Unitarism and Infinitarism

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We will examine variations of four famous arithmetical functions. For a given function χ, let χ^* denote its unitary analog, $\tilde{\chi}$ its square-free analog, and χ' its unitary square-free analog. The meanings of these phrases will be made clear in each case. At the end, the infinitary analog χ_∞ will appear as well.

0.1. Divisor Function. If $d(n)$ is the number of distinct divisors of n, then

$$\sum_{n=1}^{N} d(n) = N\ln(N) + (2\gamma - 1)N + O(\sqrt{N})$$

as $N \to \infty$, where γ is the Euler-Mascheroni constant. Let us introduce a more refined notion of divisibility. A divisor k of n is **unitary** if k and n/k are coprime, that is, if $\gcd(k, n/k) = 1$. This condition is often written as $k||n$. The number $d^*(n)$ of unitary divisors of n is $2^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of n. This fact is easily seen to be true: If $p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is the prime factorization of n, then the unitary divisors of n are of the form $p_1^{\varepsilon_1 a_1} p_2^{\varepsilon_2 a_2} \cdots p_r^{\varepsilon_r a_r}$, where each ε_s is either 0 or 1. There are 2^r possible choices for the r-tuple $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$; hence the result follows. We have [1, 2, 3, 4, 5]

$$\sum_{n=1}^{N} d^*(n) = \frac{6}{\pi^2} N\ln(N) + \frac{6}{\pi^2}\left(2\gamma - 1 - \frac{12}{\pi^2}\zeta'(2)\right)N + O(\sqrt{N}),$$

where $\zeta(x)$ is the Riemann zeta function and $\zeta'(x)$ is its derivative.

A divisor k of n is **square-free** if k is divisible by no square exceeding 1. The number $\tilde{d}(n)$ of square-free divisors of n is also $2^{\omega(n)}$; the divisors in this case are of the form $p_1^{\varepsilon_1} p_2^{\varepsilon_2} \cdots p_r^{\varepsilon_r}$. Therefore the same asymptotics apply for $\tilde{d}(n)$, but the underlying sets of numbers overlap only somewhat [6].

Define $d'(n)$ to be the number of unitary square-free divisors of n. A more complicated asymptotic formula arises here [7, 8]:

$$\sum_{n=1}^{N} d'(n) = \frac{6\alpha}{\pi^2} N\ln(N) + \frac{6\alpha}{\pi^2}\left(2\gamma - 1 - \frac{12}{\pi^2}\zeta'(2) + X\right)N + O(\sqrt{N}\ln(N))$$

where

$$\alpha = \prod_p \left(1 - \frac{1}{p(p+1)}\right) = 0.7044422009..., \qquad X = \sum_p \frac{(2p+1)\ln(p)}{(p+1)(p^2+p-1)}$$

and we agree that the product and sum extend over all primes p. The constant α is the same as what is called $\pi^2 P/6$ in [9].

We finally give corresponding reciprocal sums [10, 11, 12]:

$$\lim_{N\to\infty} \frac{\sqrt{\ln(N)}}{N} \sum_{n=1}^{N} \frac{1}{d(n)} = \frac{1}{\sqrt{\pi}} \prod_p \sqrt{p(p-1)} \ln\left(\frac{p}{p-1}\right) = \frac{0.9692769438...}{\sqrt{\pi}}$$

$$\lim_{N\to\infty} \frac{\sqrt{\ln(N)}}{N} \sum_{n=1}^{N} \frac{1}{d^*(n)} = \frac{1}{\sqrt{\pi}} \prod_p \sqrt{1 + \frac{1}{4p(p-1)}} = \frac{1.0969831191...}{\sqrt{\pi}}$$

The former sum was mentioned in [13] with regard to the arcsine law for random divisors. It is not known what constant emerges for $1/d'(n)$.

0.2. Sum-of-Divisors Function. If $\sigma(n)$ is the sum of all distinct divisors of n, then

$$\sum_{n=1}^{N} \sigma(n) = \frac{\pi^2}{12} N^2 + O(N \ln(N))$$

as $N \to \infty$. Let $\sigma^*(n)$ be the sum of unitary divisors of n and $\tilde{\sigma}(n)$ be the sum of square-free divisors of n. Although $d^*(n) = \tilde{d}(n)$ always, it is usually false that $\sigma^*(n) = \tilde{\sigma}(n)$ [14]. We have [15, 16, 17, 18]

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{n=1}^{N} \sigma^*(n) = \frac{\pi^2}{12\zeta(3)}, \qquad \lim_{N\to\infty} \frac{1}{N^2} \sum_{n=1}^{N} \tilde{\sigma}(n) = \frac{1}{2}.$$

Further, if $\sigma'(n)$ is the sum of unitary square-free divisors of n, then [15]

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{n=1}^{N} \sigma'(n) = \frac{1}{2} \prod_p \left(1 - \frac{1}{p^2(p+1)}\right) = \frac{0.8815138397...}{2},$$

a constant which appeared in [19] and turns out to be connected with class number theory [20, 21, 22].

Corresponding reciprocal sums are [23, 24]

$$\sum_{n=1}^{N} \frac{1}{\sigma(n)} \sim Y_1 \ln(N) + Y_1(\gamma + Y_2), \qquad \sum_{n=1}^{N} \frac{1}{\sigma^*(n)} \sim Y_3 \ln(N) + Y_3(\gamma + Y_4 - Y_5)$$

where

$$Y_1 = \prod_p f(p), \qquad Y_2 = \sum_p \frac{(p-1)^2 g(p)\ln(p)}{p\,f(p)},$$

$$Y_3 = \prod_p \left(1 - \frac{p}{p-1}\sum_{j=1}^{\infty}\frac{1}{p^j(p^j+1)}\right), \qquad Y_4 = \sum_p \left(\frac{p\,h(p)\ln(p)}{p-1}\sum_{j=1}^{\infty}\frac{j}{p^j(p^{j+1}+1)}\right),$$

$$Y_5 = \sum_p \left(\frac{h(p)\ln(p)}{p^2}\sum_{j=0}^{\infty}\frac{1}{p^j(p^{j+1}+1)}\right), \qquad f(p) = 1 - \frac{(p-1)^2}{p}\sum_{j=1}^{\infty}\frac{1}{(p^j-1)(p^{j+1}-1)},$$

$$g(p) = \sum_{j=1}^{\infty}\frac{j}{(p^j-1)(p^{j+1}-1)}, \qquad h(p) = 1 - \frac{p}{p-1}\sum_{j=1}^{\infty}\frac{1}{p^j(p^{j+1}+1)}.$$

No one seems to have examined $1/\tilde{\sigma}(n)$ or $1/\sigma'(n)$ yet.

0.3. Totient Function. If $\varphi(n)$ is the number of positive integers $k \le n$ satisfying $\gcd(k,n) = 1$, then [25, 26]

$$\sum_{n=1}^{N}\varphi(n) = \frac{3}{\pi^2}N^2 + O(N\ln(N))$$

as $N \to \infty$. Define $\gcd_*(k,n)$ to be the greatest divisor of k that is also a unitary divisor of n. Let $\varphi^*(n)$ be the number of positive integers $k \le n$ satisfying $\gcd_*(k,n) = 1$. Since $\gcd_*$ is never larger than gcd, it follows that φ^* is at least as large as φ. Also let $\tilde{\varphi}(n)$ be the number of positive square-free integers $k \le n$ satisfying $\gcd(k,n) = 1$. We have [15, 27]

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{n=1}^{N}\varphi^*(n) = \frac{1}{2}\alpha, \qquad \lim_{N\to\infty}\frac{1}{N^2}\sum_{n=1}^{N}\tilde{\varphi}(n) = \frac{3}{\pi^2}\alpha$$

where α is as defined earlier. The case for $\varphi'(n)$ remains open.

Corresponding reciprocal sums are [23, 24, 28]

$$\sum_{n=1}^{N}\frac{1}{\varphi(n)} \sim Z_1\ln(N) + Z_1(\gamma - Z_2), \qquad \sum_{n=1}^{N}\frac{1}{\varphi^*(n)} \sim Z_3\ln(N) + Z_3(\gamma - Z_4 + Z_5 + Z_6)$$

where

$$Z_1 = \frac{315\zeta(3)}{2\pi^4}, \qquad Z_2 = \sum_p \frac{\ln(p)}{p^2-p+1}, \qquad Z_3 = \prod_p u(p),$$

$$Z_4=\sum_p\left(\frac{(p-1)\ln(p)}{p\,u(p)}\sum_{j=1}^{\infty}\frac{j}{p^j(p^j-1)}\right),$$

$$Z_5=\sum_p\frac{\ln(p)}{p^2(p-1)u(p)},\qquad Z_6=\sum_p\frac{v(p)\ln(p)}{p^2\,u(p)},$$

$$u(p)=1+\frac{p-1}{p}\sum_{j=1}^{\infty}\frac{1}{p^j(p^j-1)},\qquad v(p)=\sum_{j=1}^{\infty}\frac{1}{p^j(p^{j+1}-1)}.$$

0.4. Square-Free Core Function. If $\tilde{\kappa}(n)$ is the maximal square-free divisor of n (also called [9] the square-free kernel of n), then [15, 17, 18, 29, 30, 31]

$$\sum_{n=1}^{N}\tilde{\kappa}(n)=\frac{\alpha}{2}N^2+O\left(N^{3/2}\right)$$

as $N\to\infty$, where α is as before. Assuming the Riemann hypothesis, the error term can be improved to $O(N^{7/5+\varepsilon})$ for any $\varepsilon>0$. If $\kappa'(n)$ is the maximal unitary square-free divisor of n, then [30, 31]

$$\sum_{n=1}^{N}\kappa'(n)=\frac{\beta}{2}N^2+O\left(N^{3/2}\right)$$

where

$$\beta=\prod_p\left(1-\frac{p^2+p-1}{p^3(p+1)}\right)=0.6496066993....$$

0.5. Infinitary Arithmetic. We continue refining the notion of divisibility [32, 33]. A divisor k of n is **biunitary** if the greatest common unitary divisor of k and n/k is 1, and **triunitary** if the greatest common biunitary divisor of k and n/k is 1. More generally, for any positive integer m, a divisor k of n is m**-ary** if the greatest common $(m-1)$**-ary** divisor of k and n/k is 1. We write $k|_m n$. Clearly $1|_m n$ and $n|_m n$.

When introducing infinitary divisors, it is best to start with prime powers. Let p be a prime, and let $x\geq 0$, $y\geq 1$ be integers. It can be proved that, for any $m\geq y-1$, $p^x|_m p^y$ if and only if $p^x|_{y-1}p^y$. Thus we define $p^x|_\infty p^y$ if $p^x|_{y-1}p^y$. For fixed y, the number of integers $0\leq x\leq y$ satisfying $p^x|_\infty p^y$ is $2^{b(y)}$, where $b(y)$ is the number of ones in the binary expansion of y. Define as well $1|_\infty 1$. The sum $\sum_{y=0}^{z-1}2^{b(y)}$ is approximately $z^{\ln(3)/\ln(2)}$ but is not well behaved asymptotically [34].

We now allow n to be arbitrary. A divisor k of n is **infinitary** if, for any prime p, the conditions $p^x||k$ and $p^y||n$ imply that $p^x|_\infty p^y$. We write $k|_\infty n$. Clearly $1|_\infty n$ and

$n|_\infty n$. Each $n > 1$ has a unique factorization as a product of distinct elements from the set

$$I = \left\{ p^{2^j} : p \text{ is prime and } j \geq 0 \right\};$$

each element of I in this product is called an I-component of n. It follows that $k|_\infty n$ if and only if every I-component of k is also an I-component of n.

Assume that $n = P_1 P_2 \cdots P_t$, where $P_1 < P_2 < \cdots < P_t$ are the I-components of n. The infinitary analogs of the functions d and σ are defined by [35, 36]

$$d_\infty(n) = 2^t, \qquad \sigma_\infty(n) = \prod_{i=1}^{t} (P_i + 1),$$

for $n > 1$; otherwise $d_\infty(1) = \sigma_\infty(1) = 1$. Two infinitary analogs of the function φ are known:

$$\varphi_\infty(n) = \text{ the number of positive integers } k \leq n \text{ satisfying } \gcd_\infty(k, n) = 1;$$

$$\hat{\varphi}_\infty(n) = \prod_{i=1}^{t} (P_i - 1) = n \prod_{i=1}^{t} \left(1 - \frac{1}{P_i}\right) \quad \text{for } n > 1, \qquad \hat{\varphi}_\infty(1) = 1.$$

It is generally untrue that $\varphi_\infty(n) = \hat{\varphi}_\infty(n)$. No similar extension of the function $\tilde{\kappa}$ is known. Cohen & Hagis [35, 37] proved that

$$\lim_{N \to \infty} \frac{1}{N^2} \sum_{n=1}^{N} \sigma_\infty(n) = \frac{A}{2} = 0.7307182421...,$$

$$\lim_{N \to \infty} \frac{1}{N^2} \sum_{n=1}^{N} \hat{\varphi}_\infty(n) = \frac{B}{2} = 0.3289358388...,$$

$$\frac{1}{N^2} \sum_{n=1}^{N} d_\infty(n) \sim CN \ln(N) + DN \sim 2(0.3666252769...)N \ln(N)$$

where

$$A = \prod_{P \in I} \left(1 + \frac{1}{P(P+1)}\right), \qquad B = \prod_{P \in I} \left(1 - \frac{1}{P(P+1)}\right), \qquad C = \prod_{P \in I} \left(1 - \frac{1}{(P+1)^2}\right)$$

but no such expression for D yet exists. It is known that $\varphi_\infty(n) = n^2/\sigma_\infty(n) + O(n^\varepsilon)$ for any $\varepsilon > 0$; reciprocal sums involving d_∞, σ_∞ and $\hat{\varphi}_\infty$ also remain open. Alternative generalizations of unitary divisor have been given [38, 39] but won't be discussed here.

0.6. Addendum. The constant X, which is associated with counting unitary square-free divisors of n, was evaluated in [40] to be 0.7483723334.... Numerical values of Y_i, Z_j remain open. An analog of $d'(n)$, corresponding to unitary cube-free divisors of n, can also be studied [8, 40].

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