is necessary that $t \leq s$ and $r \leq s$. Since the rooted decomposition tree T_1 of G_{in,D_1} and the rooted decomposition tree T_2 of G_{in,D_2} are subtrees of the tree B, each vertex of this subtrees has at most two children. Then by induction G_{in,D_1} is hamiltonian for any two boundary edges of G_{in,D_1} and G_{in,D_2} is hamiltonian for any two boundary edges of G_{in,D_2}. By Lemma 3.26 G is hamiltonian for any two boundary edges of G for the given embedding of G. ∎

With Theorem 3.27 we can extend Theorem 3.18 to exactly three separating triangles.

Theorem 3.29 (Helden and Vieten [34]) *If G is a maximal planar graph with exactly three separating triangles, then G is hamiltonian.*

Proof. Let D_1, D_2 and D_3 be the separating triangles of G. Let B be the rooted decomposition tree of an arbitrary embedding of G. Each vertex of the tree B has at most three children and the root G_{out,D_1,D_2,D_3} has at least one child. We distinguish the following three cases.

Case 1: The root G_{out,D_1,D_2,D_3} has exactly one child G_{in,D_1}. Then the graph G_{in,D_1} has two separating triangles D_2 and D_3. By Theorem 3.27 G_{in,D_1} is hamiltonian for any two boundary edges of G_{in,D_1}. The graph G_{out,D_1} has no separating triangles. Then by Theorem 3.16 G_{out,D_1} is hamiltonian for any two boundary edges of G_{out,D_1}. By Lemma 3.20 G is hamiltonian.

Case 2: The root G_{out,D_1,D_2,D_3} has exactly two children G_{in,D_1} and G_{in,D_2}. Without loss of generality, the graph G_{in,D_1} has one separating triangle D_3. Then the graph G_{out,D_1} has also one separating triangle D_2. Then by Theorem 3.17 G_{in,D_1} is hamiltonian for any two boundary edges of G_{in,D_1} and G_{out,D_1} is hamiltonian for any two boundary edges of G_{in,D_2}. By Lemma 3.20 G is hamiltonian.

Case 3: The root G_{out,D_1,D_2,D_3} has exactly three children G_{in,D_1}, G_{in,D_2} and G_{in,D_3}. The graph G_{in,D_1} has no separating triangles and the graph G_{out,D_1} has two separating triangles D_2 and D_3. Then by Theorem 3.16 G_{in,D_1} is hamiltonian for any two boundary edges of G_{in,D_1} and by Theorem 3.27 G_{out,D_1} is hamiltonian for any two boundary edges of G_{out,D_1}. By Lemma 3.20 G is hamiltonian. ∎

Analogously to Theorem 3.29 we can prove the unbounded version with Theorem 3.28. All at once this is also a generalization of Theorem 3.28.

Theorem 3.30 (Helden and Vieten [34]) *Let G be a maximal planar graph and let B be the rooted decomposition tree of the given embedding of G. If the root of the tree B has*

exactly three children and have all other vertices of the tree B at most two children, then G is hamiltonian.

Proof. Let B be the rooted decomposition tree of the given embedding of G. Let G_{in,D_1}, G_{in,D_2} and G_{in,D_3} be the three children of the root of the tree B. Since the rooted decomposition tree T_1 of G_{in,D_1} and the rooted decomposition tree T_2 of G_{out,D_1} are subtrees of the tree B, each vertex of these subtrees has at most two children. By Theorem 3.28 the graph G_{in,D_1} is hamiltonian for any two boundary edges of G_{in,D_1} and the graph G_{out,D_1} is hamiltonian for any two boundary edges of G_{out,D_1}. By Lemma 3.20 the graph G is hamiltonian. ∎

The hypothesis of Theorem 3.28 that each vertex of the rooted decomposition tree B has at most two children cannot be weakened to at most three children. To show this we consider the graph G obtained from K_4 by inserting a new vertex into each interior face of K_4 then joining the new vertex to every vertex incident with the face. Then the rooted decomposition tree of G has exactly three children and G is not immediately hamiltonian for any two boundary edges of G. In the following we extend Theorem 3.27 to four separating triangles. The first to take the position of a hamiltonian cycle into account were T. Asano, T. Nishizeki and T. Watanabe [4]. They introduced the following definition.

Definition 3.31 *Let G be a hamiltonian maximal planar graph with a hamiltonian cycle C. We say a (triangular) face of G to be* frontier *(with respect to the hamiltonian cycle C) if it has one or two edges in common with C, or else* non-frontier. *A frontier face is called a* 1-frontier *face if the face has exactly one edge in common with C, or else a* 2-frontier *face. We define $k_N(C)$ as the number of non-frontier faces and $k_{1F}(C)$ and $k_{2F}(C)$ as the number of 1-frontier and 2-frontier faces, respectively.*

Lemma 3.32 *Let G be a hamiltonian maximal planar graph on n vertices. Then*

$$n = \frac{k_{1F} + 2k_{2F}}{2} \quad \text{and} \quad k_N = k_{2F} - 4.$$

Proof. Let $G = (V, E)$ with $|V| = n$ and $|E| = m$ be a hamiltonian maximal planar graph with a hamiltonian cycle C and let $l(G)$ be the number of faces of G. Then Theorem 1.1 of Euler yields $l(G) = m - n + 2$. On the other side, counting the frontier and non-frontier faces yields $l(G) = k_{1F} + k_{2F} + k_N$. Moreover, $3l(G) = 2m$, since each edge is located on the boundary of two faces. Then with Theorem 1.1 of Euler we obtain that $m = 3n - 6$. The equation of these formulas provides $k_{1F} + k_{2F} + k_N = 2n - 4$. Together with $n = \frac{k_{1F} + 2k_{2F}}{2}$ we obtain that $k_{1F} + k_{2F} + k_N = k_{1F} + 2k_{2F} - 4$ and this yields $k_N = k_{2F} - 4$. ∎

The previous lemma provides the following immediate corollary.

Corollary 3.33 *If G is a hamiltonian maximal planar graph, then*

$$k_{2F} \geq 4.$$

For the proof of Theorem 3.36 we need the next lemma and the following definition.

Lemma 3.34 (Helden [35]) *Let $G = (V, E)$ be a maximal planar graph without separating triangles and $|V| \geq 7$. Then G has at least two vertices x and y with $d_G(x) \geq 5$ and $d_G(y) \geq 5$ and $|N(x) \cap N(y)| \geq 2$.*

Proof. If G is a maximal planar graph with $\delta(G) = 5$, then it is obvious. Therefore, let G be a maximal planar graph with $\delta(G) = 4$. We distinguish the following three cases.

Case 1: At least one component of the subgraph $G[D4]$ is a trivial graph. By Corollary 2.22 the graph G has at least two vertices x and y with $d_G(x) \geq 5$ and $d_G(y) \geq 5$ and $|N(x) \cap N(y)| \geq 2$.

Case 2: At least one component of the subgraph $G[D4]$ is a cycle C_p with $p \geq 5$. By Lemma 2.21 G has exactly two vertices x and y with $d_G(x) \geq 5$ and $d_G(y) \geq 5$ and $|N(x) \cap N(y)| \geq 2$.

Case 3: At least one component of the subgraph $G[D4]$ is a path of length q with $q \in I\!N$. By Lemma 2.18 G has at least two vertices x and y with $d_G(x) \geq 5$ and $d_G(y) \geq 5$ and $|N(x) \cap N(y)| \geq 2$.   ∎

Definition 3.35 *Let G be a maximal planar graph without separating triangles and with $n \geq 7$ vertices. Let x and y be two vertices with $d_G(x) \geq 5$ and $d_G(y) \geq 5$ and $|N(x) \cap N(y)| \geq 2$. Let $a, b \in N(x) \cap N(y)$ with ab is an edge of G. The operation T is defined as follows: T contracts the two vertices a and b to one vertex z. The edges xa (ya) and xb (yb) are contracted to one edge xz $(yz,$ respectively$)$. The new graph G' is a maximal planar graph without separating triangles and with $n - 1$ vertices, see Figure 3.2.*
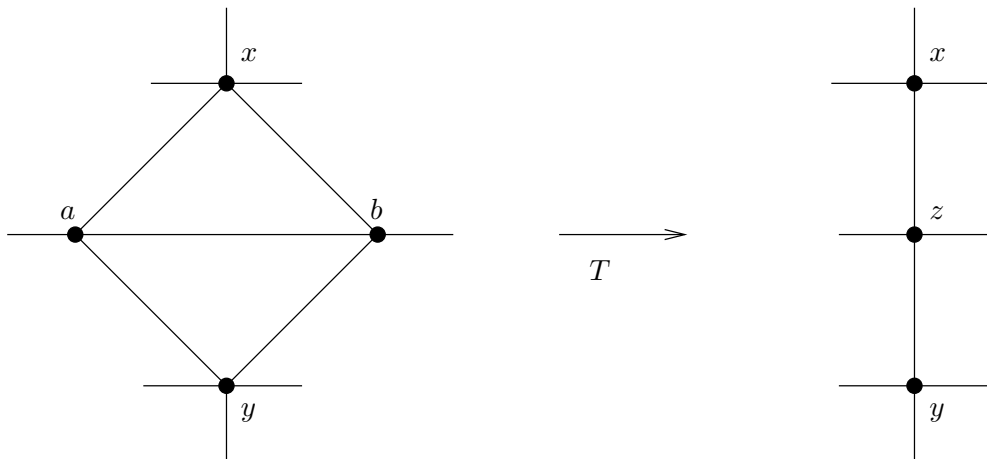

Figure 3.2: Operation T.

In general a planar graph G has many embeddings on a surface. We shall now define an equivalence relation among these embeddings. Two embeddings of a planar graph

are equivalent if the boundary of a face in one embedding always corresponds to the boundary of a face in the other. We say that the planar embedding of a graph is unique if the embeddings are all equivalent. H. Whitney [56] proved that the embedding of a 3–connected planar graph is unique.

Theorem 3.36 (Helden [35]) *If G is a maximal planar graph without separating triangles and with $n \geq 7$ vertices, then a hamiltonian cycle C of G exists so that $k_{2F} = 4$ and at least two 2-frontier faces have one edge in common.*

Proof. The proof is by induction on the number of vertices of G. Assume first $n = 7$ in order to establish the inductive base. See Figure 3.3, the thick lines represent the hamiltonian cycle.
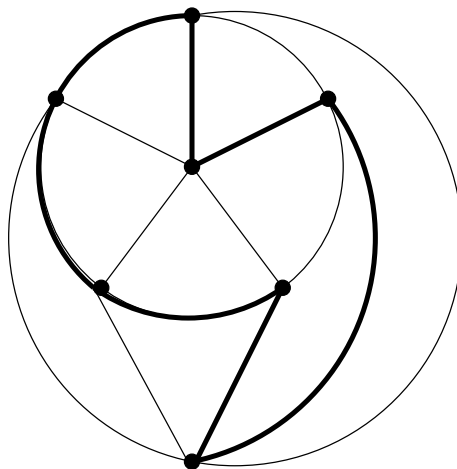

Figure 3.3: Inductive base of the proof of Theorem 3.36.

Thus we have shown that the theorem is true if $n = 7$. For the inductive step, we assume next that $n \geq 8$ and the theorem is true for any maximal planar graph with less than n vertices. Suppose that G is a maximal planar graph without separating triangles and with n vertices. Lemma 3.34 yields that G has two vertices x and y with $d_G(x) \geq 5$ and $d_G(y) \geq 5$ and $|N(x) \cap N(y)| \geq 2$. Let $a, b \in N(x) \cap N(y)$ with ab is an edge of G. Now we can apply the operation T. T contracts the two vertices a and b to one vertex z. The edges xa (ya) and xb (yb) are contracted to one edge xz (yz, respectively). The new graph G' is a maximal planar graph without separating triangles and with $n - 1$ vertices. Thus, we may apply induction to G'. Without loss of generality, there exists a hamiltonian cycle C' of G' such that $k_{2F}(C') = 4$ and at least two 2-frontier faces have one edge in common. Now we have to prove that there exists a hamiltonian cycle C of G such that $k_{2F}(C) = 4$ and at least two 2-frontier faces have one edge in common. Let u be the vertex of G which is adjacent to the vertices x and a and let v be the vertex of G which is adjacent to the vertices x and b. The vertices u, x and z are the vertices of a facial cycle F_1 in G' and the vertices v, z and x are the vertices of a facial cycle F_2 in G'. Both facial cycles have the edge xz in common. We know G' has a hamiltonian cycle C' such that at least two 2-frontier faces have one edge in common. Note that a planar graph can always be embedded on a surface such that a given face of the graph becomes the exterior face,
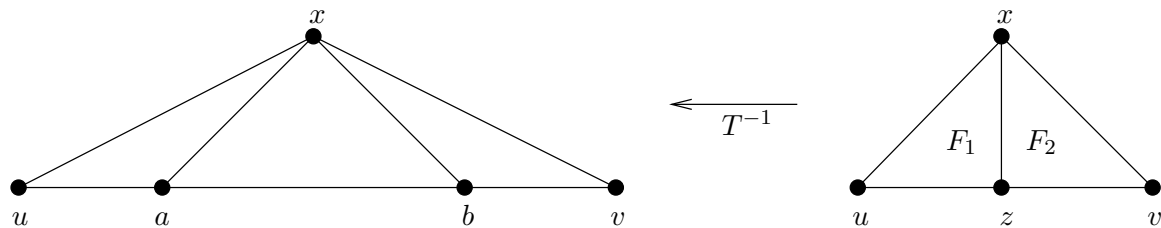
Figure 3.4: Operation T^{-1}.

see Chiba and Nishizeki [19]. Since G' is at least $3-$connected the embedding of G' is unique. Without loss of generality, we can embed the graph G' on a surface such that the two 2-frontier faces with one edge in common have the edge xz in common. Note, if we determine one face, we can always find at most three hamiltonian cycles, such that each edge of the given face can become the common edge of the 2-frontier faces with one edge in common for the corresponding hamiltonian cycle. This means F_1 and F_2 are the two 2-frontier faces in this embedding of G'. We distinguish the following two cases.

Case 1: $uxzv$ is a path of the hamiltonian cycle C'. Now we can apply the operation T^{-1}, see Figure 3.4 T^{-1} extracts the vertex z to the two vertices a and b. The edge xz is extracted to the two edges xa and xb. Then $uxabv$ is a path of the hamiltonian cycle C of G. The vertices u, x and a are the vertices of a facial cycle K_1 in G and the vertices x, a and b are the vertices of a facial cycle K_2 in G. Both facial cycles have the edge xa in common. Therefore, K_1 and K_2 are two 2-frontier faces with one edge in common in an embedding of G. The two new facial cycles with vertices x, b, v and a, b, y are both 1-frontier faces with respect to the hamiltonian cycle C.

Case 2: $uzxv$ is a path of the hamiltonian cycle C'. Now we can also apply the operation T^{-1}. T^{-1} extracts the vertex z to the two vertices a and b. The edge xz is extracted to the two edges xa and xb. Then $uabxv$ is a path of the hamiltonian cycle C of G. The vertices a, b and x are the vertices of a facial cycle K_1 in G and the vertices b, x and v are the vertices of a facial cycle K_2 in G. Both facial cycles have the edge bx in common. Therefore, K_1 and K_2 are two 2-frontier faces with one edge in common in an embedding of G. The two new facial cycles with vertices u, a, x and a, b, y are both 1-frontier faces with respect to the hamiltonian cycle C. ∎

Let G be a hamiltonian maximal planar graph without separating triangles. By Theorem 3.36 there exists a hamiltonian cycle C of G such that $k_{2F}(C) = 4$ and at least two 2-frontier faces have one edge in common. Let G_1 be the graph obtained from G by inserting a new vertex into a facial cycle F of G then joining the new vertex to every vertex a, b and c incident with the face F. G_1 is a hamiltonian maximal planar graph with one separating triangle. Now we can extend the hamiltonian cycle C of G to a hamiltonian cycle C_1 of G_1. We replace an edge of the facial cycle F which belongs to the hamiltonian cycle C by two edges which join the new vertex to the hamiltonian cycle C.

If F is a 2-frontier face with respect to the hamiltonian cycle C, which has no edge in common with another 2-frontier face, then $k_{2F}(C_1) = 5$. If F is a 2-frontier face with respect to the hamiltonian cycle C, which has an edge in common with another 2-frontier face, then $k_{2F}(C_1) = 4$. If F is a 1-frontier face with respect to the hamiltonian cycle C, then $k_{2F}(C_1) = 5$. Thus we get, $k_{2F}(C_1)$ is increased by at most one if we add a separating triangle with one vertex. By Lemma 3.32 $k_N(C_1)$ is also increased by at most one. Since G is at least $3-$connected the embedding of G is unique. Therefore, we can embed the graph G on a surface such that F is a 2-frontier face, which has no edge in common with another 2-frontier face. Then $k_{2F}(C_1) = k_{2F}(C) + 1$ and $k_N(C_1) = 1$. By the same argument, we can embed the graph G on a surface such that F is a 2-frontier face, which has an edge in common with another 2-frontier face. Then $k_{2F}(C_1) = 4 = k_{2F}(C)$ and $k_N(C_1) = 0$.

Let G_2 be the graph obtained from G by inserting two new vertices each into two distinct facial cycles F_1 and F_2 of G then joining the new vertex to every vertex incident with the faces F_1 and F_2 respectively. G_2 is a hamiltonian maximal planar graph with two separating triangles. Note that there exists a hamiltonian cycle C of G such that $k_{2F} = 4$ and at least two 2-frontier faces have one edge in common. We can extend the hamiltonian cycle C of G to a hamiltonian cycle C_2 of G_2. Since G is at least $3-$connected the embedding of G is unique. Therefore, we can embed the graph G on a surface such that F_1 and F_2 are 2-frontier faces and at least one of these has an common edge with another 2-frontier face. Then $4 \leq k_{2F}(C_2) \leq k_{2F}(C) + 1 = 5$ and $k_N(C_2) \leq 1$.

Let G_i be the graph obtained from G by inserting i new vertices each into i distinct facial cycles $F_1, F_2, ..., F_i$ of G then joining the new vertex to every vertex incident with the faces F_i respectively. G_i is a hamiltonian maximal planar graph with i separating triangles and a hamiltonian cycle C_i. We can embed the graph G on a surface such that F_s and F_t with $1 \leq s, t \leq i$ are 2-frontier faces and at least one of these has an common edge with another 2-frontier face. Then $k_{2F}(C_i) \leq 4 + i - 1 = i + 3$ and $k_N(C_i) \leq i - 1$.

We can now prove a stronger version of Theorem 3.29.

Theorem 3.37 (Helden [35]) *If G is a maximal planar graph with exactly four separating triangles, then G is hamiltonian.*

Proof. Let G be a maximal planar graph with exactly four separating triangles. D_1, D_2, D_3 and D_4 are the 4 separating triangles. Let G_{in,D_1} be the subgraph of G derived by deleting all the vertices outside the separating triangle D_1. Let G_{out,D_1} be the subgraph of G derived by deleting all the vertices inside the separating triangle D_1.

We distinguish the following four cases.

Case 1: The graph G_{in,D_1} has two separating triangles and the graph G_{out,D_1} has one separating triangle. By Theorem 3.27 G_{in,D_1} is hamiltonian for any two boundary edges of G_{in,D_1}. By Theorem 3.17 G_{out,D_1} is hamiltonian for any two boundary edges of G_{out,D_1}. By Lemma 3.20 G is hamiltonian.

Case 2: The graph G_{in,D_1} has one separating triangle and the graph G_{out,D_1} has two separating triangles. By Theorem 3.17 G_{in,D_1} is hamiltonian for any two boundary edges of G_{in,D_1}. By Theorem 3.27 G_{out,D_1} is hamiltonian for any two boundary edges of G_{out,D_1}. By Lemma 3.20 G is hamiltonian.

Case 3: The graph G_{in,D_1} has no separating triangle and the graph G_{out,D_1} has three separating triangles. By Theorem 3.16 G_{in,D_1} is hamiltonian for any two boundary edges of G_{in,D_1}. By Theorem 3.29 G_{out,D_1} is hamiltonian. Let the vertices a, b and c be the vertices of the facial cycle D_1 in G_{out,D_1}. The graph G_{out,D_1,D_2,D_3,D_4} is a subgraph of G_{out,D_1} and has no separating triangles. If G_{out,D_1,D_2,D_3,D_4} has $n < 7$ vertices, then there exists a hamiltonian cycle C of G_{out,D_1,D_2,D_3,D_4} such that the hamiltonian cycle passes through at least one edge of the edges ab, bc and ca. If G_{out,D_1,D_2,D_3,D_4} has $n \geq 7$ vertices, then by Theorem 3.36 there exists a hamiltonian cycle C of G_{out,D_1,D_2,D_3,D_4} such that $k_{2F} = 4$ and at least two 2-frontier faces have one edge in common. Now we have to prove that the hamiltonian cycle C passes through at least one edge of the edges ab, bc and ca. We insert 3 new vertices each into the facial cycles D_2, D_3 and D_4 of G then joining the new vertex to every vertex incident with the faces of D_2, D_3 and D_4 respectively. The new graph H is a hamiltonian maximal planar graph with three separating triangles. Now we can extend the hamiltonian cycle C of G_{out,D_1,D_2,D_3,D_4} to a hamiltonian cycle C' of H. Thus we get $k_{2F}(C') = 6$ and $k_N(C') = 2$ with respect to the hamiltonian cycle C'. Then we can embed the graph H on a surface such that D_1 is not a non-frontier face. Therefore, the hamiltonian cycle H passes through at least one edge of the edges ab, bc and ca. At last we have to replace the vertices in the facial cycles D_2, D_3 and D_4 by the graphs $G_{in,D_2} - D_2, G_{in,D_3} - D_3$ and $G_{in,D_4} - D_4$ respectively. Since G_{in,D_i} with $2 \leq i \leq 4$ are maximal planar graphs without separating triangles, the graphs G_{in,D_i} with $2 \leq i \leq 4$ are hamiltonian for any two boundary edges of G_{in,D_i}. Therefore, the graphs $G_{in,D_i} - D_i$ with $2 \leq i \leq 4$ have a hamiltonian path in $G_{in,D_i} - D_i$. Now we replace the two edges which join the new vertices to the hamiltonian cycle H by the hamiltonian paths in $G_{in,D_i} - D_i$ with $2 \leq i \leq 4$. Thus we get a hamiltonian cycle H' which passes through at least one edge of the edges ab, bc and ca. H' is a hamiltonian cycle of G_{out,D_1}. Without loss of generality, H' passes through the edge ca. Consider G_{in,D_1}. a, b and c are the vertices of the exterior cycle of G_{in,D_1}. Without loss of generality, G_{in,D_1} has a hamiltonian cycle $C_{in} = abcP_{in}(c, a)a$, where $P_{in}(c, a)$ is a hamiltonian path of $G_{in,D} - b$ between c and a. Then we replace the edge ca in the hamiltonian cycle H' of G_{out,D_1} by the hamiltonian path $P_{in}(c, a)$ between c and a of $G_{in,D} - b$. Then the new cycle is a hamiltonian cycle of G.

Case 4: The graph G_{in,D_1} has three separating triangles and the graph G_{out,D_1} has no separating triangle. By Theorem 3.29 G_{in,D_1} is hamiltonian. By Theorem 3.16 G_{out,D_1} is hamiltonian for any two boundary edges of G_{out,D_1}. Note that a planar graph can always be embedded in a surface such that a given face of the graph becomes the exterior face, see Chiba and Nishizeki [19]. Without loss of generality, G_{out,D_1} has a hamiltonian cycle passing through any two edges of the facial cycle D_1. Then analogically to Case 3 G is hamiltonian. ∎

With Theorem 3.38 we can extend Theorem 3.37 to exactly five separating triangles.

Theorem 3.38 (Helden [35]) *If G is a maximal planar graph with exactly five separating triangles, then G is hamiltonian.*

Proof. Let G be a maximal planar graph with exactly five separating triangles. $D_1, D_2, ..., D_5$ are the 5 separating triangles. Let G_{in,D_1} be the subgraph of G derived by deleting all the vertices outside the separating triangle D_1. Let G_{out,D_1} be the subgraph of G derived by deleting all the vertices inside the separating triangle D_1.

We distinguish the following five cases.

Case 1: The graph G_{in,D_1} has three separating triangles and the graph G_{out,D_1} has one separating triangle. Then analogically to the proof of Theorem 3.37 Case 4 G is hamiltonian.

Case 2: The graph G_{in,D_1} has two separating triangles and the graph G_{out,D_1} has two separating triangles. Then analogically to the proof of Theorem 3.37 Case 1 G is hamiltonian.

Case 3: The graph G_{in,D_1} has one separating triangle and the graph G_{out,D_1} has three separating triangles. Then analogically to the proof of Theorem 3.37 Case 3 G is hamiltonian.

Case 4: The graph G_{in,D_1} has no separating triangle and the graph G_{out,D_1} has four separating triangles. By Theorem 3.16 G_{in,D_1} is hamiltonian for any two boundary edges of G_{in,D_1}. By Theorem 3.37 G_{out,D_1} is hamiltonian. Let the vertices a, b and c be the vertices of the facial cycle D_1 in G_{out,D_1}. The graph $G_{out,D_1,D_2,...,D_5}$ is a subgraph of G_{out,D_1} and has no separating triangles. If $G_{out,D_1,D_2,...,D_5}$ has $n < 7$ vertices, then there exists a hamiltonian cycle C of $G_{out,D_1,D_2,...,D_5}$ such that the hamiltonian cycle passes through at least one edge of the edges ab, bc and ca. If $G_{out,D_1,D_2,...,D_5}$ has $n \geq 7$ vertices, then by Theorem 3.36 there exists a hamiltonian cycle C of $G_{out,D_1,D_2,...,D_5}$ such that $k_{2F} = 4$ and at least two 2-frontier faces have one edge in common. Now we have to prove that the hamiltonian cycle C passes through at least one edge of the edges ab, bc and ca. We insert 4 new vertices each into the facial cycles D_2, D_3, D_4 and D_5 of G then joining the new vertex to every vertex incident with the faces of D_2, D_3, D_4 and D_5 respectively. The new graph H is a hamiltonian maximal planar graph with 4 separating triangles. Now we can extend the hamiltonian cycle C of $G_{out,D_1,D_2,...,D_5}$ to a hamiltonian cycle C' of H. Thus we get $k_{2F}(C') = 7$ and $k_N(C') = 3$ with respect to the hamiltonian cycle C'. Then we can embed the graph H on a surface such that D_1 is not a non-frontier face. Therefore, the hamiltonian cycle C' passes through at least one edge of the edges ab, bc and ca. At last we have to replace the vertices in the facial cycles D_2, D_3, D_4 and D_5 by the graphs $G_{in,D_2} - D_2, G_{in,D_3} - D_3, G_{in,D_4} - D_4$ and $G_{in,D_5} - D_5$ respectively. Since G_{in,D_i} with $2 \leq i \leq 5$ are maximal planar graphs without separating triangles, the graphs G_{in,D_i} with $2 \leq i \leq 5$ are hamiltonian for any two boundary edges of G_{in,D_i}. Therefore, the graphs $G_{in,D_i} - D_i$ with $2 \leq i \leq 5$ have a hamiltonian path in $G_{in,D_i} - D_i$. Now we replace the two edges which join the new vertices to the hamiltonian cycle C' by the hamiltonian paths in $G_{in,D_i} - D_i$ with $2 \leq i \leq 5$. Thus we get a hamiltonian cycle C'' which passes through at

least one edge of the edges ab, bc and ca. H' is a hamiltonian cycle of G_{out,D_1}. Without loss of generality, C'' passes through the edge ca. Consider G_{in,D_1}. a, b and c are the vertices of the exterior cycle of G_{in,D_1}. Without loss of generality, G_{in,D_1} has a hamiltonian cycle $C_{in} = abcP_{in}(c,a)a$ where $P_{in}(c,a)$ is a hamiltonian path of $G_{in,D_1} - b$ between c and a. Then we replace the edge ca in the hamiltonian cycle C'' of G_{out,D_1} by the hamiltonian path $P_{in}(c,a)$ between c and a of $G_{in,D_1} - b$. Then the new cycle is a hamiltonian cycle of G.

Case 5: The graph G_{in,D_1} has four separating triangles and the graph G_{out,D_1} has no separating triangle. By Theorem 3.37 G_{in,D_1} is hamiltonian. By Theorem 3.16 G_{out,D_1} is hamiltonian for any two boundary edges of G_{out,D_1}. Note that a planar graph can always be embedded in a surface such that a given face of the graph becomes the exterior face, see Lemma 1.10. Without loss of generality, G_{out,D_1} has a hamiltonian cycle passing through any two edges of the facial cycle D_1. Then analogically to Case 4 the graph G is hamiltonian. ∎

Let G be a maximal planar graph with exactly six separating triangles. $D_1, D_2, ..., D_6$ are the 6 separating triangles. Let G_{in,D_1} be the subgraph of G derived by deleting all the vertices outside the separating triangle D_1. Let G_{out,D_1} be the subgraph of G derived by deleting all the vertices inside the separating triangle D_1. Consider the case that the graph G_{in,D_1} has no separating triangles and the graph G_{out,D_1} has five separating triangles. Assume that $G_{out,D_1,D_2,...,D_6}$ has $n \geq 7$ vertices then by Theorem 3.36 there exists a hamiltonian cycle C of $G_{out,D_1,D_2,...,D_6}$ such that $k_{2F} = 4$ and at least two 2-frontier faces have one edge in common. Now we insert 5 new vertices each into the facial cycles $D_2, ..., D_6$ of G then joining the new vertex to every vertex incident with the faces of $D_2, ..., D_6$ respectively. The new graph H is a hamiltonian maximal planar graph with 5 separating triangles. Now we can extend the hamiltonian cycle C of $G_{out,D_1,D_2,...,D_6}$ to a hamiltonian cycle C' of H. Thus we get $k_{2F}(C') = 8$ and $k_N(C') = 4$ with respect to the hamiltonian cycle C'. Consider the case that the three non-frontier faces have each one vertex in common. Then these three non-frontier faces determine a facial cycle which is the remaining non-frontier face with respect to the hamiltonian cycle C'. This leads to a contradiction. Therefore, there must be a maximal planar graph with six separating triangles which is non-hamiltonian. Finally, if G is a maximal planar graph with exactly six separating triangles, then there is a counter-example which shows that G is not immediately hamiltonian, see Figure 3.5.

Definition 3.39 *Let $G = (V, E)$ be a simple graph. We call $X \subseteq V(G)$ independent if $E(G[X]) = \emptyset$. A maximal independent vertex set is an independent set containing the largest possible number of vertices.*

Example 3.40 (Helden [35]) *The graph from Figure 3.5 is a non-hamiltonian maximal planar graph. The vertices $v_5, v_7, v_{11}, v_{13}, v_{12}, v_{14}, v_{15}, v_{17}$ form a maximal independent vertex set. If a hamiltonian cycle C exists, then these 8 vertices cannot be adjacent. Hence, only two of the remaining 9 vertices can be adjacent to at most one vertex of the maximal independent vertex set. Since the vertices v_1, v_2 and v_3 are adjacent to only one vertex of the maximal independent vertex set, there cannot exist a hamiltonian cycle.*
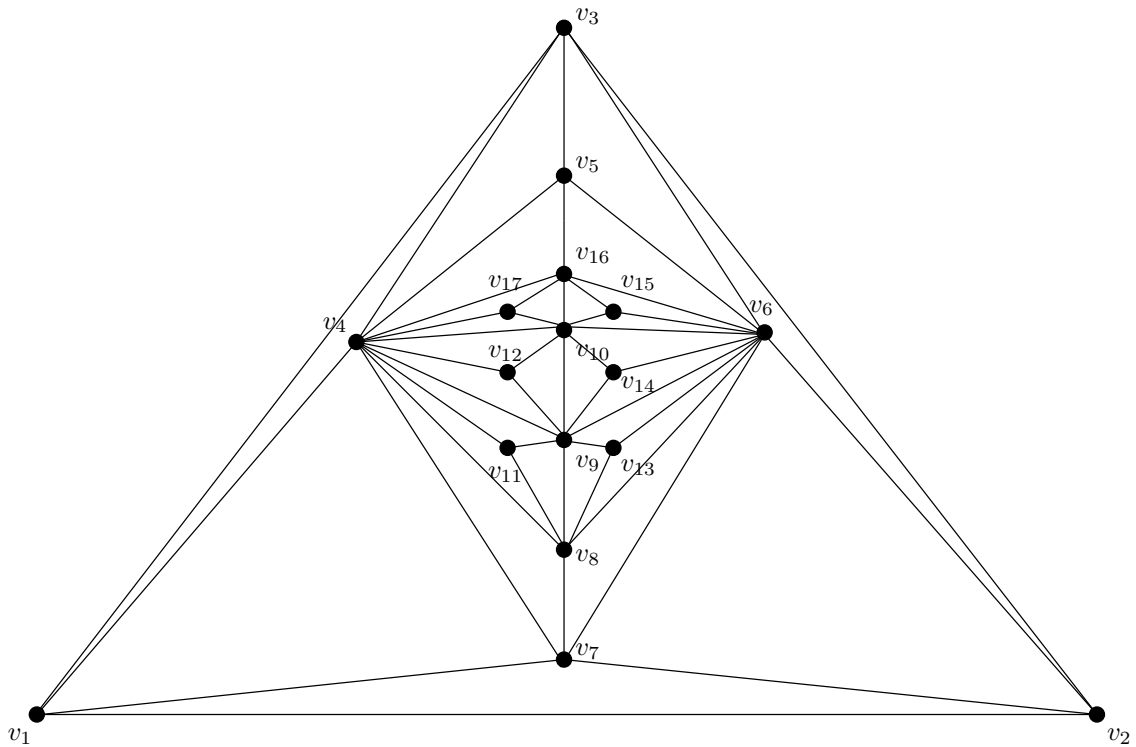
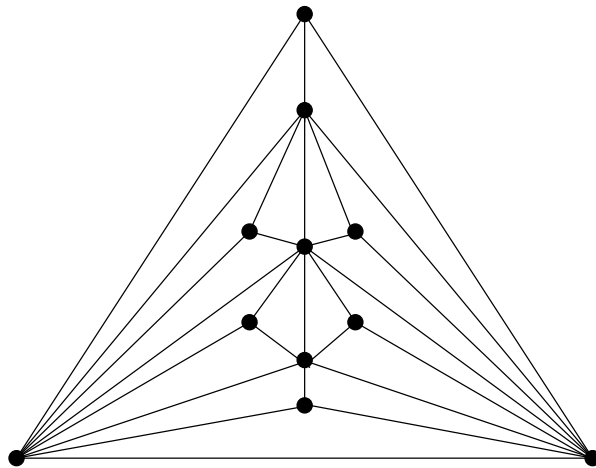Figure 3.5: Counter-example: A non-hamiltonian maximal planar graph with exactly six separating triangles.

Figure 3.6: The smallest non-hamiltonian maximal planar graph with 11 vertices.

Example 3.41 *The graph G from Figure 3.6 is the smallest non-hamiltonian maximal planar graph with* 11 *vertices. The graph G has seven separating triangles. In 1980 T. Asano, T. Nishizeki and T. Watanabe [4] showed that every non-hamiltonian maximal planar graph with* 11 *vertices is isomorphic to the graph shown in Figure 3.6.*

3.2 Hamiltonian path

Since maximal planar graphs with more than five separating triangles are not necessarily hamiltonian, we will show that maximal planar graphs with exactly six separating triangles have at least a hamiltonian path. Therefore, we need the following definition and the next lemma.

Definition 3.42 *Let G be a maximal planar graph and let $a, b \in V(G)$ with $ab \in E(G)$ and $|N(a) \cap N(b)| \geq 2$. Then we define the operation D, see Figure 3.7.*
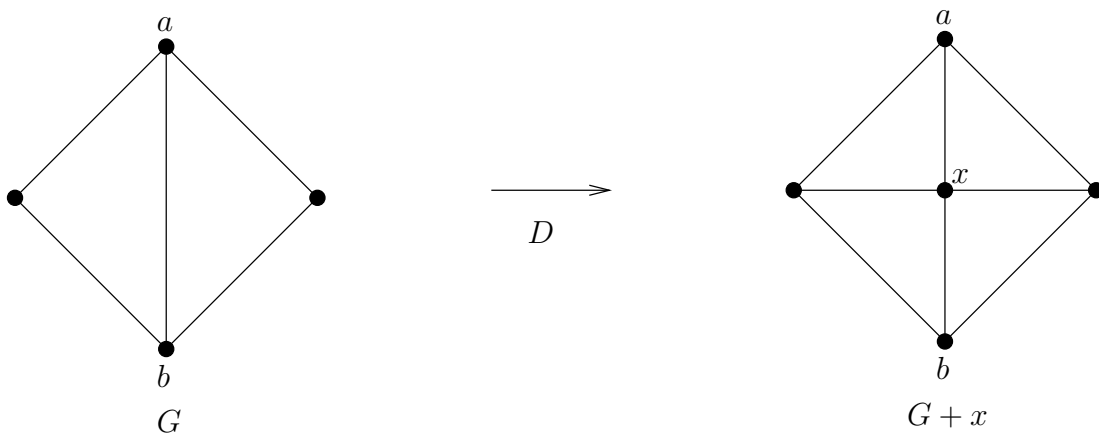

Figure 3.7: Operation D.

Remark. After applying the operation D, the graph $G + x$ is still a maximal planar graph.

Lemma 3.43 *Let G be a maximal planar graph with exactly one separating triangle. Let a, b and c be the vertices which form the separating triangle. Then we can apply the operation D to $a, b \in V(G)$ and the graph $G + x$ is a maximal planar graph without separating triangles.*

Proof. Since a, b and c form one separating triangle the vertices does not form the boundary of a face, especially not the exterior face. Because of the maximal planarity the vertices a and b have at least two common neighbors beside c, at least one vertex inside the separating triangle and one outside. Now we can apply the operation D. If we remove the vertices a, b and c, the graph $G + x$ is still connected, because the new vertex connects the inside and the outside of the separating triangle. So the resulting graph is a maximal planar graph without separating triangle. ∎

This lemma yields the next easy corollary.

Corollary 3.44 *Let G be a maximal planar graph with $k > 1$ separating triangles. Let a, b and c be the vertices which form one separating triangle. Then we can apply the operation D to $a, b \in V(G)$ and the graph $G + x$ is a maximal planar graph with at most $k - 1$ separating triangles.*

Remark. Let G be a maximal planar graph with two separating triangles. Let a, b and c be the vertices which form one separating triangle and let d, e and f be the vertices which form the other separating triangle. If the two separating triangles have one edge in common, say $ab = de$, then we can apply the operation D to $a, b \in V(G)$ only once. The resulting graph $G + x$ is a maximal planar graph without separating triangles.

The following theorem presents our main result in this section.

Theorem 3.45 *If G is a maximal planar graph with exactly six separating triangles, then G has a hamiltonian path.*

Proof. Let G be a maximal planar graph with exactly six separating triangles and let a, b and c be the vertices which form one separating triangle of the graph G. Now we can apply the operation D. The resulting graph is a maximal planar graph with at most 5 separating triangles. By Theorem 3.38, the resulting graph is hamiltonian. Now the vertex, which was added by the operation D, can be removed. Since this is only one vertex, the new graph $\overline{G}$ has a hamiltonian path. $\overline{G}$ is a subgraph of G with $|V(\overline{G})| = |V(G)|$ and $|E(\overline{G})| < |E(G)|$. Therefore, the graph G has also a hamiltonian path. ∎

With the previous remark and Theorem 3.38 the next corollary follows immediately.

Corollary 3.46 *Let G be a maximal planar graph with seven separating triangles and let two separating triangles have one edge in common. Then G has a hamiltonian path.*

Finally, if G is a maximal planar graph with exactly eight separating triangles, then there is a counter-example, which shows that G has no hamiltonian path, see Figure 3.8.

Example 3.47 *The graph from Figure 3.8 is a maximal planar graph and has no hamiltonian path. The vertices $v_4, v_6, v_8, v_{10}, v_{12}, v_{13}, v_{15}, v_{16}$ form a maximal independent vertex set. If a hamiltonian path P exists, then these 8 vertices cannot be adjacent. Hence, only two of the remaining 8 vertices can be adjacent to at most one vertex of the maximal independent vertex set. Since the vertices v_1 and v_2 are adjacent to only one vertex of the maximal independent vertex set, there cannot exist a hamiltonian path.*

3.3   Length of a hamiltonian walk

Definition 3.48 *A closed spanning walk of a graph G is a sequence $v_0 e_1 v_1 e_2 ... e_k v_k (= v_0)$, whose terms are alternately vertices and edges, such that the end vertices of e_i are v_{i-1} and v_i for each $1 \leq i \leq k$, and every vertex of G appears in this sequence at least once. The integer k is the length of the closed spanning walk. A hamiltonian walk of G is a closed spanning walk of minimum length. For a connected graph G, $h(G)$ denotes the length of a hamiltonian walk of G.*
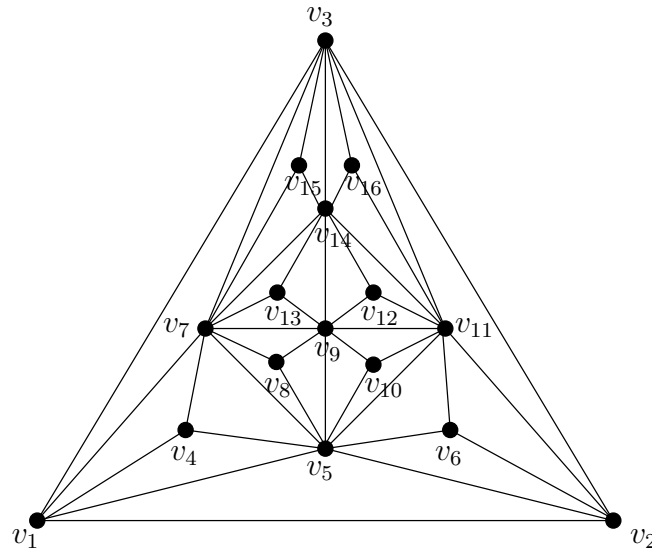
Figure 3.8: Counter-example: A maximal planar graph with exactly eight separating triangles.

Since not every maximal planar graph has a hamiltonian cycle nor a hamiltonian path, we want to determine the length of a hamiltonian walk of a maximal planar graph. A trivial lower bound and a trivial upper bound are known on the length $h(G)$ of a hamiltonian walk of a connected graph G with n vertices:

$$n \leq h(G) \leq 2(n-1).$$

These bounds follow immediately from the fact that any hamiltonian walk must pass through every vertex at least once and a closed spanning walk traversing twice every edge of any tree of a connected graph is of length $2(n-1)$.

Theorem 3.49 (Asano, Nishizeki and Watanabe [4]) *The length $h(G)$ of a hamiltonian walk of any maximal planar graph G with n vertices satisfies*

$$h(G) \begin{cases} \leq \frac{3}{2}(n-3) & \text{if } n \geq 11, \\ = n & \text{otherwise.} \end{cases} \tag{3.1}$$

3.4 Tough graphs

In this section we consider the question of which maximal planar graphs are non-hamiltonian. Sometimes there is an easily verifiable proof of the fact that a particular graph is non-hamiltonian. We would like to have a theorem that says a graph G is non-hamiltonian if G has property "Q", where "Q" can be checked in polynomial time. This motivates the definition of 1–toughness. The concept of 1–toughness was introduced by V. Chvátal [21] in 1973.

Definition 3.50 *A graph G is said to be 1–tough if for any non-empty subset S of the vertices of G,*

$$\kappa(G - S) \leq |S|.$$

Note that every hamiltonian graph is 1–tough. Thus 1–toughness is a necessary condition for a graph to be hamiltonian. Since there exist 1–tough but non-hamiltonian graphs, 1–toughness is not a sufficient condition in general.

In 1979 V. Chvátal raised the following question: Is 1–toughness a sufficient condition for a maximal planar graph to be hamiltonian? In 1980 T. Nishizeki [41] answered Chvátal's question by constructing a 1–tough non-hamiltonian maximal planar graph on 19 vertices. In [25] M. B. Dillencourt found a graph with the same property on 15 vertices. In 1994 M. Tkáč [52] found a 1–tough non-hamiltonian maximal planar graph on 13 vertices, see Figure 3.10.

3.5   Construction of maximal planar graphs

This section is concerned with the construction of hamiltonian maximal planar graphs with n vertices and k separating triangles and non-hamiltonian maximal planar graphs with n vertices and k separating triangles. Since maximal planar graphs with more than five separating triangles are not necessarily hamiltonian, we want to show that there exist at least two maximal planar graphs with the same number of vertices and the same number of separating triangles, such that one graph is hamiltonian and another graph is non-hamiltonian. We start by using Theorem 2.14 to construct a non-hamiltonian MPG-3 graph.

Corollary 3.51 *Let $n \in I\!N$ with $n \geq 13$. Then there exists a non-hamiltonian MPG-3 graph G with n vertices.*

Proof. We distinguish the following two cases.

Case 1: n is odd. As our basic graph we select an arbitrary maximal planar graph G with $\frac{n-1}{2}$ vertices which contains no separating triangles. Since $n \geq 13$ this graph G has at least eight faces. Now we select $\frac{n+1}{2}$ faces of the graph G. Now we consider the graph G' obtained from G by inserting a new vertex into each selected face of G then joining the new vertex to every vertex incident with the corresponding face. The number of vertices with degree 3 of G' is $\tau_3 = \frac{n+1}{2}$. Then G' fulfils the assumption $2\tau_3 > n$ of Theorem 2.14. Therefore, G' is non-hamiltonian.

Case 2: n is even. As our basic graph we select an arbitrary maximal planar graph G with $\frac{n-2}{2}$ vertices which contains no separating triangles. Since $n \geq 14$ this graph G has at least eight faces. Now we select $\frac{n+2}{2}$ faces of the graph G. Now we consider the graph G' obtained from G by inserting a new vertex into each selected face of G then joining the new vertex to every vertex incident with the corresponding face. The number of vertices with degree 3 of G' is $\tau_3 = \frac{n+2}{2}$. Then G' fulfils the assumption $2\tau_3 > n$ of Theorem 2.14. Therefore, G' is non-hamiltonian.   ∎

Note that the number of vertices with degree three, τ_3, is a lower bound on the number of separating triangles. If the number of vertices n is odd, we have at least $k = \frac{n+1}{2}$ separating triangles. If n is even, we have at least $k = \frac{n+2}{2}$ separating triangles. Now we consider hamiltonian MPG-3 graphs.

Theorem 3.52 *Let $d \in I\!N$. Then there exists a hamiltonian* MPG-3 *graph G with exactly* $\tau_3 = d$ *vertices of degree three.*
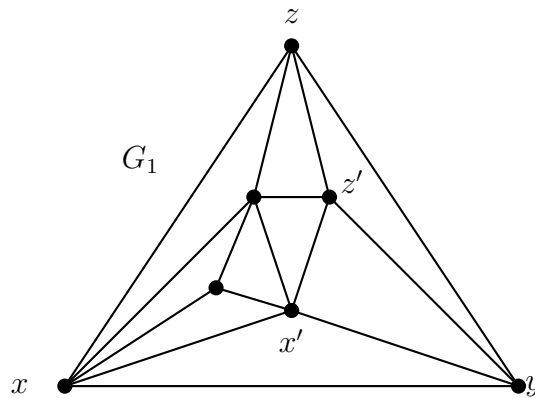

Figure 3.9: A maximal planar graph G with exactly one vertex of degree 3.

Proof. We construct an MPG-3 graph G_d with exactly d vertices of degree three inductively. Therefore, we need a basic graph G_1, see Figure 3.9. In G_1 the vertices x', y and z' in Figure 3.9 were identified with the vertices x, y and z of a copy of the graph G_1. That is, the face x', y, z' is replaced by the graph G_1 itself. The resultant graph is called G_2. This graph has exactly two vertices of degree three. Now we continue inductively. The resultant graph G_d has exactly d vertices of degree three. Let B_d be the rooted decomposition tree of the given embedding of G_d. Since each vertex of the tree B_d has at most two children, Theorem 3.28 yields that the graph G_d is hamiltonian. ∎

Now we can add the number of separating triangles of Corollary 3.51. If the number of vertices n is odd, we have $d = \frac{n+1}{2}$ separating triangles. If n is even, we have $d = \frac{n+2}{2}$ separating triangles. But we get a different number of vertices. Let n' be the number of vertices of a graph constructed by Theorem 3.52. Then

$$n' = 7 + 4 \cdot d = 7 + 4 \cdot \frac{n+1}{2} = 2n + 9$$

if n is odd and

$$n' = 7 + 4 \cdot d = 7 + 4 \cdot \frac{n+2}{2} = 2n + 11$$

if n is even. In both cases $n \neq n'$. The next example shows a non-hamiltonian maximal planar graph which was assumed for a long time to be the one with the smallest number of separating triangles. This is not the case, see Figure 3.5. But we need this graph for further constructions.

Example 3.53 *Figure 3.10 shows a non-hamiltonian maximal planar graph with 7 separating triangles. The graph G of Figure 3.10 has $n = 13$ vertices and $\tau_3 = 7$ vertices of degree 3 and hence G fulfills the assumption of Theorem 2.14 that $2\tau_3 > n$. Therefore, G is non-hamiltonian.*
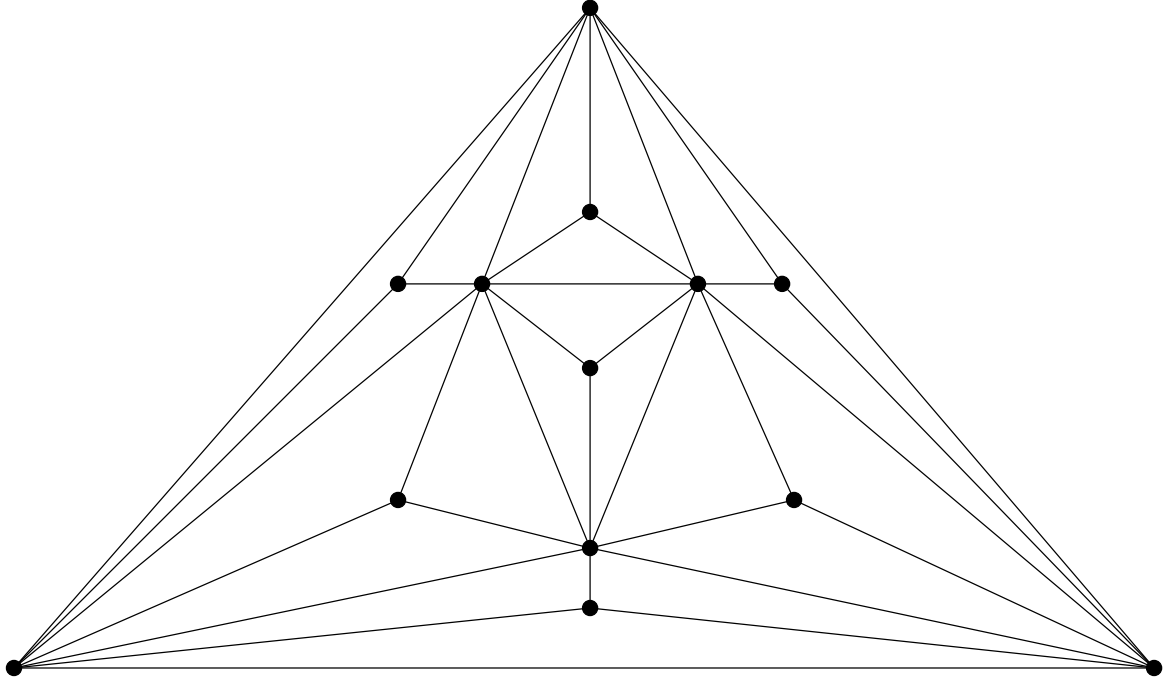

Figure 3.10: A non-hamiltonian maximal planar graph with 7 separating triangles.

In the following we try to adjust k and n. Let n be the number of vertices and k be the number of separating triangles determined. We will show that there are hamiltonian maximal planar graphs with n vertices and k separating triangles and also non-hamiltonian maximal planar graphs with n vertices and k separating triangles. First we will show which values for k are possible.

Lemma 3.54 *Let $\mathcal{G}$ be the set of all maximal planar graphs with n vertices. Then $0 \leq k \leq n - 4$ for all $G \in \mathcal{G}$.*

Proof. Theorem 3.64 of Hakimi and Schmeichel yields the following bounds on the number of cycles of length three: $2n - 4 \leq C_3 \leq 3n - 8$. On the other hand the number of faces is $l = 2n - 4$. Since a separating triangle does not form the boundary of a face, the number of separating triangles is the difference between the number of 3–cycles and the number of faces. Hence, we get: $0 \leq k \leq n - 4$. ∎

To construct a non-hamiltonian maximal planar graph G', we must find a hamiltonian maximal planar graph G that has a face which has no edge in common with every constructible hamiltonian cycle of G. That is, the face of G must be non-frontier with respect to all hamiltonian cycles of G.

Example 3.55 *The graph from Figure 3.11 is a hamiltonian maximal planar graph.* $C = v_1v_4v_5v_6v_8v_9v_{10}v_{12}v_2v_{11}v_3v_7v_1$ *is a possible hamiltonian cycle of* G
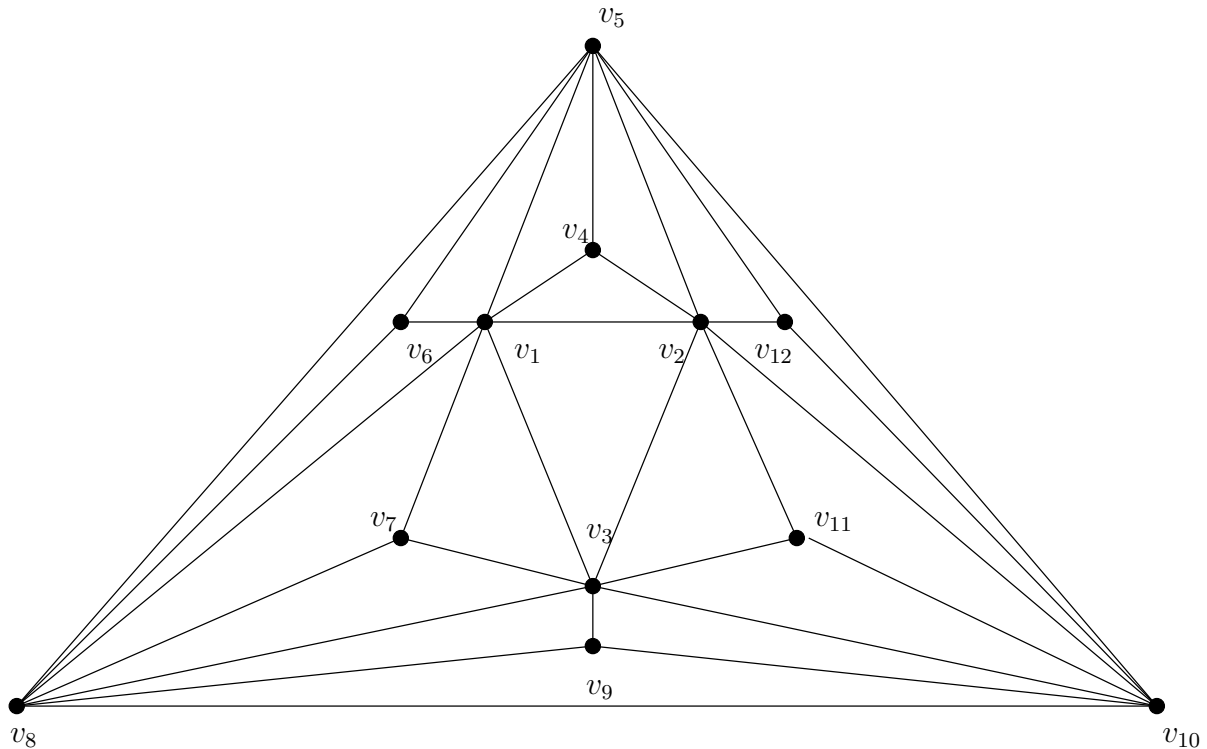

Figure 3.11: A maximal planar graph with a certain non-frontier face.

This provides the following lemma.

Lemma 3.56 *Let* G *be the graph from Figure 3.11. Then the triangle* $v_1v_2v_3$ *is a non-frontier face with respect to all hamiltonian cycles of* G.

Proof. We will show that neither two nor one of the edges of the triangle $v_1v_2v_3$ can belong to a hamiltonian cycle of G. Since the graph G is symmetrically developed around the triangle v_1, v_2, v_3, we can without loss of generality select exactly one or exactly two arbitrary edges.

Case 1: The edges v_1v_2 and v_2v_3 belong to a hamiltonian cycle C of G. Since the edge v_1v_3 do not belong to C, one of the edges v_7v_1 and v_7v_3 must belong to C. Without loss of generality, the edge v_7v_1 belongs to C. Now we consider the vertex v_4. Since the edges v_1v_4 and v_2v_4 would destroy the cycle C, these edges cannot belong to C. Therefore, only the edge v_5v_4 is incident to the vertex v_4. This is a contradiction to the assumption that there exists a hamiltonian cycle C of G which contains the edges v_1v_2 and v_2v_3.

Case 2: The edge v_1v_2 belongs to a hamiltonian cycle C of G and the edges v_2v_3 and v_3v_1 do not belong to C. Therefore, one of the edges v_4v_1 and v_4v_2 must belong to C.

Without loss of generality, the edge v_4v_1 belongs to C. Then the edge v_4v_5 must also belong to C. Now we consider the vertex v_7. Since the edge v_1v_7 would destroy the cycle C, this edge cannot belong to C. Therefore, the edges v_8v_7 and v_3v_7 belong to C. In order to merge the vertex v_6 into the cycle C, the edges v_5v_6 and v_8v_6 must also belong to C. Since v_3v_{11} and v_2v_{11} cannot both belong to the cycle C, the edge $v_{10}v_{11}$ must belong to the cycle C. Now we consider the vertex v_{12}. The vertex v_{12} can only be run through by the edges $v_{10}v_{12}$ and v_2v_{12}. So far we have not run through the vertex v_9. However, there is no possibility that the cycle C is closed and the vertex v_9 will be run through. Thus we get a contradiction and the proof is complete. ∎

Since we need a hamiltonian maximal planar graph G to construct a non-hamiltonian maximal planar graph G', we must construct a hamiltonian maximal planar graph first.

Construction 3.57

> *Input: $n \in I\!N$ and $k \in I\!N$ with $n \geq k + 6$.*
>
> *Output: A hamiltonian maximal planar graph G with n vertices and k separating triangles.*
>
> > *S1: Let $n = k + 5 + r$. Select the graph from Figure 3.12 as starting graph G with $k + 2$ interior vertices.*
> >
> > *S2: Consider the triangle abd_2. Insert the vertices $x_1, x_2, ..., x_r$ on the edge d_1b as follows. The edge d_1b is transferred into $r+1$ edges with vertices $d_1, x_1, x_2, ..., x_r$ and b. Connect the vertex c and the vertex d_2 with all vertices already inserted.*

**Proof.** Let $n \in I\!N$ and $k \in I\!N$ with $n \geq k + 6$. Let $n = k + 5 + r$. The vertices a, d_i and b for $1 \leq i \leq k+1$ form $k+1$ separating triangles in the starting graph. Since $r \geq 1$, at least the vertex x_r destroys the separating triangle ad_1b. This means we have only k separating triangles. Now we have to show that the constructed graph has a hamiltonian cycle. Begin with the edge ad_{k+2}. Then run through the vertices $d_{k+1}, ..., d_1, x_1, x_2, ..., x_r, b, c, a$. Thus we obtain a hamiltonian cycle. ∎

Now we can construct a non-hamiltonian maximal planar graph.

Construction 3.58

> *Input: $n \in I\!N$ and $k \in I\!N$ with $k > 6$ and $n \geq k + 6$.*
>
> *Output: A non-hamiltonian maximal planar graph G with n vertices and k separating triangles.*
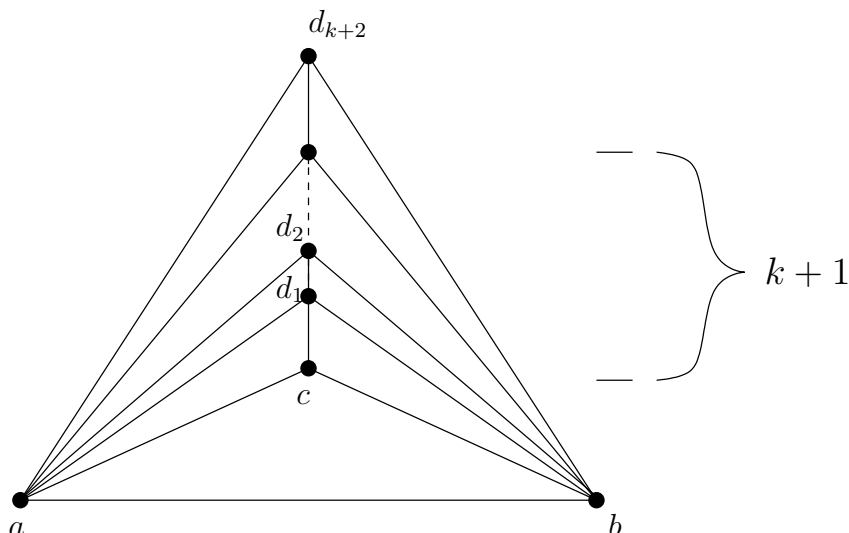
Figure 3.12: Starting graph $G = (V, E)$.

S1: Let $n = k + 5 + r$ with $k = 6 + q$. Select the graph from Figure 3.11 as starting graph G.

S2: If $q > 1$, then use Construction 3.57 to construct a hamiltonian maximal planar graph G' with $(q - 1) + r + 5$ vertices and $q - 1$ separating triangles. Else if $q = 1$ then choose G' as the K_4 graph.

S3: Replace the triangle $v_1 v_2 v_3$ by the graph G'.

Proof. Let $n \in I\!N$ and $k \in I\!N$ with $k > 6$ and $n \geq k + 6$. Let $n = k + 5 + r$ with $k = 6 + q$. If $q = 1$, then the constructed graph is the graph from Example 3.53. Hence, we have $k = 7$ separating triangles. If $q > 1$, then the starting graph G has 6 separating triangles and the constructed hamiltonian maximal planar graph G' has $q - 1$ separating triangles. Since the vertices $v_1 v_2 v_3$ form another separating triangle, we have exactly $k = 6 + q - 1 + 1 = 6 + q$ separating triangles. Now we have to show that the constructed graph has no hamiltonian cycle. If $q = 1$, then the constructed graph is the graph from Example 3.53. Hence, we know this graph has no hamiltonian cycle. If $q > 1$, then the constructed graph is also non-hamiltonian. Since there is no hamiltonian cycle if there is one vertex inside the triangle $v_1 v_2 v_3$, there is no hamiltonian cycle, especially if there are more than one vertices inside the triangle $v_1 v_2 v_3$.   ∎

With the previous constructions we are able to adjust k and n, if $n \geq 13$ and $k \geq 7$ with $n \geq k + 6$. Thus we get the next theorem.

Theorem 3.59 *Let n be the number of vertices and k be the number of separating triangles. If $n \geq 13$ and $k \geq 7$ with $n \geq k + 6$, then at least one hamiltonian maximal planar graph with n vertices and k separating triangles exists; and also at least one non-hamiltonian maximal planar graph with n vertices and k separating triangles exists.*

Note that the Construction 3.58 works starting from seven separating triangles, whereas Construction 3.57 works also without separating triangles.

Remark. The condition $n \geq k + 6$ can be intensified on $n = k + 4$ for hamiltonian maximal planar graphs. For the case $n = k + 5$ there is no construction possible.

Now we consider the case that we have exactly six separating triangles. Note that Figure 3.5 shows a non-hamiltonian maximal planar graph with six separating triangles. This graph provides the basis for our next construction.
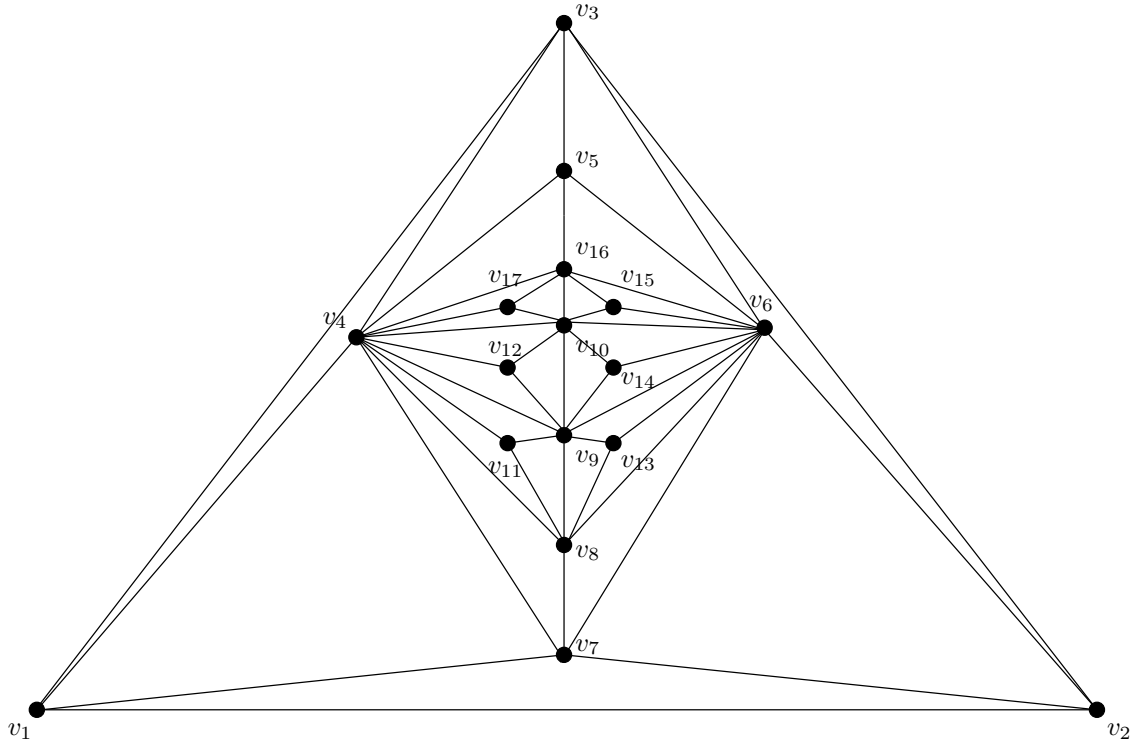

Figure 3.13: A non-hamiltonian maximal planar Graph G with $n = 17$ and $k = 6$.

Construction 3.60

Input: $n \in I\!N$ and $k \in I\!N$ with $k = 6$ and $n \geq 17$.

Output: A non-hamiltonian maximal planar graph G with n vertices and 6 separating triangles.

   S1: Let $n = 17 + r$. Select the graph from Figure 3.13 as starting graph G.

   S2: If $r \geq 2$, then use Construction 3.57 to construct a hamiltonian maximal planar graph G' with $r + 4$ vertices and no separating triangles.

> *Else if $r = 1$ then insert the vertex x_r on the edge v_7v_8 and connect the vertex v_4 and the vertex v_6 with the new vertex x_r.*

> *S3: Delete the vertex v_{17} and replace the triangle $v_4v_{10}v_{16}$ by the graph G'.*

Proof. Let $n \in I\!N$ and $k \in I\!N$ with $k = 6$ and $n \geq 17$. Let $n = 17 + r$. If $r = 0$, then the constructed graph is the graph from Example 3.40. Hence, we have $k = 6$ separating triangles. If $r = 1$, then the starting graph G has 6 separating triangles and the inserted vertex x_r yields no further separating triangle. If $r > 1$, then the starting graph G has 6 separating triangles and the inserted graph G' yields no further separating triangle.

Now we have to show that the constructed graph has no hamiltonian cycle. If $r = 0$, then the constructed graph is the graph from Example 3.40. Hence, we know this graph has no hamiltonian cycle. If $r > 1$, then the constructed graph is also non-hamiltonian. Since there is no hamiltonian cycle if there is one vertex inside the triangle $v_4v_{10}v_{16}$, there is no hamiltonian cycle, especially if there are more than one vertices inside the triangle $v_4v_{10}v_{16}$.

If $r = 1$, then the vertices $v_1, v_5, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{17}, x_r$ form a maximal independent vertex set. If a hamiltonian cycle C exists, then these 9 vertices cannot be adjacent. Hence, each of the remaining 9 vertices must be adjacent to at least two vertices of the maximal independent vertex set. Since the vertices v_2 and v_3 are adjacent to only one vertex of the maximal independent vertex set, there cannot exist a hamiltonian cycle. ∎

With the previous construction and Construction 3.57 we are able to adjust n for $k = 6$, if $n \geq 17$. Thus we get the next theorem.

Theorem 3.61 *Let n be the number of vertices and k be the number of separating triangles. If $n \geq 17$ and $k = 6$, then there exists at least one hamiltonian maximal planar graph with n vertices and six separating triangles and also at least one non-hamiltonian maximal planar graph with n vertices and six separating triangles.*

The previous remark yields us to the following.

Theorem 3.62 *Let n be the number of vertices and k be the number of separating triangles. If $n = k + 5$, then no maximal planar graph with n vertices and k separating triangles exists.*

Proof. Note that each separating triangle has at least one vertex in its interior face and at least one vertex in its exterior face. Since each separating triangle consists of three vertices, we have at most $n = 5k$ vertices. If the separating triangles are not disjoint like Figure 3.12, then we have at least $n = k + 4$ vertices. To get $n = k + 5$ vertices we have to add one more vertex. If we add this vertex to an edge, then we have to add two edges to the common neighbors of the two adjacent vertices of the added vertex. If the edge to which we add the new vertex belongs to a separating triangle, then we destroy this separating triangle and get $k - 1$ separating triangles. If the edge to which we add the new vertex does not belong to a separating triangle, then we create a new separating triangle and get $k + 1$ separating triangles. If we add the new vertex to a face, then we have to

add three edges to the three vertices of the boundary of the face. Now these three vertices of the boundary of the face form a new separating triangle and we get $k + 1$ separating triangles. ∎

Analogously to Theorem 2.14 the number of disjoint separating triangles supplies a necessary condition of whether a graph is hamiltonian.

Theorem 3.63 *Let $G = (V, E)$ be an arbitrary maximal planar graph with $\widehat{k}$ disjoint separating triangles and $n \geq 13$ vertices. If $2\widehat{k} > n$, then G is non-hamiltonian.*

Proof. Assume that G is hamiltonian. Let C be the hamiltonian cycle of G. Let $D_1, ..., D_{\widehat{k}}$ be all disjoint separating triangle of G. Let a, b and c be the vertices of the separating triangle D_1. Let the vertices a and b be the neighbors of the interior vertices of G_{in,D_1} on the hamiltonian cycle C. If we replace these edges by the edge ab, then we obtain a hamiltonian cycle C' in the maximal planar graph $G - G_{in,D_1}$. If we delete all disjoint separating triangles of G in this way, we obtain a maximal planar graph $G_{out,D_1,...,D_{\widehat{k}}}$. For two disjoint separating triangles we obtain different edges. After construction the hamiltonian cycle of $G_{out,D_1,...,D_{\widehat{k}}}$ has at least $\widehat{k}$ edges and $G_{out,D_1,...,D_{\widehat{k}}}$ consists of at most $n - \widehat{k}$ vertices. Therefore, $\widehat{k} \leq n - \widehat{k}$ if and only if $2\widehat{k} \leq n$. This is a contradiction to the assumption. ∎

3.6 Number of cycles of certain length

In this section we consider the following question. What is the number of cycles of certain length that a maximal planar graph G on n vertices could have, in terms of n?

3.6.1 Number of short cycles

Let G be a maximal planar graph on n vertices, and let $C_i(G)$ for $3 \leq i \leq n$ denote the number of cycles of length i in G. We first present tight bounds for $C_3(G)$ and $C_4(G)$ in terms of n. In 1979 S. L. Hakimi and E. F. Schmeichel proved the following.

Theorem 3.64 (Hakimi and Schmeichel [31]) *Let G be a maximal planar graph with $n \geq 6$ vertices. Then*

$$2n - 4 \leq C_3(G) \leq 3n - 8.$$

The lower bound is attained if and only if G is 4–connected, and the upper bound is attained if and only if G is obtained from K_3 by recursively placing a vertex of degree 3 inside a face, and joining this new vertex to the three vertices incident to that face. Note that no n-vertex maximal planar graph G with $C_3(G) \leq 3n - 9$ exists. For 4–cycles S. L. Hakimi and E. F. Schmeichel achieved the following result.

Theorem 3.65 (Hakimi and Schmeichel [31]) *Let G be a maximal planar graph with $n \geq 5$ vertices. Then*

$$3n - 6 \leq C_4(G) \leq \frac{1}{2}(n^2 + 3n - 22).$$

The lower bound is attained if and only if $n = 5$ or G is 5–connected. The upper bound is attained if and only if G is the graph in Figure 3.12. If the graph is additionally 4–connected, the following upper bound holds.

Lemma 3.66 (Hakimi and Schmeichel [31]) *Let G be a 4–connected maximal planar graph with $n \geq 7$ vertices. Then*

$$C_4(G) \leq \frac{1}{2}(n^2 - n - 2).$$

Note that equality in Lemma 3.66 holds if and only if G is the graph of Figure 3.15.

3.6.2 Number of longer cycles

Now we continue with the number of longer cycles. We interpret longer cycles as cycles starting from the length five. It was also Hakimi and Schmeichel who established a first lower bound for maximal planar graph on at least 8 vertices.

Lemma 3.67 (Hakimi and Schmeichel [31]) *Let G be a maximal planar graph with $n \geq 8$ vertices. Then*

$$C_5(G) \geq 6n.$$

There exists a maximal planar graph G with $C_5(G) = 6n$ for every $n \geq 14$. An upper bound was found for maximal planar graphs on at least 6 vertices.

Lemma 3.68 (Hakimi and Schmeichel [31]) *Let G be a maximal planar graph with $n \geq 6$ vertices. Then*

$$C_5(G) \leq 5n^2 - 26n.$$

Now we consider an upper bound for $C_k(G)$ for an arbitrary k.

Theorem 3.69 (Hakimi and Schmeichel [31]) *Let G be a maximal planar graph with n vertices. Then $C_k(G) \in \mathcal{O}(n^{\lfloor \frac{k}{2} \rfloor})$, for $k = 3, 4, ..., n$.*

3.6.3 Number of hamiltonian cycles

As there are maximal planar graphs that are non-hamiltonian, there are maximal planar graphs with $C_n(G) = 0$. Therefore, let G be a hamiltonian maximal planar graph on n vertices. Then the following question occurs. What is the minimum number of hamiltonian cycles that G could have in terms of n?
First we will consider 3–connected maximal planar graphs.

Theorem 3.70 (Hakimi, Schmeichel and Thomassen [32]) *Let $n \geq 12$. Form a maximal planar graph G with n vertices as follows: Begin with the graph G_2 of Figure 3.14 (with $i = n - 11$), and in the interior of the triangle $T = v_1 v_2 v_3$ place the graph G_1 of Figure 3.14 so that G_1 and G_2 have precisely the triangle T in common. Then G has exactly four hamiltonian cycles.*
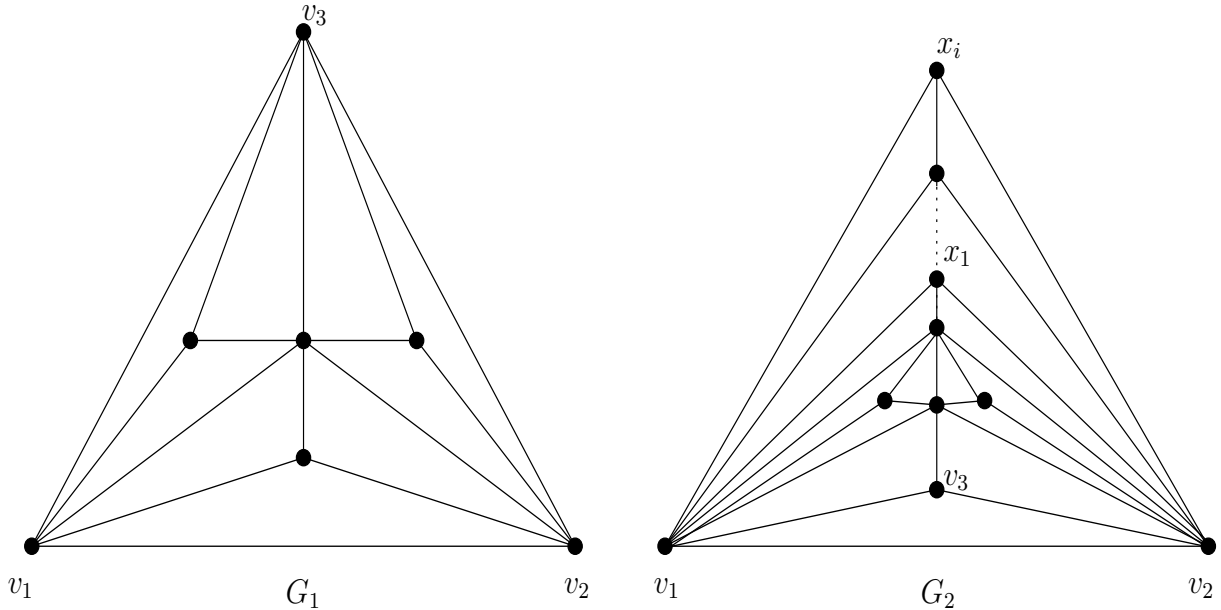
Figure 3.14: Two maximal planar graphs.

We observe that maximal planar graphs constructed in Theorem 3.70 all have connectivity three. It seems natural, therefore, to inquire about the minimum number of hamiltonian cycles in 4–connected maximal planar graphs on n vertices. It is known that every such graph has at least one hamiltonian cycle.

Theorem 3.71 (Hakimi, Schmeichel and Thomassen [32]) *Let G be a 4–connected maximal planar graph with n vertices. Then G contains at least $\frac{n}{\log_2 n}$ hamiltonian cycles.*

The lower bound given in Theorem 3.71 occurs far from tight. In fact we offer the following.

Example 3.72 *Let G be the 4–connected maximal planar graph with n vertices of Figure 3.15. Then G contains at least $2(n-2)(n-4)$ hamiltonian cycles.*
Assume that G is the 4–connected maximal planar graph of Figure 3.15 with $n = 9$ vertices. Then G contains at least 70 hamiltonian cycles. Theorem 3.71 yields G contains at least three hamiltonian cycles.

Now we prove an improved lower bound on the number of hamiltonian cycles.

Theorem 3.73 *Let G be a 4–connected maximal planar graph with $n \geq 7$ vertices. Then G contains at least $4n - 8$ hamiltonian cycles.*

Proof. Let $G = (V, E)$ with $|V| = n$ and $|E| = m$ be a 4–connected maximal planar graph and let $l(G)$ be the number of faces of G. Then Theorem 1.1 of Euler provides $l(G) = m - n + 2$. Moreover, $3l(G) = 2m$, since each edge is located on the boundary of two faces. Then with Theorem 1.1 of Euler we obtain $l = 2n - 4$. Since G is 4–connected, G contains no separating triangles. By Theorem 3.36 there exists a hamiltonian cycle C

of G such that $k_{2F} = 4$. This means each face of G is incident to at least one edge of the hamiltonian cycle C. Since the embedding of a 4–connected maximal planar graph is unique, we embed G such that three other vertices form the exterior face. If we use the same geometric structure as C, we get a new hamiltonian cycle C'. If we do this for each face of G, we get $2n - 4$ different hamiltonian cycles. Since a hamiltonian cycle has two different orientations, we get $2(2n - 4) = 4n - 8$ hamiltonian cycles. ∎

The next example will show that the lower bound given in the previous theorem appears not so far from tight.

Example 3.74 *Consider Example 3.72 again. Assume that G is a 4–connected maximal planar graph of Figure 3.15 with $n = 9$ vertices. Theorem 3.73 yields G contains at least 28 hamiltonian cycles.*
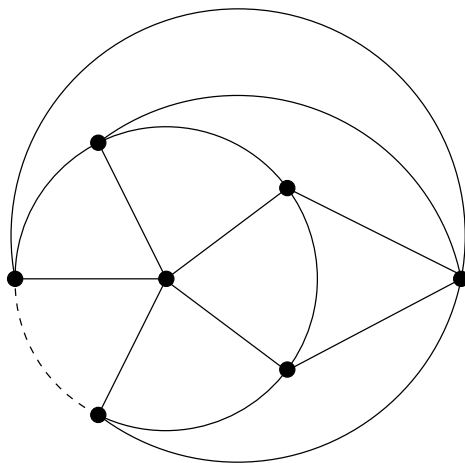

Figure 3.15: A maximal planar graph.

The previous results refer to 4–connected maximal planar graph. Now we consider 3–connected maximal planar graph with at most 5 separating triangles, since we know these graphs are hamiltonian. Let $G = (V, E)$ be a 3–connected maximal planar graph and let $D_1, D_2, ..., D_k$ for $1 \leq k \leq 5$ be the separating triangles of G. Then we define n_s as the number of all vertices inside of all separating triangles.

$$n_s = |\{v \in V(G) \mid v \in V(G) \setminus V(G_{out,D_1,D_2,...,D_k})\}|$$

for $1 \leq k \leq 5$.

Theorem 3.75 *Let G be a maximal planar graph with at most 5 separating triangles and with $n - n_s \geq 7$ vertices. Then G contains at least $4(n - n_s) - 8$ hamiltonian cycles.*

Proof. Let $G = (V, E)$ with $|V| = n$ and $|E| = m$ be a maximal planar graph with at most 5 separating triangles and let $D_1, D_2, ..., D_k$ for $1 \leq k \leq 5$ be the separating triangles of G. Now consider the graph $G_{out,D_1,D_2,...,D_k}$ with $n - n_s$ vertices and without separating triangles. By Theorem 3.36 there exists a hamiltonian cycle C of $G_{out,D_1,D_2,...,D_k}$ such

that $k_{2F} = 4$. This means each face of $G_{out,D_1,D_2,\ldots,D_k}$ is incident to at least one edge of the hamiltonian cycle C. Since the embedding of a 4–connected maximal planar graph is unique, we embed $G_{out,D_1,D_2,\ldots,D_k}$ such that three other vertices form the exterior face. If we use the same geometric structure as C, we get a new hamiltonian cycle C'. If we do this for each face of $G_{out,D_1,D_2,\ldots,D_k}$, we get $2(n-n_s)-4$ different hamiltonian cycles. Since a hamiltonian cycle has two different orientations, we get $2(2(n-n_s)-4) = 4(n-n_s)-8$ hamiltonian cycles. Now we have to insert the n_s deleted vertices. We distinguish the following five cases.

Case 1: The graph $H = G_{in,D_1}$ has four separating triangles. By Theorem 3.37 H is hamiltonian. Let the vertices a, b and c be the vertices of the separating triangle D_1. If $H_{out,D_2,\ldots,D_5}$ has $n < 7$ vertices, then there exists a hamiltonian cycle C'' of $H_{out,D_2,\ldots,D_5}$ such that the hamiltonian cycle passes through at least one of the edges ab, bc and ca. If $H_{out,D_2,\ldots,D_5}$ has $n \geq 7$ vertices, then by Theorem 3.36 there exists a hamiltonian cycle C'' of $H_{out,D_2,\ldots,D_5}$ such that $k_{2F} = 4$ and at least two 2-frontier faces have one edge in common. Now we have to prove that the hamiltonian cycle C'' passes through at least one edge of the edges ab, bc and ca. We insert 4 new vertices each into the facial cycles D_2, D_3, D_4 and D_5 of $H_{out,D_2,\ldots,D_5}$ then joining the new vertex to every vertex incident with the faces of D_2, D_3, D_4 and D_5 respectively. The new graph H' is a hamiltonian maximal planar graph with 4 separating triangles. Now we can extend the hamiltonian cycle C'' of $H_{out,D_2,\ldots,D_5}$ to a hamiltonian cycle C''' of H'. Thus we get $k_{2F}(C''') = 7$ and $k_N(C''') = 3$ with respect to the hamiltonian cycle C'''. Then we can embed the graph H' on a surface such that D_1 is not the non-frontier face. Therefore, the hamiltonian cycle C''' passes through at least one edge of the edges ab, bc and ca. At last we have to replace the vertices in the facial cycles D_2, D_3, D_4 and D_5 by the graphs $G_{in,D_2} - D_2, G_{in,D_3} - D_3, G_{in,D_4} - D_4$ and $G_{in,D_5} - D_5$ respectively. Since G_{in,D_i} with $2 \leq i \leq 5$ are maximal planar graph without separating triangles, the graphs G_{in,D_i} with $2 \leq i \leq 5$ are hamiltonian for any two boundary edges of G_{in,D_i}. Therefore, the graphs $G_{in,D_i} - D_i$ with $2 \leq i \leq 5$ have a hamiltonian path in $G_{in,D_i} - D_i$. Now we replace the two edges which join the new vertices to the hamiltonian cycle C''' by the hamiltonian paths in $G_{in,D_i} - D_i$ with $2 \leq i \leq 5$. Thus we get a hamiltonian cycle C^{iv} which passes through at least one of the edges ab, bc and ca. C^{iv} is a hamiltonian cycle of G_{in,D_1}. Without loss of generality, G_{in,D_1} has a hamiltonian cycle $C_{in} = abcP_{in}(c,a)a$, where $P_{in}(c,a)$ is a hamiltonian path of $G_{in,D} - b$ between c and a. Without loss of generality, C^{iv} passes through the edge ca. Then we replace the edge ca in the hamiltonian cycle C^{iv} of $G_{out,D_1,D_2,\ldots,D_5}$ by the hamiltonian path $P_{in}(c,a)$ between c and a of $G_{in,D_1} - b$. Then the new cycle is a hamiltonian cycle of G.

Case 2: The graph $H = G_{in,D_1}$ has three separating triangles D_2, D_3, D_4. By Theorem 3.18 H is hamiltonian. Then analogically to Case 1 the graph G has a hamiltonian cycle.

Case 3: The graph $H = G_{in,D_1}$ has two separating triangles D_2, D_3. By Theorem 3.17 H is hamiltonian. Then analogically to Case 1 the graph G has a hamiltonian cycle.

Case 4: The graph $H = G_{in,D_1}$ has one separating triangle D_2. By Theorem 3.16 H is hamiltonian. Then analogically to Case 1 the graph G has a hamiltonian cycle.

Case 5: The graph $H = G_{in,D_1}$ has no separating triangles. By Theorem 3.3 H is hamiltonian. Then analogically to Case 1 the graph G has a hamiltonian cycle.
All the other mixed cases arise from the remaining cases. The number of hamiltonian cycles does not change, because we have only one edge replaced with a hamiltonian path in all cases. ∎

Note that the previous result is not true for all maximal planar graphs with at most five separating triangles. As the condition $n - n_s \geq 7$ is very incising in the following sense. The set $\{v \in V(G) \mid v \in V(G) \setminus V(G_{out,D_1,D_2,\dots,D_k})\}$ can contain a lot of vertices, in the worst case $n_s = n - 4$. The worst case occurs if the separating triangles are not disjoint. Moreover, if there are at least three disjoint separating triangles, Theorem 3.75 always holds.

As there are maximal planar graphs with more than five separating triangles that are non-hamiltonian, there are maximal planar graphs with $C_n(G) = 0$. Therefore, let G be a hamiltonian maximal planar graph on n vertices with k separating triangles for $6 \leq k \leq 8$. Then the following question remains open. What is the minimum number of hamiltonian cycles that G could have in terms of n?

3.7 Pancyclic maximal planar graph

In this section we move away from the property that a graph contains one cycle of length only n and turn towards a problem determining when a graph G contains a cycle of each possible length l with $3 \leq l \leq n$.

Definition 3.76 *A graph $G = (V, E)$ on n vertices is called pancyclic if it contains a cycle of each possible length l with $3 \leq l \leq n$.*

First we will prove that the more general class of hamiltonian planar triangulations is pancyclic.

Theorem 3.77 *Let $G = (V, E)$ be a hamiltonian planar triangulation. Then G is pancyclic.*

Proof. For the proof we need the following two claims.

Claim 1. A planar triangulation T, whose vertices are all on X_T, has at least two vertices of degree 2.

Proof of Claim 1. Consider a chord which separates the graph T into two parts. Choose one of those subgraphs and call this graph T_1. Now consider inductively also a chord of T_1. Continuing this approach, we get a graph which is a triangle. Since one vertex has the same degree as in the graph T, this vertex has degree 2. We can find the second vertex with degree 2 if we consider the other subgraph of the first separation.

Claim 2. A planar triangulation T whose vertices are all on X_T is pancyclic.

Proof of Claim 2. For the graph $T = (V, E)$ with $|V| = |X_T| = n$ the exterior cycle X_T is also a hamiltonian cycle. It follows from Claim 1 that there are at least two vertices of degree 2. Let v be one of these vertices. Let v_1 and v_2 be the vertices which are both adjacent to v. Since the boundary of every face of T is a triangle, except possibly the exterior face, the vertices v_1 and v_2 are adjacent. We replace the edges vv_1 and vv_2 in the hamiltonian cycle C with the edge v_1v_2, then we get a cycle C' of length $n - 1$. Consider the graph $T - v$. This graph is also a planar triangulation with a hamiltonian cycle C'. Continuing this approach, we get cycles of length l with $3 \leq l \leq n$. Therefore, T is pancyclic.

Let G be a hamiltonian planar triangulation. Let C be the hamiltonian cycle of G. We generate a new graph G' by deleting all edges which are outside of the hamiltonian cycle C. We get a planar triangulation G', whose vertices are all on $X_{G'}$. It follow from Claim 2 that G' is pancyclic. Then G is also pancyclic. ∎
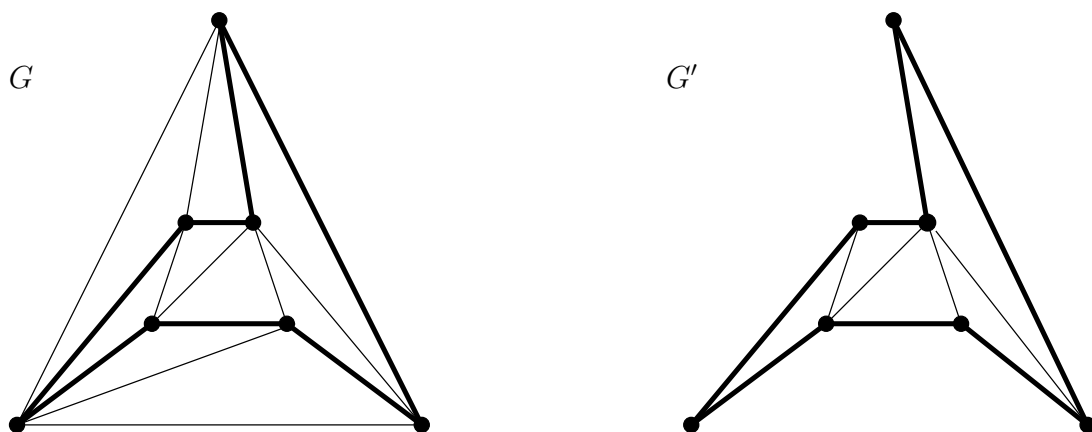

Figure 3.16: Hamiltonian maximal planar graph with related planar triangulation.

This theorem yields that the class of hamiltonian maximal planar graphs is also pancyclic.

Corollary 3.78 *Let $G = (V, E)$ be a hamiltonian maximal planar graph. Then G is pancyclic.*

As not all maximal planar graphs are hamiltonian, we seek a stronger result. In 1979 S. L. Hakimi and E. F. Schmeichel [31] proved the following.

Theorem 3.79 (Hakimi and Schmeichel [31]) *Let $G = (V, E)$ be a maximal planar graph. Let r be the length of the longest cycle in G. Then G contains a cycle of each possible length l, $3 \leq l \leq r$.*

Analogously to Definition 3.76 we consider the special case of pancyclic modulo k.

Definition 3.80 *For any integer k, we define a graph $G = (V, E)$ to be pancyclic modulo k if it contains a cycle of every length modulo k.*

In 1991 N. Dean [23] showed the next result.

Theorem 3.81 (Dean [23]) *Let $G = (V, E)$ be a 3–connected planar graph (except K_4) with minimum degree at least k. Then G is pancyclic modulo k.*

This result yields the following corollary.

Corollary 3.82   *i) Each* MPG-4 *graph is pancyclic modulo* 4.

ii) Each MPG-5 *graph is pancyclic modulo* 5.

3.8   Deletion of vertices

M. D. Plummer [42] has conjectured that any graph obtained from a 4–connected planar graph by deleting one vertex has a hamiltonian cycle. This conjecture follows from Tutte's Theorem as observed by D. A. Nelson [40]. In fact it also follows from Lemma 3.7 of C. Thomassen [51] that the deletion of any vertex from a 4–connected planar graph results in a hamiltonian graph.

Theorem 3.83 (Thomassen [51]) *Let G be a 4–connected planar graph and let x be a vertex of G. Then $G - x$ is hamiltonian.*

D. A. Nelson [40] extended Thomassen's Theorem 3.83.

Theorem 3.84 (Nelson [40]) *Let G be a 4–connected planar graph and let x and y be any two vertices of G that lie on the same face boundary. Then $G - x - y$ is hamiltonian.*

M.D. Plummer [42] also conjectured that any graph obtained from a 4–connected planar graph by deleting two vertices has a hamiltonian cycle. This conjecture was proved by R. Thomas and X. Yu [50].

Theorem 3.85 (Thomas and Yu [50]) *Let G be a 4–connected planar graph and let x and y be two vertices of G. Then $G - x - y$ is hamiltonian.*

Note that deleting three vertices from a 4–connected planar graph may result in a graph which is not 2–connected and hence has no hamiltonian cycle. However, D. P. Sanders [45] obtained a stronger version of Thomas and Yu's result. This follows from Theorem 3.11 of D. P. Sanders.

Theorem 3.86 (Sanders [45]) *Let G be a 4–connected planar graph and let x, y and z be the vertices of a triangle in G. Then $G - x - y - z$ is hamiltonian.*

Now the following question occurs; what happens if we delete one vertex from a 3–connected maximal planar graph with separating triangles?

Theorem 3.87 (Birchel and Helden [10]) *Let G be a maximal planar graph with exactly one separating triangle and let x be a vertex of G. Then $G - x$ is hamiltonian.*

Proof. Let a, b and c be the vertices of the separating triangle D.

Case 1: Let $x \notin D$.

Subcase 1.1: Let $x \in G_{out,D}$. Consider $G_{in,D}$, a, b and c are the vertices of the exterior cycle of $G_{in,D}$. Since $G_{in,D}$ is a maximal planar graph without separating triangles, it follows from Theorem 3.16 of Chen that $G_{in,D}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{in,D}$ has a hamiltonian cycle $C_{in} = abcP_{in}(c,a)a$ where $P_{in}(c,a)$ is a hamiltonian path of $G_{in,D} - b$ between c and a.
Next consider the graph $G_{out,D} - x$. $G_{out,D} - x$ is a planar triangulation without separating triangles. Since G is 3–connected, $G_{out,D} - x$ can only be 2–connected if the vertex x is adjacent to a vertex of degree three. Each vertex of degree three represents a simple separating triangle. Since there is only one separating triangle D and $x \notin D$, $G_{out,D} - x$ is 3–connected. Note that a planar graph can always be embedded in a surface such that a given face of the graph becomes the exterior face, see Chiba and Nishizeki [19]. It follows from Corollary 4.23 that there exists a hamiltonian cycle C_{out} of $G_{out,D} - x$ such that the edge ca is contained in $E(C_{out})$. Now we replace the edge ca by the path $P_{in}(c,a)$ in the graph $G - x$. Then $G - x$ has a hamiltonian cycle and $G - x$ is hamiltonian.

Subcase 1.2: Let $x \in G_{in,D}$. Consider the graph $G_{out,D}$. Note that a planar graph can always be embedded in a surface such that a given face of the graph becomes the exterior face, see Lemma 1.10. Without loss of generality, $G_{out,D}$ has a hamiltonian cycle $C_{out} = abcP_{out}(c,a)a$ where $P_{out}(c,a)$ is a hamiltonian path of $G_{out,D} - b$ between c and a. Next consider the graph $G_{in,D} - x$. $G_{in,D} - x$ is a planar triangulation without separating triangles. It follows from Corollary 4.23 that there exists a hamiltonian cycle C_{in} of $G_{in,D} - x$ such that the edge ca is contained in $E(C_{in})$. Now we replace the edge ca by the path $P_{out}(c,a)$ in the graph $G - x$. Then $G - x$ has a hamiltonian cycle and $G - x$ is hamiltonian.

Case 2: Let $x \in D$.

Subcase 2.1: The vertex x is not adjacent to a vertex of degree three. The graph $G - x$ is a 3–connected planar triangulation without separating triangles. It follows from Corollary 4.23 that $G - x$ is hamiltonian.

Subcase 2.2: The vertex x is adjacent to a vertex of degree three. The graph $G - x$ is a 2–connected planar triangulation without separating triangles. Without loss of generality, let $x = b$. Consider $G_{in,D}$, a, x and c are the vertices of the exterior cycle of $G_{in,D}$. Since $G_{in,D}$ is a maximal planar graph without separating triangles, it follows from Theorem 3.16 of Chen that $G_{in,D}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{in,D}$ has a hamiltonian cycle $C_{in} = axcP_{in}(c,a)a$ where $P_{in}(c,a)$ is a hamiltonian path of $G_{in,D} - x$ between c and a. Consider the graph $G_{out,D}$. Note that a planar graph

can always be embedded in a surface such that a given face of the graph becomes the exterior face, see Lemma 1.10. Without loss of generality, $G_{out,D}$ has a hamiltonian cycle $C_{out} = axcP_{out}(c,a)a$ where $P_{out}(c,a)$ is a hamiltonian path of $G_{out,D} - x$ between c and a. Now we connect the two paths $P_{in}(c,a)$ and $P_{out}(c,a)$ to one cycle. This is a hamiltonian cycle in the graph $G - x$ and the proof is complete ∎

Now we can extend Theorem 3.87 to exactly two separating triangles.

Theorem 3.88 (Birchel and Helden [10]) *Let G be a maximal planar graph with exactly two separating triangles and let x be a vertex of G. Then $G - x$ is hamiltonian.*

Proof. Let D and T be the two separating triangles. Let a, b and c be the vertices of the separating triangle D.

Case 1: x does not belong to any separating triangle.

Subcase 1.1: T is inside the graph $G_{out,D}$.

Subcase 1.1.1: Let $x \in G_{out,D}$. Consider $G_{in,D}$. a, b and c are the vertices of the exterior cycle of $G_{in,D}$. Since $G_{in,D}$ is a maximal planar graph without separating triangles, it follows from Theorem 3.16 of Chen that $G_{in,D}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{in,D}$ has a hamiltonian cycle $C_{in} = abcP_{in}(c,a)a$ where $P_{in}(c,a)$ is a hamiltonian path of $G_{in,D} - b$ between c and a.

Next consider the graph $G_{out,D}$. The graph $G_{out,D}$ is maximal planar with one separating triangle T. Let e, f and g be the vertices of the separating triangle T. The graph $G_{out,D,T}$ is maximal planar without separating triangles. Let u, v and x be the vertices of a triangle F of the graph $G_{out,D,T}$. It follows from Corollary 3.22 that there exists a hamiltonian cycle C_{out} of $G_{out,D,T}$ and edges $ge \in E(T)$ and $ca \in E(D)$ such that ux, xv, ge, ca are distinct and contained in $E(C_{out})$. Consider the graph $G_{in,T}$. Since $G_{in,T}$ is a maximal planar graph without separating triangles, it follows from Theorem 3.16 of Chen that $G_{in,T}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{in,T}$ has a hamiltonian cycle $C'_{in} = efgP'_{in}(g,e)e$ where $P'_{in}(g,e)$ is a hamiltonian path of $G_{in,T} - f$ between g and e. Consider the triangle F. Since the hamiltonian cycle C_{out} of $G_{out,D,T}$ contains the edges ux and xv, we can delete the vertex x. Since u and v are adjacent, we get a hamiltonian cycle C'_{out} of $G_{out,D,T} - x$ which contains the edges ca and ge. Now we replace the edge ca by the path $P_{in}(c,a)$ in the graph $G - x$ and we replace the edge ge by the path $P'_{in}(g,e)$ in the graph $G - x$. Then $G - x$ has a hamiltonian cycle and $G - x$ is hamiltonian.

Subcase 1.1.2: Let $x \in G_{in,D}$. Consider the graph $G_{out,D}$. $G_{out,D}$ is a maximal planar graph with one separating triangle. Note that a planar graph can always be embedded in a surface such that a given face of the graph becomes the exterior face, see Lemma 1.10. It follows from Theorem 3.17 that $G_{out,D}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{out,D}$ has a hamiltonian cycle $C_{out} = abcP_{out}(c,a)a$ where $P_{out}(c,a)$ is a hamiltonian path of $G_{out,D} - b$ between c and a. Next consider the graph $G_{in,D} - x$.

$G_{in,D} - x$ is a planar triangulation without separating triangles. Since G is 3–connected, $G_{in,D} - x$ can only be 2–connected if the vertex x is adjacent to a vertex of degree three. Each vertex of degree three represents a simple separating triangle. Since there is only one separating triangle D and $x \notin D$, $G_{in,D} - x$ is 3–connected. a, b and c are the vertices of the triangle D. It follows from Corollary 4.23 that there exists a hamiltonian cycle C_{in} of $G_{in,D} - x$ such that the edge ca is contained in $E(C_{in})$. Now we replace the edge ca by the path $P_{out}(c, a)$ in the graph $G - x$. Then $G - x$ has a hamiltonian cycle and $G - x$ is hamiltonian.

Subcase 1.2: T is inside the graph $G_{in,D}$.

Subcase 1.2.1: Let $x \in G_{out,D}$. Consider $G_{in,D}$, a, b and c are the vertices of the exterior cycle of $G_{in,D}$. Since $G_{in,D}$ is a maximal planar graph with one separating triangle, it follows from Theorem 3.17 that $G_{in,D}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{in,D}$ has a hamiltonian cycle $C_{in} = abcP_{in}(c,a)a$ where $P_{in}(c,a)$ is a hamiltonian path of $G_{in,D} - b$ between c and a.

Next consider the graph $G_{out,D}$. $G_{out,D} - x$ is a planar triangulation without separating triangles. Since G is 3–connected, $G_{out,D} - x$ can only be 2–connected if the vertex x is adjacent to a vertex of degree three. Each vertex of degree three represents a simple separating triangle. Since there is only one separating triangle D and $x \notin D$, $G_{out,D} - x$ is 3–connected. a, b and c are the vertices of the triangle D. Note that a planar graph can always be embedded in a surface such that a given face of the graph becomes the exterior face, see Lemma 1.10. It follows from Corollary 4.23 that there exists a hamiltonian cycle C_{out} of $G_{out,D} - x$ such that the edge ca is contained in $E(C_{out})$. Now we replace the edge ca by the path $P_{in}(c, a)$ in the graph $G - x$. Then $G - x$ has a hamiltonian cycle and $G - x$ is hamiltonian.

Subcase 1.2.2: Let $x \in G_{in,D}$. Consider $G_{out,D}$. Since $G_{out,D}$ is a maximal planar graph without separating triangles, it follows from Theorem 3.16 of Chen that $G_{out,D}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{out,D}$ has a hamiltonian cycle $C_{out} = abcP_{out}(c,a)a$ where $P_{out}(c,a)$ is a hamiltonian path of $G_{out,D} - b$ between c and a.

Next consider the graph $G_{in,D}$. Note that $G_{in,D}$ is a maximal planar graph with one separating triangle T. Let e, f and g be the vertices of the separating triangle T.

Define $H = G_{in,D}$ and $G_{in,D,T} = H_{out,T}$. Then the graph $G_{in,D,T}$ is a maximal planar graph without separating triangles. Let $F = \{u, v, x\}$ be a triangle of the graph $G_{in,D,T}$. It follows from Corollary 3.22 that there exists a hamiltonian cycle C_{in} of $G_{in,D,T}$ and edges $ge \in E(T)$ and $ca \in E(D)$ such that ux, xv, ge, ca are distinct and contained in $E(C_{in})$. Consider the graph $G_{in,T}$. Since $G_{in,T}$ is a maximal planar graph without separating triangles, it follows from Theorem 3.16 of Chen that $G_{in,T}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{in,T}$ has a hamiltonian cycle $C'_{in} = efgP'_{in}(g,e)e$ where $P'_{in}(g,e)$ is a hamiltonian path of $G_{in,T} - f$ between g and e. Consider the triangle F. Since the hamiltonian cycle C_{in} of $G_{in,D,T}$ contains the edges ux and xv, we can delete the vertex x. Since u and v are adjacent, we get a hamiltonian cycle C'_{in} of $G_{in,D,T} - x$ which contains the edges ca and ge. Now we replace the

edge ca by the path $P_{out}(c, a)$ in the graph $G - x$ and replace the edge ge by the path $P'_{in}(g, e)$ in the graph $G - x$. Then $G - x$ has a hamiltonian cycle and $G - x$ is hamiltonian.

Case 2: Without loss of generality, $x \in D$.

Subcase 2.1: T is inside the graph $G_{out,D}$. Without loss of generality, let $x = b$. Consider $G_{in,D}$, a, x and c are the vertices of the exterior cycle of $G_{in,D}$. Since $G_{in,D}$ is a maximal planar graph without separating triangles, it follows from Theorem 3.16 of Chen that $G_{in,D}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{in,D}$ has a hamiltonian cycle $C_{in} = axcP_{in}(c, a)a$ where $P_{in}(c, a)$ is a hamiltonian path of $G_{in,D} - x$ between c and a. Consider the graph $G_{out,D}$. $G_{out,D}$ is a maximal planar graph with one separating triangle. Note that a planar graph can always be embedded in a surface such that a given face of the graph becomes the exterior face, see Chiba and Nishizeki [19]. It follows from Theorem 3.17 that $G_{out,D}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{out,D}$ has a hamiltonian cycle $C_{out} = axcP_{out}(c, a)a$ where $P_{out}(c, a)$ is a hamiltonian path of $G_{out,D} - x$ between c and a. Now we connect the two paths $P_{in}(c, a)$ and $P_{out}(c, a)$ to one cycle. This is a hamiltonian cycle in the graph $G - x$ and $G - x$ is hamiltonian.

Subcase 2.2: T is inside the graph $G_{in,D}$. Without loss of generality, let $x = b$. Consider $G_{in,D}$, a, x and c are the vertices of the exterior cycle of $G_{in,D}$. Since $G_{in,D}$ is a maximal planar graph with one separating triangle, it follows from Theorem 3.17 that $G_{in,D}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{in,D}$ has a hamiltonian cycle $C_{in} = axcP_{in}(c, a)a$ where $P_{in}(c, a)$ is a hamiltonian path of $G_{in,D} - x$ between c and a. Consider the graph $G_{out,D}$. Note that a planar graph can always be embedded in a surface such that a given face of the graph becomes the exterior face, see Chiba and Nishizeki [19]. Without loss of generality, $G_{out,D}$ has a hamiltonian cycle $C_{out} = axcP_{out}(c, a)a$ where $P_{out}(c, a)$ is a hamiltonian path of $G_{out,D} - x$ between c and a. Now we connect the two paths $P_{in}(c, a)$ and $P_{out}(c, a)$ to one cycle. This is a hamiltonian cycle in the graph $G - x$ and the proof is complete. ∎

We can extend Theorem 3.88 to exactly three separating triangles, if we assume that the deleted vertex x belongs to one of the separating triangles. The problem of whether $G - x$ is hamiltonian if the deleted vertex x belongs to the interior or exterior of one separating triangle is still open.

Theorem 3.89 (Birchel and Helden [10]) *Let G be a maximal planar graph with exactly three separating triangles and let x be a vertex of one of these separating triangles of G. Then $G - x$ is hamiltonian.*

Proof. Let D be one separating triangle and let a, b and c be the vertices of the separating triangle D. Without loss of generality, $x \in D$ and without loss of generality, let $x = b$. Consider $G_{in,D}$. a, x and c are the vertices of the exterior cycle of $G_{in,D}$. Since $G_{in,D}$ is a maximal planar graph with at most two separating triangles, it follows from Theorem 3.27 that $G_{in,D}$ is hamiltonian for any two boundary edges. Without loss of

generality, $G_{in,D}$ has a hamiltonian cycle $C_{in} = axcP_{in}(c,a)a$ where $P_{in}(c,a)$ is a hamiltonian path of $G_{in,D} - x$ between c and a. Next consider the graph $G_{out,D}$. $G_{out,D}$ is a maximal planar graph with at most two separating triangles. Note that a planar graph can always be embedded in a surface such that a given face of the graph becomes the exterior face, see Lemma 1.10. It follows from Theorem 3.27 that $G_{out,D}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{out,D}$ has a hamiltonian cycle $C_{out} = axcP_{out}(c,a)a$ where $P_{out}(c,a)$ is a hamiltonian path of $G_{out,D} - x$ between c and a. Now we connect the two paths $P_{in}(c,a)$ and $P_{out}(c,a)$ to one cycle. This is a hamiltonian cycle in the graph $G - x$ and $G - x$ is hamiltonian. ∎

The next example shows that the deletion of one vertex from a maximal planar graph with four separating triangles may result in a graph which has no hamiltonian cycle.

Example 3.90 (Birchel and Helden [10]) *The left side of Figure 3.17 shows a hamiltonian maximal planar graph G with four separating triangles. The right side of Figure 3.17 shows a non-hamiltonian planar triangulation $G - x$ with one separating triangle after the deletion of the vertex x. Since the adjacent vertices a, b and c have degree two, these vertices imply a cycle. Thus it is not possible to form a hamiltonian cycle in the graph $G - x$. Note that the vertex x belongs to three separating triangles.*
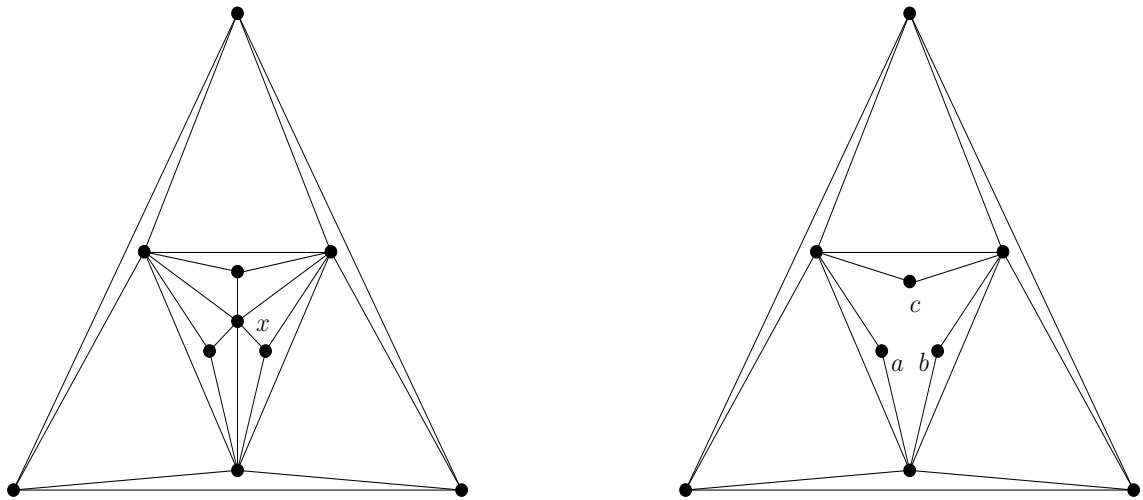

Figure 3.17: Hamiltonian maximal planar graph G and non-hamiltonian graph $G - x$.

Now the following question occurs; what happens if we delete two vertices from a 3–connected maximal planar graph with separating triangles?

Theorem 3.91 (Birchel and Helden [10]) *Let $G = (V, E)$ be a hamiltonian maximal planar graph with at least one separating triangle and let x and y be two vertices of one of these separating triangles of G. Then $G - x - y$ is non-hamiltonian.*

Proof. Let x, y and z be the vertices of the separating triangle T. Then there is a subgraph $T_{in} = G_{in,T} - V(T)$, see Figure 3.18. After the deletion of x and y there is only

the possibility by z on a hamiltonian cycle to get into the subgraph T_{in}. After running through the subgraph we must leave it again. However, this is possible only by the vertex z. This leads to a contradiction. ∎
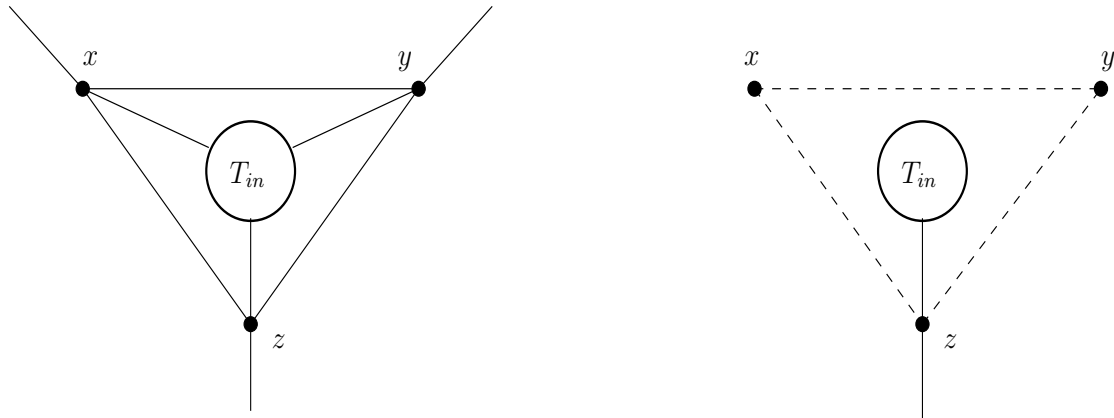

Figure 3.18: Deletion of an edge of a separating triangle.

However, if we assume that exactly one vertex belongs to the separating triangle, we can prove the following.

Theorem 3.92 (Birchel and Helden [10]) *Let $G = (V, E)$ be a maximal planar graph with exactly one separating triangle. If exactly one vertex of two vertices x and y belongs to the separating triangle, then $G - x - y$ is hamiltonian.*

Proof. Let u, v and x be the vertices of the separating triangle T.

Case 1: The vertex y is inside the graph $G_{in,T}$. Consider the graph $G_{in,T} - y$. $G_{in,T} - y$ is a planar triangulation without separating triangles. Since G is 3–connected, $G_{in,T} - y$ can only be 2–connected if the vertex y is adjacent to a vertex of degree three. Each vertex of degree three represents a simple separating triangle. Since there is only one separating triangle T and y does not belong to T, $G_{in,T} - y$ is 3–connected. u, v and x are the vertices of the triangle T. It follows from Corollary 4.23 that there exists a hamiltonian cycle C_{in} of $G_{in,T} - y$ such that the edge uv is contained in $E(C_{in})$. Next consider the graph $G_{out,T}$. $G_{out,T}$ is a maximal planar graph without separating triangles. Note that a planar graph can always be embedded in a surface such that a given face of the graph becomes the exterior face, see Chiba and Nishizeki [19]. It follows from Theorem 3.16 that $G_{out,T}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{out,T}$ has a hamiltonian cycle $C_{out} = uxvP_{out}(v, u)u$ where $P_{out}(v, u)$ is a hamiltonian path of $G_{out,T} - x$ between v and u. Now we replace the edge uv by the path $P_{out}(v, u)$ in the graph $G - x - y$. This is a hamiltonian cycle in the graph $G - x - y$ and $G - x - y$ is hamiltonian.

Case 2: The vertex y is inside the graph $G_{out,T}$. Consider the graph $G_{out,T} - y$. $G_{out,T} - y$ is a planar triangulation without separating triangles. Since G is 3–connected, $G_{out,T} - y$ can only be 2–connected if the vertex y is adjacent to a vertex of degree three.

Each vertex of degree three represents a simple separating triangle. Since there is only one separating triangle T and y does not belong to T, $G_{out,T} - y$ is 3–connected. u, v and x are the vertices of the triangle T. It follows from Corollary 4.23 that there exists a hamiltonian cycle C_{out} of $G_{out,T} - y$ such that the edge uv is contained in $E(C_{out})$. Next consider the graph $G_{in,T}$. $G_{in,T}$ is a maximal planar graph without separating triangles. Note that a planar graph can always be embedded in a surface such that a given face of the graph becomes the exterior face, see Chiba and Nishizeki [19]. It follows from Theorem 3.16 that $G_{in,T}$ is hamiltonian for any two boundary edges. Without loss of generality, $G_{in,T}$ has a hamiltonian cycle $C_{in} = uxvP_{in}(v, u)u$ where $P_{in}(v, u)$ is a hamiltonian path of $G_{in,T} - x$ between v and u. Now we replace the edge uv by the path $P_{in}(v, u)$ in the graph $G - x - y$. This is a hamiltonian cycle in the graph $G - x - y$ and $G - x - y$ is hamiltonian. ∎

The problem of whether $G - x - y$ is hamiltonian, if the deleted vertices x and y both do not belong to the separating triangle is still open. Theorem 3.92 provides the next immediate corollary, since the vertex which belongs to the separating triangle destroys this separating triangle and we get a planar triangulation without separating triangles.

Corollary 3.93 (Birchel and Helden [10]) *Let G be a 3–connected planar triangulation without separating triangles and let x be a vertex of G. Then $G - x$ is hamiltonian.*

If for a maximal planar graph G with exactly one separating triangle and for two non-adjacent vertices x and y which do not belong to the separating triangle, the graph $G - x - y$ is hamiltonian, then for a 3–connected planar triangulation G with exactly one separating triangle and for a vertex x, the graph $G - x$ is hamiltonian.

Now we consider two separating triangles. The next example shows that the deletion of two vertices from a maximal planar graph with exactly two separating triangles may result in a graph which has no hamiltonian cycle.

Example 3.94 (Birchel and Helden [10]) *The left side of Figure 3.19 shows a hamiltonian maximal planar graph G with two separating triangles. The right side of Figure 3.19 shows a non-hamiltonian planar triangulation $G - u - v$ without separating triangles after the deletion of the non-adjacent vertices u and v. Note that both vertices u and v belong to a separating triangle.*

Anyway the following result can be proved.

Theorem 3.95 (Birchel and Helden [10]) *Let G be a hamiltonian maximal planar graph with at most two separating triangles T_1 and T_2 and let x and y be two adjacent vertices of G that do not lie on any separating triangle. If the graph $G - x - y$ is 3–connected, then $G - x - y$ is hamiltonian.*

Proof. Since x and y are adjacent, we get a planar triangulation after the deletion of both vertices. If x and y lie on a separating 4-cycle, then $G - x - y$ is only 2–connected. Note that each vertex cut of a maximal planar graph is a cycle. If we assume that there is a chordal 4-cut, a cycle arises, because both edges which lie on the exterior face are
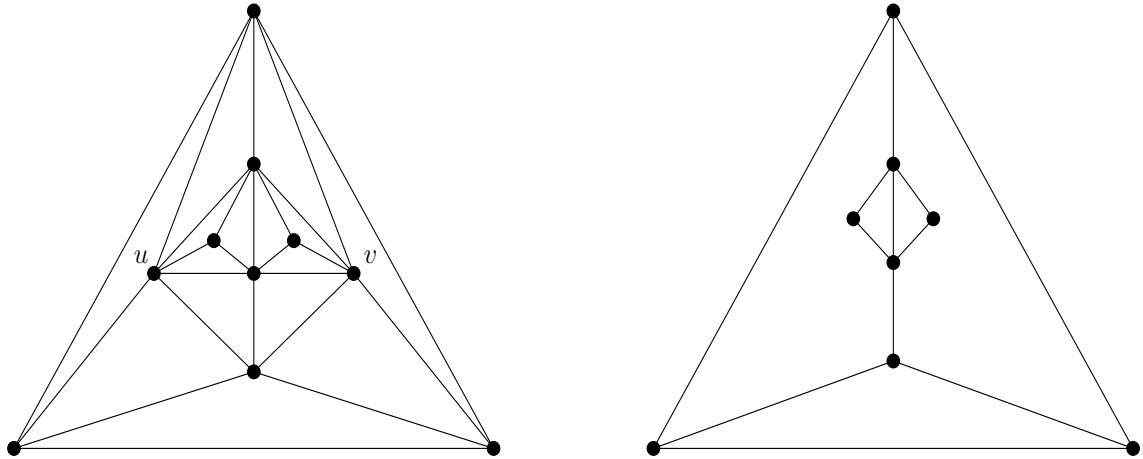
Figure 3.19: Deletion of two vertices of a maximal planar graph with two separating triangles.

incident anyway, because the exterior face is bounded by a 3-cycle. Since $G - x - y$ is 3–connected, $G - x - y$ is hamiltonian by Theorem 4.27. ∎

Now we consider three separating triangles. The next example shows that the deletion of two vertices from a maximal planar graph with exactly three separating triangles may result in a graph which has no hamiltonian cycle.

Example 3.96 (Birchel and Helden [10]) *Figure 3.20 shows a hamiltonian maximal planar graph G with three separating triangles. The left side of Figure 3.21 shows a non-hamiltonian planar triangulation $G - u - v$ with one separating triangle after the deletion of the adjacent vertices u and v. The right side of Figure 3.21 shows a non-hamiltonian planar triangulation $G - u - w$ without separating triangles after the deletion of the non-adjacent vertices u and w. Note that all three vertices u, v and w belong to a separating triangle.*

An additional condition which we could demand would be that we have only simple separating triangles T_i in the graph. This means G_{in,T_i} would be the K_4, or in other words only vertices of degree three form separating triangles. Now we determine one of these vertices of degree three as one of the vertices to be deleted. Then we can consider that the deletion of a vertex of degree three does not transform a maximal planar graph into a planar triangulation. The consideration for the second vertex, which we would like to delete, can be led back in the case of deletion of only one vertex of a maximal planar graph with $i - 1$ separating triangles.

3.9   Claw-free maximal planar graph

Definition 3.97 *A graph G is claw-free if it contains no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$.*
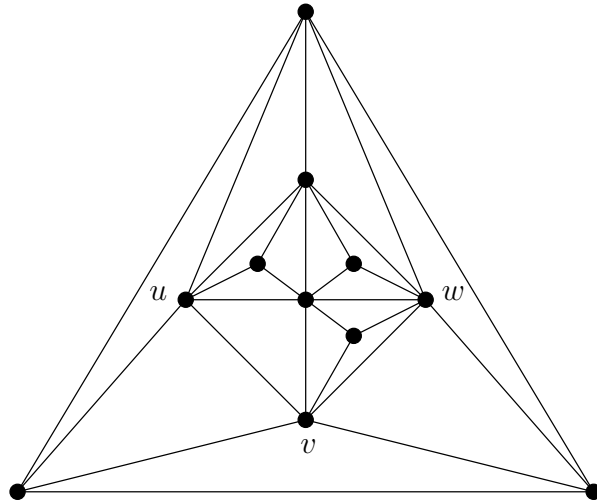
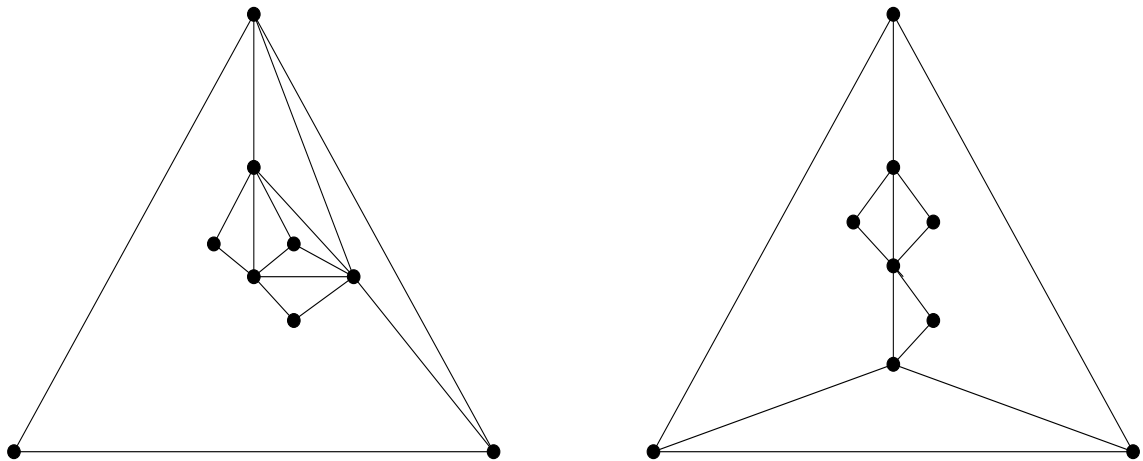Figure 3.20: A maximal planar graph with three separating triangles.

Figure 3.21: $G - u - v$ $G - u - w$.

In 1990 M. D. Plummer showed that every arbitrary 3–connected claw-free and planar graph G has maximum degree at most six.

Theorem 3.98 (Plummer [43]) *If G is 3–connected claw-free and planar, then*

i) $\Delta(G) \leq 6$, *and*

ii) *if a vertex v has degree 6 in G then v lies on at least two separating triangles.*

Note that the graph need not be maximal planar. This theorem yields the following immediate corollary.

Corollary 3.99 *If G is a 3–connected claw-free and planar graph without separating triangles, then $\Delta(G) \leq 5$.*

We are now prepared to find all claw-free maximal planar graphs containing no separating triangle.

Theorem 3.100 (Plummer [43]) *Let G be a claw-free maximal planar graph without separating triangles.*

> *i) If $\Delta(G) = 2$, then $G = K_3$.*
>
> *ii) If $\Delta(G) = 3$, then $G = K_4$.*
>
> *iii) If $\Delta(G) = 4$, then G is the octahedron, see Figure 2.2.*
>
> *iv) If $\Delta(G) = 5$, then G is one of the five graphs $G(7), G(8), G(9), G(10)$ shown in Figure 3.22 or G is the icosahedron, see Figure 2.3.*

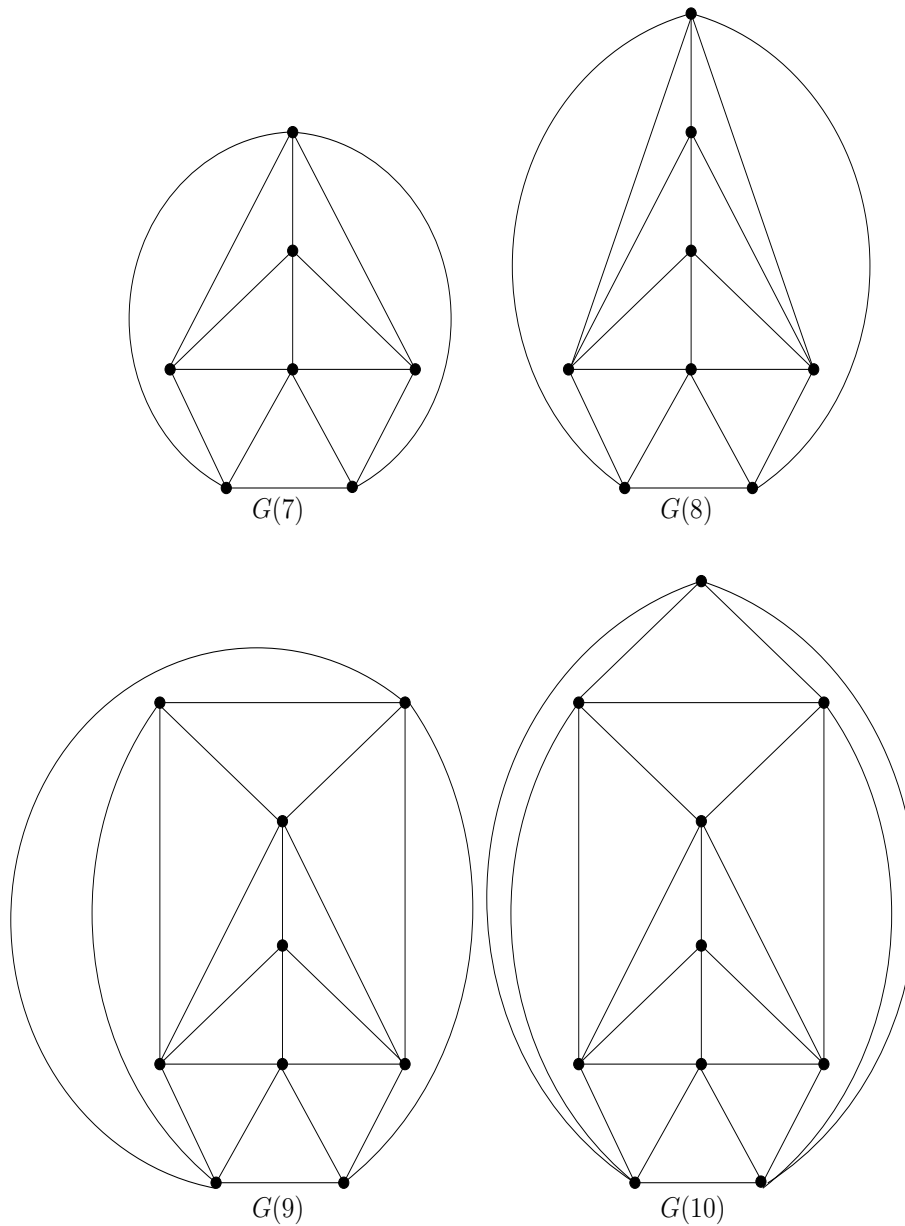Figure 3.22: Claw-free maximal planar graphs without separating triangles for $7 \leq n \leq 10$.

Now we want to know whether claw-free maximal planar graphs are hamiltonian. In 1973 F. Ewald [28] proved the following result.

Theorem 3.101 (Ewald [28]) *If G is a planar triangulation with $\Delta(G) \leq 6$, then G is hamiltonian.*

Using this result of F. Ewald, our next result is immediate.

Theorem 3.102 *If G is a claw-free maximal planar graph, then G is hamiltonian.*

By using the fact that maximal planar graphs are locally 2–connected, M. D. Plummer was able to prove a stronger result.

Theorem 3.103 (Plummer [43]) *If G is a claw-free maximal planar graph, then G is hamiltonian-connected.*

At the end of this chapter we will summarise some of the results we have achieved, see Table 3.1. First we consider the results of Section 3.6, Furthermore, we summarise some of the results we have achieved in Section 3.8 and in Section 3.1. In Table 3.1. $G = (V, E)$ is a maximal planar graph with k separating triangles $D_1, D_2, ..., D_k$ and $x, y, z \in V(G)$.

k	G	G − x	G − x − y	G − x − y − z	C_n(G)
0	Thm 3.3 hamiltonian	Thm 3.83 hamiltonian	Thm 3.85 hamiltonian	Thm 3.86 hamiltonian if x, y, z one face	Thm 3.73 $4n - 8$
1	Thm 3.15 hamiltonian	Thm 3.87 hamiltonian	Figure 3.18 non-hamiltonian	—	Thm 3.75 $4(n - n_s) - 8$
2	Thm 3.18 hamiltonian	Thm 3.88 hamiltonian	Figure 3.19 non-hamiltonian	—	Thm 3.75 $4(n - n_s) - 8$
3	Thm 3.29 hamiltonian	Thm 3.89 hamiltonian if $x \in D_k$	Figure 3.21 non-hamiltonian	—	Thm 3.75 $4(n - n_s) - 8$
4	Thm 3.37 hamiltonian	Figure 3.17 non-hamiltonian	—	—	Thm 3.75 $4(n - n_s) - 8$
5	Thm 3.38 hamiltonian	—	—	—	Thm 3.75 $4(n - n_s) - 8$

Table 3.1: Summary of some of the results of this chapter

Chapter 4

Hamiltonicity of planar triangulations

In this chapter we turn to general planar triangulations. Certainly each maximal planar graph is a planar triangulation. Before discussing results related to planar triangulation in more detail in the later sections, I will briefly explain some properties of planar triangulations in the first section. The second section of this chapter deals with general results on 2–connected planar triangulations. In the third section we focus on 3–connected planar triangulations and the last section concentrates on 4–connected planar triangulations.

4.1   Properties of planar triangulations

In this section we study some properties of planar triangulations. We start this section with a simple, but very useful lemma.

Lemma 4.1 (Birchel and Helden [10]) *Let T be a planar triangulation. If a minimal k–cut S of T divides T into two components, then S is connected.*

Proof. Since the exterior cycle X_T of T is 2–connected, the minimal k–cut S contains at least two vertices. Consider one vertex $x \in S$. Since the k–cut is minimal, x is connected to each component of $T - S$. If this was not the case, this would mean that this vertex of S could be deleted in S, this would lead to a contradiction. We denote the two components of $T - S$ with Y and Z, see Figure 4.1. Let $y_1, y_2, ..., y_k \in Y$ and $z_1, z_2, ..., z_m \in Z$ be the neighbors of x. Since the boundary of every face of T is a triangle, except possibly the exterior face, the vertices y_k and z_1 are adjacent. This would mean that Y and Z are not two components, which would prove a contradiction. Therefore, there must be a further neighbor of between y_k and z_1 which belongs to the k–cut S. The desired result follows inductively. ∎

We proceed with an easy observation.

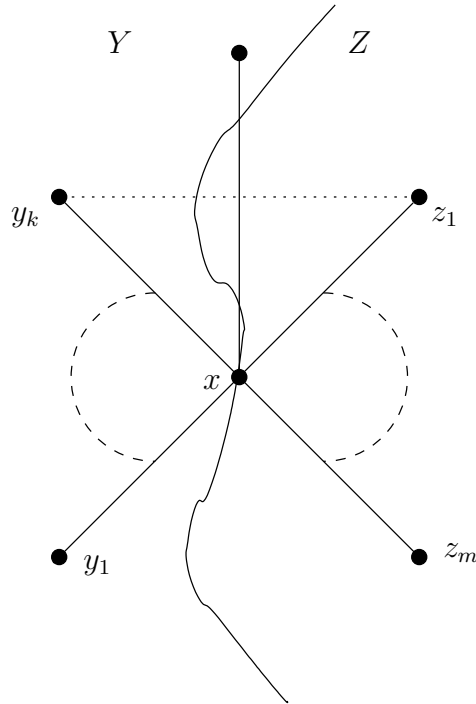**Observation 4.2** *A planar triangulation is at least 2–connected.*

Figure 4.1: A separating subgraph.

Proof. Let T be a planar triangulation. Since the exterior cycle X_T of T is a cycle, T is at least 2–connected. ∎

Note that maximal planar graphs on at least four vertices have minimal degree at least three or minimal degree at most five. Since the deletion of a vertex of a maximal planar graph provides a planar triangulation, we get the following observation.

Observation 4.3 *Let T be a planar triangulation then*

$$\delta(T) \in \{2, 3, 4, 5\}.$$

This observation provides a very interesting corollary.

Corollary 4.4 *Let T be a planar triangulation. Then T is at most 5–connected.*

Moreover, we get the following result.

Lemma 4.5 *Let T be a planar triangulation and let $x \in V(T)$ be a vertex of degree two. Then x is on the exterior cycle X_T of T.*

Proof. Let T be a planar triangulation and let $x \in V(T)$ be a vertex of degree two. Assume x is not on the exterior cycle X_T of T. Then x belongs to a facial cycle F. Since T is a planar triangulation, each facial cycle is a triangle. Since x has degree 2, F is not a triangle. This contradiction provides the claim. ∎

4.2   2–connected planar triangulations

In 1990 M. B. Dillencourt [24] generalized Whitney's Theorem 3.3 in a different direction, to some of the graphs that he called NST–triangulations. M. B. Dillencourt defined NST–triangulations to be those planar graphs that are almost triangulations in the sense that every face is a triangle except one, the exterior face, and those which have no separating triangles. Note that an NST–triangulation need not even be 3–connected. For Theorem 4.7 we need the following definition.

Definition 4.6 *We define the boundary graph derived from a planar triangulation T, denoted BG(T), to be the subgraph induced by the boundary vertices of T.*

Then M. B. Dillencourt [24] showed the following.

Theorem 4.7 (Dillencourt [24]) *If T is an NST–triangulation such that the boundary of each face of the boundary graph $\mathrm{BG}(T)$ of T has at most three chordal edges, then T is hamiltonian.*

The previous Theorem 4.7 provides the following two corollaries.

Corollary 4.8 *If an NST–triangulation T has at most three chords, then T is hamiltonian.*

Corollary 4.9 *If an NST–triangulation T has at most seven boundary vertices, then T is hamiltonian.*

Proof. Suppose T is a non-hamiltonian NST–triangulation. By Theorem 4.7 some face F of the boundary graph $\mathrm{BG}(T)$ must have at least four chordal edges. Each of these chordal edges is a chord of T, and hence separates F from some boundary vertex, a different boundary vertex for each chord. These four vertices plus the at least four endpoints of the chords, constitute at least eight distinct boundary vertices. This is a contradiction. ■

Note that a 2–cut of a 2–connected planar triangulation is always connected. Therefore, 2–cuts are always chords. Now we study how many chords are allowed to occur and their location, so that a hamiltonian cycle can be found. Therefore, we give a new definition.

Definition 4.10 *Let T be a 2–connected planar triangulation and $C(T)$ be the set of vertices of chords of G. Consider $H = T − C(T)$. H decomposes into several components. Now we add the chords to each corresponding component in the following sense. A component K of $T − C(T)$ together with all vertices of H adjacent to vertices of K and all edges with one end in H and the other in K. Now we add the chords to each component. We call the subgraphs obtained from this decomposition the chordal-subgraphs of T.*

Note that chordal-subgraphs are also planar triangulations, whereas the boundary graphs are almost 2–connected planar graphs. Now we have to analyse the different forms of the chordal-subgraphs.

Lemma 4.11 (Birchel and Helden [10]) *Let T be a 2–connected NST–triangulation. The chordal-subgraphs of T can have the following structures.*

a) *The subgraph is 3–connected.*

b) *The subgraph is a 3–cycle whose vertices are boundary vertices. Then there are three possibilities.*

   i) *Exactly one edge is a chord, see Figure 4.2, 1).*

   ii) *Exactly two edges are chords, see Figure 4.2, 2).*

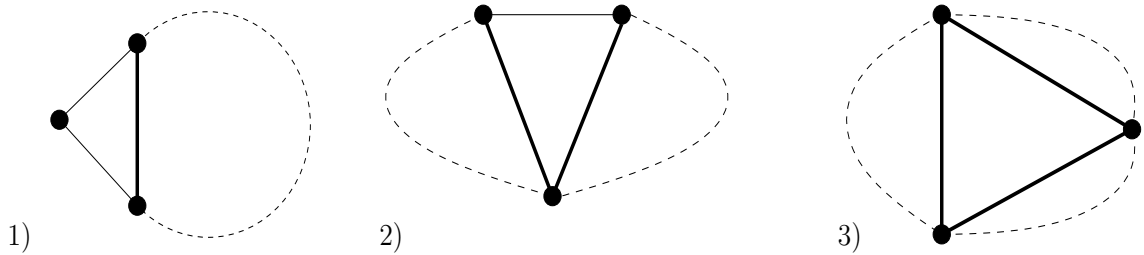   iii) *All three edges are chords, see Figure 4.2, 3).*

Figure 4.2: Three 3–cycles as chordal-subgraphs.

Now we can prove the following result.

Theorem 4.12 (Birchel and Helden [10]) *If T is a 2–connected NST–triangulation such that each chordal-subgraph of T has at most three chordal edges, then T is hamiltonian.*

Proof. Theorem 4.7 of Dillencourt shows that a NST–triangulation T is hamiltonian if the boundary of each face of the boundary graph $BG(T)$ of T has at most three chordal edges. The vertices of one face of the boundary graph $BG(T)$ correspond to the vertices of the exterior cycle X_U of one chordal-subgraph U. Since the chordal-subgraphs are constructed with the aid of the chords we can transfer the claim. ∎

The statement of Theorem 4.12 can be extended under certain conditions.

Theorem 4.13 (Birchel and Helden [10]) *Let T be a 2–connected NST–triangulation. Let $k = xy$ be a chordal edge of T such that k is a boundary edge of a chordal-subgraph U with $d_T(x) > 3$ and $d_T(y) > 3$. If each chordal-subgraph has at most three such chordal edges of T in their exterior cycles, then T is hamiltonian.*

Proof. We consider two cases.

Case 1: In the first case we consider all graphs whose chordal-subgraphs have only chordal edges whose incident vertices have a degree of at least four. The claim follows from Theorem 4.12.

Case 2: In the second case we consider all graphs which have at least one chordal-subgraph with at least one chordal edge and at least one incident vertex to the chordal edge has degree three. Consider this chordal-subgraph U. Let without loss of generality x be one vertex with degree three $d_T(x) = 3$, see Figure 4.3. It is necessary that $d_U(x) = 2$. Lemma 4.11 provides that U is a 3–cycle, since U is 2–connected. It follows from Theorem 4.12 that U has a hamiltonian cycle. This means the chord k belongs to the hamiltonian cycle. The other chordal-subgraphs also have hamiltonian cycles. If the chordal-subgraph is 2–connected, the claim also follows from Theorem 4.12. If the chordal-subgraph is 3–connected, the claim follows from Corollary 4.23. The exterior cycle of this chordal-subgraph has at least four vertices. Therefore, there are hamiltonian cycles containing three edges of the exterior cycles of these chordal-subgraphs. Now we join all hamiltonian cycles of the chordal-subgraphs together. We delete all chords of the chordal-subgraphs in the hamiltonian cycles. Instead we use the adjacent sub-cycles of the adjacent chordal-subgraphs. Since we have at most three chordal edges in 3–connected chordal-subgraphs, we get a hamiltonian cycle C of T. ∎
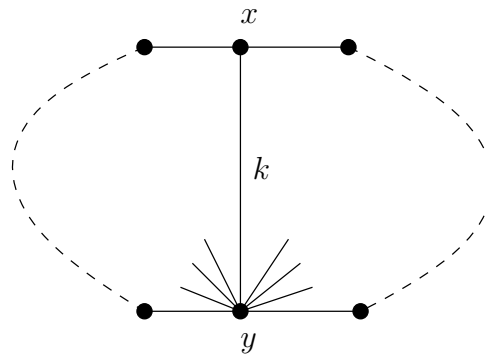

Figure 4.3: A vertex of a chordal edge with degree three.

Thus we obtain the following easy corollary.

Corollary 4.14 (Birchel and Helden [10]) *Let T be a 2–connected NST–triangulation. Let $k = xy$ be a chordal edge of T such that k is a boundary edge of a chordal-subgraph U with $d_T(x) = 3$ or $d_T(y) = 3$. Each chordal-subgraph U can have infinitely many such chords in X_U. If each chordal-subgraph has at most three other chordal edges of T in their exterior cycles, then T is hamiltonian.*

The example of Figure 4.4 shows that the theorems above cannot be improved. The example has exactly four chords and eight boundary vertices.

4.3   3–connected planar triangulations

In this section we will consider planar triangulations without chords.

Lemma 4.15 *Let G be a planar triangulation without chords. Then G is 3–connected.*
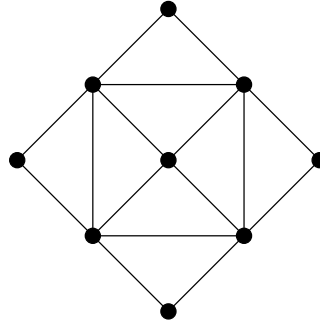
Figure 4.4: A non-hamiltonian NST–triangulation.

If a graph G is 3–connected and planar, then Lemma 4.15 holds reversely. Naturally, for a planar triangulation without chords it is of significance to consider all possible 3–cuts. So we give the following definition.

Definition 4.16 *Let $T = (V, E)$ be a planar triangulation and let a subset $S \subset V(G)$ with $|S| = 3$ be not an induced cycle. If $T - S$ is not connected, then S is called chordal 3–cut.*

Note that a 3–cut in a planar triangulation is always a separating triangle or a chordal 3–cut.

Lemma 4.17 (Birchel and Helden [10]) *Let $T = (V, E)$ be a planar triangulation and let S be a chordal 3–cut. Then exactly two vertices of S are boundary vertices of T.*

Proof. The exterior cycle X_T of a planar triangulation is at least 2–connected. So we need at least two non-adjacent boundary vertices whose removal causes X_T to become disconnected. ∎

In 1990 M. B. Dillencourt [24] proved the next result.

Theorem 4.18 (Dillencourt [24]) *If an NST–triangulation T has no chords, then T is hamiltonian.*

Theorem 4.18 provides the following corollary.

Corollary 4.19 *Let $G = (V, E)$ be a 3–connected NST–triangulation. Then G is hamiltonian.*

In 1996 D. P. Sanders introduced the following definition.

Definition 4.20 *Let G be a planar graph. An interior component 3–cut S of G is a minimal k–cut of G such that $|S| = 3$ and that there is a component of $G - S$ that contains no boundary vertices of G.*

Then D. P. Sanders [46] showed the following.

Theorem 4.21 (Sanders [46]) *Let G be a 3–connected planar graph without interior component 3–cuts. Given any two edges of X_G, G has a hamiltonian cycle containing those edges.*

Furthermore he showed the next result.

Theorem 4.22 (Sanders [46]) *Let G be a 3–connected planar graph without interior component 3–cuts and $|V(X_G)| \geq 4$. Given any three edges of X_G, G has a hamiltonian cycle containing those edges.*

Note that graphs without interior component 3–cuts include 4–connected planar graphs and NST–triangulations. Thus we get an improvement of the result of M. D. Dillencourt.

Corollary 4.23 *Let G be a 3–connected NST–triangulation. Given any two edges of X_G, G has a hamiltonian cycle containing those edges.*
Let G be a 3–connected NST–triangulation and $|V(X_G)| \geq 4$. Given any three edges of X_G, G has a hamiltonian cycle containing those edges.

Until now we have considered only 3–connected NST–triangulations. Now we will consider 3–connected planar triangulations with separating triangles.

Definition 4.24 *Let T be a planar triangulation and let $D_1, D_2, ..., D_k$ be k separating triangles. Let T_{in,D_1} ($T_{in,D_2}, ..., T_{in,D_k}$, respectively) be the subgraph of T derived by deleting all the vertices outside the separating triangle D_1 ($D_2, ..., D_k$, respectively). Let $T_{out,D_1,D_2,...,D_k}$ be the subgraph of T derived by deleting all the vertices inside the separating triangles $D_1, D_2, ..., D_k$.*

Let T be a 3–connected planar triangulation. Suppose that D is a separating triangle in T. Then we define a rooted decomposition tree B of T as the equivalent definition for maximal planar graphs in Chapter 3.

Theorem 4.25 (Birchel and Helden [10]) *Let T be a 3–connected planar triangulation with $|X_T| \geq 4$ and let B be the rooted decomposition tree of the given embedding of T. If the root of the tree B has at most three children whose related disjoint separating triangles have one edge in common with X_T and if all other vertex of the tree B has at most two children, then T is hamiltonian.*

Proof. Let $D_1 = a_1 b_1 c_1$, $D_2 = a_2 b_2 c_2$ and $D_3 = a_3 b_3 c_3$ be the three separating triangles of the root. Without loss of generality let $k_i = a_i b_i$, $1 \leq i \leq 3$ be the edges which are in common with X_T. The graph T_{out,D_1,D_2,D_3} is a NST-triangulation with $|X_{T_{out,D_1,D_2,D_3}}| > 3$. It follows from Corollary 4.23 that the graph T_{out,D_1,D_2,D_3} has a hamiltonian cycle C which contains the edges $k_i = a_i b_i$, $1 \leq i \leq 3$. It follows from Theorem 3.28 that T_{in,D_i} are hamiltonian for any two boundary edges of T_{in,D_i}, $1 \leq i \leq 3$. Without loss of generality, T_{in,D_i} has a hamiltonian cycle

$$C_{in,D_i} = a_i c_i b_i P_{in,D_i}(b_i, a_i) a_i$$
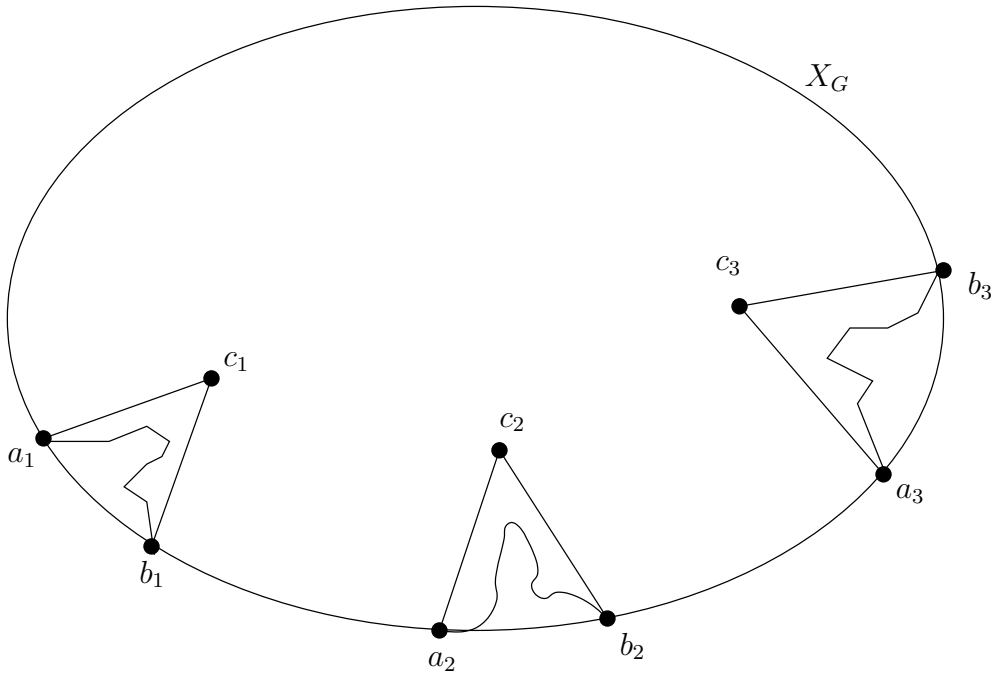
Figure 4.5: Three separating triangles on X_T.

where $P_{in,D_i}(b_i, a_i)$ is a hamiltonian path of $T_{in,D_i} - c_i$ between b_i and a_i. Now we replace the edges $a_i b_i$ by the paths $P_{in,D_i}(b_i, a_i)$ in the graph T. Then T is hamiltonian. ∎

Moreover, we can prove the following result.

Theorem 4.26 (Birchel and Helden [10]) *Let G be a 3–connected planar triangulation and let B be the rooted decomposition tree of the given embedding of G. If each vertex of the tree B has at most two children and if the two separating triangles D_1 and D_2 of the root have each one edge which are both located on the same face, then G is hamiltonian.*
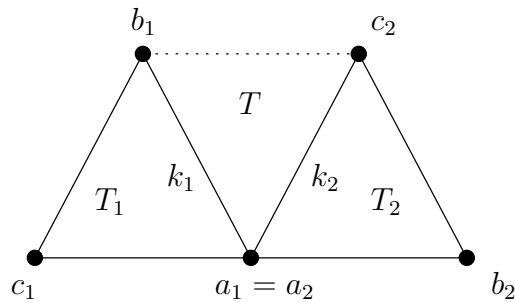

Figure 4.6: Two separating triangles with a common vertex.

Proof. Let $D_1 = a_1 b_1 c_1$ and $D_2 = a_2 b_2 c_2$ be the two separating triangles of G. Without loss of generality let $k_1 = a_1 b_1$ and $k_2 = a_2 c_2$ with $a_1 = a_2$ be the edges which are both located on the same face T. Note that a planar graph can always be embedded

in the plane such that a given face of the graph becomes the exterior face (see [19]). Therefore, we can embed G in the plane such that T becomes the exterior face of G. We call this graph G'. The graph G'_{out,D_1,D_2} is a 3–connected planar graph without interior component 3-cuts. Therefore, it follows from Theorem 4.21 of Sanders that G'_{out,D_1,D_2} has a hamiltonian cycle C which contains the edges k_1 and k_2. It follows from Theorem 3.28 that G_{in,D_i} are hamiltonian for any two boundary edges of G_{in,D_i}, $1 \leq i \leq 2$. Without loss of generality, G_{in,D_1} has a hamiltonian cycle

$$C_{in,D_1} = a_1 c_1 b_1 P_{in,D_1}(b_1, a_1) a_1$$

where $P_{in,D_1}(b_1, a_1)$ is a hamiltonian path of $G_{in,D_1} - c_1$ between b_1 and a_1. Now we replace the edge $k_1 = a_1 b_1$ by the path $P_{in,D_1}(b_1, a_1)$ in the graph G'. Also without loss of generality, G_{in,D_2} has a hamiltonian cycle

$$C_{in,D_2} = a_2 b_2 c_2 P_{in,D_2}(c_2, a_2) a_2$$

where $P_{in,D_2}(c_2, a_2)$ is a hamiltonian path of $G_{in,D_2} - b_2$ between c_2 and a_2. Now we replace the edge $k_2 = a_2 c_2$ by the path $P_{in,D_i}(c_2, a_2)$ in the graph G'. Then G' has a hamiltonian cycle and G is hamiltonian. ∎

With Theorem 3.87 we can extend the result of M. D. Dillencourt to at most two separating triangles.

Theorem 4.27 (Birchel and Helden [10]) *Let T be a 3–connected planar triangulation with at most two separating triangles, then T is hamiltonian.*

Proof. We insert a new vertex v in the exterior face which is bounded by X_T, and connect this vertex with every vertex of X_T. By insertion from the vertex v one obtains no new separating triangles. If we obtain a new separating triangle, v must be one of three vertices. Thus this would mean that the two remaining vertices would be a chord in T, which would lead to a contradiction.

A chordal 3–cut could only be extended to a separating triangle if both vertices on X_T were connected. As, however, after construction only the boundary vertices are connected with the new vertex v, no new separating triangle can arise from this. This means the graph $T + v$ is a maximal planar graph with at most 2 separating triangles. It follows from Theorem 3.87 that the graph $T + v - v = T$ has a hamiltonian cycle. ∎

The problem of whether T is hamiltonian if T is a 3–connected planar triangulation with exactly three separating triangles is still open.

4.4 4–connected planar triangulations

In this section we consider 4–connected planar triangulations.

Lemma 4.28 *Let T be a planar triangulation without chords, separating triangles and chordal 3-cuts. Then T is 4–connected.*

Proof. Each planar triangulation is at least 2–connected. Since there is no chord there is no 2-cut whose removal causes the graph to become disconnected. Moreover, since there are no separating triangles and chordal 3-cuts there is no 3-cut whose removal causes the graph to become disconnected. Therefore, T is 4–connected. ∎

As a simple consequence of Theorem 3.4 we obtain the following corollary.

Corollary 4.29 *If T is a 4–connected planar triangulation, then G is hamiltonian.*

Chapter 5

Applications of planar triangulations and maximal planar graphs

5.1 Algorithms

In this section we will turn towards algorithms which might be useful for applications. First we will show a simple algorithm to count the number of separating triangles of a maximal planar graph. Since a separating triangle does not form the boundary of a face, it is sufficient to know the number of triangles of a maximal planar graph. Therefore, the number k of separating triangles results from the difference between the number of triangles t and the number of faces l.

$$k = t - l = t - (2n - 4) = t + 4 - 2n.$$

Algorithm 5.1

Input: *Adjacency matrix A of a maximal planar graph $G = (V, E)$.*

Output: *The number k of separating triangles.*

S1: Sort the vertices $v_1, v_2, ..., v_n$ of G in such a way that $d(v_1) \geq d(v_2) \geq ... \geq d(v_n)$. Set $t = 0$.

S2: Take the vertex v_n with minimum degree. Prove for each pair of neighbors of v_n whether these vertices are adjacent. If two neighbors of v_n are adjacent, then increase the variable t by one. $t = t + 1$. Delete v_n from G. Let G be the resulting graph.

S3: Sort the vertices $v_1, v_2, ..., v_{n-1}$ of G in such a way that $d(v_1) \geq d(v_2) \geq ... \geq d(v_{n-1})$. If $d(v_{n-1}) > 1$, then go to [S2].

S4: $k = t + 4 - 2n$.

We have the following result for the algorithm.

Theorem 5.2 *Algorithm 5.1 counts the number of separating triangles of a maximal planar graph in $\mathcal{O}(n^2)$ time.*

Proof.

S1: Clearly the degrees of vertices can be computed in $\mathcal{O}(m)$ time. Since the degree of any vertex is at most $n-1$, one can sort the vertices in $\mathcal{O}(n)$ time.

S2: Since the degree of at least one vertex in a planar graph is $d(v) \leq 5$, and since after deleting one vertex of a planar graph the remaining graph is also planar, the vertex with minimum degree has at most degree 5. Therefore, no more than ten pairs of vertices are to be proved.

S3: Since [S2] runs through at most n times, the total running time of [S2] and [S3] is at most $(n+10)n = n^2 + 10n$ and therefore, $\mathcal{O}(n^2)$ time.

S4: The complexity is constant.

Therefore, the total running time of the algorithm is bounded by $\mathcal{O}(n^2)$ time. ∎

Algorithm 5.1 is conceptually very simple and easy to implement. The disadvantage of Algorithm 5.1 is that you do not know the vertices which build the separating triangles. Therefore, we present the following algorithm based on an idea of the following lemma.

Lemma 5.3 *Let $G = (V, E)$ be a planar triangulation and let $x, y \in V$ be two adjacent vertices of G.*

a) *Let $x, y \in X_G$ and let $e \in E$ be the edge which is incident to x and y.*

   i) *If the edge e is a chord of G, then the two vertices x and y have at least two common neighbors. Each further common neighbor implies a separating triangle.*

   ii) *If $e \in X_G$, then the two vertices x and y have at least one common neighbor. Each further common neighbor implies a separating triangle.*

b) *Let $x, y \notin X_G$. The two vertices x and y have at least two common neighbors. Each further common neighbor implies a separating triangle.*

Proof.

a) Let $x, y \in X_G$ and let $e \in E$ be the edge which is incident to x and y.

   i) Let e be a chord of G and let $e \notin X_G$. Assume x and y have exactly one common neighbor. Since G is a planar triangulation $e \in X_G$. This contradicts $e \notin X_G$.
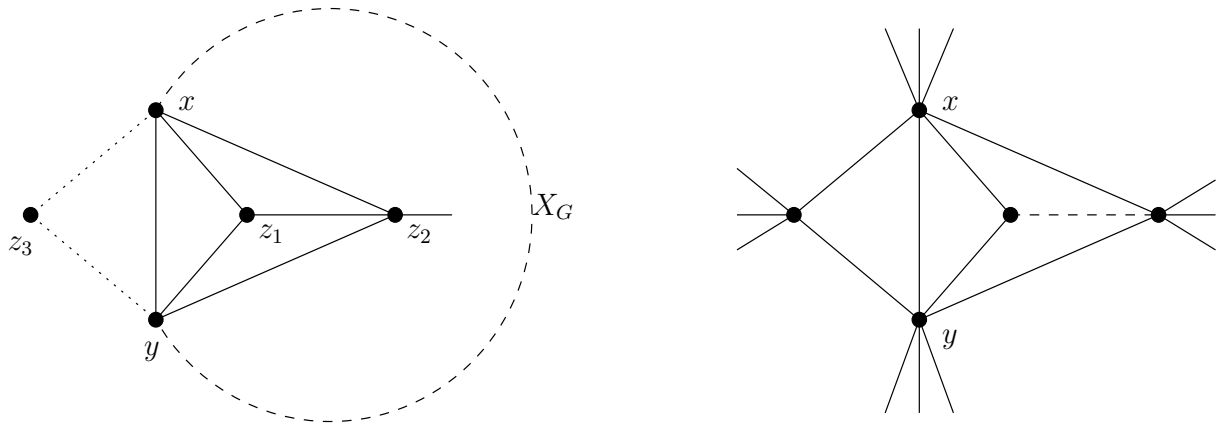
Figure 5.1: Common neighbors.

ii) Let $e \in X_G$. Since G is a planar triangulation the two vertices x and y have at least one common neighbor and each further common neighbor implies a separating triangle. Otherwise there would be a common neighbor z_3 embedded on the other side of e, see Figure 5.1. This contradicts $e \in X_G$.

b) Let $x, y \notin X_G$. There exist at least two common neighbors which are embedded on different sides of e. Otherwise $e \in X_G$. Since G is a planar triangulation each further common neighbor implies a separating triangle, see Figure 5.1. ∎

The following algorithm counts the number of all separating triangles of a planar triangulation as Algorithm 5.1 does. Furthermore, Algorithm 5.4 determines a base edge e of each separating triangle, see Figure 5.1. This is useful for applications, see Section 5.3. Note that the input is a planar triangulation.

Algorithm 5.4

Input: *Adjacency matrix A of a 3–connected planar triangulation $T = (V, E)$ with $|X_T| = t$ and the vertices of X_T.*

Output: *The number k of separating triangles. All base edges of all separating triangles of T.*

S1: *Sort the vertices $v_1, v_2, ..., v_n$ of T in such a way that the vertices $v_i \in X_T$ are labeled from 1 to t.*

S2: *Build a new adjacency matrix $A'_G \in \{0, 1\}^{n \times n}$ such that the first t entries are the exterior vertices. $k=0$.*

S3:

$$for \quad i = 1 \; to \quad n - 1 \tag{5.1}$$

$\quad if \quad i \leq t - 3 \; then \quad d1 = -1$

$$\quad\quad for \quad j = t + 1 \; to \quad n \tag{5.2}$$

$\quad\quad\quad if \quad (A[i,j] = A[(i+1) \mod t, j] = 1) \; then \quad d1 + +$

$\quad\quad if \quad (d2 > 0) \; then$

$\quad\quad\quad k = k + d1$

$\quad\quad\quad A[i, \; (i+1) \mod t] = 0$

$\quad\quad\quad and \; push \; stack \; [i, \; (i+1) \mod t]$

$$\quad\quad for \quad j = i + 2 \; to \quad t - 1 \tag{5.3}$$

$\quad\quad\quad if \quad (A[i,j] = 1) \; then \quad d2 = -2$

$$\quad\quad\quad\quad for \quad l = 1 \; to \quad n \tag{5.4}$$

$\quad\quad\quad\quad\quad if \quad (A[i,l] = A[j,l] = 1) \; then \quad d2 + +$

$\quad\quad\quad\quad if \quad (d2 > 0) \quad then$

$\quad\quad\quad\quad\quad k = k + d2$

$\quad\quad\quad\quad\quad A[i,j] = 0$

$\quad\quad\quad\quad\quad and \; push \; stack \; [i,j]$

$\quad if \quad t - 2 \leq i \leq t \; then \quad d1 = -1$

$$\quad\quad for \quad j = t + 1 \; to \quad n \tag{5.5}$$

$\quad\quad\quad if \quad (A[i,j] = A[(i+1) \mod t, j] = 1) \; then \quad d1 + +$

$\quad\quad if \quad (d1 > 0) \; then$

$\quad\quad\quad k = k + d1$

$\quad\quad\quad A[i, \; (i+1) \mod t] = 0$

$\quad\quad\quad and \; push \; stack \; [i, \; (i+1) \mod t]$

$\quad if \quad i \geq t \; then$

$$\quad\quad for \quad j = i + 1 \; to \quad n \tag{5.6}$$

$\quad\quad\quad if \quad (A[i,j] = 1) \; then \quad d2 = -2$

$$\quad\quad\quad\quad for \quad l = 1 \; to \quad n \tag{5.7}$$

$\quad\quad\quad\quad\quad if \quad (A[i,l] = A[j,l] = 1) \; then \quad d2 + +$

$\quad\quad\quad\quad if \quad (d2 > 0) \quad then$

$\quad\quad\quad\quad\quad k = k + d2$

$\quad\quad\quad\quad\quad A[i,j] = 0$

$\quad\quad\quad\quad\quad and \; push \; stack \; [i,j]$

This algorithm provides the following time complexity.

Theorem 5.5 *Algorithm 5.4 determines all separating triangles of a planar triangulation in $\mathcal{O}(n^3)$ time.*

Proof. Let $|V(T)| = n$.

S1: One can sort the vertices in $\mathcal{O}(n)$ time.

S2: One can build a new adjacency matrix $A'_T \in \{0,1\}^{n \times n}$ in $\mathcal{O}(n)$ time.

S3: We consider all for loops for the determination of the complexity. The first for loop (5.1) consists of n calls of entries of the adjacency matrix. This can be done in $\mathcal{O}(n)$ time. The same applies to the for loops (5.4) and (5.7). Since in the worst case one of the two for loops is used in each step, this can be done in $\mathcal{O}(n^2)$ time. Now we consider the for loops (5.2), (5.5) and (5.6). These for loops consist of $n - t$ calls of entries of the adjacency matrix. This can be done in $\mathcal{O}(n - t)$ time. The last for loop (5.3) consists of t calls. This can be done in $\mathcal{O}(t)$ time. Since in the worst case all vertices are exterior vertices, t can lie in the size of n. Therefore, the for loops (5.1), (5.2), (5.3) and (5.4) in the first segment have a total running time of at most n^3. The for loops (5.1) and (5.5) in the second segment have a total running time of at most n and the for loops (5.1), (5.6) and (5.7) in the third segment have a total running time of at most n^2. Therefore, the total running time of the algorithm is bounded by $\mathcal{O}(n^3)$ time. ∎

Algorithm 5.4 determines all base edges e_i of each separating triangle. Therefore, deleting these base edges will destroy all separating triangles. Note that the number of base edges could be much smaller than the number of separating triangles.

In the following we will present an algorithm to determine all chords of a planar triangulation.

Algorithm 5.6

Input: *Adjacency matrix $A \in \{0,1\}^{n \times n}$ of a planar triangulation $T = (V, E)$ with $|X_T| = t$ and the vertices of X_T.*

Output: All chords of T.

S1: Sort the vertices $v_1, v_2, ..., v_n$ of T in such a way that the vertices $v_i \in X_T$ are labeled from 1 to t.

S2: Build a new adjacency matrix $A_{X_T} \in \{0,1\}^{t \times t}$.

S3: Consider all entries of $A_{X_T}[i,j]$ with $j > i + 1$. If the entry is equal to 1, then the edge $v_i v_j$ is a chord. If $i = 1$, then j is at most $t - 1$, since the edge $v_1 v_t$ is no chord.

S4: If $i < t$, then $i = i + 1$ and go to [S3].

The algorithm described here determines only all the chords of the graph. In this algorithm the properties are not determined, which could be of interest for the use of Theorem 4.13. We can improve this, however, with a simple modification, by inserting the following step after step [S3].

S3a: If the edge $v_i v_j$ is a chord, then determine the degrees of v_i and v_j.

By this additive step one can consider directly whether the current chord is of interest in application. The previous algorithm provides the following time complexity.

Theorem 5.7 *Algorithm 5.6 determines all chords of a planar triangulation in $\mathcal{O}(n^2)$ time.*

Proof. Let $|V(T)| = n$.

S1: One can sort the vertices in $\mathcal{O}(n)$ time.

S2: One can build a new adjacency matrix $A_{X_T} \in \{0,1\}^{t \times t}$ in $\mathcal{O}(n)$ time.

S3: All t lines must be run through. In each line i one must scan $t - i$ entries. This can be done in $\mathcal{O}(t^2)$ time.

S3a: For each chord one must scan $2n$ entries. The number of chords lies in the size of $\mathcal{O}(t)$. This can be done in $\mathcal{O}(tn)$ time.

S4: The complexity is constant.

Since in the worst case all vertices are exterior vertices, t can lie in the size of n. Therefore, the total running time of the algorithm is bounded by $\mathcal{O}(n^2)$ time. ∎

The previous Algorithm 5.6 is conceptually very simple and easy to implement.

Now we will consider a planar embedding algorithm. In 1974 the first linear time planarity testing algorithm was developed by J. E. Hopcroft and R. E. Tarjan [36]. They achieved linear running time by extensive use of depth first search on graphs. The most existing algorithms for planarity testing and planar embedding can be grouped into two principal classes. Either, they run in linear time, but at the expense of complex algorithmic concepts, or they are easy to understand and implement, but require more than linear time.

In 1993 H. Stamm-Wilbrandt [48] found a linear time algorithm for embedding maximal planar graphs. This algorithm is both easy to understand and easy to implement. This algorithm will be used later in one of our applications in Section 5.3.

Theorem 5.8 (Stamm-Wilbrandt [48]) *There exists a linear time algorithm for embedding maximal planar graphs.*

In addition to planar embedding, this algorithm allows to test graphs for maximal planarity. The algorithm for embedding maximal planar graphs consists of 3 phases. During the first phase the graph $G = (V, E)$ is reduced to the K_4 by repeated reductions. In the second phase the resulting graph, the K_4, is embedded. In the third phase all reductions of the first phase are undone by the corresponding inverse reductions in reverse order. This results in an embedding of each graph between K_4 and G and therefore in an embedding of G itself.

In 1984 T. Asano, S. Kikuchi and N. Saito [5] described a linear time algorithm for finding hamiltonian cycles in 4–connected maximal planar graphs. This algorithm will be used later in one of our applications in Section 5.3.

Theorem 5.9 (Asano et. al. [5]) *There exists a linear time algorithm for finding hamiltonian cycles in 4–connected maximal planar graphs.*

Although such a 4–connected maximal planar graph G always has a hamiltonian cycle, it is not an easy matter to actually find a hamiltonian cycle of G. One can easily design an $\mathcal{O}(n^2)$ time algorithm to find a hamiltonian cycle in a 4–connected maximal planar graph G with n vertices, entirely based on Whitney's proof of his Theorem 3.2.

5.2   Dual graphs of maximal planar graphs

Let G be a maximal planar graph. As we will see, the dual graph of a maximal planar graph is of interest for applications.

Definition 5.10 *A graph is called cyclically k-edge-connected if at least k edges must be deleted to disconnect a component so that every remaining component contains a cycle.*

Lemma 5.11 *The dual graph of a 5–connected maximal planar graph G is a planar cyclically 5–edge-connected cubic graph G^*.*

The dual graph of a 4–connected maximal planar graph G is a planar cyclically 4–edge-connected cubic graph G^*.

The dual graph of a 3–connected maximal planar graph G is a planar cyclically 3–edge-connected cubic graph G^*.

The dual graph of a 5–connected maximal planar graph G with minimum degree $\delta = 5$ and maximum degree $\Delta = 6$ is a fullerene graph G^*.

We consider the following problem towards the goal of increasing the efficiency of rendering in computer graphics: Given a maximal planar graph, find a hamiltonian cycle for the dual graph. In 2000 R. E. L. Aldred, S. Bau, D. A. Holten and B. D. McKay proved the following results.

Theorem 5.12 (Aldred, Bau, Holten, McKay [2]) *All planar cyclically 3–edge-connected cubic graphs with at most 36 vertices are hamiltonian.*

Theorem 5.13 (Aldred, Bau, Holten, McKay [2]) *All planar cyclically 4–edge-connected cubic graphs with at most 40 vertices are hamiltonian.*

Theorem 5.14 (Aldred, Bau, Holten, McKay [2]) *All planar cyclically 5–edge-connected cubic graphs with at most 44 vertices are hamiltonian.*

In 2002 T. Doŝlić studied fullerene graphs and found the following result.

Theorem 5.15 (Doŝlić [26]) *All fullerene graphs with at most 176 vertices are hamiltonian.*

Thus we know that the dual of a planar graph G is a graph with a vertex for each face in G. We can calculate the number of vertices of the planar graph G, such that the dual graph G^* is still hamiltonian.

Corollary 5.16 *Every 3–connected maximal planar graph G with at most 20 vertices has a hamiltonian dual graph G^*.*

Every 4–connected maximal planar graph G with at most 22 vertices has a hamiltonian dual graph G^*.

Every 5–connected maximal planar graph G with at most 24 vertices has a hamiltonian dual graph G^*.

Every 5–connected maximal planar graph G with minimum degree $\delta = 5$ and maximum degree $\Delta = 6$ with at most 90 vertices has a hamiltonian dual graph G^*.

Proof. Let $G = (V, E)$ with $|V| = n$ and $|E| = m$ be a maximal planar graph and let $l(G)$ be the number of faces of G. Then Theorem 1.1 of Euler provides $l(G) = m - n + 2$. Moreover, $3l(G) = 2m$, since each edge is located on the boundary of two faces. Then with Euler's formula we obtain $l = 2n - 4$. Since G^* has $n^* = 2n - 4$ vertices, we get $n = \frac{n^*}{2} + 2$. ∎

Definition 5.17 *A tree partition of a graph $G = (V, E)$ is a pair (T_1, T_2) such that $(V_1, V_2) = (V(T_1), V(T_2))$ is a partition of V and both $T_1 = G[V_1]$ and $T_2 = G[V_2]$ are trees.*

Note that not every graph has a tree partition, for example K_6. T. Böhme, H. J. Broersma and H. Tuinstra [11] gave a characterization of graphs that have a tree partition.

Lemma 5.18 (Böhme, Broersma, Tuinstra [11]) *A planar graph is hamiltonian if and only if the dual graph has a tree partition.*

A natural class of graphs that admits a tree partition is obtained from the class of 4–connected maximal planar graphs by taking their dual graphs.

Theorem 5.19 (Böhme, Broersma, Tuinstra [11]) *A planar cyclically 4–edge-connected cubic graph has a tree partition.*

The results of Chapter 3 provide the following result.

Theorem 5.20 *The class of graphs obtained from the class of 3–connected maximal planar graphs with at most 5 separating triangles by taking their dual graphs admits a tree partition.*

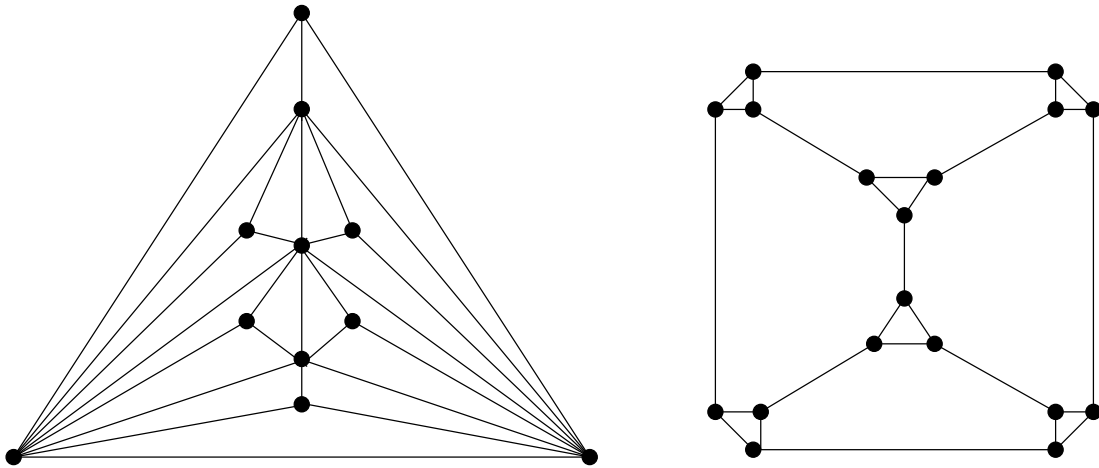The next example shows that there exist non-hamiltonian maximal planar graphs, which do not admit a tree partition.

Figure 5.2: The right graph is the dual graph of the left graph.

Example 5.21 *The left graph G from Figure 5.2 is a non-hamiltonian maximal planar graph with 11 vertices and 18 faces. The right graph G^* with 18 vertices from Figure 5.2 is the dual graph of the left graph G and admits no tree partition.*

5.3 Applications in computer graphics

In this section we consider some applications of hamiltonian maximal planar graphs and planar triangulations. In mathematics, discretization concerns the process of transferring continuous models into discrete counterparts. This process is usually carried out as a first step towards making them suitable for implementation on digital computers. In order to be processed on a digital computer another process has become of interest 3–dimensional computer graphics, based on vector graphics. Instead of the computer storing information about points, lines, and curves on a 2–dimensional surface, the computer stores the location of points, lines, and, typically, faces to construct a polygon in the 3–dimensional space.

3–dimensional polygons are the lifeblood of virtually all 3–dimensional computer graphics. As a result, most 3–dimensional graphics engines are based around storing points

(single 3–dimensional coordinates), lines that connect those points together, faces defined by the lines, and then a sequence of faces to create 3–dimensional polygons.

The idea, which is described in the next section, consists of the search for a minimum data set, in order to describe a 3–dimensional polygon uniquely. The speed of high-performance rendering engines on triangular meshes in computer graphics can be bounded by the rate at which triangulation data is sent into the machine. Obviously, each triangle can be specified by three data points, but to reduce the data rate, it is desirable to order the triangles so that consecutive triangles share a face. With such an ordering, only the incremental change of one vertex per triangle need to be specified. This potentially reduces the rendering time by a factor of three because it avoids redundant clipping and transformation computations.

Even if a given planar triangulation $T = (V, E)$ of n vertices and l faces is hamiltonian, the topology of the planar triangulation is not necessarily specified by the encoding sequences of vertices $v_0, v_1, ..., v_{l+1}$ defining the incremental changes. The problem is that you do not know which of the two vertices of the last triangle form the new triangle together with the new inserted vertex. If the vertices v_{i-2}, v_{i-1} and v_i form the last active triangle, then the question arises which is the next triangle. Since, in general, v_i can form a triangle with either v_{i-1}, v_{i+1} or v_{i-2}, v_{i+1}, see figure 5.3.
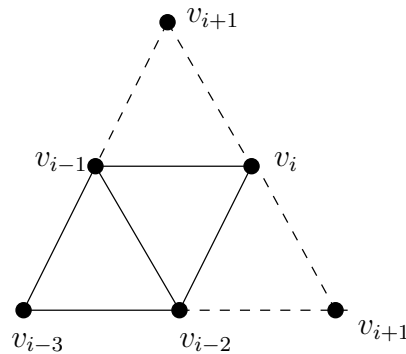

Figure 5.3: A non-specified topology of a planar triangulation.

Therefore, to completely specify the topology of the planar triangulation, we must specify the insertion order of the new vertices.

Definition 5.22 *We call the insertion order left-right turns, if the order inserting new vertices alternates at the left edge and in the next turn at the right edge.*

At least two models are currently in use, both assuming that all turns alternate from left to right.

  i) The Silicon Graphics triangular-mesh renderer OpenGL [47].

  ii) The Interactive graphics library (IGL) [15].

The Silicon Graphics triangular-mesh renderer OpenGL [47] demands that the user issue a swaptmesh call whenever the insertion order deviates from alternating left- right turns. In general, this topology can be transmitted at the cost of one extra bit per triangle. The Interactive graphics library (IGL) [15] expects triangulations as vertex strips, lists of vertices without turn specifications. To get two consecutive left or right turns, the vertex must be sent twice, creating an empty triangle, which is discarded. For efficiency, it would be highly desirable to eliminate these extra bits or vertices by finding a path that uses an implied turn order.

A perfect ordering exists if and only if the dual graph of the planar triangulation contains a hamiltonian path. We need the following definitions.

Definition 5.23 *Let G be a planar triangulation. We call G_* the interior dual graph of G if G_* is the dual graph of G without the exterior face. We say that a planar triangulation G is interior dual hamiltonian if its interior dual graph G_* contains a hamiltonian path. We say a planar triangulation is sequential if its interior dual graph contains a hamiltonian path such that this embedding correspond to left-right turns.*
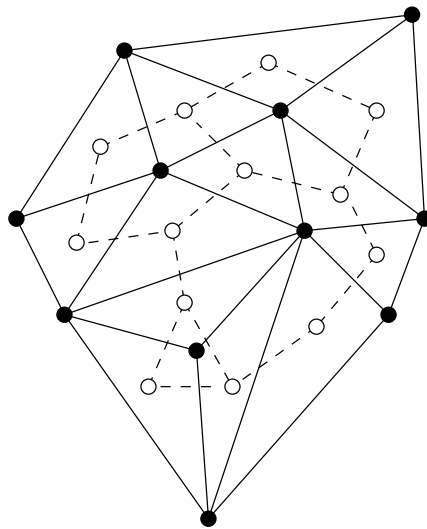

Figure 5.4: A planar triangulation and an affiliated interior dual graph.

Note that a sequential planar triangulation has an implied left-right turn order.

We consider the problem of constructing hamiltonian and sequential planar triangulations, towards the goal of increasing the efficiency of rendering in computer graphics. Two classes of problems are of interest:

1. Given a point set or a polygon, construct a hamiltonian or a sequential planar triangulation for it.

2. Given a planar triangulation T, find a hamiltonian path for the interior dual graph T_*, or failing that, add Steiner points to T such that a hamiltonian path of T_* can be generated.

Definition 5.24

a) *A point set is called in general position if three points do not lie on a single straight line.*

b) *A polygon is called in general position if three points do not lie on a single straight line.*

5.3.1   Interior dual hamiltonian planar triangulations of polygons

In this section we consider the problem of constructing interior dual hamiltonian planar triangulations.

Definition 5.25 *Let G be a planar triangulation. We call a face of G an ear of the planar triangulation if two of its three edges belong to the exterior face X_G.*

We start with a simple, but very useful result.

Theorem 5.26 *A planar triangulation of a polygon is interior dual hamiltonian if and only if it contains exactly two ears.*

Proof. Let G be a planar triangulation of a polygon P and let G_* be the interior dual graph of G.

Claim 1: The interior dual graph of a planar triangulation of a polygon is a tree.

Proof of Claim 1. The connectivity of the interior dual graph G_* is clear. Therefore, we have to show that there exists no cycle in the interior dual graph. We assume the opposite. Let $x_1, x_2, ...x_p$ be the vertices of the interior dual graph which form a cycle C. Each of these vertices stands for an interior face of G. Since there exists at least one face inside the cycle C, there exists at least one vertex x which lies within this cycle in the graph G. However, this is a contradiction, because x would lie within X_G. This cannot be the case, because all vertices of P lie on X_G.

Claim 2: The ears of the planar triangulation G are the leaves of the interior dual graph G_*.

Proof of Claim 2. Let T be an ear of G and let k_1, k_2, k_3 be the affiliated edges of T. Let k_1 be the edge which does not lie on X_G and let z be the vertex belonging to T in the interior dual graph G_*. As T has only one interior face as neighbor, which has k_1 as an edge, there exists only one edge in the interior dual graph G_* , which is incident to the vertex z.

According to Definition 5.23 a planar triangulation is interior dual hamiltonian if the interior dual graph has a hamiltonian path. Because the interior dual graph is a tree, this interior dual graph can only have one hamiltonian path if the interior dual graph has exactly two leaves. If the tree has none or only one leaf, there exists a cycle in G. This is, however, a contradiction to claim 1. If there exists three or more leaves, exactly two can be traversed, because it is impossible to go through the additional leaves with a path.∎

The following statement arises from the proof of the previous Theorem 5.26.

Lemma 5.27 *The following propositions are equivalent.*

a) *A planar triangulation of a polygon is interior dual hamiltonian.*

b) *The interior dual graph G_* of a planar triangulation G is a path.*

While not all polygons admit an interior dual hamiltonian planar triangulation, we must consider under which conditions there is always an interior dual hamiltonian planar triangulation for a polygon. Moreover, we must consider by which means we can modify the polygon in order to get the desired result. First one can note that for a convex polygon there always exists an interior dual hamiltonian planar triangulation, by selecting one point of the polygon and then connecting this point with every other point. The hamiltonian path through all triangles is trivial, see Figure 5.5. As a counter-example we consider the right graph in Figure 5.5, which is also a counter-example to Theorem 5.26, since it possesses three ears. This does not mean, however, that every non-convex polygon does not have an interior dual hamiltonian planar triangulation.
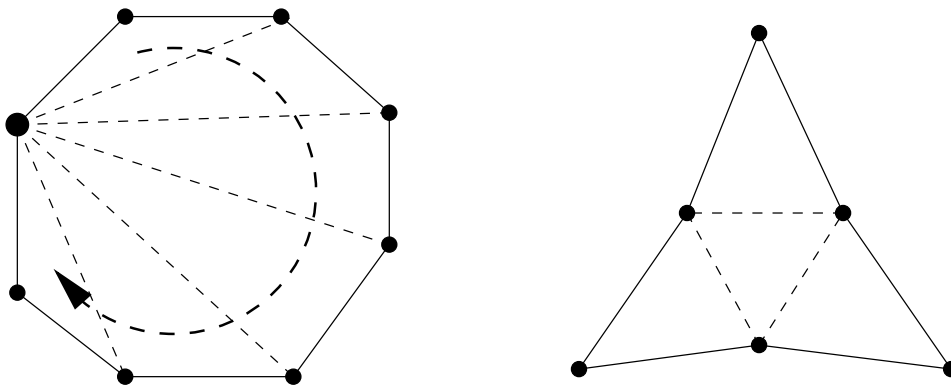

Figure 5.5: A convex polygon and a planar triangulation with three ears.

Theorem 5.28 (Arkin, Held, Mitchell, Skiena [3]) *The problem of testing whether a given polygon P in general position has an interior dual hamiltonian triangulation $T = (V, E)$ can be solved in $\mathcal{O}(|E|)$ time.*

While not all polygons admit an interior dual hamiltonian planar triangulation, it turns out that any planar triangulation can be made interior dual hamiltonian by adding Steiner points.

Theorem 5.29 (Arkin, Held, Mitchell, Skiena [3]) *Every planar triangulation T con-taining l faces can be converted into an interior dual hamiltonian planar triangulation by adding l Steiner points.*

Note that the hamiltonian path of the interior dual graph is even a hamiltonian cycle.

5.3.2 Interior dual hamiltonian planar triangulations of point sets

We now consider the question of whether a set of points has a interior dual hamiltonian planar triangulation.

Theorem 5.30 (Arkin, Held, Mitchell, Skiena [3]) *Let S be a finite set of points in the plane with $|S| = n$. If S is convex, then there is an interior dual hamiltonian planar triangulation. If S is not convex, and if the points are in general position, then there is also an interior dual hamiltonian planar triangulation and the interior dual graph is even a hamiltonian cycle. Such an interior dual hamiltonian planar triangulation can be computed in time $\mathcal{O}(n \log n)$.*

In contrast to the polygons which one must modify, perhaps, with the help of Steiner points to obtain an interior dual hamiltonian planar triangulation, it is with point sets in such a way that for every point set not in general position there is always an interior dual hamiltonian planar triangulation.

5.3.3 Sequential planar triangulation

Sequential planar triangulations represent a significant restriction. For example, convex polygons have an exponential number of interior dual hamiltonian triangulations, but have only a linear number of sequential planar triangulations. Also, point sets that admit interior dual hamiltonian planar triangulations may not have sequential planar triangulations.

Lemma 5.31 *The following propositions are equivalent.*

   a) *G is a sequential planar triangulation.*

   b) *No three edges of the planar triangulation consecutively crossed by the hamiltonian path are incident upon the same vertex of the planar triangulation.*

This implies that all turns alternate left and right, so sequential planar triangulations can be described without the extra bit-per-face needed to specify a general dual hamil-tonian planar triangulation. While trying to answer the question of whether sequential planar triangulations for point sets always exist, E. Arkin, M. Held, J. Mitchell, S. Skiena proved the following.

Theorem 5.32 (Arkin, Held, Mitchell, Skiena [3]) *For any $n \geq 9$, there is a set of n points in general position that does not admit a sequential planar triangulation.*

But with an additional condition they could prove the following result.

Theorem 5.33 (Arkin, Held, Mitchell, Skiena [3]) *Let S be a point set in general position. If at most two points of S lie strictly in the interior of the convex hull of S, then S has a sequential planar triangulation.*

The decision problem of whether a given planar triangulation on n points is sequential yields the following time complexity.

Theorem 5.34 (Arkin, Held, Mitchell, Skiena [3]) *Testing whether a given planar triangulation on n points is sequential can be done in $\mathcal{O}(n)$ time.*

5.3.4 Graham triangulation

Now we will present another possibility to construct an interior dual hamiltonian planar triangulation of a given point set in general position. Under certain conditions there even exists a hamiltonian cycle in the interior dual graph. We call such a triangulation Graham triangulation. First we describe the algorithm which constructs such a triangulation, the socalled Graham Scan.

Algorithm 5.35

Input: A point set P in general position.

Output: A interior dual hamiltonian triangulation $T = (V, E)$ with $P = V$.

S1: Find the point p_0 of P with the smallest x-coordinate.

S2: Draw the line segments connecting p_0 to the elements of P.

S3: Sort the remaining elements of P with respect to the slope of the line segments connecting them to p_0. Relabel the elements of $P - p_0$ $p_1, p_2, ..., p_{n-1}$ according to this order.

S4: Once the convex hull of $\{p_0, p_1, ..., p_{k-1}\}$ has been calculated, calculate the convex hull of $\{p_0, p_2, ..., p_{k-1}, p_k\}$ recursively as follows: Draw the line segments connecting p_k to the elements of P in the convex hull of $\{p_0, p_1, ..., p_{k-1}\}$ visible from p_k, that is, the elements in the convex hull of $\{p_0, p_1, ..., p_{k-1}\}$ such that the line segments connecting them to p_k do not intersect the interior of the convex hull of $\{p_0, p_1, ..., p_{k-1}\}$.

R. Monroy and J. Urrutia proved the following result.

Theorem 5.36 (Monroy and Urrutia [39]) *Let S be a point set on a surface in general position. The planar triangulation $GT(S)$ produced by applying Grahams Scan to S is interior dual hamiltonian.*

With the next definition R. Monroy and J. Urrutia were able to show Theorem 5.38.

Definition 5.37 *If a planar triangulation $T = (V, E)$ has a vertex v adjacent to all the vertices of $V - v$, we call such a vertex a center of T. If the center lies on the exterior cycle X_T, we call such a planar triangulation, a planar triangulation with an exterior center.*

This provides the following generalization.

Theorem 5.38 (Monroy, Urrutia [39]) *A planar triangulation with a center is interior dual hamiltonian.*

5.3.5   Hamiltonian cycles in 3–dimensional polygons

In this section we assume that the polygon points contain data. Thus we cannot reduce the data rate and we have to store three data points for each triangle. Now it is necessary to run through all polygon points but not to visit one point twice.

We assume that the 3–dimensional polygon is a maximal planar graph G on n vertices. Now we look for a hamiltonian cycle C of G.

First of all we use Algorithm 5.4 to count the number k of all separating triangles of the maximal planar graph G. Furthermore, Algorithm 5.4 determines a base edge e of each separating triangle, see Figure 5.1. Then we can apply the operation D to $a, b \in V(G)$ with $ab = e \in E(G)$, see Corollary 3.44. The graph $G + x$ is a maximal planar graph with at most $k - 1$ separating triangles and $n + 1$ vertices. Now we repeat this approach at least $k - 1$ times until we get a maximal planar graph G' with no separating triangles and $n + k$ vertices. Since $k \leq n - 4$, the new graph G' has at most $2n - 4$ vertices. Moreover, G' is 4–connected, since G' has no separating triangles. Actually such a 4–connected maximal planar graph G' always has a hamiltonian cycle. Now we use a linear time algorithm of Asano et. al. [5] to find a hamiltonian cycle C of G'.

Note that an enlargement of the number of vertices entails that the resolution of the surface of a continuous model becomes better. A disadvantage is that we must store more data and the required storage space increases. As the enlargement of the number of vertices happens in a scale of n, the increase of storage space is non-binding. Therefore, we did not get a hamiltonian cycle C of G, but a hamiltonian cycle C of a maximal planar graph G' with $G \subseteq G'$. Since G' entails a better resolution, this does not seem to be a problem.

Now we describe an algorithm to find a hamilton cycle C of a maximal planar graph G' with $G \subseteq G'$.

Algorithm 5.39

Input: *Adjacency matrix $A \in \{0,1\}^{n \times n}$ of a maximal planar graph $G = (V, E)$ with n vertices.*

Output: *A hamiltonian cycle C of a maximal planar graph G' with $G \subseteq G'$.*

S1: *Use the Algorithm 5.4 to count the number k of all separating triangles of the maximal planar graph G.*

S2: *For $i = 1$ to k apply the operation D to get a maximal planar graph G' with no separating triangles and $n + k$ vertices.*

S3: *Use the linear time Algorithm of Asano et. al. [5] to find a hamiltonian cycle C of G'.*

We have the following result on the algorithm.

Theorem 5.40 *Let $G = (V, E)$ be a maximal planar graph with n vertices. Determining a hamiltonian cycle C of a maximal planar graph G' with $G \subseteq G'$ can be done in $\mathcal{O}(n^3)$ time.*

Proof. Let $|V(G)| = n$.

S1: Determining all separating triangles of a maximal planar graph can be done in $\mathcal{O}(n^3)$ time.

S2: Applying the operation D can be done in $\mathcal{O}(n)$ time.

S3: Find a hamiltonian cycle in a 4–connected maximal planar graph G' with $2n - 4$ vertices can be done in $\mathcal{O}((2n - 4)^2)$ time.

Therefore, the total running time of the algorithm is bounded by $\mathcal{O}(n^3)$ time. ∎

5.4 Applications in Chemistry

An important subclass of all 5–connected MPG-5 graphs with many practical applications are those with maximum degree 6, best known via their duals, the fullerene graphs.

Definition 5.41 *A fullerene graph is a planar, cubic and 3–connected graph, twelve of whose faces are pentagons, and any remaining face is a hexagon.*

These objects are often referred to as fullerenes or by the more general term C_{60} molecule in chemical literature. C_{60} is a molecule that consists of 60 carbon atoms, arranged as twelve pentagons and twenty hexagons. The shape is the same as that of a soccer ball. The black pieces of the leather are the pentagons, the hexagons are white. There are 60 different points where three of the leather patches meet. Imagine a carbon atom sitting at each of these points, and you have a model of the C_{60} molecule. That model, however, is vastly out of scale. If the C_{60} molecule were the size of the soccer ball, then the soccer ball in turn would be roughly the size of the earth. They are called fullerenes after the American architect Richard Buckminster Fuller, who was renowned for his geodesic domes, that are based on hexagons and pentagons. Soon after the establishment of the C_{60} structure and the birth of fullerene chemistry, the underlying graphs became a subject of increasing scientific interest. As the main focus of the current chemical research is on fullerene and its close relatives with isolated pentagons, there are many results concerning the structure and the graphical invariants of the corresponding graphs.

In 2002 T. Doŝlić [26] studied fullerene graphs and found the following result.

Theorem 5.42 (Doŝlić [26]) *Every fullerene graph is cyclically 4–edge-connected.*

For Theorem 5.44 we need the next definition.

Definition 5.43 *The tree-partition (V_1, V_2) of a graph $G = (V, E)$ is called balanced if $|V_1| = |V_2|$.*

Note that not every graph has a tree-partition, for example, K_n for $n \geq 5$. The example of two copies of K_4, connected by an edge, shows that the existence of a tree-partition is not guaranteed even in planar graphs.

Theorem 5.44 *Every fullerene graph has a balanced tree-partition.*

Proof. The existence of a tree-partition follows from the fact that the dual of a cubic cyclically 5–edge-connected planar graph is a 5–connected maximal planar graph, and hence has a hamiltonian cycle. It remains to prove that this tree-partition is balanced. It follows from [11]. We reproduce the proof here for the convenience of the reader. Let $G = (V, E)$ be a fullerene graph with a tree-partition (V_1, V_2). Then the graphs $G[V_i]$ induced by V_i are trees, so $G[V_i]$ is connected $(i = 1, 2)$. Define $n := |V|$, $T_i := G[V_i]$, $n_i := |V_i|$. Denote by x_i , y_i and z_i the numbers of vertices in T_i with degree $1, 2$ and 3, respectively $(i = 1, 2)$. Then $n_1 + n_2 = n$, $|E| = 3n/2$, $|E(T_i)| = n_i - 1$, for $i = 1, 2$. Because $G[V_i]$ is a tree for $i = 1, 2$, all edges of G not in T_1 or T_2 are between V_1 and V_2, and there are $|E| - (n_1 - 1) - (n_2 - 1) = n/2 + 2$ of them. It is easily seen that the number of edges between T_1 and T_2 is equal to $2x_1 + y_1 = 2x_2 + y_2$. Furthermore, since $n_i \geq 2$, $x_i = 2 + z_i$ for $i = 1, 2$, hence,

$$n_1 = x_1 + y_1 + z_1 = x_1 + y_1 + x_1 - 2 = 2x_1 + y_1 - 2$$
$$= 2x_2 + y_2 - 2 = x_2 + y_2 + x_2 - 2 = x_2 + y_2 + z_2 = n_2.$$

∎

Finally, we observe a very interesting real-world application.

Observation 5.45 *The dual graph of an ordinary soccer ball has a hamiltonian cycle.*

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Zusammenfassung

Die topologische Graphentheorie nahm ihren Ursprung im Jahre 1750, als L. Euler seine berühmte Polyederformel entdeckte. Seitdem sind vielfältige Resultate über planare Graphen erzielt worden. Im Jahr 1931 zeigte H. Whitney, dass jeder maximal planare Graph ohne trennende Dreiecke hamiltonsch ist, wobei ein trennendes Dreieck ein Dreieck eines planaren Graphen ist, das nicht die Grenzen eines Landes bildet und dessen Eliminierung den Graphen trennt. Insbesondere sind alle 4–fach zusammenhängenden maximal planaren Graphen hamiltonsch. In der Klasse der 3–fach zusammenhängenden maximal planaren Graphen wurden zahlreiche Beispiele gefunden, die nicht hamiltonsch sind. Im Allgemeinen kann man über die Existenz eines Hamiltonkreises in einem planaren Graphen nur wenig aussagen, wenn man nicht weitere Informationen über den Graphen zur Verfügung hat. Im Jahr 2003 konnte C. Chen das Ergebnis von Whitney verbessern. Chen zeigte, dass jeder maximal planare Graph mit nur einem trennenden Dreieck hamiltonsch ist. Eine interessante Fragestellung in Bezug auf die Klasse der 3–fach zusammenhängenden maximal planaren Graphen ist die folgende, wie groß ist die maximale Zahl α, so dass jeder maximal planare Graph mit höchstens α trennenden Dreiecken hamiltonsch ist?

Im ersten Teil dieser Arbeit untersuchen wir die Existenz von Hamiltonkreisen und Hamiltonwegen in maximal planaren Graphen. Eine Idee von C. Chen aufgreifend, zeigen wir, dass jeder maximal planare Graph mit höchstens zwei trennenden Dreiecken hamiltonsch ist. Eine Strukturanalyse zeigt sogar, dass jeder maximal planare Graph mit höchstens fünf trennenden Dreiecken hamiltonsch ist. Im Fall von mehr als fünf trennenden Dreiecken zeigen wir eine spezielle Struktur der Position der trennenden Dreiecke. Diese Struktur garantiert ebenfalls Hamiltonkreise. Wir geben ein Gegenbeispiel für den Fall von sechs trennenden Dreiecken an. Da maximal planare Graphen mit mehr als fünf trennenden Dreiecken nicht hamiltonsch sein müssen, zeigen wir, dass in maximal planaren Graphen mit genau sechs trennenden Dreiecken mindestens ein Hamiltonweg existiert. Des Weiteren konstruieren wir spezifische hamiltonsche maximal planare Graphen mit n Ecken und k trennenden Dreiecken und nicht hamiltonsche maximal planare Graphen mit n Ecken und k trennenden Dreiecken. Einer Idee von C. Thomassen folgend, zeigen wir, wie viele Ecken eines hamiltonschen maximal planaren Graphen entfernt werden können, so dass der restliche Graph noch hamiltonsch ist.

Im zweiten Teil dieser Arbeit wird die Existenz von Hamiltonkreisen in planaren Triangulationen untersucht. Eine planare Triangulation ist ein planarer Graph, in dessen Einbettung jedes Land – außer vielleicht das äußere – von einem Dreieck berandet wird. Es ist klar, dass jeder maximal planare Graph eine planare Triangulation ist. Im Jahr 1990 verallgemeinerte M.B. Dillencourt das Ergebnis von H. Whitney auf eine andere Weise. Dillencourt zeigte, dass eine 3–fach zusammenhängende planare Triangulation ohne trennende Dreiecke und ohne Sehnen hamiltonsch ist. In der vorliegenden Arbeit sehen wir, dass das Entfernen von Ecken eines hamiltonschen maximal planaren Graphen zu einigen

interessanten Ergebnissen in der Klasse von 3–fach zusammenhängenden planaren Triangulationen führt. Wir zeigen, dass jede 3–fach zusammenhängende planare Triangulation mit höchstens zwei trennenden Dreiecken hamiltonsch ist. Im Fall von mehr trennenden Dreiecken zeigen wir eine spezielle Struktur der Position der trennenden Dreiecke, welche ebenfalls Hamiltonkreise garantiert.

Im letzten Teil dieser Arbeit befassen wir uns mit einigen Anwendungen von hamiltonschen maximal planaren Graphen und planaren Triangulationen in der Computergrafik und in der Chemie.

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