Singularities in Calabi-Yau varieties

Mee Seong Im
The University of Illinois Urbana-Champaign

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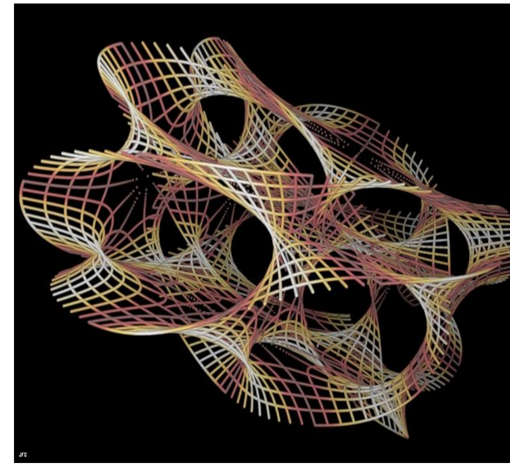
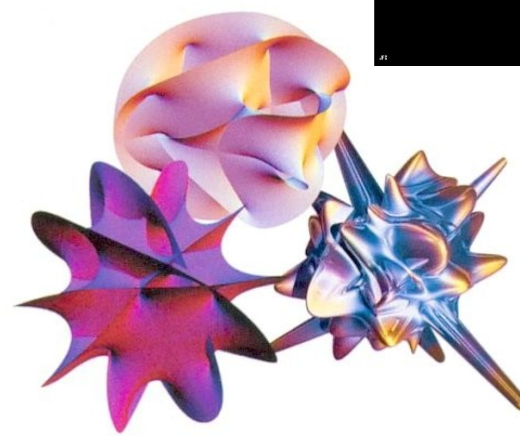
What are Calabi-Yau varieties?

- Complex Kahler manifolds with a vanishing first Chern class
- In other words, they have trivial canonical bundle. Canonical bundle: top exterior power of the cotangent bundle (also known as invertible bundle).
- Reasons why physicists are interested in C-Y varieties are because Kahler manifolds admit a Kahler metric and they have nowhere vanishing volume form. So we can do integration on them.
- May not necessary be differential manifolds, so the key to studying C-Y varieties is by algebraic geometry, rather than differential geometry.
- Nice results in derived categories and a concept of duality in string theory.

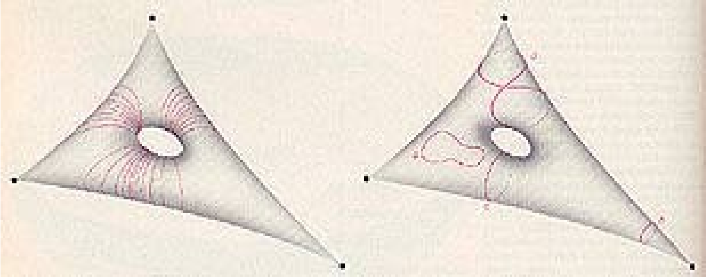




A manifold with Hermitian metric is called an **almost Hermitian manifold**. A **Kahler manifold** is a manifold with a Hermitian metric satisfying an integrability condition.

What are orbifold singularities and why do they occur?

- Orbifold singularity is the simplest singularity arising on a Calabi-Yau manifold. An orbifold is a quotient of a smooth Calabi-Yau manifold by a discrete group action with fixed points.
- For a finite subgroup G of SL(n,C), an orbifold is locally C^n/G



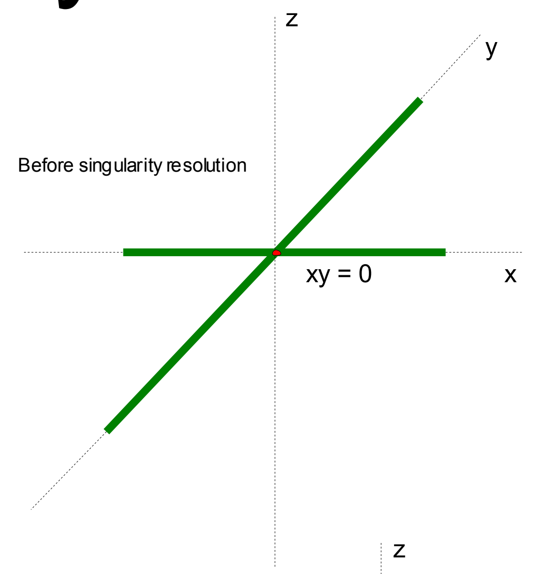
- Example: consider the real line R. Then $R \to R/\{\pm 1\}$
- This looks like a positive real axis together with the origin, so it's not a smooth one-dimensional manifold.
- This is generically a 2:1 map at every point except at the origin. At the origin, the map is 1:1 map, so the quotient space has a singularity at the origin. So a singularity formed from a quotient by a group action is a fixed point in the original space.

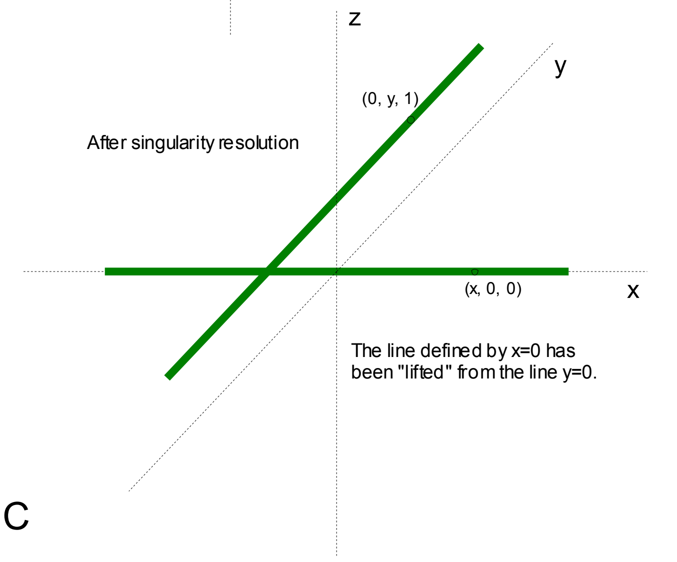
Orbifold space is a highly curved 6-dim space in which strings move. The diagram above is a closed, 2-dim surface which has been stretched out to form 3 sharp points, which are the conical isolated singularities. The orbifold is flat everywhere except at the singularities; the curvature there is infinite.
See Scientific American: http://www.damtp.cam.ac.uk/user/mbg15/superstrings/superstrings.html

Mee Seong Im, UIUC

How do we resolve an orbifold singularity?

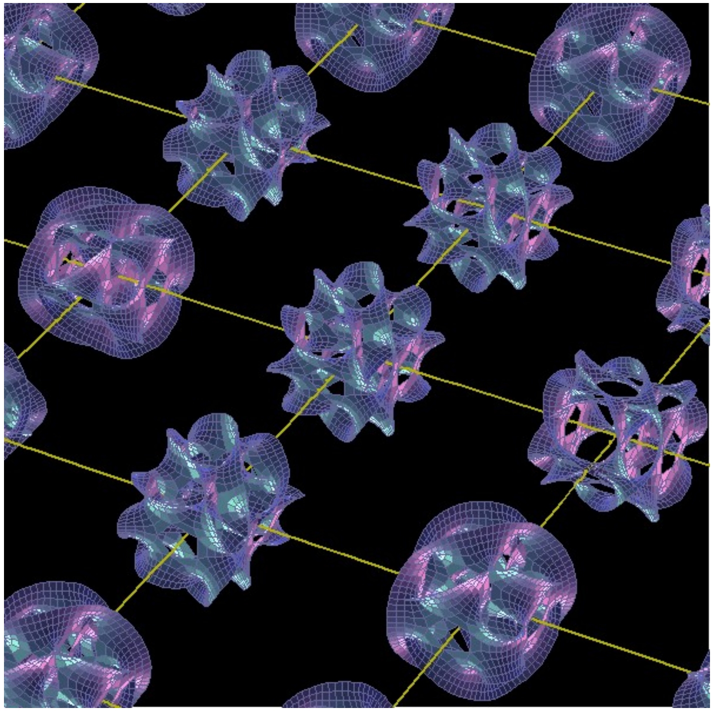
- We resolute singularities because smooth manifolds are "nicer" to work with, e.g., they have well-defined notion of dimension, cohomology and derived categories, etc.
- Example: Consider the zero set of xy=0 in the affine two-dim space. We have a singularity at the origin. To desingularize, consider the singular curve in 3-dimensional affine space where the curve is sitting in the x-y plane. We "remove" the origin, pull the two lines apart, and then go back and fill in the holes.





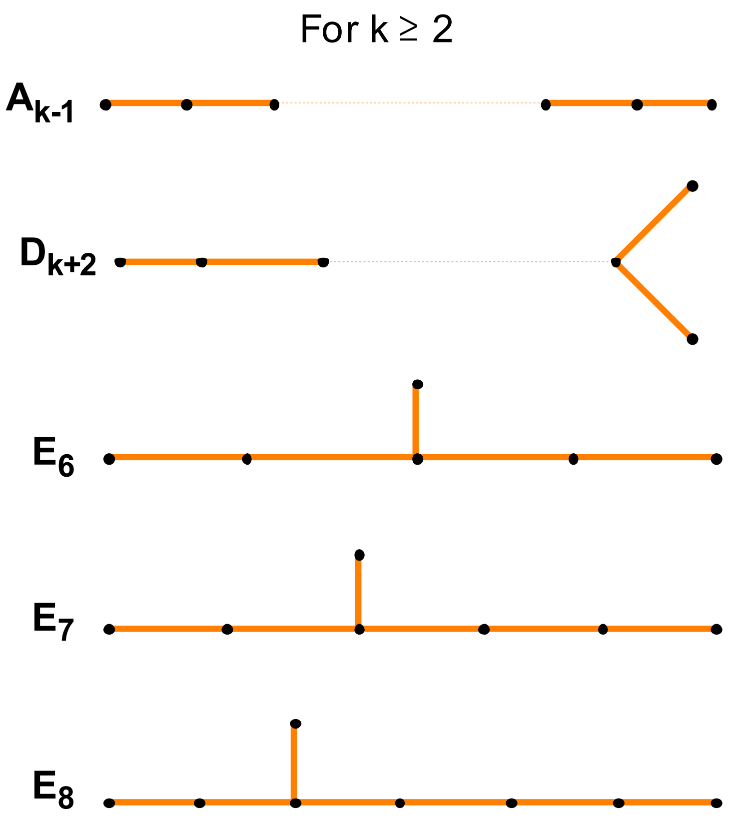
- A crepant resolution (X, p) of C^n/G is a nonsingular complex manifold X of dimension n with a proper biholomorphic map p: X→ C^n/G that induces a biholomorphism between dense open sets.
- The space X is a crepant resolution of C^n/G if the canonical bundles are isomorphic; i.e., p*($K_{C^n/G}$) is isomorphic to K_X.
- Choose crepant resolutions of singularities to obtain a Calabi-Yau structure on X.
- We'll see that the amount of information we obtain from the resolution will depend on the dimension n of the orbifold.
- The mathematics behind the resolution is found in “Platonic solids, binary polyhedral groups, Kleinian singularities and Lie algebras of type A, D, E” by Joris van Hoboken.

A six real dimensional Calabi-Yau manifold



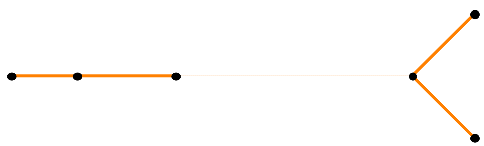
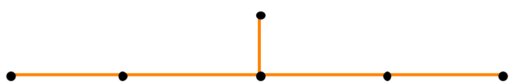
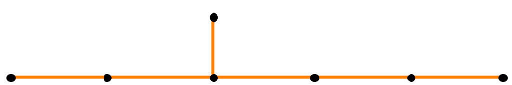
McKay Correspondence

- There is a 1-1 correspondence between irreducible representations of G and vertices of an extended Dynkin diagram of type ADE.
- The topology of the crepant resolution is described completely by the finite group G, the Dynkin diagram, and the Cartan matrix.
- The extended Dynkin diagram of type ADE are the Dynkin diagrams corresponding to the Lie algebras of types $A_{k-1}, D_{k+2}, E_6, E_7,$ and E_8.



- Simple Lie groups and simple singularities classified in the same way by Dynkin diagrams
- Platonic solids ←→ binary polyhedral groups ←→ Kleinian (simple) singularities ←→ Dynkin diagrams ←→ representations of binary poly groups ←→ Lie algebras of type A, D, E
- Reference for more details: Joris van Hoboken's thesis

For $k \geq 2$

A_{k-1}		degenerate	Cyclic subgroup of order k
D_{k+2}		< 2, 2, k >	Generalized quaterions Q_k of order 4k
E_6		< 2, 3, 3 >	2 Alt_4 binary Tetrahedral group of order 24
E_7		< 2, 3, 4 >	2 Symm_4 binary Octahedron group of order 48
E_8		< 2, 3, 5 >	2 Alt_5 binary Icosahedron group of order 120

where $\langle a, b, c\rangle := \langle x, y, z : x^a = y^b = z^c = xyz \rangle$

Singularity for the < 2, 3, 5 > case is $x^2 + y^3 + z^5 = 0$ but not true in general.

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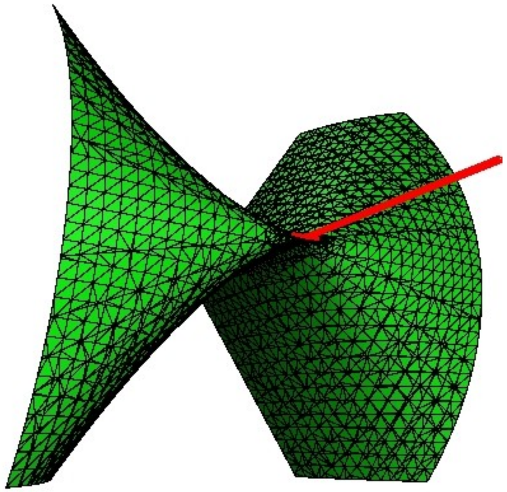
Case when n=2

- A unique crepant resolution always exists for surface singularities.
- Felix Klein (1884) first classified the quotient singularities C^2/G for a finite subgroup G of SL(2,C). Known as Kleinian or Du Val singularities or rational double points.
- A unique crepant resolution exists for the 5 families of finite subgroups of SL(2,C): cyclic subgroups of order k, binary dihedral groups of order 4k, binary tetrahedral groups of order 24, binary octahedral group of order 48, binary icosahedral group of order 120.
- How do we prove that only 5 finite subgroups exist in SL(2, C)? Natural embedding of binary polyhedral groups in SL(2, C) or its compact version SU(2,C), a double cover of SO(3, R).

Reference: Joris van Hoboken's thesis



Case when n=3

- A crepant resolution always exists but it is not unique as they are related by flops.
- Blichfeldt (1917) classified the 10 families of finite subgroups of SL(3, C).
- S. S. Roan (1996) considered cases and used analysis to construct a crepant resolution of C^3/G with given stringy Euler and Betti numbers. So the resolution has the same Euler and Betti numbers as the ones of the orbifold.
- Nakamura (1995) conjectured that Hilb(C^3) being fixed by G is a crepant resolution of C^3/G based on his computations for n=2. He later proved when n=3 for abelian groups.
- Bridgeland, King and Reid (1999) proved the conjecture for n=3 for all finite groups using derived category techniques.
- Craw and Ishii (2002) proved that all the crepant resolutions arise as moduli spaces in the case when G is abelian.

Further Investigation

- When n=3, no information about the multiplicative structures in cohomology or K-theory.
- Consider cases when n>3. Crepant resolutions exist under special/particular conditions.
- In higher dimensional cases (n>3), many singularities are found to be terminal, i.e., crepant resolutions do not exist.