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Brief Paper

Guaranteed Properties of Gain Scheduled Control for Linear Parameter-varying Plants*

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Key Words—Gain scheduling; parameter-varying systems; robustness; time-varying systems.

Abstract—Gain scheduling has proven to be a successful design methodology in many engineering applications. However in the absence of a sound theoretical analysis, these designs come with no guarantees on the robustness, performance, or even nominal stability of the overall gain scheduled design.

This paper presents such an analysis for one type of gain scheduled system, namely, a linear parameter-varying plant scheduling on its exogenous parameters. Conditions are given which guarantee that the stability, robustness, and performance properties of the fixed operating point designs carry over to the global gain scheduled design. These conditions confirm and formalize popular notions regarding gain scheduled design, such as the scheduling variable should "vary slowly."

1. *Introduction*

1.1. *Problem statement.* Gain scheduling (see e.g. Stein, 1980) is a popular engineering method used to design controllers for systems with widely varying nonlinear and/or parameter dependent dynamics, i.e. systems for which a single linear time-invariant model is insufficient. The idea is to select several operating points which cover the range of the plant's dynamics. Then, at each of these points, the designer makes a linear time-invariant approximation to the plant and designs a linear compensator for each linearized plant. In between operating points, the parameters (i.e. "gains") of the compensators are then interpolated, or "scheduled," thus resulting in a global feedback compensator.

Despite the lack of a sound theoretical analysis, gain scheduling is a design methodology which is known to work in a myriad of operating control systems (e.g. jet engines, submarines, and aircraft). However, in the absence of such an analysis, these designs come with no guarantees. More precisely, even though the local point designs may have excellent feedback properties, the global gain scheduled design need not have any of these properties, even nominal stability. In other words, one typically cannot assess *a priori* the guaranteed stability, robustness and performance properties of gain scheduled designs. Rather, any such properties are inferred from extensive computer simulations.

This paper addresses this issue of guaranteed properties for one class of gain scheduled control systems, namely, linear parameter-varying plants. This class of systems is important since it can be shown that gain scheduled control of nonlinear plants takes the form of a linear parameter-varying plant where the "parameter" is actually a reference trajectory or some endogenous signal such as the plant output (cf. Shamma and Athans, 1988, 1989). One example of a physical system whose (linearized) dynamics take the form of a parameter-varying plant is an aircraft, where the time-varying parameter is typically dynamic pressure (e.g. Stein *et al.*, 1977).

Consider a plant of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\theta(t))\mathbf{x}(t) + \mathbf{B}(\theta(t))\mathbf{u}(t),$$
$$\mathbf{y}(t) = \mathbf{C}(\theta(t))\mathbf{x}(t).$$

These equations represent a linear plant whose dynamics depend on a vector of time-varying exogenous parameters, θ, which take their values in some prescribed set $\theta(t) \in \Theta$. Gain scheduled controllers for such plants typically are designed as follows. First, the designer selects a set of parameter values, $\{\theta_i\}$, which represent the range of the plant's dynamics, and designs a linear time-invariant compensator for each. Then, in between operating points, the compensators are interpolated such that for all frozen values of the parameters, the closed loop system has desirable feedback properties, such as nominal stability, robustness to unmodeled dynamics, and robust performance (Fig. 1).

Since the parameters are actually time-varying, none of these properties need carry over to the overall time-varying closed loop system. Even in the simplest case of nominal stability (i.e. no unmodeled dynamics), parameter time-variations can be destabilizing.

In this paper, conditions are given which guarantee that the closed loop system will retain the feedback properties of the frozen-time designs. These conditions formalize various heuristic ideas which have guided successful gain scheduled designs. For example, one primary guideline is "the scheduling variables should vary slowly with respect to the system dynamics." Note that this idea is simply a reminder that the original designs were based on linear time-invariant approximations to the actual plant. In this sense, these approximations must be sufficiently faithful to the true plant if one expects the global design to exhibit the desired feedback properties. In fact, it is this idea which proves most fundamental in the forthcoming analysis.

The remainder of this paper is organized as follows. This section closes with the mathematical notation to be used throughout the paper. Section 2 addresses the issues of robust stability and robust performance. The formal problem statement is given in Section 2.1. Section 2.2 presents background material on Volterra integrodifferential equations. In Section 2.3, conditions are given which guarantee time-varying robustness/performance given frozen-time robustness/performance. The conditions are presented from both a state-space and input–output viewpoint. Finally, concluding remarks are given in Section 3.

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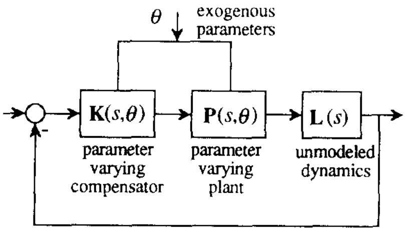

FIG. 1. A linear plant scheduling on exogenous parameters.

1.2. *Mathematical notation.* Some notation regarding standard concepts for analysis of feedback systems (e.g. Desoer and Vidyasagar, 1975; Willems, 1971) is established.

$\mathcal{R}$ denotes the field of real numbers, $\mathcal{R}^+$ the set $\{t \in \mathcal{R} \mid t \geq 0\}$, $\mathcal{R}^n$ the set of $n \times 1$ vectors with elements in $\mathcal{R}$, and $\mathcal{R}^{n \times m}$ the set of $n \times m$ matrices with elements in $\mathcal{R}$. A_{ij} denotes the ijth element of the matrix A. $|\cdot|$ denotes both a vector norm and its induced matrix norm.

Let $\mathbf{f}: \mathcal{R}^+ \to \mathcal{R}^n$. $\hat{\mathbf{f}}$ denotes the Laplace transform of $\mathbf{f}$. $\mathcal{W}_{T,\sigma}$ denotes the truncation and exponential weighting operator on $\mathbf{f}$ defined by

$$\mathcal{W}_{T,\sigma}\mathbf{f}(t) = \begin{cases} e^{-\sigma(T-t)}\mathbf{f}(t) & t \leq T; \\ 0, & t > T. \end{cases}$$

$\mathcal{L}_1$ and $\mathcal{L}_\infty$ denote the standard Lebesgue function spaces of integrable and essentially bounded measurable functions, respectively. $\mathcal{B}$ denotes the set of measurable functions, $\mathbf{f}: \mathcal{R}^+ \to \mathcal{R}^n$, such that

$$\|\mathbf{f}\|_{\mathcal{B}} \stackrel{\text{def}}{=} \sup_{t \in \mathcal{R}^+} |\mathbf{f}(t)| < \infty.$$

$\mathcal{A}(\sigma)$ denotes the set whose elements are of the form

$$f(t) = \begin{cases} f_a(t) + \sum_{i=0}^{\infty} f_i \delta(t - t_i), & t \geq 0; \\ 0, & t < 0, \end{cases}$$

where $f_a: \mathcal{R}^+ \to \mathcal{R}$, $t_i \geq 0$, $f_i \in \mathcal{R}$, and

$$\|f\|_{\mathcal{A}(\sigma)} \stackrel{\text{def}}{=} \int_0^\infty |f_a(t)e^{-\sigma t}| \, dt + \sum_{i=0}^{\infty} |f_i e^{-\sigma t_i}| < \infty.$$

$\mathcal{A}^{n \times m}(\sigma)$ denotes the set of $n \times m$ matrices whose elements are in $\mathcal{A}(\sigma)$. Let $\Delta \in \mathcal{A}^{n \times m}(\sigma)$ and let $\Delta' \in \mathcal{R}^{n \times m}$ be defined as $\Delta'_{ij} = \|\Delta_{ij}\|_{\mathcal{A}(\sigma)}$. Then $\|\Delta\|_{\mathcal{A}(\sigma)} \stackrel{\text{def}}{=} |\Delta'|$. $\hat{\mathcal{A}}(\sigma)$ and $\hat{\mathcal{A}}^{n \times m}(\sigma)$ are defined as the set of Laplace transforms of elements of $\mathcal{A}(\sigma)$ and $\mathcal{A}^{n \times m}(\sigma)$, respectively. For further details on $\mathcal{A}(\sigma)$ and $\mathcal{A}^{n \times m}(\sigma)$, see Callier and Desoer (1978) and Desoer and Vidyasagar (1975).

2. *Robust stability and robust performance*

2.1. *Problem statement.* Suppose that one has carried out the gain scheduled design procedure discussed in the introduction for some linear parameter-varying plant. Then for any *frozen* value of the parameter-vector, one has designed a feedback system which has desirable nominal stability, robust stability, and robust performance properties. Since the parameters are actually time-varying, these properties may be lost.

Now a standard practice in robust control theory is to express robust stability and robust performance requirements as the maintaining of stability in the presence of stable linear uncertainties throughout the feedback loop (e.g. Doyle, 1982; Doyle *et al.*, 1982). The original control system block diagram then may be transformed into the form of Fig. 2. In this figure, $\mathbf{H}(\theta)$ represents a finite-dimensional parameter-varying linear system, and Δ represents a block diagonal—possibly infinite-dimensional—stable linear system which depends on only the uncertainties. In this framework, satisfying the various robust stability and robust performance specifications is equivalent to the feedback system of Fig. 2 being stable for an appropriate class of admissible uncertainties (see Doyle, 1982; Doyle *et al.*, 1982 for details).

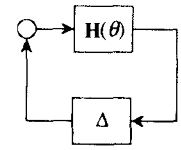

FIG. 2. General block diagram for robustness/performance analysis.

Employing this equivalent representation of design specifications, it then follows that the feedback diagram of Fig. 2, being a product of a gain scheduled design, is stable for all *frozen* parameter values. Furthermore, stability of Fig. 2 for *time-varying* parameters implies that the robustness and performance properties are maintained in the presence of parameter time-variations.

Let $\mathbf{H}(\theta)$ have the following state-space realization:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\theta(t))\mathbf{x}(t) + \mathbf{B}(\theta(t))\mathbf{e}(t),$$
$$\mathbf{y}(t) = \mathbf{C}(\theta(t))\mathbf{x}(t).$$

Furthermore, let the input/output relationship of Δ be given by

$$\mathbf{y}'(t) = \int_0^t \Delta(t - \tau)\mathbf{y}(\tau) \, d\tau.$$

Then the feedback equations are

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\theta(t))\mathbf{x}(t) + \int_0^t \mathbf{B}(\theta(t))\Delta(t - \tau)\mathbf{C}(\theta(\tau))\mathbf{x}(\tau) \, d\tau. \tag{1}$$

This equation represents a type of linear Volterra integrodifferential equation (VIDE). In this section it is shown that the stability of (1), hence the desired robustness and performance properties, is maintained in the presence of sufficiently slow parameter time-variations. This generalizes a well known result for ordinary differential equations (e.g. Desoer, 1969).

2.2. *Volterra integrodifferential equations.* Before time-varying robustness and performance are discussed, some facts are presented regarding equations of the form in (1). Evaluating (1) along any parameter vector trajectory, one has that

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \int_0^t \mathbf{B}(t)\Delta(t - \tau)\mathbf{C}(\tau)\mathbf{x}(\tau) \, d\tau, \tag{2}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ have been appropriately redefined. This is the general form of time-varying VIDEs and will be the object of all of the forthcoming analysis. Note that any conditions imposed on (2) can be translated immediately into conditions on the parameter-varying (1).

It was stated that equation (2) falls under the class of linear VIDEs. In fact, under assumptions to be stated on Δ, (2) actually represents a combination of VIDEs and linear delay-differential equations. Thus, both types of equations are treated under the same framework. VIDEs and their stability have been studied in, for example, Burton (1983), Corduneanu and Laksmikantham (1980), Miller (1971), and delay-differential equations in Corduneanu and Laksmikantham (1980), Driver (1977) and Hale (1977).

In this section, assumptions on (2) are given, a definition of exponential stability is introduced, a sufficient condition for exponential stability in the case of time-invariant $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ matrices is given, and a perturbational result time-invariant VIDEs is presented.

Consider the VIDE

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \int_0^t \mathbf{B}(t)\Delta(t - \tau)\mathbf{C}(\tau)\mathbf{x}(\tau) \, d\tau, \quad t > t_0, \tag{3}$$

with initial condition

$$\begin{cases} \mathbf{x}(t) = \phi(t), \\ \mathbf{x}(t_0^+) = \phi(t_0). \end{cases} \quad 0 \leq t \leq t_0, \quad \phi \in \mathcal{B}; \tag{4}$$

Note that an initial condition for (3) consists of both an initial time, t_0, *and* an initial function, ϕ.

The following assumptions are made on (3):

Assumption 1. The matrices $\mathbf{A}: \mathcal{R}^+ \to \mathcal{R}^{n\times n}$, $\mathbf{B}: \mathcal{R}^+ \to \mathcal{R}^{n\times m}$ and $\mathbf{C}: \mathcal{R}^+ \to \mathcal{R}^{p\times n}$ are globally bounded and Lipschitz continuous with Lipschitz constants L_A, L_B, and L_C, respectively.

Assumption 2. For some $\sigma \geq 0$, $\Delta \in \mathcal{A}^{m\times p}(-\sigma)$.

VIDEs containing an integral operator as in assumption 2 have been studied in Corduneanu (1973), Corduneanu and Luca (1975) and Luca (1979) and references are contained in Corduneanu and Laksmikantham (1980). In case of time-invariant **A**, **B** and **C** matrices, solutions to (3) can be explicitly characterized as follows:

Theorem 1. Consider the time-invariant VIDE

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \int_0^t \mathbf{B}\Delta(t-\tau)\mathbf{C}\mathbf{x}(\tau)\,\mathrm{d}\tau + \mathbf{f}(t), \quad t > t_0, \quad \mathbf{f} \in \mathcal{L}_\infty, \tag{5}$$

with initial condition (4) under assumption 2. The unique solution to (5) is given by

$$\mathbf{x}(t+t_0) = \mathbf{R}(t)\mathbf{x}(t_0) + \int_0^t \mathbf{R}(t-\tau)(\mathbf{f}(\tau+t_0) + \mathbf{F}(\tau+t_0))\,\mathrm{d}\tau, \quad t > 0,$$

where

$$\mathbf{F}(t+t_0) \stackrel{\text{def}}{=} \int_0^{t_0} \mathbf{B}\Delta(t+t_0-\tau)\mathbf{C}\phi(\tau)\,\mathrm{d}\tau, \quad t > 0,$$

and **R**, known as the resolvent matrix, is the unique matrix satisfying

$$\mathbf{R}(t) = \mathbf{I} + \int_0^t \left(\mathbf{A}\mathbf{R}(\tau) + \int_0^\tau \mathbf{B}\Delta(\tau-\xi)\mathbf{C}\mathbf{R}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\tau, \quad t > 0, \quad \mathbf{R}(0^+) = \mathbf{I}.$$

Proof. See Corduneanu (1973) and Corduneanu and Luca (1975). ■

A definition of exponential stability for VIDEs is now introduced.

Definition 1. Consider the VIDE (3) with initial condition (4). This VIDE is said to be exponentially stable if there exists constants m, λ, $\beta > 0$ where $\beta \geq \lambda$ such that

$$|\mathbf{x}(t)| \leq m\mathrm{e}^{-\lambda(t-t_0)} \, \|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}}, \quad \forall t \geq t_0.$$

It is stressed that the constants m, λ and β are *independent* of the initial condition (ϕ, t_0).

This definition implies that not only does the state decay exponentially, but also with a magnitude which is proportional to an exponentially forgotten initial function. The convention that $\beta \geq \lambda$ implies that the solutions cannot decay faster than they are forgotten. Furthermore, this inequality will be needed in subsequent proofs (cf. Theorem 2).

The following theorem gives a sufficient condition for exponential stability for time-invariant VIDEs.

Theorem 2. Consider the time-invariant VIDE (5) with initial condition (4). A sufficient condition for exponential stability is that there exist a constant $\beta > 0$ such that

$$s \mapsto (s\mathbf{I} - \mathbf{A} - \mathbf{B}\hat{\Delta}(s)\mathbf{C})^{-1} \in \hat{\mathcal{A}}^{n\times n}(-2\beta), \tag{6}$$

$$\hat{\Delta} \in \hat{\mathcal{A}}^{m\times p}(-2\beta). \tag{7}$$

Proof. It is first shown that the resolvent matrix **R** is bounded by a decaying exponential. From the definition of **R** in Theorem 1, one has that **R** satisfies almost everywhere

$$\dot{\mathbf{R}}(t) = \mathbf{A}\mathbf{R}(t) + \int_0^t \mathbf{B}\Delta(t-\tau)\mathbf{C}\mathbf{R}(\tau)\,\mathrm{d}\tau, \quad t > 0. \tag{8}$$

Taking the Laplace transform of (8) shows that

$$\hat{\mathbf{R}}(s) = (s\mathbf{I} - \mathbf{A} - \mathbf{B}\hat{\Delta}(s)\mathbf{C})^{-1}.$$

It follows by hypothesis that $\mathbf{R} \in \mathcal{A}^{n\times n}(-2\beta)$. Furthermore, it may be seen from the definition of **R** in Theorem 1 that **R** contains no impulses, hence $\mathbf{R} \in \mathcal{L}_1$. From (8), it follows that $\dot{\mathbf{R}} \in \mathcal{L}_1$. These two imply that $\mathbf{R} \in \mathcal{L}_\infty$. Now define

$$\mathbf{R}'(t) \stackrel{\text{def}}{=} \mathbf{R}(t)\mathrm{e}^{\beta t}.$$

Then $\mathbf{R}' \in \mathcal{A}^{n\times n}(-\beta)$ because $\mathbf{R} \in \mathcal{A}^{n\times n}(-2\beta)$. Using the same arguments as above along with

$$\dot{\mathbf{R}}(t) = (\mathbf{A} + \beta\mathbf{I})\mathbf{R}'(t) + \int_0^t \mathbf{B}\Delta(t-\tau)\mathrm{e}^{\beta(t-\tau)}\mathbf{C}\mathbf{R}'(\tau)\,\mathrm{d}\tau, \quad t > 0,$$

it follows that $\mathbf{R}'$ and $\dot{\mathbf{R}}' \in \mathcal{L}_1$, hence $\mathbf{R}' \in \mathcal{L}_\infty$. Thus, it follows that there exists a constant k_1, for example $k_1 = \|\mathbf{R}'\|_{\mathcal{L}_\infty}$, such that

$$|\mathbf{R}(t)| \leq k_1\mathrm{e}^{-\beta t}, \quad t \geq 0.$$

Now recall that the solution to (5) is given by

$$\mathbf{x}(t+t_0) = \mathbf{R}(t)\mathbf{x}(t_0) + \int_0^t \mathbf{R}(t-\tau)\mathbf{F}(\tau+t_0)\,\mathrm{d}\tau, \quad t > 0,$$

where

$$\mathbf{F}(t+t_0) \stackrel{\text{def}}{=} \int_0^{t_0} \mathbf{B}\Delta(t+t_0-\tau)\mathbf{C}\phi(\tau)\,\mathrm{d}\tau, \quad t > 0.$$

It is now shown that **F** is also bounded by a decaying exponential. Rewriting the definition of **F**,

$$\begin{aligned}\mathbf{F}(t+t_0) &= \int_0^t \mathbf{B}\Delta(t+t_0-\tau)\mathrm{e}^{\beta(t+t_0-\tau)}\mathbf{C}\mathrm{e}^{-\beta(t+t_0-\tau)}\phi(\tau)\,\mathrm{d}\tau \\ &= \mathrm{e}^{-\beta t}\int_0^{t_0} \mathbf{B}\Delta(t+t_0-\tau)\mathrm{e}^{\beta(t+t_0-\tau)}\mathbf{C}\mathrm{e}^{-\beta(t_0-\tau)}\phi(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Since $\Delta \in \mathcal{A}^{n\times n}(-2\beta)$, it follows that there exists a constant k_2, for example $k_2 = |\mathbf{B}| \, \|\Delta\|_{\mathcal{A}(-\beta)} \, |\mathbf{C}|$, such that

$$|\mathbf{F}(t+t_0)| \leq k_2\mathrm{e}^{-\beta t} \, \|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}}.$$

Using the exponential bounds on **R** and **F** to bound **x**,

$$\begin{aligned}|\mathbf{x}(t+t_0)| &\leq k_1\mathrm{e}^{-\beta t}|\mathbf{x}(t_0)| + \int_0^t k_1\mathrm{e}^{-\beta(t-\tau)}k_2\mathrm{e}^{-\beta\tau} \, \|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}}\,\mathrm{d}\tau \\ &\leq k_1\mathrm{e}^{\beta t}(1 + k_2 t) \, \|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}} \\ &\leq k_1\left(1 + \frac{2k_2}{\beta e}\right)\mathrm{e}^{-\beta t/2} \, \|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}},\end{aligned}$$

which completes the proof. ■

Theorem 2 is novel in that it takes a state-space approach, rather than an input/output approach, to the robust stability of time-invariant linear systems. This approach is chosen since it corresponds to the original motivation of parameter-varying gain scheduled systems. Nevertheless, standard results on robust stability can be obtained from this theorem. Rewriting $\hat{\mathbf{R}}(s)$, one has that

$$\hat{\mathbf{R}}(s) = (\mathbf{I} - (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\Delta}(s)\mathbf{C})^{-1}(s\mathbf{I} - \mathbf{A})^{-1}.$$

Now suppose that **A** is a stable matrix; thus $s \mapsto (s\mathbf{I} - \mathbf{A})^{-1} \in \hat{\mathcal{A}}^{n\times n}(-2\beta)$ for some $\beta > 0$. Assume further that $\hat{\Delta} \in \hat{\mathcal{A}}^{m\times p}(-2\beta)$. Then

$$s \mapsto (\mathbf{I} - (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\Delta}(s)\mathbf{C}) \in \hat{\mathcal{A}}^{n\times n}(-2\beta).$$

Under these conditions, $\hat{\mathbf{R}} \in \hat{\mathcal{A}}^{n\times n}(-2\beta)$ if (Desoer and Vidyasagar, 1975)

$$\begin{aligned}&\inf_{\mathrm{Re}\, s \geq -2\beta} |\det(\mathbf{I} - (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\Delta}(s)\mathbf{C})| \\ &\quad = \inf_{\mathrm{Re}\, s \geq -2\beta} |\det(\mathbf{I} - \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\Delta}(s))| > 0.\end{aligned}$$

However, a sufficient condition for the above equation is that

$$|\mathbf{C}((-2\beta + \mathrm{j}\omega)\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\Delta}(-2\beta + \mathrm{j}\omega)| \leq \gamma < 1, \quad \forall \omega \in \mathcal{R}.$$

As $\beta \to 0$, this condition approaches the standard small-gain robustness condition for time-invariant linear systems (e.g. Chen and Desoer, 1982; Doyle and Stein, 1981). Unlike previous results, however, Theorem 2 gives a quantitative indication of the rate of exponential decay of the state-variables.

This section concludes with a presentation of a perturbational result for time-invariant VIDEs.

Theorem 3. Consider the following perturbation of the VIDE (5):

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \int_0^t \mathbf{B}\Delta(t-\tau)\mathbf{C}\mathbf{x}(\tau)\,d\tau + (\mathbf{g}\mathbf{x})(t), \quad t > t_0, \quad (9)$$

where $\mathbf{g}$ is an integral operator on $\mathbf{x}$. Let

$$s \mapsto (s\mathbf{I} - \mathbf{A} - \mathbf{B}\hat{\Delta}(s)\mathbf{C})^{-1} \in \hat{\mathcal{A}}^{n\times n}(-2\beta),$$
$$\hat{\Delta} \in \hat{\mathcal{A}}^{m\times p}(-2\beta),$$

for some $\beta > 0$. Assume further that there exist constants $k > 0$ and $\alpha \ge \beta$ such that

$$|(\mathbf{g}\mathbf{x})(t)| \le k\,\|\mathcal{W}_{t,\alpha}\mathbf{x}\|_{\mathcal{B}}, \quad \forall t \ge 0, \quad \forall \mathbf{x} \in \mathcal{B}.$$

Under these conditions, there exists a $\gamma > 0$ such that the VIDE (9) is exponentially stable for $k < \gamma$.

Proof. Let (9) have an initial condition (4). Define $\mathbf{z}(t) \stackrel{\text{def}}{=} \mathbf{x}(t+t_0)$. As in Theorem 1, one has that

$$\mathbf{z}(t) = \mathbf{R}(t)\mathbf{z}(t) + \int_0^t \mathbf{R}(t-\tau)(\mathbf{F}(\tau+t_0) + (\mathbf{g}\mathbf{x})(\tau+t_0))\,d\tau, \quad t > 0,$$

where $\mathbf{R}$ is the resolvent matrix and

$$\mathbf{F}(t+t_0) = \int_0^{t_0} \mathbf{B}\Delta(t+t_0-\tau)\mathbf{C}\phi(\tau)\,d\tau, \quad t > 0.$$

As in the proof of Theorem 2, there exist k_1 and k_2 such that

$$|\mathbf{R}(t)| \le k_1 e^{-\beta t},$$
$$|\mathbf{F}(t+t_0)| \le k_2 e^{-\beta t}\,\|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}}.$$

Using these bounds to bound $\mathbf{z}$,

$$|\mathbf{z}(t)| \le k_1 e^{-\beta t}|\mathbf{z}(0)| + \int_0^t k_1 e^{-\beta(t-\tau)} \times (k_2 e^{-\beta t}\,\|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}} + k\,\|\mathcal{W}_{\tau+t_0,\alpha}\mathbf{x}\|_{\mathcal{B}})\,d\tau.$$

By definition of the $\mathcal{W}$ operator

$$\|\mathcal{W}_{\tau+t_0,\alpha}\mathbf{x}\|_{\mathcal{B}} \le \|\mathcal{W}_{\tau+t_0,\beta}\mathbf{x}\|_{\mathcal{B}} \stackrel{\text{def}}{=} \sup_{\xi\in[0,\tau+t_0]} |e^{-\beta(\tau+t_0-\xi)}\mathbf{x}(\xi)|$$
$$\le e^{-\beta\tau}\Big(\sup_{\xi\in[0,t_0]} |e^{-\beta(t_0-\xi)}\phi(\xi)| + \sup_{\xi\in[t_0,\tau+t_0]} |e^{-\beta(t_0-\xi)}\mathbf{x}(\xi)|\Big).$$

Thus,

$$e^{\beta t}|\mathbf{z}(t)| \le k_1(1+(k+k_2)t)\,\|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}} + \int_0^t k_1 k \sup_{\xi\in[0,\tau]} |e^{\beta\xi}\mathbf{z}(\xi)|\,d\tau.$$

Since the right-hand side of the above equation is a nondecreasing function of time,

$$\sup_{\xi\in[0,t]} |e^{\beta\xi}\mathbf{z}(\xi)| \le k_1(1+(k+k_2)t)\,\|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}} + \int_0^t k_1 k \sup_{\xi\in[0,\tau]} |e^{\beta\xi}\mathbf{z}(\xi)|\,d\tau.$$

Rewriting this equation yields

$$f(t) \le \kappa_1 + \kappa_2 t + \kappa_3\int_0^t f(\tau)\,d\tau,$$

where κ_1, κ_3 and f are defined in the obvious manner. Applying the Bellman–Gronwall inequality (e.g. Desoer and Vidyasagar, 1975),

$$f(t) \le \left(\kappa_1 + \frac{\kappa_2}{\kappa_3}\right)e^{\kappa_3 t} - \frac{\kappa_2}{\kappa_3}.$$

Thus,

$$|\mathbf{z}(t)| \le \left(\left(k_1 + 1 + \frac{k_2}{k}\right)e^{-(\beta-k_1k)t} - \left(1+\frac{k_2}{k}\right)e^{-\beta t}\right)\|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}}$$
$$= \left(k_1 + \left(\frac{k+k_2}{k}\right)(1-e^{-k_1kt})\right)e^{-(\beta-k_1k)t}\,\|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}}$$
$$\le k_1(1+(k_2+k)t)e^{-(\beta-k_1k)t}\,\|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}}.$$

Thus, it is seen that $k < \gamma \stackrel{\text{def}}{=} \beta/k_1$ implies exponential stability. Furthermore,

$$|\mathbf{z}(t)| \le k_1\left(1 + \frac{2(k_2+k)}{e(\beta-k_1k)}\right)e^{-((\beta-k_1k)/2)t}\,\|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}}. \quad (10)$$

■

2.3. *Robustness and performance of slowly-varying linear systems.* In this section, it is shown that if the time-varying VIDE (3) is exponentially stable for all frozen values of time, then it is exponentially stable for sufficiently slow time-variations. In terms of the original motivation of guaranteed properties of gain scheduled control systems, this means that robust stability and robust performance are maintained provided that parameter variations are sufficiently slow.

Before proceeding with the main theorem, some assumptions and definitions are given.

Assumption 3. Consider the time-varying VIDE (3) under assumptions 1–2. The matrix functions $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and Δ are such that for each $\xi \in \mathcal{R}^+$,

$$s \mapsto (s\mathbf{I} - \mathbf{A}(\xi) - \mathbf{B}(\xi)\hat{\Delta}(s)\mathbf{C}(\xi))^{-1} \in \hat{\mathcal{A}}^{n\times n}(-2\beta),$$
$$\hat{\Delta} \in \hat{\mathcal{A}}^{m\times p}(-2\beta).$$

Via Theorem 2, these two conditions imply that for each $\xi \in \mathcal{R}^+$, the time-invariant VIDE

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\xi)\mathbf{x}(t) + \int_0^t \mathbf{B}(\xi)\Delta(t-\tau)\mathbf{C}(\xi)\mathbf{x}(\tau)\,d\tau, \quad t > t_0, \quad (11)$$

is exponentially stable as in Definition 1. It is further assumed that this exponential stability is uniform in ξ. That is, there exist exponential stability constants (m, λ, β) for (11) which are independent of ξ.

Note that in the case of gain scheduling, one may use Theorem 2 to verify exponential stability for all frozen parameter values, hence for all time. Furthermore, if the parameters take their values in some compact set, then the exponential stability is uniform.

Definition 2. Let assumptions 1–2 hold. Under assumption 1, let k_B and k_C satisfy $\forall t \in \mathcal{R}^+$

$$|\mathbf{B}(t)| \le k_B, \quad |\mathbf{C}(t)| \le k_C.$$

Then the measure, K, of the rate of time-variations of (3) is defined as

$$K \stackrel{\text{def}}{=} L_A + L_B\,\|\Delta\|_{\mathcal{A}(-\beta)}k_C + k_B\,\|\Delta\|_{\mathcal{A}(-\beta)}\,L_C.$$

The question of slowly time-varying stability of linear VIDE's is now addressed.

Theorem 4. Consider the time-varying VIDE (3) under assumptions 1–3. Under these conditions, (3) is exponentially stable for sufficiently small K, or equivalently, for sufficiently slow time-variations in $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

Proof. Let $\mathbf{R}_\xi$ denote the resolvent matrix associated with the frozen-time VIDE (11). From assumption 2 of uniform frozen-time exponential stability, there exists a constant K_1 such that

$$|\mathbf{R}_\xi(t)| \le K_1 e^{-\beta t}, \quad \forall \xi \in \mathcal{R}^+.$$

Similarly, define

$$K_2 \stackrel{\text{def}}{=} k_B\,\|\Delta\|_{\mathcal{A}(-\beta)}\,k_C.$$

Note that the constants K_1 and K_2 represent worst case values of their frozen-ξ analogs in the proof of Theorem 2.

Let the initial condition of (3) be (4), and let t_n denote $t_0 + nT$, where T is some constant interval to be chosen. Approximating $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ by piecewise constant matrices, one has that

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t_n)\mathbf{x}(t) + \int_{t_n}^{t} \mathbf{B}(t_n)\Delta(t-\tau)\mathbf{C}(t_n)\mathbf{x}(\tau)\,\mathrm{d}\tau + \int_{0}^{t_n} \mathbf{B}(t)\Delta(t-\tau)\mathbf{C}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau + (\mathbf{g}_n\mathbf{x})(t),$$

where

$$(\mathbf{g}_n\mathbf{x})(t) = (\mathbf{A}(t) - \mathbf{A}(t_n))\mathbf{x}(t) + \int_{t_n}^{t} (\mathbf{B}(t) - \mathbf{B}(t_n))\Delta(t-\tau)\mathbf{C}(t_n)\mathbf{x}(\tau)\,\mathrm{d}\tau + \int_{t_n}^{t} \mathbf{B}(t)\Delta(t-\tau)(\mathbf{C}(\tau) - \mathbf{C}(t_n))\mathbf{x}(\tau)\,\mathrm{d}\tau.$$

Then

$$|(\mathbf{g}_n\mathbf{x})(t)| \le KT\,\|\mathcal{W}_{t,\beta}\mathbf{x}\|_{\mathcal{B}}, \quad t_n \le t \le t_{n+1}.$$

Choose any $\eta \in (0, \beta)$. Then using arguments exactly parallel to those of the proof of Theorem 3, it can be shown that

$$KT \le \frac{\beta - \eta}{2K_1} \tag{12}$$

implies that

$$|\mathbf{x}(t)| \le K_1\left(1 + \frac{2\left(K_2 + \dfrac{\beta-\eta}{2K_1}\right)}{\mathrm{e}\left(\beta - K_1\dfrac{\beta-\eta}{2K_1}\right)}\right)\mathrm{e}^{-[(\beta - K_1(\beta-\eta)/(2K_1))/2](t-t_n)}\,\|\mathcal{W}_{t_n,\beta}\mathbf{x}\|_{\mathcal{B}}$$
$$= M\mathrm{e}^{-[(\beta+\eta)/4](t-t_n)}\,\|\mathcal{W}_{t_n,\beta}\mathbf{x}\|_{\mathcal{B}},$$

where M is defined in the obvious manner [cf. (10)]. Note that since $K_1 \ge 1$ by definition, one has $M > 1$ in general. This bound on $\mathbf{x}$ further implies that

$$\|\mathcal{W}_{t_{n+1},\beta}\mathbf{x}\|_{\mathcal{B}} \stackrel{\text{def}}{=} \sup_{\xi\in[0,t_{n+1}]} |\mathrm{e}^{-\beta(t_{n+1}-\xi)}\mathbf{x}(t)|$$
$$= \max\left(\sup_{\xi\in[0,t_n]} |\mathrm{e}^{-\beta(t_{n+1}-\xi)}\mathbf{x}(\xi)|,\ \sup_{\xi\in(t_n,t_{n+1})} |\mathrm{e}^{-\beta(t_{n+1}-\xi)}\mathbf{x}(\xi)|\right)$$
$$\le \max\left(\mathrm{e}^{-\beta T}\sup_{\xi\in[0,t_n]} |\mathrm{e}^{-\beta(t_n-\xi)}\mathbf{x}(\xi)|,\ \sup_{\xi\in(t_n,t_{n+1}]} \mathrm{e}^{-\beta(t_{n+1}-\xi)}M\mathrm{e}^{-[(\beta+\eta)/4](\xi-t_n)}\,\|\mathcal{W}_{t_n,\beta}\mathbf{x}\|_{\mathcal{B}}\right)$$
$$= \max\left(\mathrm{e}^{-\beta T}\,\|\mathcal{W}_{t_n,\beta}\mathbf{x}\|_{\mathcal{B}},\ \sup_{\xi\in(t_n,t_{n+1}]} M\mathrm{e}^{(\beta-(\beta+\eta)/4)\xi}\mathrm{e}^{-\beta t_{n+1}}\mathrm{e}^{[(\beta+\eta)/4]t_n}\,\|\mathcal{W}_{t_n,\beta}\mathbf{x}\|_{\mathcal{B}}\right)$$
$$= M\mathrm{e}^{-(1/2)(\beta+\eta/2)T}\,\|\mathcal{W}_{t_n,\beta}\mathbf{x}\|_{\mathcal{B}}.$$

In order to guarantee (12), choose

$$T = 4\ln M/(\beta - \eta).$$

Then

$$K \le \frac{(\beta-\eta)^2}{8K_1\ln M}$$

implies the desired (12). Furthermore, recursively applying the bound on $\|\mathcal{W}_{t_n,\beta}\mathbf{x}\|_{\mathcal{B}}$ shows that

$$|\mathbf{x}(t)| \le M\mathrm{e}^{-[(\beta+\eta)/4](t-t_n)}\,\|\mathcal{W}_{t_n,\beta}\mathbf{x}\|_{\mathcal{B}}$$
$$\le M\mathrm{e}^{-(1/2)(\eta+(\beta-\eta)/2))(t-t_0-nT)} \times (M\mathrm{e}^{-(1/2)(\eta+(\beta-\eta)/2))T})^n\,\|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}}$$
$$= M\mathrm{e}^{-(\eta/2)(t-t_0)}\mathrm{e}^{-(1/2)((\beta-\eta)/2))(t-t_n)} \times (M\mathrm{e}^{-(1/2)((\beta-\eta)/2))T})^n\,|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}}.$$

Substituting the choice of T into the above equation yields

$$|\mathbf{x}(t)| \le M\mathrm{e}^{-(\eta/2)(t-t_0)}\,|\mathcal{W}_{t_0,\beta}\phi\|_{\mathcal{B}},$$

which completes the proof. ∎

The main idea behind Theorem 4 is as follows. The time-varying VIDE (3) is approximated by a piecewise constant VIDE which is piecewise exponentially stable. Thus on each interval, the time-varying VIDE is decomposed into a constant part and a time-varying perturbation. Using Theorem 3, the solution will decay provided that the piecewise constant approximations are sufficiently accurate over sufficiently long intervals, which, in turn, is guaranteed by sufficiently slow time-variations.

3. *Concluding remarks*

This paper has addressed the robust stability and robust performance of parameter-varying linear systems in the context of gain-scheduling. The results may be summarized as follows. Essentially, it was shown that a gain scheduled system which has desirable feedback properties for all *frozen* values of the parameters maintains these properties provided that the parameter time-variations are sufficiently slow. Explicit sufficient conditions on the parameter time-variations were given. Thus the heuristic guidelines of "scheduling on a slow variable" has been transformed into quantitative statements.

Unfortunately, the actual bounds on the parameter time-variations may be difficult—at best—to compute. For example, to verify the sufficient conditions for frozen-parameter exponential stability (cf. Theorem 2) would require satisfying a small-gain condition *off* the $\mathrm{j}\omega$-axis. Once these conditions are verified, one can then use Theorem 4 to guarantee time-varying stability. However, this requires computation of the measure of time-variations in Definition 2 (K), a bound on the resolvent matrix for the frozen-parameter systems (K_1), and a bound on the exponentially weighted input/output norm of the linear uncertainties ($\|\Delta\|_{\mathcal{A}(-\beta)}$). Furthermore, even if verified these results are apt to be conservative.

In spite of these limitations, the value of the results is that they lead to new insights into gain-scheduled systems. For example, in performing the frozen parameter designs, one must guarantee some degree of internal exponential stability in addition to the input/output stability of standard robustness tests. Furthermore, the sufficiency of the conditions is simply a reminder that the designs were based on time-invariant approximations to the actual time-varying plant. If these approximations are inaccurate, then one should not demand guarantees on the overall gain scheduled system.

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