LIMIT THEOREMS FOR CONTINUOUS TIME RANDOM WALKS

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1. Introduction

Continuous time random walks (CTRW) were introduced in [26] to study random walks on a lattice. They are now used in physics to model a wide variety of phenomena connected with anomalous diffusion [15, 32, 38]. A CTRW is a random walk subordinated to a renewal process. The random walk increments represent the magnitude of particle jumps, and the renewal epochs represent the times of the particle jumps. If the time between renewals has finite mean, the renewal process is asymptotically equivalent to a constant multiple of the time variable, and the CTRW behaves like the original random walk for large time [2, 17]. In many physical applications, the waiting time between renewals has infinite mean [34]. In this paper, we derive the scaling limit of a CTRW with infinite mean waiting time. The limit process is an operator Lévy motion subordinated to the hitting time process of a classical stable subordinator. This limit process is also the stochastic solution to a fractional kinetic equation for Hamiltonian chaos [40].

The model: Let $J_1, J_2, \ldots$ be nonnegative independent and identically distributed (i.i.d) random variables which model the waiting times between jumps of a particle. We set $T(0) = 0$ and $T(n) = \sum_{j=1}^n J_j$, the time of the n-th jump. The particle jumps are given by i.i.d. random vectors $Y_1, Y_2, \ldots$ on $\mathbb{R}^d$ which are assumed independent of (J_i). Let $S_0 = 0$ and $S_n = \sum_{i=1}^n Y_i$, the position of the particle after the n-the jump.

For $t \geq 0$ let

$$N_t = \max\{n \geq 0 : T(n) \leq t\}, \tag{1.1}$$

the number of jumps up to time t and we define the stochastic process $\{X(t)\}_{t\geq 0}$ by

$$X(t) = S_{N_t} = \sum_{i=1}^{N_t} Y_i. \tag{1.2}$$

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Then $X(t)$ is the position of the particle at time t. We call $\{X(t)\}_{t\geq 0}$ a *continuous time random walk* (CTRW).

In this paper we show that under certain assumptions on the distribution of J_1 and Y_1, respectively, scaling limits of the CTRW $\{X(t)\}_{t\geq 0}$ exist. These assumption are that J_1 and Y_1 belong to some domains of attraction, that is, T_n and S_n normalized appropriately converge in distribution to some nondegenerate limit. See section 2 for details.

Under these conditions the scaling limit $\{M(t)\}_{t\geq 0}$ of the CTRW has some very interesting properties. We show that this process is operator selfsimilar, however it is not a Gaussian or operator stable process, and it does not have stationary increments. Moreover the distribution of $M(t)$ has a Lebesgue density which solves a fractional kinetic equation for Hamiltonian chaos [40]. Previously Kotulski [17] and Saichev and Zaslavsky [28] have computed the limit distribution for scalar CTRW models at one fixed point in time. In this paper, we derive the entire stochastic process limit in the J_1-topology on the Skorodhod space $D([0,\infty),\mathbb{R}^d)$ for both scalar and vector CTRW models, and we elucidate the nature of the limit process as a subordinated operator Lévy motion.

2. Distributional Assumptions

As in section 1 let (J_i) be i.i.d., nonnegative and assume that J_1 belongs to the strict domain of attraction of some stable law with index $0<\beta<1$. This means that there exist $b_n>0$ such that

$$b_n(J_1+\cdots+J_n)\Rightarrow D \tag{2.1}$$

where $D>0$ almost surely. Here $\Rightarrow$ denotes convergence in distribution. The distribution ρ of D is stable with index β, meaning that $\rho^t=t^{1/\beta}\rho$ for all $t>0$, where ρ^t is the t-th convolution power of the infinitely divisible law ρ and $(a\rho)\{dx\}=\rho\{a^{-1}dx\}$ is the probability distribution of aD for $a>0$. Moreover ρ has a Lebesgue density g_β which is a C^∞-function. Note that by Theorem 4.7.1 and (4.7.13) of [38] it follows that there exists a constant $K>0$ such that

$$g_\beta(x)\leq Kx^{(1-\beta/2)/(\beta-1)}\exp\{-|1-\beta|\big(\frac{x}{\beta}\big)^{\beta/(\beta-1)}\} \tag{2.2}$$

for all $x>0$ sufficiently small.

For $t\geq 0$ let $T(t)=\sum_{j=1}^{[t]}J_j$ and let $b(t)=b_{[t]}$, where $[t]$ denotes the integer part of t. Then $b(t)=t^{-1/\beta}L(t)$ for some slowly varying function $L(t)$ (so that $L(\lambda t)/L(t)\to 1$ as $t\to\infty$ for any $\lambda>0$, see for example [11]) and it follows from Example 11.2.18 of [25] that

$$\{b(c)T(ct)\}_{t\geq 0}\stackrel{f.d.}{\Longrightarrow}\{D(t)\}_{t\geq 0}\quad\text{as } c\to\infty, \tag{2.3}$$

where $\stackrel{f.d.}{\Longrightarrow}$ denotes convergence in distribution of all finite dimensional marginal distributions. The process $\{D(t)\}_{t\geq 0}$ has stationary independent increments and since the distribution ρ of $D(1)$ is strictly stable, $\{D(t)\}_{t\geq 0}$ is called a strictly stable Lévy process. Moreover

$$\{D(ct)\}_{t\geq 0} \stackrel{f.d.}{=} \{c^{1/\beta}D(t)\}_{t\geq 0} \tag{2.4}$$

for all $c > 0$, where $\stackrel{f.d.}{=}$ denotes equality of all finite dimensional marginal distributions. Hence by Definition 13.4 of [30] the process $\{D(t)\}_{t\geq 0}$ is selfsimilar with exponent $H = 1/\beta > 1$. See [30] for more details on stable Lévy processes and selfsimilarity. Note that by Example 21.7 of [30] the sample paths of $\{D(t)\}_{t\geq 0}$ are almost surely increasing. Moreover, since $D(t) \stackrel{d}{=} t^{1/\beta}D$, where $\stackrel{d}{=}$ means equal in distribution, it follows that

$$D(t) \to \infty \quad \text{in probability as } t \to \infty. \tag{2.5}$$

Then it follows from Theorem I.19 in [6] that $D(t) \to \infty$ almost surely as $t \to \infty$. Furthermore, note that since $b(c) \to 0$ as $c \to \infty$ it follows that

$$b(c)T([cx] + k(c)) \Rightarrow D(x) \quad \text{as } c \to \infty$$

for any $x \geq 0$ as long as $|k(x)| \leq M$ for all $c > 0$ and some constant M. Hence it follows along the same lines as Example 11.2.18 in [25] that

$$\{b(c)T([cx] + k(c))\}_{x\geq 0} \stackrel{f.d.}{\Longrightarrow} \{D(x)\}_{x\geq 0} \quad \text{as } c \to \infty, \tag{2.6}$$

as long as $|k(c)| \leq M$ for all $c > 0$ and some constant M. Furthermore, since (J_j) are i.i.d. it follows that the process $\{T(k) : k = 0, 1, \dots\}$ has stationary increments, that is for any nonnegative integer ℓ we have

$$\{T(k + \ell) - T(\ell) : k = 0, 1, \dots\} \stackrel{f.d.}{=} \{T(k) : k = 0, 1, \dots\}. \tag{2.7}$$

In view of our general CTRW model introduced in section 1 let (Y_i) be i.i.d. $\mathbb{R}^d$-valued random variables independent of (J_i) and assume that Y_1 belongs to the strict generalized domain of attraction of some full operator stable law ν, where full means that ν is not supported on any proper hyperplane of $\mathbb{R}^d$. By Theorem 8.1.5 of [25] there exists a function $B \in \mathrm{RV}(-E)$ (that is, $B(c)$ is invertible for all $c > 0$ and $B(\lambda c)B(c)^{-1} \to \lambda^{-E}$ as $c \to \infty$ for any $\lambda > 0$), E being a $d \times d$ matrix with real entries, such that

$$B(n)\sum_{i=1}^{n} Y_i \Rightarrow A \quad \text{as } n \to \infty, \tag{2.8}$$

where A has distribution ν. Then $\nu^t = t^E\nu$ for all $t > 0$, where $T\nu\{dx\} = \nu\{T^{-1}dx\}$ is the probability distribution of TA for any Borel measurable function $T : \mathbb{R}^d \to \mathbb{R}^m$. Note that by Theorem 7.2.1 of [25] the real parts of the eigenvalues of E are greater than or equal to $1/2$.

Moreover, if we define the stochastic process $\{S(t)\}_{t\geq 0}$ by $S(t) = \sum_{i=1}^{[t]} Y_i$ it follows from Example 11.2.18 in [25] that

$$\{B(c)S(ct)\}_{t\geq 0} \stackrel{f.d.}{\Longrightarrow} \{A(t)\}_{t\geq 0} \quad \text{as } c \to \infty, \tag{2.9}$$

where $\{A(t)\}_{t\geq 0}$ has stationary independent increments with $P_{A(t)} = \nu^t = t^E \nu$ for all $t > 0$; P_X denoting the distribution of X. Then $\{A(t)\}$ is continuous in law, and it follows that

$$\{A(ct)\}_{t\geq 0} \stackrel{f.d.}{=} \{c^E A(t)\}_{t\geq 0} \tag{2.10}$$

so by Definition 11.1.2 of [25] $\{A(t)\}_{t\geq 0}$ is operator selfsimilar with exponent E. The stochastic process $\{A(t)\}$ is called an *operator Lévy motion*. If the exponent $E = aI$ a constant multiple of the identity, then ν is a stable law with index $\alpha = 1/a$, and $\{A(t)\}$ is a classical d–dimensional Lévy motion. In the special case $a = 1/2$ the process $\{A(t)\}_{t\geq 0}$ is a d–dimensional Brownian motion.

Since we are interested in convergence of stochastic processes in Skorodhod spaces we need some further notation. If S is a complete separable metric space, let $D([0,\infty), S)$ denote the space of all right-continuous S-valued functions on $[0,\infty)$ with limits from the left and endow $D([0,\infty), S)$ with the usual J_1-topology introduced in [37]. Note that in view of [6] p.197, we can assume without loss of generality that all the sample path of the processes $\{T(t)\}_{t\geq 0}$ and $\{D(t)\}_{t\geq 0}$ belong to $D([0,\infty),[0,\infty))$ as well as all the sample path of $\{S(t)\}_{t\geq 0}$ and $\{A(t)\}_{t\geq 0}$ belong to $D([0,\infty), \mathbb{R}^d)$

3. The time process

In this section we investigate the limiting behavior of the counting process $\{N_t\}_{t\geq 0}$ defined by (1.1). It turns out that the scaling limit of this process is the hitting time process for the Lévy motion $\{D(x)\}_{x\geq 0}$. This hitting time process is also selfsimilar with exponent β. We will use the results of this section in the next section to derive limit theorems for the CTRW.

Recall that all the sample paths of the Lévy motion $\{D(x)\}_{x\geq 0}$ are continuous from the right, with left hand limits, strictly increasing, and that $D(0) = 0$ and $D(x) \to \infty$ as $x \to \infty$. Now the hitting time process

$$E(t) = \inf\{x : D(x) > t\} \tag{3.1}$$

is well–defined. If $D(x) \geq t$ then $D(y) > t$ for all $y > x$ so that $E(t) \leq x$. On the other hand, if $D(x) < t$ then $D(y) < t$ for all $y > x$ sufficiently close to x, so that $E(t) > x$. Then it follows easily that for any $0 \leq t_1 < \cdots < t_m$ and $x_1, \ldots, x_m \geq 0$ we have

$$\{E(t_i) \leq x_i \text{ for } i = 1, \ldots, m\} = \{D(x_i) \geq t_i \text{ for } i = 1, \ldots, m\}. \tag{3.2}$$

Proposition 3.1. *The process* $\{E(t)\}_{t\geq 0}$ *defined by* (3.1) *is selfsimilar with exponent* $\beta \in (0,1)$*, that is for any* $c>0$ *we have*

$$\{E(ct)\}_{t\geq 0} \stackrel{f.d.}{=} \{c^\beta E(t)\}_{t\geq 0}.$$

Proof. Note that it follows directly from (3.1) that $\{E(t)\}_{t\geq 0}$ has continuous sample path and hence is continuous in probability and so in law. Now fix any $0 < t_1 < \cdots < t_m$ and $x_1, \ldots, x_m \geq 0$. Then by (2.4) and (3.2) we obtain

$$\begin{aligned} P\{E(ct_i) \leq x_i \text{ for } i = 1,\ldots,m\} &= P\{D(x_i) \geq ct_i \text{ for } i=1,\ldots,m\} \\ &= P\{(c^{-\beta})^{1/\beta} D(x_i) \geq t_i \text{ for } i=1,\ldots,m\} \\ &= P\{D(c^{-\beta}x_i) \geq t_i \text{ for } i=1,\ldots,m\} \\ &= P\{E(t_i) \leq c^{-\beta}x_i \text{ for } i=1,\ldots,m\} \\ &= P\{c^\beta E(t_i) \leq x_i \text{ for } i=1,\ldots,m\}. \end{aligned}$$

Hence by Definition 11.1.2 of [25] the assertion follows. □

We now collect some further properties of the process $\{E(t)\}_{t\geq 0}$.

For a real valued random variable X let $\mathbb{E}(X)$ denote its expectation, whenever it exists, and $\mathrm{Var}(X)$ denotes the variance of X whenever it exists.

Corollary 3.2. *For any* $t > 0$ *we have*

(a) $E(t) \stackrel{d}{=} (D/t)^{-\beta}$*, where* D *is as in* (2.1).

(b) *For any* $\gamma > 0$ *the* γ*-moment of* $E(t)$ *exists and there exists a positive finite constant* $C(\beta,\gamma)$ *such that*

$$\mathbb{E}(E(t)^\gamma) = C(\beta,\gamma)t^{\beta\gamma},$$

especially

$$\mathbb{E}(E(t)) = C(\beta,1)t^\beta. \tag{3.3}$$

(c) *The random variable* $E(t)$ *has density*

$$f_t(x) = \frac{t}{\beta} x^{-1-1/\beta} g_\beta(tx^{-1/\beta})$$

where g_β *is the density of the limit* D *in* (2.1).

Proof. (a) Note that $D(x) \stackrel{d}{=} x^{1/\beta}D$. In view of (3.2) we have for any $x > 0$

$$P\{E(t) \leq x\} = P\{D(x) \geq t\} = P\{x^{1/\beta}D \geq t\} = P\{(D/t)^{-\beta} \leq x\},$$

proving part (a).

For the proof of (b) let $H_t(y) = (y/t)^{-\beta}$. Since D has distribution ρ it follows from part (a) that $E(t)$ has distribution $H_t(\rho)$. Recall that ρ has a C^∞ density

g_β. For $\gamma > 0$ we then get

$$\mathbb{E}(E(t)) = \int_0^\infty x^\gamma dH_t(\rho)(x) = \int_0^\infty (H_t(x))^\gamma d\rho(x)$$
$$= t^{\beta\gamma} \int_0^\infty x^{-\beta\gamma} g_\beta(x) dx = C(\beta,\gamma) t^{\beta\gamma},$$

where $C(\beta,\gamma) = \int_0^\infty x^{-\beta\gamma} g_\beta(x)dx$ is finite since g_β is a density function satisfying (2.2) for some $0 < \beta < 1$, so that $g_\beta(x) \to 0$ at an exponential rate as $x \to 0$.

Since $E(t) \stackrel{d}{=} H_t(D)$ by (a) and $H_t^{-1}(x) = tx^{-1/\beta}$, (c) follows by a change of variables. □

Corollary 3.3. *For any $t > 0$ the variance of $E(t)$ exists and* $\mathrm{Var}(E(t)) = \big(C(\beta,2) - C(\beta,1)^2\big)t^{2\beta}$ *for any $t > 0$.*

Proof. It follows from Corollary 3.2(b) that $\mathbb{E}(E(t)^2)$ exists and hence the result follows from Corollary 3.2(b). □

Corollary 3.4. $\{E(t)\}_{t\geq 0}$ *is not a process with stationary increments.*

Proof. Suppose that $\{E(t)\}_{t\geq 0}$ is a process with stationary increments. Then for any integer t we have

$$\mathbb{E}(E(t)) = \mathbb{E}(E(1) + (E(2) - E(1)) + \cdots + (E(t) - E(t-1)) = t\mathbb{E}(E(1))$$

which contradicts (3.3). □

Theorem 3.5. *The process* $\{E(t)\}_{t\geq 0}$ *does not have independent increments.*

Proof. Assume the contrary. The process $\{E(t)\}$ is the inverse of the stable subordinator $\{D(x)\}$, in the sense of Bingham [8]. Then Proposition 1(a) of [8] implies that for any $0 < t_1 < t_2$ we have

$$\frac{\partial^2 \mathbb{E}(E(t_1)E(t_2))}{\partial t_1 \partial t_2} = \frac{1}{\Gamma(\beta)^2 [t_1(t_2 - t_1)]^{1-\beta}} \tag{3.4}$$

Moreover, by Corollary 3.2 we know that for some positive constant C we have

$$\mathbb{E}(E(t)) = Ct^\beta \tag{3.5}$$

Since $E(t)$ has moments of all orders we have for $0 < t_1 < t_2 < t_3$ by independence of the increments and (3.5) that

$$\mathbb{E}((E(t_3) - E(t_2)) \cdot (E(t_2) - E(t_1))) = \mathbb{E}(E(t_3) - E(t_2)) \cdot \mathbb{E}(E(t_2) - E(t_1))$$
$$= C^2 \big\{ (t_2t_3)^\beta - (t_1t_3)^\beta - t_2^{2\beta} + (t_1t_2)^\beta \big\}$$
$$= R(t_1, t_2, t_3)$$

On the other hand

$$\begin{aligned}\mathbb{E}((E(t_3)-E(t_2))\cdot(E(t_2)-E(t_1))) &= \mathbb{E}(E(t_2)E(t_3)) - \mathbb{E}(E(t_1)E(t_3)) \\ &\quad - \mathbb{E}(E(t_2)^2) + \mathbb{E}(E(t_1)E(t_2)) \\ &= L(t_1,t_2,t_3)\end{aligned}$$

so that $R(t_1,t_2,t_3) = L(t_1,t_2,t_3)$ whenever $0 < t_1 < t_2 < t_3$.

Computing the derivatives of R directly and applying (3.4) to L gives

$$\begin{aligned}\frac{\partial^2 R(t_1,t_2,t_3)}{\partial t_1 \partial t_2} &= C^2\beta^2(t_1t_2)^{\beta-1} \\ \frac{\partial^2 L(t_1,t_2,t_3)}{\partial t_1 \partial t_2} &= \Gamma(\beta)^{-2}(t_1t_2)^{\beta-1}\left\{1-\left(\frac{t_1}{t_2}\right)\right\}^{\beta-1}\end{aligned}$$

for all $0 < t_1 < t_2 < t_3$. Since the left hand sides are equal, so are the right hand sides of the equations above, which gives a contradiction. □

Recall from section 2 that the function b in (2.3) is regularly varying with index $-1/\beta$. Hence b^{-1} is regularly varying with index $1/\beta > 0$ so by property 1.5.5 of [36] there exists a regularly varying function $\tilde{b}$ with index β such that $1/b(\tilde{b}(c)) \sim c$ as $c \to \infty$. Here we use the notation $f \sim g$ for positive functions f, g if and only if $f(c)/g(c) \to 1$ as $c \to \infty$. Equivalently we have

$$b(\tilde{b}(c)) \sim \frac{1}{c} \quad \text{as } c \to \infty. \tag{3.6}$$

Furthermore note that for (1.1) it follows easily that for any integer $n \geq 0$ and any $t \geq 0$ we have

$$\{T(n) \leq t\} = \{N_t \geq n\}. \tag{3.7}$$

With the use of the function $\tilde{b}$ defined above we now prove a limit theorem for $\{N_t\}_{t\geq 0}$.

Theorem 3.6.

$$\Big\{\frac{1}{\tilde{b}(c)}N_{ct}\Big\} \overset{f.d.}{\Longrightarrow} \{E(t)\}_{t\geq 0} \quad \textit{as } c \to \infty.$$

Proof. Fix any $0 < t_1 < \cdots < t_m$ and $x_1, \ldots, x_m \geq 0$. Let $\forall i$ mean for $i = 1, \ldots, m$. Note that by (3.7) we have

$$\{N_t \geq x\} = \{T(\lceil x \rceil) \leq t\} \tag{3.8}$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x. This is equivalent to $\{N_t < x\} = \{T(\lceil x \rceil) > t\}$, and then (2.6) together with (3.2) and (3.6) imply

$$\begin{aligned}
P\Big\{\frac{1}{\tilde b(c)}N_{ct_i} < x_i\ \forall i\Big\} &= P\{N_{ct_i} < \tilde b(c)x_i\ \forall i\}\\
&= P\{T(\lceil \tilde b(c)x_i\rceil) > ct_i\ \forall i\}\\
&= P\{b(\tilde b(c))T(\lceil \tilde b(c)x_i\rceil) > b(\tilde b(c))ct_i\ \forall i\}\\
&\to P\{D(x_i) > t_i\ \forall i\} = P\{D(x_i) \ge t_i\ \forall i\}\\
&= P\{E(t_i) \le x_i\ \forall i\} = P\{E(t_i) < x_i\ \forall i\}
\end{aligned}$$

as $c \to \infty$ since both $E(t)$ and $D(x)$ have a density. Hence

$$\Big(\frac{1}{\tilde b(c)}N_{ct_i} :\ i = 1,\dots,m\Big) \Rightarrow (E(t_i) :\ i = 1,\dots,m)$$

and the proof is complete. □

As a corollary we get convergence in the Skorohod space $D([0,\infty),[0,\infty))$.

Corollary 3.7.

$$\Big\{\frac{1}{\tilde b(c)}N_{ct}\Big\}_{t\ge 0} \Rightarrow \{E(t)\}_{t\ge 0} \quad \textit{in } D([0,\infty),[0,\infty)) \textit{ as } c \to \infty.$$

Proof. Note that the sample path of $\{N_t\}_{t\ge0}$ and $\{E(t)\}_{t\ge0}$ are increasing and that by the proof of Proposition 3.1 the process $\{E(t)\}_{t\ge0}$ is continuous in probability. Then Theorem 3.6 together with Theorem 3 of [8] yields the assertion. □

Remark 3.8. The hitting time $E(t) = \inf\{x : D(x) > t\}$ is also called a *first passage time*. A general result of Port [27] implies that $P\{E(t) \ge x\} = o(x^{1-1/\beta})$ as $x \to \infty$, but in view of Corollary 3.2 that tail bound can be considerably improved in this special case. Getoor [12] computes the first and second moments of the hitting time for a symmetric stable process, but the moment results of Corollary 3.2 and 3.3 are apparently new. The hitting time process $\{E(t)\}$ is also a *local time* for the Markov process $R(t) = \inf\{D(x)-t : D(x) > t\}$, see [6] Exercise 6.2. This means that the jumps of the *inverse local time* $\{D(x)\}$ for the Markov process $\{R(t)\}_{t\ge0}$ coincide with the lengths of the excursion intervals during which $R(t) > 0$. Since $R(t) = 0$ when $D(x) = t$, the lengths of the excursion intervals for $\{R(t)\}$ equal the size of the jumps in the process $\{D(x)\}$, and $E(t)$ represents the time it takes until the sum of the jump sizes (the sum of the lengths of the excursion intervals) exceeds t.

4. Limit Theorems

In this section we present the main result of this paper. Namely we prove a functional limit theorem for the CTRW $\{X(t)\}_{t\ge0}$ defined in (1.2) under the distributional assumptions of section 2. The limiting process $\{M(t)\}$ is a subordination of the operator stable Lévy process $\{A(t)\}_{t\geq 0}$ in (2.9) by the process $\{E(t)\}_{t\geq 0}$ introduced in section 3. We show that $\{M(t)\}_{t\geq 0}$ is operator selfsimilar with exponent βE. Here β and E are as in section 2.

Our method of proof is to use the continuity of the composition in Skorodhod spaces valid under certain conditions obtained in [39]. In order to do so, we first need to prove $D([0,\infty),\mathbb{R}^d)$-convergence in (2.9), which is apparently not available in the literature. The following result, which is of independent interest, closes this gap.

Theorem 4.1. *Under the assumptions of section 2 we have*

$$\{B(n)S(nt)\}_{t\geq 0} \Rightarrow \{A(t)\}_{t\geq 0} \quad \text{in } D([0,\infty),\mathbb{R}^d) \text{ as } n\to\infty.$$

Proof. In view of (2.9) and the theorem in Stone [37] it suffices to check that that

$$\lim_{h\downarrow 0}\limsup_{n\to\infty}\sup_{|s-t|\leq h} P\left\{\|\xi_n(t)-\xi_n(s)\| > \varepsilon\right\} = 0 \tag{4.1}$$

for any $\varepsilon > 0$ and $T > 0$, where $0 \leq s,t \leq T$ and

$$\xi_n(t) = B(n)\sum_{i=1}^{[nt]} X_i.$$

Recall that $S(j) = X_1 + \cdots + X_j$. Given any $\varepsilon > 0$ and $\delta > 0$, using the fact that $\{B(j)S(j)\}$ is uniformly tight, there exists a $R > 0$ such that

$$\sup_{j\geq 1} P\{\|B(j)S(j)\| > R\} < \delta. \tag{4.2}$$

In view of the symmetry in s,t in (4.1) we can assume $0 \leq s < t \leq T$ and $t - s \leq h$. Let

$$r_n(t,s) = \|B(n)B([nt]-[ns])^{-1}\|$$

and note that

$$\begin{aligned} r_n(t,s) &= \left\|B(n)B(n\cdot([nt]-[ns])/n)^{-1}\right\| \\ &\leq \sup_{0\leq\lambda\leq h+\frac{1}{n}} \|B(n)B(n\lambda)^{-1}\| \\ &= \|B(n)B(n\lambda_n)^{-1}\| \end{aligned}$$

for some $0 \leq \lambda_n \leq h + 1/n$.

Now choose $h_0 > 0$ such that $\|h^E\| < \varepsilon/(2R)$ for all $0 < h \leq h_0$ and assume in the following that $0 < h \leq h_0$. Given any subsequence, there exists a further subsequence (n') such that $\lambda_n \to \lambda \in [0,h]$ along (n'). We have to consider several cases separately:

Case 1: If $\lambda > 0$ then, in view of the uniform convergence on compact sets we have
$$\|B(n)B(n\lambda_n)^{-1}\| \to \|\lambda^E\|$$
and hence there exists a n_0 such that $r_n(t,s) < \varepsilon/R$ for all $n' > n_0$.

Case 2: If $\lambda = 0$ and $n\lambda_n \le M$ for all (n') then, since $B(n) \to 0$ as $n \to \infty$, there exists a n_0 such that
$$\|B(n)B(n\lambda_n)^{-1}\| \le \|B(n)\| \sup_{1\le j\le M} \|B(j)^{-1}\| < \varepsilon/R$$
for all $n' \ge n_0$ and hence $r_n(t,s) < \varepsilon/R$ for all $n' \ge n_0$.

Case 3: If $\lambda = 0$ and $n\lambda_n \to \infty$ along (n'), let $m = n\lambda_n$ and $\lambda(m) = n/m$ so that $m \to \infty$ and $\lambda(m) \to \infty$. Choose $\lambda_0 > 1$ large enough to make $\|\lambda_0^{-E}\| < 1/4$. Then choose m_0 large enough to make $\|B(\lambda m)B(m)^{-1} - \lambda^{-E}\| < 1/4$ for all $m \ge m_0$ and all $1 \le \lambda \le \lambda_0$. Now $\|B(m\lambda_0)B(m)^{-1}\| < 1/2$ for all $m \ge m_0$, and since $\|\lambda^{-E}\|$ is a continuous function of $\lambda > 0$ this ensures that for some $C > 0$ we have $\|B(\lambda m)B(m)^{-1}\| \le C$ for all $m \ge m_0$ and all $1 \le \lambda \le \lambda_0$. Given $m \ge m_0$ write $\lambda(m) = \mu\lambda_0^k$ for some integer k and some $1 \le \mu < \lambda_0$. Then
$$\begin{aligned}\|B(n)B(n\lambda_n)^{-1}\| &= \|B(m\lambda(m))B(m)^{-1}\|\\ &\le \|B(\mu\lambda_0^k m)B(\lambda_0^k m)^{-1}\| \cdots \|B(\lambda_0 m)B(m)^{-1}\|\\ &\le C(1/2)^k\end{aligned}$$
and since $\lambda(m) \to \infty$ this shows that $\|B(n)B(n\lambda_n)^{-1}\| \to 0$ as $n \to \infty$ along (n'). Hence there exists an n_0 such that
$$\|B(n)B(n\lambda_n)^{-1}\| < \varepsilon/R$$
for all $n' \ge n_0$ and hence $r_n(t,s) < \varepsilon/R$ for all $n' \ge n_0$.

Now it follows easily that there exist $h_0 > 0$ and n_0 such that
$$r_n(t,s) < \frac{\varepsilon}{R} \tag{4.3}$$
for all $n \ge n_0$ and $|t-s| \le h < h_0$. Finally, we obtain for $n \ge n_0$ and $|t-s| \le h < h_0$, in view of (4.2) and (4.3) that
$$\begin{aligned}P\{\|\xi_n(t) - \xi_n(s)\| > \varepsilon\} &= P\{\|B(n)S([nt]-[ns])\| > \varepsilon\}\\ &\le P\{r_n(t,s)\|B([nt]-[ns])S([nt]-[ns])\| > \varepsilon\}\\ &\le \sup_{j\ge 1} P\{\|B_jS(j)\| > R\} < \delta\end{aligned}$$
which proves (4.1). □

Recall from the paragraph before (3.6) that $\tilde{b}$ is regularly varying with index β and that the norming function B in (2.9) is $\mathrm{RV}(-E)$. We define $\tilde{B}(c) = B(\tilde{b}(c))$. Then $\tilde{B} \in \mathrm{RV}(-\beta E)$.

Theorem 4.2. *Under the assumptions of section 2 we have*

$$\{\tilde{B}(c)X(ct)\}_{t\geq 0} \Rightarrow \{M(t)\}_{t\geq 0} \quad \text{in } D([0,\infty),\mathbb{R}^d) \text{ as } c\to\infty,$$

where $\{M(t)\}_{t\geq 0} = \{A(E(t))\}_{t\geq 0}$ *is a subordinated process with*

$$P_{(M(t_1),\dots,M(t_n))} = \int_{\mathbb{R}^m_+} P_{(A(x_i):1\leq i\leq m)} dP_{(E(t_i):1\leq i\leq m)}(x_1,\dots,x_m)$$

for any $0 < t_1 < \cdots < t_m$.

Proof. Note that since (J_i) and (Y_i) are independent, the processes $\{S(t)\}_{t\geq 0}$ and $\{N_t\}_{t\geq 0}$ are independent and hence it follows from Corollary 3.7 together with Theorem 4.1 then we also have

$$\big\{(\tilde{B}(c)S(\tilde{b}(c)t), \frac{1}{\tilde{b}(c)}N_{ct})\big\}_{t\geq 0} \Rightarrow \{(A(t),E(t))\}_{t\geq 0} \quad \text{as } c\to\infty$$

in $D([0,\infty),\mathbb{R}^d)\times D([0,\infty),[0,\infty))$. Then the continuous mapping theorem, see e.g. [7], Theorem 5.1 and 5.5, together with Theorem 3.1 in [39] yields

$$\{\tilde{B}(c)S(N_{ct})\}_{t\geq 0} \Rightarrow \{A(E(t))\}_{t\geq 0} \quad \text{in } D([0,\infty),\mathbb{R}^d) \text{ as } c\to\infty,$$

which is the first part of the assertion. The formula for the finite dimensional distributions of $\{M(t)\}_{t\geq 0}$ follows easily, since $\{A(t)\}_{t\geq 0}$ and $\{E(t)\}_{t\geq 0}$ are independent. This concludes the proof. □

As a first corollary to Theorem 4.2 we see that the limiting process is operator selfsimilar.

Corollary 4.3. *Then limiting process* $\{M(t)\}_{t\geq 0}$ *obtained in Theorem 4.2 above is operator selfsimilar with exponent* βE*, that is for all* $c>0$

$$\{M(ct)\}_{t\geq 0} \stackrel{f.d.}{=} \{c^{\beta E}M(t)\}_{t\geq 0}.$$

Proof. We first show that $\{M(t)\}_{t\geq 0}$ is continuous in law. Assume $t_n\to t\geq 0$ and let f be any bounded continuous function on $\mathbb{R}^d$. Since $\{A(x)\}_{x\geq 0}$ is continuous in law the function $x\mapsto \int f(y)dP_{A(x)}(y)$ is continuous and bounded. Recall from section 3 that $\{E(t)\}_{t\geq 0}$ is continuous in law and hence $E(t_n)\Rightarrow E(t)$ as $n\to\infty$. Then

$$\begin{aligned}\int f(y)dP_{M(t_n)}(y) &= \int\Big(\int f(y)dP_{A(x)}(y)\Big)dP_{E(t_n)}(x)\\ &\to \int\Big(\int f(y)dP_{A(x)}(y)\Big)dP_{E(t)}(x) = \int f(y)dP_{M(t)}(y)\end{aligned}$$

as $n\to\infty$, showing that $\{M(t)\}_{t\geq 0}$ is continuous in law. It follows from Theorem 4.2 that for any $c>0$

$$\{\tilde{B}(s)X(s(ct))\}_{t\geq 0} \stackrel{f.d.}{\Longrightarrow} \{M(ct)\}_{t\geq 0}$$

as $s \to \infty$ and since $\tilde{B} \in \mathrm{RV}(-\beta E)$ we also get

$$\{\tilde{B}(s)X((sc)t)\}_{t\geq 0} = \{(\tilde{B}(s)\tilde{B}(sc)^{-1})\tilde{B}(sc)X((sc)t)\}_{t\geq 0} \stackrel{f.d.}{\Longrightarrow} \{c^{\beta E}M(t)\}_{t\geq 0}$$

as $s \to \infty$. Hence

$$\{M(ct)\}_{t\geq 0} \stackrel{f.d.}{=} \{c^{\beta E}M(t)\}_{t\geq 0}$$

so the proof is complete. □

For later use we collect some further properties of the limiting process $\{M(t)\}_{t\geq 0}$. Recall from [14] Theorem 4.10.2 that the distribution ν^t of $A(t)$ in (2.9) has a C^∞ density $p(x,t)$, so that $d\nu^t(x) = p(x,t)dx$, and that g_β is the density of the limit D in (2.1).

Corollary 4.4. *Let $\{M(t)\}_{t\geq 0}$ be the limiting process obtained in Theorem 4.2. Then for any $t > 0$*

$$P_{M(t)} = \int_0^\infty \nu^{(t/s)^\beta} g_\beta(s)ds = \frac{t}{\beta}\int_0^\infty \nu^\xi g_\beta(\frac{t}{\xi^{1/\beta}})\xi^{-1/\beta-1}d\xi. \tag{4.4}$$

Moreover $P_{M(t)}$ has the density

$$h(x,t) = \int_0^\infty p(x,(t/s)^\beta)g_\beta(s)ds = \frac{t}{\beta}\int_0^\infty p(x,\xi)g_\beta(\frac{t}{\xi^{1/\beta}})\xi^{-1/\beta-1}d\xi. \tag{4.5}$$

Proof. Let $H_t(y) = (y/t)^{-\beta}$. Then by Corollary 3.2 we know $E(t) \stackrel{d}{=} H_t(D)$ and since $P_{A(s)} = \nu^s$ we obtain

$$\begin{aligned} P_{M(t)} = \int_0^\infty P_{A(s)}dP_{E(t)}(s) &= \int_0^\infty \nu^s dP_{H_t(D)}(s) \\ &= \int_0^\infty \nu^{H_t(s)}dP_D(s) = \int_0^\infty \nu^{(t/s)^\beta}g_\beta(s)ds \end{aligned}$$

and the second equality follows from a simple substitution. The assertion on the density follows immediately. □

Corollary 4.5. *The limiting process $\{M(t)\}_{t\geq 0}$ obtained in Theorem 4.2 does not have stationary increments.*

Proof. Suppose $t > 0$ and $h > 0$. In view of (3.1) and (3.2) the events

$$\{E(t+h) = E(h)\} = \{E(t+h) \leq E(h)\} = \{D(E(h)) \geq t+h\}$$

so

$$\begin{aligned} P\{M(t+h) - M(h) = 0\} &= P\{A(E(t+h)) = A(E(h))\} \\ &\geq P\{E(t+h) = E(h)\} = P\{D(E(h)) \geq t+h\} > 0 \end{aligned}$$

for all $t > 0$ sufficiently small, since $D(E(h)) > h$ almost surely by Theorem III.4 of [6]. But $P\{M(t) = 0\} = 0$ since $M(t)$ has a density, hence $M(t)$ and $M(t+h) - M(h)$ are not identically distributed. □

Theorem 4.6. *Let $\{M(t)\}_{t\geq 0}$ be the limiting process obtained in Theorem 4.2. Then the distribution of $M(t)$ is not operator stable for any $t>0$.*

Proof. The distribution ν of the limit A in (2.8) is infinitely divisible, hence its characteristic function $\hat\nu(k)=e^{-\psi(k)}$ for some continuous complex–valued function $\psi(k)$, see for example [25] Theorem 3.1.2. Since $|\hat\nu(k)|=|e^{-\operatorname{Re}\psi(k)}|\leq 1$ we must have $F(k)=\operatorname{Re}\psi(k)\geq 0$ for all k. Since ν is operator stable, Corollary 7.1.2 of [25] implies that $|\hat\nu(k)|<1$ for all $k\neq 0$ so in fact $F(k)>0$ for all $k\neq 0$. Since $\nu^t=t^E\nu$ we also have $tF(k)=F(t^{E^*}k)$ for all $t>0$ and $k\neq 0$, which implies that F is a regularly varying function in the sense of [25] Definition 5.1.2. Then Theorems 5.3.14 and 5.3.18 of [25] imply that for some positive real constants a,b_i,c_i we have

$$c_1\|k\|^{b_1}\leq F(k)\leq c_2\|k\|^{b_2} \tag{4.6}$$

for all $\|k\|\geq a$. In fact, we can take $b_1=1/a_p-\delta$ and $b_2=1/a_1+\delta$ where $\delta>0$ is arbitrary and the real parts of the eigenvalues of E are $a_1<\cdots<a_p$. In view of (4.4) the random vector $M(t)$ has characteristic function

$$\varphi_t(k)=\int_0^\infty e^{-(t/s)^\beta\psi(k)}g_\beta(s)ds \tag{4.7}$$

where g_β is the density of the limit D in (2.1). Using the well–known series expansion for this stable density, equation (4.2.4) in [38], it follows that for some positive constants c_0,s_0 we have

$$g_\beta(s)\geq c_0 s^{-\beta-1} \tag{4.8}$$

for all $s\geq s_0$. Take $r_0=\max\{a,s_0^{\beta/b_2}t^{-\beta/b_2}c_2^{-1/b_2}\}$. Then for $\|k\|\geq r_0$ and $s_1=tc_2^{1/\beta}\|k\|^{b_2/\beta}$ we have that (4.6) holds and, since $s_1\geq s_0$ we also have that (4.8) holds for all $s\geq s_1$. Then in view of (4.7) and the fact that $e^{-u}\geq 1-u$

for all real u we have

$$\begin{aligned}
\operatorname{Re}\varphi_t(k) &= \int_0^\infty e^{-(t/s)^\beta \operatorname{Re}\psi(k)} g_\beta(s)ds \\
&\geq \int_{s_1}^\infty e^{-(t/s)^\beta F(k)} g_\beta(s)ds \\
&\geq \int_{s_1}^\infty [1-(t/s)^\beta F(k)] c_0 s^{-\beta-1} ds \\
&\geq \int_{s_1}^\infty [1-(t/s)^\beta c_2\|k\|^{b_2}] c_0 s^{-\beta-1} ds \\
&= (c_0/\beta)s_1^{-\beta} - (c_0/2\beta)t^\beta c_2 \|k\|^{b_2} s_1^{-2\beta} \\
&= (c_0/\beta)s_1^{-\beta}[1 - t^\beta c_2\|k\|^{b_2} s_1^{-\beta}/2] \\
&= (c_0/\beta)t^{-\beta} c_2^{-1}\|k\|^{-b_2}[1-1/2] \\
&= C\|k\|^{-b_2}
\end{aligned} \tag{4.9}$$

where $C > 0$ does not depend on the choice of $\|k\| > r_0$. But if $M(t)$ is operator stable, then the same argument as before shows that $\operatorname{Re}\varphi_t(k) = e^{-F(k)}$ for some F satisfying (4.6) for all $\|k\|$ large (for some positive real constants a, b_i, c_i), so that $\operatorname{Re}\varphi_t(k) \leq e^{-c_1\|k\|^{b_1}}$ for all $\|k\|$ large, which is a contradiction. □

5. Remarks

Saichev and Zaslavsky [28] state that in the case where $\{A(t)\}$ is scalar Brownian motion, the limiting process $\{M(t)\}$ is "fractal Brownian motion." Theorem 4.6 shows that the limiting process is not Gaussian, since $M(t)$ cannot have a normal distribution. Therefore this process cannot be a fractional Brownian motion. Furthermore, in view of Corollary 4.5 the process $\{M(t)\}_{t\geq 0}$ does not have stationary increments in contrast to fractional Brownian motion.

For a one dimensional CTRW with infinite mean waiting time, Kotulski [17] derives the results of Theorem 4.2 and Corollary 4.4 at one fixed time $t > 0$. Our stochastic process results, concerning $D([0,\infty), \mathbb{R}^d)$-convergence, seem to be new even in the one dimensional case. Kotulski [17] also considers coupled CTRW models in which the jump times and lengths are dependent. These coupled CTRW models occur in many physical applications [9, 15, 16, 34]. It would be interesting to extend the results of this section to coupled models, but this is still an open problem.

In the one dimensional situation $d = 1$ Corollary 4.4 implies that $M(t) \stackrel{d}{=} (t/D)^{\beta/\alpha}A$ where D is the limit in (2.1) and A is the limit in (2.8). If A is a nonnormal stable random variable with index $0 < \alpha < 2$ then $M(t)$ has a ν–stable distribution [22], i.e., a random mixture of stable laws. Corollary 3.2 shows that the mixing variable $(t/D)^{\beta/\alpha}$ has moments of all orders, and then Proposition 4.1 of [22] shows that $P(M(t) > x) \sim Cpx^{-\alpha}$ and $P(M(t) < -x) \sim Cqx^{-\alpha}$ as $x \to \infty$ for some $C > 0$ and some $0 \le p, q \le 1$ with $p + q = 1$. Proposition 5.1 of [22] shows that $\mathbb{E}|M(t)|^p$ exists for $0 < p < \alpha$ and diverges for $p \ge \alpha$. If A is a nonnormal stable random vector then $M(t)$ has a multivariate ν–stable distribution [21], and again the moment and tail behavior of $M(t)$ are similar to that of A. The ν–stable laws are the limiting distributions of random sums, so their appearance in the limit theory for a CTRW is natural. It may also be possible to consider ν–operator stable laws, but this is an open problem.

If A is normal then the density $h(x, t)$ of $M(t)$ is a mixture of normal densities. In some cases, mixtures of normal densities take a familiar form. If A is normal and D is the limit in (2.1), then the density of $D^{\gamma/2}A$ is stable with index γ when $0 < \gamma < 2$, see [29]. If D is exponential then $D^{1/2}A$ has a Laplace distribution. More generally, if A is any stable random variable or random vector, and D is exponential, then $D^{1/2}A$ has a geometric stable distribution [19]. Geometric stable laws have been applied in finance [18, 20].

The limit process $\{M(t)\}$ in Theorem 4.2 is also the stochastic solution to a fractional kinetic equation for Hamiltonian chaos. Specialize to $\mathbb{R}^1$ so that the limit process $\{A(t)\}_{t \ge 0}$ in (2.9) is a stable Lévy motion with some index $0 < \alpha \le 2$, the case $\alpha = 2$ corresponding to a Brownian motion. When $1 < \alpha \le 2$ the densities $p(x, t)$ of the process $\{A(t)\}$ solve a fractional diffusion equation

$$\frac{\partial p(x,t)}{\partial t} = Lp(x,t) \tag{5.1}$$

where the linear operator

$$Lp(x,t) = -v\frac{\partial p(x,t)}{\partial x} + qa\frac{\partial^\alpha p(x,t)}{\partial(-x)^\alpha} + (1-q)a\frac{\partial^\alpha p(x,t)}{\partial x^\alpha} \tag{5.2}$$

for some $v \in \mathbb{R}^1$, $a > 0$, and $0 \le q \le 1$ [5, 10]. Using the Fourier transform $\hat{p}(k, t) = \int e^{-ik\cdot x} p(x, t)dx$, so that $\hat{p}(-k, t)$ is the characteristic function of $A(t)$, the fractional space derivative $\partial^\alpha p(x, t)/\partial(\pm x)^\alpha$ is defined as the inverse Fourier transform of $(\pm ik)^\alpha \hat{p}(k, t)$, extending the familiar formula where α is a positive integer, see for example Samko, Kilbas and Marichev [31]. If $\alpha = 2$ then (5.1) reduces to the classical diffusion equation. Equation (5.1) has been applied to problems in physics [33] and hydrology [3, 4] where diffusion occurs more rapidly than the classical Brownian motion model predicts.

Zaslavsky introduced the fractional kinetic equation

$$\frac{\partial^\beta h(x,t)}{\partial t^\beta} = Lh(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{5.3}$$

for Hamiltonian chaos [28, 40]. The fractional time derivative $\partial^\beta h(x,t)/\partial t^\beta$ for $0 < \beta < 1$ is the inverse Laplace transform of $s^\beta \tilde{h}(x,s)$, where $\tilde{h}(x,s) = \int_0^\infty e^{-st} h(x,t)\,dt$ is the usual Laplace transform, and $\delta(x)$ is the Dirac delta function. Saichev and Zaslavsky [28] use Laplace–Fourier transform methods to argue that for a one dimensional CTRW with symmetric jumps and infinite mean waiting times, the probability densities $h(x,t)$ of the scaling limit solve the fractional kinetic equation (5.3) in the symmetric case $q = 1/2$. In view of Theorem 4.2 and Corollary 4.4, this means that (5.3) governs the CTRW scaling limit $\{A(E(t))\}$ when (5.1) governs the Lévy process $\{A(t)\}$. In other words, invoking a fractional time derivative results in subordination of the Lévy process $\{A(t)\}$ by the hitting time process $\{E(t)\}$. Corollary 4.4 also gives an explicit formula for the solution to (5.3).

Physically the fractional space derivative models high velocity particle jumps [35]. The order α of the fractional space derivative corresponds to the stable index, and it also governs the probability tails of the jump sizes [29]. The fractional time derivative models particle sticking or trapping, and the order β of the fractional time derivative governs the power law tail of the delay between jumps. The fractional diffusion equation (5.1) also describes (operator) Lévy motions on $\mathbb{R}^d$ [24, 23]. An interesting open question is whether subordination of these operator Lévy motions by the hitting time process $\{E(t)\}$ solves the fractional kinetic equation (5.3) on $\mathbb{R}^d$.

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