Lecture 6

Relevant sections in text: §1.4

Complete set of commuting observables

Only in the simplest physical systems will the measurement of a single observable suffice to determine the state vector of the system. Of course, the spin 1/2 system has this property: if you measure any component of the spin the state vector is fixed (up to an irrelevant phase factor*). More complicated systems need more observables to characterize the state. For example, recall that the energy levels of a hydrogen atom can be uniquely specified by the energy, orbital angular momentum magnitude, one component of the angular momentum and the electron spin state. Other choices are possible. After measuring an observable and getting a particular outcome – an eigenvalue – the state of the system is the corresponding eigenvector. It may happen that more than one state vector is associated with that measurement outcome. Mathematically, the eigenvalue is degenerate. To pin down the state uniquely, other observables must be measured. Evidently, if we have a set of observables whose measurement uniquely determines the state (vector – up to a phase), then that state vector must be a simultaneous eigenvector of all the observables in question. If we demand that the possible outcomes of measurements of this set of observables defines a basis for the Hilbert space, then the observables must be compatible, *i.e.,* their operator represntatives must all commute. Such a set of variables is called a *complete set of commuting observables* (CSCO). For a hydrogen atom, modeled as an electron in a Coulomb potential, the energy, orbital angular momentum magnitude, one component of the orbital angular momentum and one component of the electron spin constitute a CSCO. For a single spin 1/2 system, any component of spin defines a CSCO. Evidently, a CSCO is not unique.

Given a CSCO, $A, B, \ldots$ we can label the elements of the ON basis it determines by the eigenvalues of the CSCO, $|a, b, \ldots\rangle$. These are states in which the elements of the CSCO are known with certainty to have the values $a, b, \ldots$. The vectors satisfy

$$\langle a, b, \ldots | a', b', \ldots \rangle = \delta_{aa'} \delta_{bb'} \ldots.$$

Incompatible observables. The uncertainty principle.

Incompatible observables are represented by operators which do not commute. As we have seen, incompatible observables are such that there exist states where both observables

---

* Two states that differ by a phase factor (e^{ia}) will give the same expectation value to all observables – exercise. This means that all probability distributions are insensitive to the phase factors, and hence all physical predictions are as well.

cannot be determined with certainty. We will now make this more precise and give a very general version of the celebrated uncertainty principle — probably better called the uncertainty *theorem.*

Given an observable A and a state $|\psi\rangle$, we define the *dispersion of A in the state $|\psi\rangle$* to be

$$\langle (\Delta A)^2 \rangle := \langle (A - \langle A \rangle)^2 \rangle = \langle \psi | A^2 | \psi \rangle - \langle \psi | A | \psi \rangle^2 = \langle A^2 \rangle - \langle A \rangle^2.$$

The dispersion (also called the variance) is a non-negative real number which characterizes the statistical uncertainty of the observable A in the state $|\psi\rangle$. To see that the dispersion is indeed non-negative, note that the expectation value of the square of any Hermitian operator C is a positive number:

$$\langle \psi | C^2 | \psi \rangle = \langle \psi | C^\dagger C | \psi \rangle;$$

the right hand side is just the length-squared of the vector $C|\psi\rangle$, which must be non-negative. Setting $C = A - \langle A \rangle I$, and noting that $\langle A \rangle$ must be a real number (exercise), we conclude that $\langle (\Delta A)^2 \rangle \geq 0$.

Note the dispersion vanishes if and only if the state $|\psi\rangle$ is an eigenvector of A. You can easily verify the "if" part. Let me show you the "only if" part. Write

$$\begin{aligned} \langle (\Delta A)^2 \rangle &= \langle \psi | (A - \langle A \rangle I)^2 | \psi \rangle \\ &= \langle \psi | (A - \langle A \rangle I)^\dagger (A - \langle A \rangle I) | \psi \rangle \\ &= || (A - \langle A \rangle I) | \psi \rangle ||^2. \end{aligned}$$

The norm of a vector vanishes if and only if the vector is the zero vector, so if the dispersion vanishes we have

$$(A - \langle A \rangle I)|\psi\rangle = 0,$$

which is the eigenvector condition (exercise).

Using the Schwarz inequality the following relation between the dispersions of 2 obervables can be established (see your text for the proof):

$$\langle \Delta A^2 \rangle \langle \Delta B^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2.$$

This is the general form of the *uncertainty relation. In a given state* it relates the product of the statistical uncertainty of a pair of observables to the expectation value of the commutator of the observables. If the commutator vanishes, or if its expectation value does in the given state, then the uncertainty relation has no content. Otherwise, it provides information about the effect of the incompatibility of the observables. In general, this "effect" depends upon the state that is chosen. You can see this from the fact that the expectation values occurring in the inequality above are defined by a given state. This is

important to keep in mind. In certain cases (*e.g.,* the position-momentum relation, to be studied later) the uncertainty relation turns out to be state independent and hence it is much more dramatic — and famous.

For a spin 1/2 system it is straightforward to compute the uncertainty relation for various observables. To do it, you need the following commutation relations (which you derive in your homework):

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y.$$

Let let $|\psi\rangle = a|+\rangle + b|-\rangle$, $|a|^2 + |b|^2 = 1$ (which is a general state vector). We consider the uncertainty relation for S_x and S_z. We have (exercise)

$$\langle \Delta S_z \rangle^2 \langle \Delta S_x \rangle^2 \geq \hbar^4 [\mathrm{Im}(ab^*)]^2.$$

Of course, if a or b vanish the state is an eigenstate of S_z and the uncertainty relation has no import. Otherwise, though, you can see how the lower limit for the product of uncertainties varies with the choice of state. For example, if $a = 1/\sqrt{2}$ and $b = i\sqrt{2}$ (so that the state is an eigenstate of S_y) we have

$$\langle \Delta S_z \rangle^2 \langle \Delta S_x \rangle^2 \geq \frac{\hbar^4}{4}.$$