Composition of Time-Consistent Dynamic Monetary Risk Measures in Discrete Time*

Patrick Cheridito†
Princeton University
Princeton, NJ 08544, USA

Michael Kupper‡
Humboldt University Berlin
10099 Berlin, Germany.

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Abstract

In discrete time, every time-consistent dynamic monetary risk measure can be written as a composition of one-step risk measures. We exploit this structure to give new dual representation results for time-consistent convex monetary risk measures in terms of one-step penalty functions. We first study risk measures for random variables modelling financial positions at a fixed future time. Then we consider the more general case of risk measures that depend on stochastic processes describing the evolution of financial positions or cumulated cash flows. In both cases the new representations allow for a simple composition of one-step risk measures in the dual. We discuss several explicit examples and provide connections to the recently introduced class of dynamic variational preferences.

Key words: Dynamic risk measures; time-consistency; dual representations.

1 Introduction

Following the introduction of coherent, convex and monetary risk measures in [2, 3, 19, 20, 21], different dynamic extensions have been proposed. This has led to the study of conditional representations and time-consistency properties of dynamic risk measures in various setups. We refer to [4, 32, 33, 36, 14, 10, 34, 7, 23, 25, 18, 1] for the discrete time case and [22, 12, 31, 5, 6, 28, 13] for risk measures in continuous time; see also [16] and [29] for related results for dynamic preferences in discrete time.

In this paper we provide representations of time-consistent dynamic monetary risk measures in discrete time that are similar in spirit to the continuous-time representations of [22, 31, 5, 6, 13]. Rather than looking at general dynamic monetary risk measures and trying to establish conditions for time-consistency, we here only consider time-consistent ones and view them as compositions of one-step risk measures. For time-consistent dynamic convex monetary risk measures, we exploit this structure to derive new dual representations in terms of the penalty functions of the one-step risk measures. These representations permit a simple construction of time-consistent dynamic convex monetary risk measures by composing one-step risk measures in the dual.

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The structure of the paper is as follows. In Section 2, we study time-consistent dynamic monetary risk measures for random variables. They can be written as simple concatenations of one-step risk measures. In Theorem 2.4, Lemma 2.8 and Corollary 2.9 we give dual representations for time-consistent dynamic convex monetary risk measures for random variables. The time-consistency is reflected by an additive structure in the dual. We illustrate this with examples and provide connections to the dynamic variational preferences of [29]. In Section 3, we consider dynamic monetary risk measures that depend on stochastic processes describing the evolution of financial positions over time. In this case, the composition of one-step risk measures involves the aggregation of current and future risk. For time-consistent dynamic convex monetary risk measures, we translate this structure into a dual representation in terms of supermartingales; see Theorem 3.4, Lemma 3.8 and Corollary 3.9. We conclude by introducing a special class of one-step aggregators of composed form and discussing several related examples of risk measures that depend on the whole path of a stochastic process.

2 Dynamic risk measures for bounded random variables

We fix a finite time horizon $T \in \mathbb{N}$ and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ be a filtered probability space such that $\mathbb{P}[A] \in \{0,1\}$ for all $A \in \mathcal{F}_0$. $\mathbb{P}$ is not necessarily understood as a physical probability measure. We use it as reference measure that specifies the negligible events. Equalities and inequalities between random variables as well as equalities and inclusions betweens events are understood in the $\mathbb{P}$-almost sure sense. For instance, $A \subset B$ for $A, B \in \mathcal{F}$ means $\mathbb{P}[A \setminus B] = 0$. $L^\infty(\mathcal{F}_t)$ is the space of essentially bounded $\mathcal{F}_t$-measurable random variables. By $\mathcal{P}^a$ we denote the set of all probability measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to $\mathbb{P}$.

In this section the risky objects are financial positions at time T modelled by the set $L^\infty(\mathcal{F}_T)$ of essentially bounded $\mathcal{F}_T$-measurable random variables. We assume that there exists a money market account and use it as numeraire, that is, money at later times is expressed in multiples of the value of one dollar put into the money market account at time 0. A risk measure at time t is a mapping $\rho_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$. $\rho_t(X)$ is interpreted as a capital requirement at time t for the financial position X conditional on the information given by $\mathcal{F}_t$. For the study of dynamic risk measures, it is more convenient to work with the negative $\phi_t = -\rho_t$ of a monetary risk measure. We call ϕ_t a monetary utility function. Alternative names are risk adjusted valuation ([4]) or acceptance measure ([33]).

Definition 2.1 *Let $t \in \{0, \dots, T\}$. We call a mapping $\phi_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$ a monetary utility function at time t, if it has the following properties:*

(N) **Normalization:** $\phi_t(0) = 0$

(M) **Monotonicity:** $\phi_t(X) \geq \phi_t(Y)$ *for all* $X, Y \in L^\infty(\mathcal{F}_T)$ *such that* $X \geq Y$

(T) **Translation property:** $\phi_t(X + m) = \phi_t(X) + m$ *for all* $X \in L^\infty(\mathcal{F}_T)$ *and* $m \in L^\infty(\mathcal{F}_t)$

We call ϕ_t a concave monetary utility function at time t, if it also satisfies

(C) $\mathcal{F}_t$**-concavity:** $\phi_t(\lambda X + (1-\lambda)Y) \geq \lambda\phi_t(X) + (1-\lambda)\phi_t(Y)$
for all $X, Y \in L^\infty(\mathcal{F}_t)$ *and* $\lambda \in L^\infty(\mathcal{F}_t)$ *such that* $0 \leq \lambda \leq 1$.

A dynamic monetary utility function is a family of monetary utility functions $(\phi_t)_{t=0}^T$. If all ϕ_t are concave, then we call $(\phi_t)_{t=0}^T$ a dynamic concave monetary utility function.

The normalization property (N) is convenient for the study of time-consistency questions. Every function $\phi_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$ satisfying (M) and (T) can readily be normalized by passing to $\phi_t(.) - \phi_t(0)$. Note that the properties (M) and (T) imply the

(LP) **Local property:**

$$\phi_t(1_A X + 1_{A^c} Y) = 1_A \phi_t(X) + 1_{A^c} \phi_t(Y)$$

for all $X, Y \in L^\infty(\mathcal{F}_T)$ and $A \in \mathcal{F}_t$.

Indeed, it follows from (M) and (T) that

$$\begin{aligned} \phi_t(1_A X) - 1_{A^c} \|X\|_\infty &= \phi_t(1_A X - 1_{A^c} \|X\|_\infty) \le \phi_t(X) \\ &\le \phi_t(1_A X + 1_{A^c} \|X\|_\infty) = \phi_t(1_A X) + 1_{A^c} \|X\|_\infty . \end{aligned}$$

By multiplying through with 1_A, one obtains $1_A \phi_t(X) = 1_A \phi_t(1_A X)$, which is equivalent to (LP).

Definition 2.2 *We call a dynamic monetary utility function $(\phi_t)_{t=0}^T$ time-consistent if*

$$\phi_{t+1}(X) \ge \phi_{t+1}(Y) \quad \textit{implies} \quad \phi_t(X) \ge \phi_t(Y) \tag{2.1}$$

for all $X, Y \in L^\infty(\mathcal{F}_T)$ and $t = 0, \dots, T-1$.

Due to the properties (N), (M) and (T), time-consistency of dynamic monetary utility functions on $L^\infty(\mathcal{F}_T)$ is equivalent to the dynamic programming principle

$$\phi_t(X) = \phi_t(\phi_{t+1}(X)) \quad \text{for all} \quad X \in L^\infty(\mathcal{F}_T) \text{ and } t = 0, \dots, T-1. \tag{2.2}$$

Concepts equivalent or similar to (2.1) or (2.2) have been studied in different contexts, see for instance, [26, 27, 17, 15, 35, 16, 12, 4, 32, 31, 5, 6, 33, 36, 14, 10, 34, 25, 28, 18, 29].

2.1 Generators

For a dynamic monetary utility function $(\phi_t)_{t=0}^T$, we denote by φ_t the restriction of ϕ_t to $L^\infty(\mathcal{F}_{t+1})$ and call $(\varphi_t)_{t=0}^{T-1}$ the generator of $(\phi_t)_{t=0}^T$. It follows from (2.2) that a time-consistent dynamic monetary utility function is uniquely given by its generator. One can also start with an arbitrary family

$$\varphi_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t), \quad t = 0, \dots, T-1,$$

of monetary utility functions and define the time-consistent monetary utility function $(\phi_t)_{t=0}^T$ by backwards induction:

$$\phi_T(X) = X \quad \text{and} \quad \phi_t(X) = \varphi_t(\phi_{t+1}(X)), \quad t \le T-1.$$

It is clear that every ϕ_t is $\mathcal{F}_t$-concave if and only if all φ_t are so.

2.2 Duality

In this section we provide duality results for time-consistent dynamic concave monetary utility functions on $L^\infty(\mathcal{F}_T)$ in terms of one-step penalty functions. We need the sets of one-step transition densities

$$\mathcal{D}_t := \left\{ \xi \in L^1_+(\mathcal{F}_t) : \mathrm{E}_\mathbb{P}\left[\xi \mid \mathcal{F}_{t-1}\right] = 1 \right\}, \quad t = 1, \dots, T.$$

Every sequence $(\xi_1, \dots, \xi_T) \in \mathcal{D}_1 \times \dots \times \mathcal{D}_T$ defines a probability measure $\mathbb{Q}^\xi$ in $\mathcal{P}^a$ with density

$$\frac{d\mathbb{Q}^\xi}{d\mathbb{P}} = \xi_1 \cdots \xi_T.$$

On the other hand, every probability measure $\mathbb{Q}$ in $\mathcal{P}^a$ induces a non-negative martingale

$$M_t^\mathbb{Q} := \mathrm{E}_\mathbb{P}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t\right], \quad t = 0, \dots, T.$$

One has

$$\left\{M^{\mathbb{Q}}_{t-1} = 0\right\} \subset \left\{M^{\mathbb{Q}}_t = 0\right\} \quad \text{for all } 1 \le t \le T,$$

and the sequence

$$\xi^{\mathbb{Q}}_t := \begin{cases} \frac{M^{\mathbb{Q}}_t}{M^{\mathbb{Q}}_{t-1}} & \text{on } \left\{M^{\mathbb{Q}}_{t-1} > 0\right\} \\ 1 & \text{on } \left\{M^{\mathbb{Q}}_{t-1} = 0\right\} \end{cases} \quad \text{for } t = 1, \dots, T,$$

is an element in $\mathcal{D}_1 \times \dots \times \mathcal{D}_T$ with the property

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi^{\mathbb{Q}}_1 \cdots \xi^{\mathbb{Q}}_T.$$

We will work with the convention

$$\mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] := \mathrm{E}_{\mathbb{P}}\left[\xi^{\mathbb{Q}}_{t+1} \cdot \dots \cdot \xi^{\mathbb{Q}}_T X \mid \mathcal{F}_t\right], \quad X \in L^\infty(\mathcal{F}_T), \quad t = 0, \dots T-1.$$

This is consistent with the standard definition of the conditional expectation. But according to the standard definition, $\mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right]$ is only defined up to $\mathbb{Q}$-almost sure equality, whereas $\mathrm{E}_{\mathbb{P}}\left[\xi^{\mathbb{Q}}_{t+1} \cdot \dots \cdot \xi^{\mathbb{Q}}_T X \mid \mathcal{F}_t\right]$ is a version of $\mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right]$ which is defined up to $\mathbb{P}$-almost sure equality. This will be important for formulas like (2.3) and (2.11) below, which involve the essential infimum or supremum of random variables. We denote the essential infimum by $\operatorname{ess\,inf}$ and the essential supremum by $\operatorname{ess\,sup}$. Both are understood with respect to the reference measure $\mathbb{P}$; for the definition and properties of the essential infumum and supremum, see for instance, Proposition VI.1.1 of Neveu [30] or the Appendix in Föllmer and Schied [20].

By $\bar{L}_+(\mathcal{F}_t)$ we denote all $\mathcal{F}_t$-measurable functions $X : \Omega \to [0, \infty]$. The conditional expectation of $X \in \bar{L}_+(\mathcal{F}_t)$ is, as usual, understood as

$$\mathrm{E}_{\mathbb{P}}\left[X \mid \mathcal{F}_t\right] := \lim_{n \to \infty} \mathrm{E}_{\mathbb{P}}\left[X \wedge n \mid \mathcal{F}_t\right].$$

Definition 2.3 *For $t \in \{0, \dots, T-1\}$, we call a mapping*

$$\psi_t : \mathcal{D}_{t+1} \to \bar{L}_+(\mathcal{F}_t)$$

a one-step penalty function if it satisfies the following two conditions:

(i) $\operatorname{ess\,inf}_{\xi \in \mathcal{D}_{t+1}} \psi_t(\xi) = 0$

(ii) $\psi_t(1_A \xi + 1_{A^c} \xi') = 1_A \psi_t(\xi) + 1_{A^c} \psi_t(\xi')$ *for all $\xi, \xi' \in \mathcal{D}_{t+1}$ and $A \in \mathcal{F}_t$.*

For $\mathbb{Q} \in \mathcal{P}^a$, we set

$$\psi_t(\mathbb{Q}) := \psi_t(\xi^{\mathbb{Q}}_{t+1}).$$

A dynamic penalty function consists of a sequence $(\psi_t)_{t=0}^{T-1}$ of one-step penalty functions. It induces the mapping $\Psi : \mathcal{P}^a \to \bar{L}_+(\mathcal{F}_{T-1})$ given by $\Psi(\mathbb{Q}) = \sum_{t=0}^{T-1} \psi_t(\mathbb{Q})$.

In the following theorem we provide different dual representations through which a dynamic penalty function induces a time-consistent dynamic concave monetary utility function. A new feature of Theorem 2.4 compared to other representations of time-consistent dynamic convex monetary risk measures is that (2.5) provides a dual representation of a whole family of risk measures $(\phi_t)_{t=0}^T$ in terms of the single penalty function $\Psi = \sum_{j=0}^{T-1} \psi_j$. This extends the dual representation of a time-consistent dynamic coherent risk measure with one m-stable (or rectangular) set of probability measures (see [4, 12, 32, 16, 10]). Formula (2.4) can be seen as a discrete version of the continuous-time representation (37) in [6].

Theorem 2.4 *Let $(\psi_t)_{t=0}^{T-1}$ be a dynamic penalty function. Then*

$$\varphi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}^a} \left\{ \mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \psi_t(\mathbb{Q}) \right\}, \quad t = 0, \dots, T-1, \quad X \in L^\infty(\mathcal{F}_{t+1}), \tag{2.3}$$

defines a generator of a time-consistent concave monetary utility function $(\phi_t)_{t=0}^T$ with the following representations:

$$\phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}^a} \mathrm{E}_{\mathbb{Q}}\left[X + \sum_{j=t+1}^{T} \psi_{j-1}(\mathbb{Q}) \mid \mathcal{F}_t\right] \tag{2.4}$$

$$= \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}^a} \mathrm{E}_{\mathbb{Q}}\left[X + \Psi(\mathbb{Q}) \mid \mathcal{F}_t\right] \tag{2.5}$$

for all $t \le T-1$ and $X \in L^\infty(\mathcal{F}_T)$.

Proof. It can easily be checked that for all $t \le T-1$,

$$\varphi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}^a} \left\{ \mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \psi_t(\mathbb{Q}) \right\}$$

defines a concave monetary utility function from $L^\infty(\mathcal{F}_{t+1})$ to $L^\infty(\mathcal{F}_t)$. Therefore, $(\varphi_t)_{t=0}^{T-1}$ is the generator of a time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^T$. It remains to show (2.4) and (2.5). To do this we denote (2.5) by $\tilde{\phi}_t(X)$. If $X \in L^\infty(\mathcal{F}_s)$ for $0 \le t < s \le T$, then

$$\begin{aligned}
\tilde{\phi}_t(X) &= \operatorname*{ess\,inf}_{(\xi_1,\dots,\xi_T)\in\mathcal{D}_1\times\cdots\times\mathcal{D}_T} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_T\left(X + \sum_{j=1}^{T}\psi_{j-1}(\xi_j)\right) \mid \mathcal{F}_t\right] \\
&= \sum_{j=1}^{t} \operatorname*{ess\,inf}_{\xi_j\in\mathcal{D}_j} \psi_{j-1}(\xi_j) \qquad (2.6) \\
&+ \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_s)\in\mathcal{D}_{t+1}\times\cdots\times\mathcal{D}_s} \Bigg\{ \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_s\left(X + \sum_{j=t+1}^{s}\psi_{j-1}(\xi_j)\right) \mid \mathcal{F}_t\right] \\
&\qquad + \operatorname*{ess\,inf}_{(\xi_{s+1},\dots,\xi_T)\in\mathcal{D}_{s+1}\times\cdots\times\mathcal{D}_T} \sum_{j=s+1}^{T} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_{j-1}\;\psi_{j-1}(\xi_j) \mid \mathcal{F}_t\right] \Bigg\}. \qquad (2.7)
\end{aligned}$$

The terms (2.6) and (2.7) are both equal to 0. For (2.6) this follows directly from condition (i) of Definition 2.3. For (2.7) we prove it by induction over T: Fix $(\xi_{t+1}, \dots, \xi_s) \in \mathcal{D}_{t+1} \times \cdots \times \mathcal{D}_s$. If $T = s$, then the sum in (2.7) contains no terms and is equal to zero. If $T = s+1$, then (2.7) equals

$$\operatorname*{ess\,inf}_{\xi_{s+1}\in\mathcal{D}_{s+1}} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_s\;\psi_s(\xi_{s+1}) \mid \mathcal{F}_t\right]. \tag{2.8}$$

By condition (ii) of Definition 2.3, the family $\{\psi_s(\xi_{s+1}) : \xi_{s+1} \in \mathcal{D}_{s+1}\}$ is directed downwards. Therefore, the ess inf in (2.8) can be taken over a decreasing sequence and, by Beppo Levi's dominated convergence theorem, commutes with the conditional expectation. By condition (i) of Definition 2.3, this shows that (2.8) is equal to zero. Now, assume $T \ge s+2$ and

$$\operatorname*{ess\,inf}_{(\xi_{s+1},\dots,\xi_{T-1})\in\mathcal{D}_{s+1}\times\cdots\times\mathcal{D}_{T-1}} \sum_{j=s+1}^{T-1} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_{j-1}\;\psi_{j-1}(\xi_j) \mid \mathcal{F}_t\right] = 0.$$

Then, to prove that (2.7) is equal to zero, it is enough to show that for fixed $(\xi_{t+1},\dots,\xi_{T-1}) \in \mathcal{D}_{t+1}\times\cdots\times\mathcal{D}_{T-1}$, the term

$$\operatorname*{ess\,inf}_{\xi_T\in\mathcal{D}_T} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_{T-1}\ \psi_{T-1}(\xi_T) \mid \mathcal{F}_t\right]$$

is zero. As above, this follows because ψ_{T-1} satisfies condition (ii) of Definition 2.3 and therefore, the ess inf can be taken inside the conditional expectation.

Since (2.6) and (2.7) are both equal to zero, one has

$$\begin{aligned}\tilde{\phi}_t(X) &= \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}^a} \mathrm{E}_{\mathbb{Q}}\left[X+\sum_{j=t+1}^{T}\psi_{j-1}(\mathbb{Q}) \mid \mathcal{F}_t\right]\\ &= \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}^a} \mathrm{E}_{\mathbb{Q}}\left[X+\sum_{j=t+1}^{s}\psi_{j-1}(\mathbb{Q}) \mid \mathcal{F}_t\right].\end{aligned}\tag{2.9}$$

Next, we show $\phi_t=\tilde{\phi}_t$ by induction over s. If $s=t+1$, then we obtain from (2.9) that

$$\phi_t(X)=\varphi_t(X)=\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}^a}\mathrm{E}_{\mathbb{Q}}\left[X+\psi_t(\mathbb{Q})\mid\mathcal{F}_t\right]=\tilde{\phi}_t(X)\quad\text{for all } X\in L^\infty(\mathcal{F}_s).$$

Now, assume $s\geq t+2$ and $\phi_t(Y)=\tilde{\phi}_t(Y)$ for all $Y\in L^\infty(\mathcal{F}_{s-1})$. If $X\in L^\infty(\mathcal{F}_s)$, then $\varphi_{s-1}(X)\in L^\infty(\mathcal{F}_{s-1})$, and we get

$$\begin{aligned}&\phi_t(X)=\phi_t(\varphi_{s-1}(X))=\tilde{\phi}_t(\varphi_{s-1}(X))\\ =&\operatorname*{ess\,inf}_{\xi_{t+1},\dots,\xi_{s-1}\in\mathcal{D}_{t+1}\times\cdots\times\mathcal{D}_{s-1}}\mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_{s-1}\left(\varphi_{s-1}(X)+\sum_{j=t+1}^{s-1}\psi_{j-1}(\xi_j)\right)\mid\mathcal{F}_t\right]\\ =&\operatorname*{ess\,inf}_{\xi_{t+1},\dots,\xi_{s-1}\in\mathcal{D}_{t+1}\times\cdots\times\mathcal{D}_{s-1}}\mathrm{E}_{\mathbb{P}}\Bigg[\xi_{t+1}\cdot\dots\cdot\xi_{s-1}\\ &\qquad\operatorname*{ess\,inf}_{\xi_s\in\mathcal{D}_s}\left\{\mathrm{E}_{\mathbb{P}}\left[\xi_sX\mid\mathcal{F}_{s-1}\right]+\sum_{j=t+1}^{s}\psi_{j-1}(\xi_j)\right\}\mid\mathcal{F}_t\Bigg].\end{aligned}\tag{2.10}$$

By condition (ii) of Definition 2.3, the family

$$\mathrm{E}_{\mathbb{P}}\left[\xi_sX\mid\mathcal{F}_{s-1}\right]+\psi_{s-1}(\xi_s),\quad \xi_s\in\mathcal{D}_s$$

is directed downwards. Therefore, we can take the ess inf in (2.10) outside of the conditional expectation and arrive at

$$\phi_t(X)=\operatorname*{ess\,inf}_{\xi_{t+1},\dots,\xi_s\in\mathcal{D}_{t+1}\times\cdots\times\mathcal{D}_s}\mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_s\left(X+\sum_{j=t+1}^{s}\psi_{j-1}(\xi_j)\right)\mid\mathcal{F}_t\right]=\tilde{\phi}_t(X),$$

which concludes the proof. □

Remark 2.5 The generator $(\varphi_t)_{t=0}^{T-1}$ and dynamic utility function $(\phi_t)_{t=0}^{T-1}$ in Theorem 2.4 can also be represented by the variants of (2.3)–(2.5), where $\mathcal{P}^a$ is replaced with the smaller set

$$\mathcal{P}^\Psi:=\left\{\mathbb{Q}\in\mathcal{P}^a:\Psi(\mathbb{Q})<\infty\right\}.$$

Indeed, it follows from conditions (i) and (ii) of Definition 2.3 that for all $t \le T-1$, there exists a sequence $(\xi^n_{t+1})_{n\ge 1}$ in $\mathcal{D}_{t+1}$ such that $\psi_t(\xi^n_{t+1}) \downarrow 0$ $\mathbb{P}$-almost surely. Set $A^n_t := \left\{\psi_t(\xi^n_{t+1}) < \infty\right\}$ and define

$$\xi^*_{t+1} := \xi^1_{t+1} 1_{A^1_t} + \sum_{n\ge 2} \xi^n_{t+1} 1_{\left\{A^n_t \setminus A^{n-1}_t\right\}}.$$

Then

$$\xi^*_{t+1} \in \mathcal{D}^{\psi_t}_{t+1} := \left\{\xi_{t+1} \in \mathcal{D}_{t+1} : \psi_t(\xi_{t+1}) < \infty\right\}.$$

For arbitrary $\xi_{t+1} \in \mathcal{D}_{t+1}$, the element $\tilde{\xi}_{t+1} := \xi_{t+1} 1_{\{\psi_t(\xi_{t+1})<\infty\}} + \xi^*_{t+1} 1_{\{\psi_t(\xi_{t+1})=\infty\}}$ is in $\mathcal{D}^{\psi_t}_{t+1}$, and

$$\mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} X \mid \mathcal{F}_t\right] + \psi_t(\xi_{t+1}) \ge \mathrm{E}_{\mathbb{P}}\left[\tilde{\xi}_{t+1} X \mid \mathcal{F}_t\right] + \psi_t(\tilde{\xi}_{t+1}) \quad \text{for } X \in L^\infty(\mathcal{F}_{t+1}).$$

Since

$$\mathcal{P}^\Psi = \mathcal{D}^{\psi_0}_1 \cdots \mathcal{D}^{\psi_{T-1}}_T,$$

this shows that

$$\varphi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}^\Psi} \left\{\mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \psi_t(\mathbb{Q})\right\}, \quad X \in L^\infty(\mathcal{F}_{t+1}).$$

From here, exactly the same arguments as in the proof of Theorem 2.4 lead to

$$\phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}^\Psi} \mathrm{E}_{\mathbb{Q}}\left[X + \sum_{j=t+1}^{T} \psi_{j-1}(\mathbb{Q}) \mid \mathcal{F}_t\right] = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}^\Psi} \mathrm{E}_{\mathbb{Q}}\left[X + \Psi(\mathbb{Q}) \mid \mathcal{F}_t\right]$$

for $t \le T-1$ and $X \in L^\infty(\mathcal{F}_T)$. Note that the second identity can be written in the alternative form

$$\phi_t(X) = \operatorname*{ess\,inf}_{(\mathbb{Q},q)\in\mathcal{Q}^\Psi} \mathrm{E}_{\mathbb{Q}}\left[X + q \mid \mathcal{F}_t\right],$$

for the set

$$\mathcal{Q}^\Psi := \left\{(\mathbb{Q}, q) \in \mathcal{P}^a \times L^0_+(\mathcal{F}_T) : q \ge \Psi(\mathbb{Q})\right\}.$$

It can easily be checked that generators with a dual representation of the form (2.3) and the corresponding dynamic monetary utility functions (2.4)–(2.5) have the following continuity property:

Definition 2.6 *For $0 \le t \le s \le T$, we call a mapping $I : L^\infty(\mathcal{F}_s) \to L^\infty(\mathcal{F}_t)$ continuous from above if*

$$I(X^n) \to I(X) \quad \mathbb{P}\text{-almost surely}$$

for every sequence $(X^n)_{n\in\mathbb{N}}$ in $L^\infty(\mathcal{F}_s)$ that decreases $\mathbb{P}$-almost surely to $X \in L^\infty(\mathcal{F}_s)$. We call a generator $(\varphi_t)_{t=0}^{T-1}$ or a dynamic monetary utility function $(\phi_t)_{t=0}^{T}$ on $L^\infty(\mathcal{F}_T)$ continuous from above if all φ_t or ϕ_t are continuous from above, respectively.

Next we are going to show that every time-consistent dynamic concave monetary utility function with generator that is continuous from above has representations of the form (2.4)–(2.5). This will be shown in Corollary 2.9 below. First we need the following definition and lemma.

Definition 2.7 *For a time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^{T-1}$ with generator $(\varphi_t)_{t=0}^{T-1}$ that is continuous from above we define for all $t = 0, \dots, T-1$,*

$$\varphi^\#_t(\xi_{t+1}) := \operatorname*{ess\,sup}_{X\in L^\infty(\mathcal{F}_{t+1})} \left\{\varphi_t(X) - \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} X \mid \mathcal{F}_t\right]\right\}, \quad \xi_{t+1} \in \mathcal{D}_{t+1}. \tag{2.11}$$

Up to a sign, $\varphi_t^{\#}$ is the conditional concave conjugate of φ_t. In the next lemma we provide a conditional dual representation result for concave generators φ_t in terms of $\varphi_t^{\#}$. Similar results are proved in [32, 4, 14, 10, 7, 34, 28]. Since our setup is slightly different, we provide a proof.

Lemma 2.8 *Let $(\phi_t)_{t=0}^T$ be a time-consistent dynamic concave monetary utility function on $L^\infty(\mathcal{F}_T)$ with generator $(\varphi_t)_{t=0}^{T-1}$ that is continuous from above. Then $(\varphi_t^{\#})_{t=0}^{T-1}$ is the smallest dynamic penalty function such that*

$$\varphi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}^a} \left\{ \mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \varphi_t^{\#}(\mathbb{Q}) \right\} \tag{2.12}$$

for all $t = 0, \dots, T-1$ and $X \in L^\infty(\mathcal{F}_{t+1})$.

Proof. We fix $t \in \{0, \dots, T-1\}$ and introduce the sets

$$\mathcal{B}_t := \{X \in L^\infty(\mathcal{F}_{t+1}) : \varphi_t(X) \geq 0\}$$

and

$$\mathcal{C}_t := \left\{ X \in L^\infty(\mathcal{F}_{t+1}) : \mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \varphi_t^{\#}(\mathbb{Q}) \geq 0 \text{ for all } \mathbb{Q} \in \mathcal{P}^a \right\}.$$

It follows directly from the definition of $\varphi_t^{\#}$ that

$$\mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \varphi_t^{\#}(\mathbb{Q}) \geq \varphi_t(X)$$

for all $X \in L^\infty(\mathcal{F}_{t+1})$ and $\mathbb{Q} \in \mathcal{P}^a$. This shows that $\mathcal{B}_t \subset \mathcal{C}_t$. In the following we are going to prove $\mathcal{C}_t \subset \mathcal{B}_t$. Assume this is not the case. Then there exists $X^* \in \mathcal{C}_t \setminus \mathcal{B}_t$. Hence, the set $A := \{\varphi_t(X^*) < 0\}$ has positive $\mathbb{P}$-measure and $1_A X^*$ is still in $\mathcal{C}_t \setminus \mathcal{B}_t$. The mapping

$$X \mapsto I(X) := \mathrm{E}\left[\varphi_t(X)\right]$$

is a concave monetary utility function from $L^\infty(\mathcal{F}_{t+1})$ to $\mathbb{R}$ that is continuous from above. Hence, it can be deduced from the Krein–Šmulian theorem that

$$\mathcal{B} := \{X \in L^\infty(\mathcal{F}_{t+1}) : I(X) \geq 0\}$$

is $\sigma(L^\infty(\mathcal{F}_{t+1}), L^1(\mathcal{F}_{t+1}))$-closed, see for instance, the proof of Theorem 3.2 in Delbaen [11] or Remark 4.3 in Cheridito et al. [10]. Since it does not contain $1_A X^*$, it follows from the separating hyperplane theorem that there exists a $\mathbb{Q} \in \mathcal{P}^a$ such that

$$\mathrm{E}_{\mathbb{Q}}\left[1_A X^*\right] < \inf_{X \in \mathcal{B}} \mathrm{E}_{\mathbb{Q}}\left[X\right] \leq \inf_{X \in \mathcal{B}_t} \mathrm{E}_{\mathbb{Q}}\left[X\right]. \tag{2.13}$$

Since the family $\{\mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] : X \in \mathcal{B}_t\}$ is directed downwards, we obtain from Beppo Levi's monotone convergence theorem that

$$\inf_{X \in \mathcal{B}_t} \mathrm{E}_{\mathbb{Q}}\left[X\right] = \mathrm{E}_{\mathbb{Q}}\left[\operatorname*{ess\,inf}_{X \in \mathcal{B}_t} \mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right]\right].$$

Moreover, by the translation property of φ_t, $\varphi_t^{\#}$ can be written as

$$\varphi_t^{\#}(\mathbb{Q}) = \operatorname*{ess\,sup}_{X \in \mathcal{B}_t} \mathrm{E}_{\mathbb{Q}}\left[-X \mid \mathcal{F}_t\right].$$

Therefore, it follows from (2.13) that

$$\mathrm{E}_{\mathbb{Q}}\left[1_A X^*\right] < \mathrm{E}_{\mathbb{Q}}\left[-\varphi_t^{\#}(\mathbb{Q})\right],$$

and hence,

$$\mathrm{E}_{\mathbb{Q}}\left[\mathrm{E}_{\mathbb{Q}}\left[1_A X^* \mid \mathcal{F}_t\right] + \varphi_t^{\#}(\mathbb{Q})\right] < 0.$$

But this contradicts $1_A X^* \in \mathcal{C}_t$. Thus, we must have $\mathcal{B}_t = \mathcal{C}_t$. Now note that for all $X \in L^\infty(\mathcal{F}_{t+1})$,

$$\varphi_t(X) = \operatorname{ess\,sup}\left\{m \in L^\infty(\mathcal{F}_t) : X - m \in \mathcal{B}_t\right\},$$

and therefore,

$$\begin{aligned}
\varphi_t(X) &= \operatorname{ess\,sup}\left\{m \in L^\infty(\mathcal{F}_t) : \mathrm{E}_{\mathbb{Q}}\left[X - m \mid \mathcal{F}_t\right] + \varphi_t^{\#}(\mathbb{Q}) \geq 0 \text{ for all } \mathbb{Q} \in \mathcal{P}^a\right\} \\
&= \operatorname{ess\,sup}\left\{m \in L^\infty(\mathcal{F}_t) : \mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \varphi_t^{\#}(\mathbb{Q}) \geq m \text{ for all } \mathbb{Q} \in \mathcal{P}^a\right\} \\
&= \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{P}^a}\left\{\mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \varphi_t^{\#}(\mathbb{Q})\right\}.
\end{aligned}$$

Finally, observe that for every function ψ_t from $\mathcal{D}_{t+1}$ to $\bar{L}_+(\mathcal{F}_t)$ satisfying

$$\varphi_t(X) = \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{D}_{t+1}}\left\{\mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} X \mid \mathcal{F}_t\right] + \psi_t(\xi_{t+1})\right\}$$

for all $X \in L^\infty(\mathcal{F}_{t+1})$, one has

$$\varphi_t(X) - \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} X \mid \mathcal{F}_t\right] \leq \psi_t(\xi_{t+1})$$

for all $X \in L^\infty(\mathcal{F}_{t+1})$ and $\xi_{t+1} \in \mathcal{D}_{t+1}$, and hence, $\varphi_t^{\#} \leq \varphi_t$. □

The following corollary is an immediate consequence of Theorem 2.4 and Lemma 2.8.

Corollary 2.9 *Let $(\phi_t)_{t=0}^T$ be a time-consistent dynamic concave monetary utility function that is continuous from above. Then*

$$\phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{P}^a} \mathrm{E}_{\mathbb{Q}}\left[X + \sum_{j=t+1}^{T} \varphi_{j-1}^{\#}(\mathbb{Q}) \mid \mathcal{F}_t\right] = \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{P}^a} \mathrm{E}_{\mathbb{Q}}\left[X + \phi^{\#}(\mathbb{Q}) \mid \mathcal{F}_t\right]$$

for all $t \leq T-1$ and $X \in L^\infty(\mathcal{F}_T)$, where $\phi^{\#}(\mathbb{Q}) = \sum_{j=1}^{T} \varphi_{j-1}^{\#}(\mathbb{Q})$.

2.3 Examples

2.3.1 Dynamic Average-Value-at-Risk

For every $t = 0, \ldots, T-1$, let α_t be an element of $L^\infty(\mathcal{F}_t)$ such that $0 < \alpha_t \leq 1$ and consider the generators

$$\varphi_t(X) = \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{D}_{t+1},\, \xi_{t+1} \leq \alpha_t^{-1}} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} X \mid \mathcal{F}_t\right], \quad X \in L^\infty(\mathcal{F}_{t+1}).$$

Then, $-\varphi_t$ is a conditional Average-Value-at-Risk on $L^\infty(\mathcal{F}_{t+1})$ at the level α_t; see [20] for the definition of the unconditional Average-Value-at-Risk. The minimal dynamic penalty function $(\varphi_t^{\#})_{t=0}^{T-1}$ of the induced time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^T$ is given by

$$\varphi_t^{\#}(\xi_{t+1}) = \operatorname*{ess\,sup}_{X \in L^\infty(\mathcal{F}_{t+1})}\left\{\varphi_t(X) - \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} X \mid \mathcal{F}_t\right]\right\} = \left\{\begin{array}{ll} 0 & \text{on } \left\{\mathbb{P}[\xi_{t+1} > \alpha_t^{-1} \mid \mathcal{F}_t] = 0\right\} \\ \infty & \text{on } \left\{\mathbb{P}[\xi_{t+1} > \alpha_t^{-1} \mid \mathcal{F}_t] > 0\right\} \end{array}\right. .$$

Hence,

$$\phi^{\#}(\mathbb{Q}) = \sum_{j=1}^{T} \varphi^{\#}_{j-1}(\mathbb{Q}) = \begin{cases} 0 & \text{if } \mathbb{P}[\xi_j^{\mathbb{Q}} > \alpha_{j-1}^{-1} \mid \mathcal{F}_{j-1}] = 0 \text{ for all } j = 1, \dots, T \\ \infty & \text{else} \end{cases},$$

and

$$\phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}^a} \mathrm{E}_{\mathbb{Q}} \left[X + \phi^{\#}(\mathbb{Q}) \mid \mathcal{F}_t \right] = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}} \mathrm{E}_{\mathbb{Q}} \left[X \mid \mathcal{F}_t \right],$$

where

$$\mathcal{Q} := \left\{ \mathbb{Q} \in \mathcal{P}^a : \xi_j^{\mathbb{Q}} \le \alpha_{j-1}^{-1} \text{ for all } j = 1, \dots, T \right\}.$$

$(\rho_t)_{t=0}^T = (-\phi_t)_{t=0}^T$ is a time-consistent dynamic Average-Value-at-Risk at the dynamic level $(\alpha_0, \dots, \alpha_{T-1})$.

2.3.2 Dynamic entropic risk measure

For all $t = 0, \dots, T-1$, let $\alpha_t \in L^\infty(\mathcal{F}_t)$ with $\alpha_t > 0$ and define φ_t by

$$\varphi_t(X) = -\alpha_t^{-1} \log \mathrm{E}_{\mathbb{P}} \left[\exp(-\alpha_t X) \mid \mathcal{F}_t \right], \quad X \in L^\infty(\mathcal{F}_{t+1}).$$

Then, $-\varphi_t$ is a conditional entropic risk measure on $L^\infty(\mathcal{F}_{t+1})$ with risk aversion parameter α_t; see [19, 20, 5, 6, 14, 10]. It is well known that the minimal dynamic penalty function $(\varphi_t^{\#})_{t=0}^{T-1}$ of the induced time-consistent concave monetary utility function $(\phi_t)_{t=0}^T$ is given by

$$\varphi_t^{\#}(\xi_{t+1}) = \alpha_t^{-1} \mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} \, \log(\xi_{t+1}) \mid \mathcal{F}_t \right].$$

Hence,

$$\phi^{\#}(\mathbb{Q}) = \sum_{j=1}^{T} \varphi^{\#}_{j-1}(\mathbb{Q}) = \sum_{j=1}^{T} \frac{1}{\alpha_j} \mathrm{E}_{\mathbb{Q}} \left[\log\left(\xi_{j+1}^{\mathbb{Q}} \right) \mid \mathcal{F}_j \right],$$

and

$$\phi_t(X) \quad = \quad \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}^a} \mathrm{E}_{\mathbb{Q}} \left[X + \phi^{\#}(\mathbb{Q}) \mid \mathcal{F}_t \right], \quad X \in L^\infty(\mathcal{F}_T), \quad t = 0, \dots, T.$$

2.3.3 Dynamic variational preferences

Denote by $\mathcal{R}^\infty$ the space of all essentially bounded adapted processes $(X_t)_{t=0}^T$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$. We here understand X_t as a (not necessarily discounted) cashflow at time t. Let $(\phi_t)_{t=0}^T$ be a time-consistent dynamic concave monetary utility function on $L^\infty(\mathcal{F}_T)$ and $\beta > 0$ a depreciation factor. The transform

$$W_t(.) := \beta^{-t} \phi_t(\beta^t .), \quad t = 0, \dots, T,$$

is still a dynamic concave monetary utility function on $L^\infty(\mathcal{F}_T)$. It satisfies the β-time-consistency condition

$$W_{t+1}(Y) \ge W_{t+1}(Z) \quad \text{implies} \quad W_t(\beta Y) \ge W_t(\beta Z) \quad \text{for } Y, Z \in L^\infty(\mathcal{F}_T).$$

or equivalently, the β-dynamic programming principle

$$W_t(Y) = W_t(\beta W_{t+1}(\beta^{-1} Y)) \quad \text{for } Y \in L^\infty(\mathcal{F}_T).$$

Now, let u be an increasing continuous function from $\mathbb{R}$ to $\mathbb{R}$. Then, the functionals

$$V_t(X) := W_t\left(\sum_{j=t}^{T} \beta^{j-t} u(X_j)\right), \quad X \in \mathcal{R}^\infty,\ t = 0, \ldots, T,$$

satisfy the recursive relation

$$V_t(X) = u(X_t) + W_t(\beta V_{t+1}(X)), \quad t = 0, \ldots, T-1.$$

In the simpler framework of finite sample spaces, this class of dynamic preferences is axiomatized in [29], where they are called dynamic variational preferences.

3 Dynamic monetary risk measures for stochastic processes

In this section the risky objects are stochastic processes $X \in \mathcal{R}^\infty$ modelling discounted value processes or discounted cumulated cash flows; for instance, the discounted market value of a portfolio, the discounted equity value of a firm or the discounted surplus of an insurance company. This interpretation of $X \in \mathcal{R}^\infty$ is the same as in [4, 8, 9, 10, 25] but different from the one in Subsection 2.3.3 above, where X is understood as a sequence of cash flows. However, in discrete time it is easy to pass from cash flows to cumulated cash flows and back (see Subsection 3.1 below). As before, we are interested in monetary risk measures ρ_t but find it more convenient to work with the corresponding monetary utility functions $\phi_t = -\rho_t$. In the following, we generalize the definitions of Section 2 to this more general setup. For $0 \le t \le s \le T$, we define the projection $\pi_{t,s} : \mathcal{R}^\infty \to \mathcal{R}^\infty$ by

$$\pi_{t,s}(X)_r := 1_{\{t \le r\}} X_{r \wedge s}, \quad r = 0, \ldots, T.$$

and denote

$$\mathcal{R}^\infty_{t,s} := \pi_{t,s}(\mathcal{R}^\infty).$$

Definition 3.1 *Let $t \in \{0, \ldots, T\}$. A monetary utility function on $\mathcal{R}^\infty_{t,T}$ is a mapping $\phi_t : \mathcal{R}^\infty_{t,T} \to L^\infty(\mathcal{F}_t)$ with the following properties:*

(N) **Normalization:** $\phi_t(0) = 0$

(M) **Monotonicity:** $\phi_t(X) \ge \phi_t(Y)$ *for all* $X, Y \in \mathcal{R}^\infty_{t,T}$ *such that* $X \ge Y$

(T) **Translation property:** $\phi_t(X + m1_{[t,T]}) = \phi_t(X) + m$ *for all* $X \in \mathcal{R}^\infty_{t,T}$ *and* $m \in L^\infty(\mathcal{F}_t)$.

We call ϕ_t $\mathcal{F}_t$-concave if it satisfies

(C) $\mathcal{F}_t$**-concavity:** $\phi_t(\lambda X + (1-\lambda)Y) \ge \lambda\phi_t(X) + (1-\lambda)\phi_t(Y)$ *for all* $X, Y \in \mathcal{R}^\infty_{t,T}$ *and* $\lambda \in L^\infty(\mathcal{F}_t)$ *such that* $0 \le \lambda \le 1$

For $X \in \mathcal{R}^\infty$ we set

$$\phi_t(X) := \phi_t \circ \pi_{t,T}(X).$$

A dynamic monetary utility function on $\mathcal{R}^\infty$ is a family $(\phi_t)_{t=0}^T$ such that each ϕ_t is a monetary utility function on $\mathcal{R}^\infty_{t,T}$. If all ϕ_t satisfy (C)*, then we call $(\phi_t)_{t=0}^T$ a dynamic concave monetary utility function on $\mathcal{R}^\infty$.*

As in the case of risk measures for random variables, it can be deduced from (M) and (T) that ϕ_t satisfies the

(LP) **Local property:** $\phi_t(1_A X + 1_{A^c} Y) = 1_A \phi_t(X) + 1_{A^c} \phi_t(Y)$ for all $X, Y \in \mathcal{R}^\infty$ and $A \in \mathcal{F}_t$.

Definition 3.2 *We call a dynamic monetary utility function* $(\phi_t)_{t=0}^T$ *on* $\mathcal{R}^\infty$ *time-consistent if for all* $X, Y \in \mathcal{R}^\infty$ *and* $t = 0, \ldots, T-1$,

$$X_t = Y_t \quad \text{and} \quad \phi_{t+1}(X) \geq \phi_{t+1}(Y)$$

imply

$$\phi_t(X) \geq \phi_t(Y).$$

It can easily be deduced from (N), (M) and (T) that time-consistency of a dynamic monetary risk measure on $\mathcal{R}^\infty$ is equivalent to the following dynamic programming principle:

$$\phi_t(X) = \phi_t(X_t 1_{\{t\}} + \phi_{t+1}(X) 1_{[t+1,T]}) \quad \text{for all } X \in \mathcal{R}^\infty \text{ and } t = 0, \ldots, T-1. \tag{3.1}$$

3.1 Cash flow streams

In discrete time, one can easily pass from discounted value processes or cumulated discounted cash flows $X \in \mathcal{R}^\infty$ to discounted increments or discounted cash flows $C_0 = X_0$, $C_t = \Delta X_t = X_t - X_{t-1}$, $t \geq 1$. A monetary utility function $\phi_t : \mathcal{R}^\infty_{t,T} \to L^\infty(\mathcal{F}_t)$ induces the following functional for future discounted cash flows

$$\tilde{\phi}_t(C_{t+1}, \ldots, C_T) = \phi_t\left(0, \ldots, 0, C_{t+1}, C_{t+1} + C_{t+2}, \ldots, \sum_{j=t+1}^T C_j\right).$$

The dynamic programming principle (3.1) translates into

$$\tilde{\phi}_t\left(C_{t+1}, \ldots, C_T\right) = \tilde{\phi}_t\left(C_{t+1} + \tilde{\phi}_{t+1}\left(C_{t+2}, \ldots, C_T\right), 0, \ldots, 0\right),$$

and ϕ_t can be recovered from $\tilde{\phi}_t$ through

$$\phi_t(X) = X_t + \tilde{\phi}_t(\Delta X_{t+1}, \ldots, \Delta X_T).$$

So in discrete time, the two formulations are equivalent.

3.2 Aggregators and generators

For a time-consistent dynamic monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$, we define the aggregators

$$G_t : L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t), \quad t = 0, \ldots, T-1$$

by

$$G_t(X_t, X_{t+1}) := \phi_t(X),$$

where X is the process in $\mathcal{R}^\infty_{t,t+1}$ given by

$$X_r := \begin{cases} 0 & \text{for } r < t \\ X_t & \text{for } r = t \\ X_{t+1} & \text{for } r \geq t+1 \end{cases}.$$

Clearly, G_t has the following three properties:

(G1) $G_t(0,0) = 0$
(G2) $G_t(X_t, X_{t+1}) \geq G_t(Y_t, Y_{t+1})$ if $X_t \geq Y_t$ and $X_{t+1} \geq Y_{t+1}$
(G3) $G_t(X_t + m, X_{t+1} + m) = G_t(X_t, X_{t+1}) + m$ for all $m \in L(\mathcal{F}_t)$,

and it can be seen from (3.1) that the whole dynamic functional $(\phi_t)_{t=0}^T$ is uniquely determined by the aggregators $(G_t)_{t=0}^{T-1}$. In fact, every sequence of functionals $(G_t)_{t=0}^{T-1}$ satisfying (G1)–(G3) defines a time-consistent dynamic monetary utility function $(\phi_t)_{t=0}^T$ by

$$\begin{aligned} \phi_T(X) &= X \\ \phi_t(X) &= G_t(X_t, \phi_{t+1}(X)), \quad t \leq T-1. \end{aligned}$$

It is clear that $(\phi_t)_{t=0}^T$ is concave if and only if all G_t satisfy

(G4) $G_t(\lambda X_t + (1-\lambda)Y_t, \lambda X_{t+1} + (1-\lambda)Y_{t+1}) \geq \lambda G_t(X_t, X_{t+1}) + (1-\lambda)G_t(Y_t, Y_{t+1})$ for all $X_t, Y_t \in L^\infty(\mathcal{F}_t)$, $X_{t+1}, Y_{t+1} \in L^\infty(\mathcal{F}_{t+1})$ and $\lambda \in L^\infty(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$.

By (G3), we can write G_t as

$$G_t(X_t, X_{t+1}) = X_t + G_t(0, X_{t+1} - X_t) = X_t + H_t(X_{t+1} - X_t), \tag{3.2}$$

for the mapping $H_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$ given by

$$H_t(X) := G_t(0, X).$$

It follows from (G1)–(G3) that H_t has the following three properties:

(H1) $H_t(0) = 0$

(H2) $H_t(X) \geq H_t(Y)$ for $X, Y \in L^\infty(\mathcal{F}_{t+1})$ with $X \geq Y$

(H3) $H_t(X + m) \leq H_t(X) + m$ for all $X \in L^\infty(\mathcal{F}_{t+1})$ and $m \in L^\infty_+(\mathcal{F}_t)$

(H1) and (H2) are clear, and (H3) holds because for $X \in L^\infty(\mathcal{F}_{t+1})$ and $m \in L^\infty_+(\mathcal{F}_t)$ one has

$$H_t(X + m) = G_t(0, X + m) = m + G_t(-m, X) \leq m + G_t(0, X) = m + H_t(X).$$

On the other hand, every sequence $(H_t)_{t=0}^{T-1}$ of mappings satisfying (H1)–(H3) induces aggregators $(G_t)_{t=0}^{T-1}$ of a time-consistent dynamic monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$. Indeed, if H_t satisfies (H1)–(H3), then

$$G_t(X_t, X_{t+1}) = X_t + H_t(X_{t+1} - X_t)$$

satisfies (G1)–(G3). (G1) and (G3) are clear, and (G2) holds because for $X_t, Y_t \in L^\infty(\mathcal{F}_t)$ and $X_{t+1}, Y_{t+1} \in L^\infty(\mathcal{F}_{t+1})$ such that $X_t \geq Y_t$ and $X_{t+1} \geq Y_{t+1}$, one obtains from (H2) and (H3) that

$$G_t(X_t, X_{t+1}) = X_t + H_t(X_{t+1} - X_t) \geq X_t + H_t(Y_{t+1} - X_t) \geq Y_t + H_t(Y_{t+1} - Y_t).$$

We call $(H_t)_{t=0}^{T-1}$ the generators of $(\phi_t)_{t=0}^T$. It is clear that G_t satisfies (G4) if and only if H_t fulfils

(H4) $H_t(\lambda X + (1-\lambda)Y) \geq \lambda H_t(X) + (1-\lambda)H_t(Y)$ for all $\lambda \in L^\infty(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$.

3.3 Duality

For $t = 1, \dots, T$, we define the set

$$\mathcal{E}_t := \left\{ \xi \in L^1_+(\mathcal{F}_t) : \mathrm{E}_{\mathbb{P}}\left[\xi \mid \mathcal{F}_{t-1}\right] \leq 1 \right\}.$$

Every sequence $(\xi_{t+1}, \dots, \xi_T) \in \mathcal{E}_{t+1} \times \dots \times \mathcal{E}_T$ induces a $\mathbb{P}$-supermartingale $(M^\xi_r)_{r=0}^T$ by

$$M^\xi_r := \begin{cases} 1 & \text{for } r \leq t \\ \xi_{t+1} \cdots \xi_r & \text{for } r = t+1, \dots, T. \end{cases}$$

Definition 3.3 *A one-step penalty function on $\mathcal{E}_{t+1}$ is a mapping*

$$\psi_t : \mathcal{E}_{t+1} \to \bar{L}_+(\mathcal{F}_t)$$

that satisfies the two properties:

(i) $\operatorname{ess\,inf}_{\xi \in \mathcal{E}_{t+1}} \psi_t(\xi) = 0$

(ii) $\psi_t(1_A \xi + 1_{A^c} \xi') = 1_A \psi_t(\xi) + 1_{A^c} \psi_t(\xi')$ *for all $\xi, \xi' \in \mathcal{E}_{t+1}$ and $A \in \mathcal{F}_t$*

A dynamic penalty function on $\mathcal{E}$ is a sequence $(\psi_t)_{t=0}^{T-1}$ of one-step penalty functions.

Theorem 3.4 *Let $(\psi_t)_{t=0}^{T-1}$ be a dynamic penalty function on $\mathcal{E}$. Then*

$$H_t(X) = \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{E}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} X \mid \mathcal{F}_t \right] + \psi_t(\xi_{t+1}) \right\}, \quad t = 0, \dots, T-1, \tag{3.3}$$

defines generators of a time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$ with the following representation:

$$\phi_t(X) = X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_T) \in \mathcal{E}_{t+1} \times \dots \times \mathcal{E}_T} \mathrm{E}_{\mathbb{P}} \left[\sum_{j=t+1}^T M_j^\xi \Delta X_j + M_{j-1}^\xi \psi_{j-1}(\xi_j) \mid \mathcal{F}_t \right] \tag{3.4}$$

for all $t \le T-1$ and $X \in \mathcal{R}^\infty$.

Proof. It can easily be checked that for every $t = 0, \dots, T-1$,

$$H_t(X) = \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{E}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} X \mid \mathcal{F}_t \right] + \psi_t(\xi_{t+1}) \right\}$$

defines a mapping from $L^\infty(\mathcal{F}_{t+1})$ to $L^\infty(\mathcal{F}_t)$ satisfying (H1)–(H4). Therefore, the family $(H_t)_{t=0}^{T-1}$ induces a time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$. Let $X \in \mathcal{R}^\infty_{t,s}$ for some $t < s \le T$ and denote

$$\phi_{t,s}(X) = X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_s) \in \mathcal{E}_{t+1} \times \dots \times \mathcal{E}_s} \mathrm{E}_{\mathbb{P}} \left[\sum_{j=t+1}^s M_j^\xi \Delta X_j + M_{j-1}^\xi \psi_{j-1}(\xi_j) \mid \mathcal{F}_t \right].$$

Since

$$\operatorname*{ess\,inf}_{(\xi_{s+1},\dots,\xi_T) \in \mathcal{E}_{s+1} \times \dots \times \mathcal{E}_T} \mathrm{E}_{\mathbb{P}} \left[\sum_{j=s+1}^T M_{j-1}^\xi \psi_{j-1}(\xi_j) \mid \mathcal{F}_t \right] = 0,$$

for all $s \le T-1$, one has

$$\phi_{t,s}(X) = X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_T) \in \mathcal{E}_{t+1} \times \dots \times \mathcal{E}_T} \mathrm{E}_{\mathbb{P}} \left[\sum_{j=t+1}^T M_j^\xi \Delta X_j + M_{j-1}^\xi \psi_{j-1}(\xi_j) \mid \mathcal{F}_t \right].$$

So it is enough to show that $\phi_t(X) = \phi_{t,s}(X)$. We prove this by induction over s. First, assume that $X \in \mathcal{R}^\infty_{t,t+1}$. Then,

$$\begin{aligned} & \phi_t(X) = G_t(X_t, \phi_{t+1}(X)) = G_t(X_t, X_{t+1}) = X_t + H_t(\Delta X_{t+1}) \\ = \; & X_t + \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{E}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} \Delta X_{t+1} \mid \mathcal{F}_t \right] + \psi_t(\xi_{t+1}) \right\} = \phi_{t,t+1}(X). \end{aligned}$$

Now, assume $X \in \mathcal{R}^\infty_{t,s}$ for $s \geq t+2$ and $\phi_t(Y) = \phi_{t,s-1}(Y)$ for all $Y \in \mathcal{R}^\infty_{t,s-1}$. By (3.1), we have $\phi_t(X) = \phi_t(Y)$ for

$$Y = 1_{[t,s-1)}X + 1_{[s-1,T]}\phi_{s-1}(X) \in \mathcal{R}^\infty_{t,s-1},$$

and therefore,

$$\begin{aligned}
& \phi_t(X) = \phi_t(Y) = \phi_{t,s-1}(Y) \\
= \; & X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_{s-1}) \in \mathcal{E}_{t+1}\times\cdots\times\mathcal{E}_{s-1}} \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{s-1} M^\xi_j \Delta Y_j + M^\xi_{j-1}\psi_{j-1}(\xi_j) \mid \mathcal{F}_t\right] \\
= \; & X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_{s-1}) \in \mathcal{E}_{t+1}\times\cdots\times\mathcal{E}_{s-1}} \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{s-2} M^\xi_j \Delta X_j + M^\xi_{s-1}[\phi_{s-1}(X) - X_{s-2}]\right. \\
& \qquad \left. + \sum_{j=t+1}^{s-1} M^\xi_{j-1}\psi_{j-1}(\xi_j) \mid \mathcal{F}_t\right] \\
= \; & X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_{s-1}) \in \mathcal{E}_{t+1}\times\cdots\times\mathcal{E}_{s-1}} \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{s-2} M^\xi_j \Delta X_j\right. \\
& \qquad + M^\xi_{s-1}[\Delta X_{s-1} + \operatorname*{ess\,inf}_{\xi_s \in \mathcal{E}_s}\{\mathrm{E}_{\mathbb{P}}[\xi_s \Delta X_s \mid \mathcal{F}_{s-1}] + \psi_{s-1}(\xi_s)\}] \\
& \qquad \left. + \sum_{j=t+1}^{s-1} M^\xi_{j-1}\psi_{j-1}(\xi_j) \mid \mathcal{F}_t\right],
\end{aligned} \tag{3.5}$$

where for $s = t+2$, the term $\sum_{j=t+1}^{s-2} M^\xi_j \Delta X_j$ is understood as 0. By condition (ii) of Definition 3.3, the family

$$\mathrm{E}_{\mathbb{P}}[\xi_s \Delta X_s \mid \mathcal{F}_{s-1}] + \psi_{s-1}(\xi_s), \quad \xi_s \in \mathcal{E}_s,$$

is directed downwards. Therefore, we can take the ess inf in (3.5) outside of the conditional expectation $\mathrm{E}_{\mathbb{P}}[. \mid \mathcal{F}_t]$ and arrive at

$$\phi_t(X) = X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_s) \in \mathcal{E}_{t+1}\times\cdots\times\mathcal{E}_s} \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{s} M^\xi_j \Delta X_j + M^\xi_{j-1}\psi_{j-1}(\xi_j)\right] = \phi_{t,s}(X).$$

□

It is easy to see that aggregators of the form (3.3) are continuous from above in the sense of Definition 2.6 and utility functions of the form (3.4) are continuous from above in the following more general sense:

Definition 3.5 *Let $0 \leq t \leq s \leq T$. We call a mapping $I : \mathcal{R}^\infty_{t,s} \to L^\infty(\mathcal{F}_t)$ continuous from above if $I(X^n) \to I(X)$ $\mathbb{P}$-almost surely for all $(X^n)_{n\in\mathbb{N}}$ and X in $\mathcal{R}^\infty_{t,s}$ such that X^n_r decreases to X_r $\mathbb{P}$-almost surely for all $r = t,\dots,s$. We call a dynamic monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$ continuous from above if every ϕ_t is continuous from above.*

Lemma 3.6 *A time-consistent dynamic monetary utility function $(\phi_t)_{t=0}^T$ is continuous from above if and only if all of the corresponding aggregators $(G_t)_{t=0}^{T-1}$ are continuous from above, which is the case if and only if all of the associated generators $(H_t)_{t=0}^{T-1}$ are continuous from above.*

Proof. It is obvious that $(\phi_t)_{t=0}^T$ is continuous from above if and only if all aggregators $(G_t)_{t=0}^{T-1}$ are continuous from above. Now, fix t and assume that G_t is continuous from above. If $(X^n)_{n\in\mathbb{N}}$ and X are in $L^\infty(\mathcal{F}_{t+1})$ such that $(X^n)_{n\in\mathbb{N}}$ decreases to X $\mathbb{P}$-almost surely, then we have

$$H_t(X^n) = G_t(0, X^n) \downarrow G_t(0, X) = H_t(X) \quad \mathbb{P}\text{-almost surely.}$$

Hence, H_t is continuous from above. On the other hand, if we assume that H_t is continuous from above and $(X_t^n, X_{t+1}^n)_{n\in\mathbb{N}}$ is a sequence in $L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1})$ that decreases to $(X_t, X_{t+1}) \in L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1})$ $\mathbb{P}$-almost surely, then

$$G_t(X_t^n, X_{t+1}^n) = X_t^n + H_t(X_{t+1}^n - X_t^n) \le X_t^n + H_t(X_{t+1}^n - X_t) \downarrow X_t + H_t(X_{t+1} - X_t) = G_t(X_t, X_{t+1})$$

$\mathbb{P}$-almost surely. This shows that G_t is continuous from above. □

Definition 3.7 *For a time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$ with generators $(H_t)_{t=0}^{T-1}$ that are continuous from above, we define for $\xi_{t+1} \in \mathcal{E}_{t+1}$ and $t = 0, \dots, T-1$,*

$$H_t^{\#}(\xi_{t+1}) := \operatorname*{ess\,sup}_{X \in L^\infty(\mathcal{F}_{t+1})} \left\{ H_t(X) - \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} X \mid \mathcal{F}_t\right] \right\}.$$

Lemma 3.8 *Let $(\phi_t)_{t=0}^T$ be a time-consistent dynamic concave monetary utility function on $\mathcal{R}^\infty$ with generators $(H_t)_{t=0}^{T-1}$ that are continuous from above. Then $(H_t^{\#})_{t=0}^{T-1}$ is the smallest dynamic penalty function on $\mathcal{E}$ such that*

$$H_t(X) = \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{E}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} X \mid \mathcal{F}_t\right] + H_t^{\#}(\xi_{t+1}) \right\} \tag{3.6}$$

for all $t = 0, \dots, T-1$ and $X \in L^\infty(\mathcal{F}_{t+1})$.

Proof. Fix $t \in \{0, \dots, T-1\}$ and consider $\hat{\Omega} := \{t, t+1\} \times \Omega$ with the σ-algebra $\hat{\mathcal{F}}_{t+1}$ generated by all sets of the form $\{j\} \times A_j$ for $j = t, t+1$ and $A_j \in \mathcal{F}_j$. Let $\hat{\mathbb{P}}$ be the probability measure on $(\hat{\Omega}, \hat{\mathcal{F}}_{t+1})$ given by $\hat{\mathbb{P}}[\{j\} \times A_j] := \frac{1}{2}\mathbb{P}[A_j]$ for $j = t, t+1$ and $A_j \in \mathcal{F}_j$. By $\hat{\mathcal{F}}_t$ we denote the σ-algebra on $\hat{\Omega}$ generated by all sets of the form $\{t, t+1\} \times A_t$ for $A_t \in \mathcal{F}_t$. Then one has

$$L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1}) = L^\infty(\hat{\Omega}, \hat{\mathcal{F}}_{t+1}, \hat{\mathbb{P}}),$$

and the aggregator G_t can be viewed as a concave monetary utility function from $L^\infty(\hat{\Omega}, \hat{\mathcal{F}}_{t+1}, \hat{\mathbb{P}})$ to $L^\infty(\hat{\Omega}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}})$. Clearly, it is continuous from above. Therefore, it follows from Lemma 2.8 that

$$G_t(X_t, X_{t+1}) = \operatorname*{ess\,inf}_{(a,\xi_{t+1}) \in L^\infty_{[0,1]}(\mathcal{F}_t) \times \mathcal{D}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}}\left[(1-a)X_t + a\xi_{t+1} X_{t+1} \mid \mathcal{F}_t\right] + \zeta_t(a, \xi_{t+1}) \right\}, \tag{3.7}$$

where

$$L^\infty_{[0,1]}(\mathcal{F}_t) := \{a \in L^\infty(\mathcal{F}_t) : 0 \le a \le 1\}$$

and

$$\begin{aligned}
\zeta_t(a, \xi_{t+1}) &:= \operatorname*{ess\,sup}_{(X_t, X_{t+1}) \in L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1})} \left\{ G_t(X_t, X_{t+1}) - \mathrm{E}_{\mathbb{P}}\left[(1-a)X_t + a\,\xi_{t+1} X_{t+1} \mid \mathcal{F}_t\right] \right\} \\
&= \operatorname*{ess\,sup}_{(X_t, X_{t+1}) \in L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1})} \left\{ G_t(0, \Delta X_{t+1}) - \mathrm{E}_{\mathbb{P}}\left[a\,\xi_{t+1} \Delta X_{t+1} \mid \mathcal{F}_t\right] \right\} \\
&= \operatorname*{ess\,sup}_{X \in L^\infty(\mathcal{F}_{t+1})} \left\{ H_t(X) - \mathrm{E}_{\mathbb{P}}\left[a\,\xi_{t+1} X \mid \mathcal{F}_t\right] \right\}.
\end{aligned}$$

Hence,

$$H_t(X) \quad = \quad G_t(0,X) = \operatorname*{ess\,inf}_{\xi_{t+1}\in\mathcal{E}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}X \mid \mathcal{F}_t\right] + H_t^{\#}(\xi_{t+1}) \right\}$$

for all $X \in L^\infty(\mathcal{F}_{t+1})$. The minimality of $H_t^{\#}$ follows because every function $\psi_t : \mathcal{E}_{t+1} \to \bar{L}_+(\mathcal{F}_t)$ fulfilling

$$H_t(X) = \operatorname*{ess\,inf}_{\xi_{t+1}\in\mathcal{E}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}X \mid \mathcal{F}_t\right] + \psi_t(\xi_{t+1}) \right\} \quad \text{for all } X \in L^\infty(\mathcal{F}_{t+1})$$

must also satisfy

$$H_t(X) - \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}X \mid \mathcal{F}_t\right] \le \psi_t(\xi_{t+1}) \quad \text{for all } X \in L^\infty(\mathcal{F}_{t+1}) \text{ and } \xi_{t+1} \in \mathcal{E}_{t+1},$$

and therefore, $H_t^{\#} \le \psi_t$. □

The following corollary is an immediate consequence of Theorem 3.4 and Lemma 3.8.

Corollary 3.9 *Let $(\phi_t)_{t=0}^T$ be a time-consistent dynamic concave monetary utility function that is continuous from above. Then*

$$\phi_t(X) = X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_T)\in\mathcal{E}_{t+1}\times\dots\times\mathcal{E}_T} \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{T} M_j^\xi \Delta X_j + M_{j-1}^\xi H_{j-1}^{\#}(\xi_j) \mid \mathcal{F}_t\right]$$

for all $t \le T-1$ and $X \in R^\infty$.

3.4 Composed generators

We now consider generators of the special form

$$H_t(X) = h_t(\varphi_t(X)), \tag{3.8}$$

where $\varphi_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$ is a monetary utility function from $L^\infty(\mathcal{F}_{t+1})$ to $L^\infty(\mathcal{F}_t)$ and $h_t : \mathbb{R} \to \mathbb{R}$ is a function satisfying

(h1) $h_t(0) = 0$
(h2) $h_t(x) \ge h_t(y)$ for $x \ge y$
(h3) $|h_t(x) - h_t(y)| \le |x-y|$ for all $x, y \in \mathbb{R}$.

Then $H_t = h_t \circ \varphi_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$ satisfies the properties (H1), (H2) and (H3). Hence, $(h_t, \varphi_t)_{t=0}^{T-1}$ induces a time-consistent dynamic monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$. If in addition to (h1)–(h3), h_t is concave and φ_t $\mathcal{F}_t$-concave, then H_t satisfies (H4), and $(\phi_t)_{t=0}^T$ is a time-consistent dynamic concave monetary utility function on $\mathcal{R}^\infty$.

By standard convex duality, every concave function $h : \mathbb{R} \to \mathbb{R}$ can be represented as

$$h(x) = \min_{y\in\mathbb{R}} xy - h^*(y),$$

where h^* is the concave conjugate given by

$$h^*(y) := \inf_{x\in\mathbb{R}} xy - h(x).$$

As a consequence, the following representation result holds:

Proposition 3.10 *If $\varphi_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$ is given by*

$$\varphi_t(X) = \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{D}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} X \mid \mathcal{F}_t \right] + \eta_t(\xi_{t+1}) \right\}$$

for a one-step penalty function η_t on $\mathcal{D}_{t+1}$ and $h_t : \mathbb{R} \to \mathbb{R}$ is a concave function satisfying (h1)–(h3)*. Then $H_t = h_t \circ \varphi_t$ can be represented as*

$$H_t(X) = \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{E}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} X \mid \mathcal{F}_t \right] + \psi_t(\xi_{t+1}) \right\}, \quad X \in L^\infty(\mathcal{F}_{t+1}),$$

for the one-step penalty function ψ_t on $\mathcal{E}_{t+1}$ given by

$$\psi_t(\xi_{t+1}) = \mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} \mid \mathcal{F}_t \right] \, \eta_t \left(\frac{\xi_{t+1}}{\mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} \mid \mathcal{F}_t \right]} \right) - h_t^*(\mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} \mid \mathcal{F}_t \right]).$$

Proof. Since h_t satisfies (h2) and (h3), we have $h_t^*(y) = -\infty$ for $y \notin [0,1]$, and therefore

$$\begin{aligned}
& H_t(X) = h_t(\varphi_t(X)) \\
= \; & h_t \left(\operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{D}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} X \mid \mathcal{F}_t \right] + \eta_t(\xi_{t+1}) \right\} \right) \\
= \; & \operatorname*{ess\,inf}_{0 \le y \le 1 \,,\, \xi_{t+1} \in \mathcal{D}_{t+1}} \left\{ y \left[\mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} X \mid \mathcal{F}_t \right] + \eta_t(\xi_{t+1}) \right] - h_t^*(y) \right\} \\
= \; & \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{E}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} X \mid \mathcal{F}_t \right] + \mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} \mid \mathcal{F}_t \right] \, \eta_t \left(\frac{\xi_{t+1}}{\mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} \mid \mathcal{F}_t \right]} \right) - h_t^*(\mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} \mid \mathcal{F}_t \right]) \right\}.
\end{aligned}$$

□

In the following we are going to discuss different specifications of the function h_t. This leads to extensions of some of the examples of Section 5 of Cheridito et al. [10].

3.4.1 Risk measures which only depend on the final value

If $h_t(x) = x$, then the aggregators reduce to

$$G_t(X_t, X_{t+1}) = X_t + h_t(\varphi_t(X_{t+1} - X_t)) = \varphi_t(X_{t+1}).$$

So one has

$$\phi_t(X) = G_t(X_t, \phi_{t+1}(X)) = \varphi_t(\phi_{t+1}(X)) = \varphi_t \circ \cdots \circ \varphi_{T-1}(X_T),$$

that is, $(\phi_t)_{t=0}^T$ is a time-consistent dynamic utility function which only depends on the final value X_T of X.

3.4.2 Risk measures that depend on a weighted average over time

If $h_t(x) = \gamma_t x$ for $\gamma_t \in L^\infty(\mathcal{F}_t)$ such that $0 < \gamma_t \le 1$, then

$$G_t(X_t, X_{t+1}) = X_t + h_t(\varphi_t(X_{t+1} - X_t)) = (1 - \gamma_t) X_t + \gamma_t \varphi_t(X_{t+1}).$$

Consider the mappings $\phi_t^\gamma : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$ given by

$$\phi_t^\gamma(X) = \gamma_t \varphi_t \circ \gamma_{t+1} \varphi_{t+1} \circ \cdots \circ \gamma_{T-1} \varphi_{T-1} \left(\frac{X}{\gamma_t \cdots \gamma_{T-1}} \right).$$

Then the time-consistent dynamic monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$ induced by the aggregators $(G_t)_{t=0}^T$ is given by

$$\phi_t(X) = \phi_t^\gamma\left(\sum_{j=t}^T \delta_j^t X_j\right), \quad t = 0, \dots T-1,$$

where

$$\delta_j^t = \begin{cases} 1-\gamma_t & \text{for } j = t \\ \gamma_t \cdot \dots \cdot \gamma_{j-1}(1-\gamma_j) & \text{for } t < j < T \\ \gamma_t \cdot \dots \cdot \gamma_{T-1} & \text{for } j = T \end{cases}.$$

In particular, for $\gamma_t = \frac{T-t}{T-t+1}$, one obtains

$$\phi_t(X) = \phi_t^\gamma\left(\frac{1}{T-t+1}\sum_{j=t}^T X_j\right).$$

Observe that for the special case, where all φ_t have the scaling property $\varphi_t(\gamma_t X) = \gamma_t \varphi_t(X)$, the ϕ_t^γ are of the form $\phi_t^\gamma = \varphi_t \circ \dots \circ \varphi_{T-1}$.

3.4.3 Risk measures defined by worst stopping

For $h_t(x) = x \wedge 0$, the aggregators become

$$G_t(X_t, X_{t+1}) = X_t + h_t(\varphi_t(X_{t+1} - X_t)) = X_t \wedge \varphi_t(X_{t+1}), \tag{3.9}$$

and the ϕ_t are of the form

$$\phi_t(X) = \operatorname*{ess\,inf}_{\tau \in \Theta_t} \tilde{\phi}_t(X_\tau), \quad X \in \mathcal{R}^\infty,$$

where Θ_t is the set of all $\{t, \dots, T\}$-valued stopping times and $(\tilde{\phi}_t)_{t=0}^T$ is the time-consistent dynamic concave monetary utility function on $L^\infty(\mathcal{F}_T)$ given by

$$\tilde{\phi}_t(X) = \varphi_t \circ \dots \circ \varphi_{T-1}(X), \quad X \in L^\infty(\mathcal{F}_T).$$

This can be seen by checking that

$$\operatorname*{ess\,inf}_{\tau \in \Theta_t} \tilde{\phi}_t(X_\tau), \quad t = 0, \dots, T$$

is a time-consistent dynamic monetary utility function on $\mathcal{R}^\infty$ whose aggregators are given by (3.9).

3.4.4 Trade-off functions

Instead of specifying the function h_t directly, one can start with a continuous decreasing function $g_t : \mathbb{R} \to \mathbb{R}$ such that $g_t(0) = 0$ and define the corresponding aggregator G_t by

$$G_t(X_t, X_{t+1}) = \operatorname{ess\,sup}\left\{m \in L^\infty(\mathcal{F}_t) : (X_t - m, X_{t+1} - m) \in \mathcal{B}_t\right\},$$

where the one-step acceptance set $\mathcal{B}_t$ is given by

$$\mathcal{B}_t = \left\{(X_t, X_{t+1}) \in L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1}) : g_t(X_t) \le \varphi_t(X_{t+1})\right\}.$$

The function g_t specifies the trade-off between risk at time t and $t+1$ of an acceptable process $X \in \mathcal{R}^\infty_{t,t+1}$. It can easily be checked that the inverse h_t of the strictly increasing function $g_t(-x)+x$ satisfies (h1)–(h3), and $\mathcal{B}_t$ can be written as

$$\mathcal{B}_t = \{(X_t, X_{t+1}) \in L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1}) : X_t + h_t \circ \varphi_t(X_{t+1} - X_t) \geq 0\}.$$

Hence, the generator H_t is given by $h_t \circ \varphi_t$. Note h_t is concave if and only if g_t is convex. For $g_t(x) = 0$ we get $h_t(x) = x$, and we are back in the case of Subsection 3.4.1. The case $g_t(x) = (1 - 1/\gamma_t)x$ for $0 < \gamma_t < 1$ corresponds to $h_t(x) = \gamma_t x$ of Subsection 3.4.2. The function $h_t(x) = x \wedge 0$ of Subsection 3.4.3 is not bijective. Therefore, it cannot be obtained from a trade-off function g_t.

Example 3.11 Our last example is built on trade-off functions of the form $g_t(x) := \exp(-\gamma_t x) - 1$ for $\gamma_t > 0$. In this case there exists no closed form expression for h_t. But one has the following relation between h_t^* and the convex conjugate $g_t^*(y) = \sup_{x \in \mathbb{R}} xy - g_t(x)$ of g_t:

$$h_t^*(y) = \begin{cases} -y g_t^*\left(1 - 1/y\right) & \text{for } y \in (0,1] \\ -\infty & \text{for } y \notin (0,1] \end{cases}.$$

Indeed, $h_t^*(y) = -\infty$ for $y \notin (0,1]$ is an immediate consequence of the fact that h_t satisfies $\lim_{x\to\infty} h_t(x) = \infty$ as well as (h2) and (h3). For $y \in (0,1]$, one can write

$$\begin{aligned} & h_t^*(y) = \inf_{x \in \mathbb{R}} xy - h_t(x) = \inf_{x \in \mathbb{R}} h_t^{-1}(-x)y - h_t(h_t^{-1}(-x)) \\ = \; & y \inf_{x \in \mathbb{R}} h_t^{-1}(-x) + x/y = y \inf_{x \in \mathbb{R}} h_t^{-1}(-x) + x + (1/y - 1)\, x \\ = \; & -y \sup_{x \in \mathbb{R}} (1 - 1/y)\, x - g_t(x) = -y g_t^*\left(1 - 1/y\right). \end{aligned}$$

For $z < 0$, one has

$$g_t^*(z) = \frac{z}{\gamma_t}\left(1 - \log\left(-\frac{z}{\gamma_t}\right)\right) + 1,$$

and therefore,

$$h_t^*(y) = \frac{1-y}{\gamma_t}\left[1 - \log\left(\frac{1-y}{\gamma_t y}\right)\right] - y.$$

We now combine h_t with the entropic generator

$$\varphi_t(X) = -\alpha_t^{-1} \log \mathrm{E}_{\mathbb{P}}\left[\exp(-\alpha_t X) \mid \mathcal{F}_t\right], \quad X \in L^\infty(\mathcal{F}_{t+1})$$

with minimal penalty function

$$\varphi_t^{\#}(\xi_{t+1}) = \alpha_t^{-1} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} \log(\xi_{t+1}) \mid \mathcal{F}_t\right].$$

It follows from Proposition 3.10 that the minimal dynamic penalty function of the dynamic concave monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$ induced by $(h_t, \varphi_t)_{t=0}^{T-1}$ is given by

$$\psi_t(\xi_{t+1}) = \alpha_t^{-1} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} \log(\xi_{t+1}) \mid \mathcal{F}_t\right] - \alpha_t^{-1} \lambda \log(\lambda) + \lambda + \frac{1-\lambda}{\gamma_t} \log\left(\frac{1-\lambda}{\gamma_t \lambda}\right) - \frac{1-\lambda}{\gamma_t},$$

for $\xi_{t+1} \in \mathcal{E}_{t+1}$ and $\lambda = \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} \mid \mathcal{F}_t\right]$.

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