$$P_5^{\pm}=\frac{384A}{\pi}\frac{r}{D}\cos(\theta+\phi)$$
$$\times\int_0^{2\pi}d\delta\left\{H_3^{\pm}\cos 2\delta+2\frac{r}{D}H_2^{\pm}\cos(\delta+\theta)\pm\frac{\rho}{D}H_2^{\pm}\cos(\delta+\phi)\right.$$
$$\left.+\left[\left(\frac{r}{D}\right)^2\cos 2\theta+\frac{1}{4}\left(\frac{\rho}{D}\right)^2\cos 2\phi\right]H_1^{\pm}\pm\frac{\rho}{D}\frac{r}{D}H_1^{\pm}\cos(\theta+\phi)\right\}, \tag{B14}$$

$$P_6^{\pm}=\frac{768A}{\pi}\frac{r}{D}\sin(\theta+\phi)$$
$$\times\int_0^{2\pi}d\delta\left\{\tfrac{1}{2}H_3^{\pm}\sin 2\delta+\frac{r}{D}H_2^{\pm}\sin(\delta+\theta)\pm\frac{1}{2}\frac{\rho}{D}H_2^{\pm}\sin(\delta+\phi)\right.$$
$$\left.+\left[\frac{1}{2}\left(\frac{r}{D}\right)^2\sin 2\theta+\frac{1}{8}\left(\frac{\rho}{D}\right)^2\sin 2\phi\right]H_1^{\pm}\pm\frac{1}{2}\frac{\rho}{D}\frac{r}{D}H_1^{\pm}\sin(\theta+\phi)\right\}, \tag{B15}$$

where

$$A=6.88\left(\frac{D}{r_0}\right)^{5/3}\frac{\rho}{D}, \tag{B16}$$

and

$$H_n^{\pm}=\frac{(\eta^{\pm})^{8/3+n}}{8/3+n}, \quad n=1,\ 2,\ 3. \tag{B17}$$

In Eq. (B17) the quantity $\eta^{\pm}$ has been normalized by the lens diameter.

*This work was supported by General Dynamics Research funds.

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Bandwidth specification for adaptive optics systems*

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A simplified expression for the bandwidth of an adaptive optics system is found to depend on a weighted path integral of the turbulence strength, where the weighting is transverse wind velocity to the 5/3 power. The wave-front corrector is conservatively assumed to match the phase perfectly, at least spatially, if not temporally. For the case of astronomical imaging from a mountaintop observatory, the necessary bandwidth is found to be less than 200 Hz.

In an earlier paper,[1] the power spectra describing corrector motion were considered as necessary elements of a complete servo system design for an adaptive optics system. However, for a preliminary design, a more simplified handbook-type formula for bandwidth is more desirable. This paper presents such a formula for the servo cutoff frequency without making any significantly limiting assumptions. Basically, the result depends on a weighted integral of the turbulence strength C_n^2, where the weighting is the transverse wind velocity to the $\frac{5}{3}$ power. Even that integral may be easily evaluated analytically when C_n^2 is a constant and the wind speed is composed of a constant plus a pseudowind due to slew. The result of this final simplification may be easily programmed on a hand-held scientific calculator.

Rather than investigate the way a wave-front corrector would respond to a phase aberration, we have considered the statistics of the phase itself, so the resultant bandwidth is conservative in that we assume the corrector

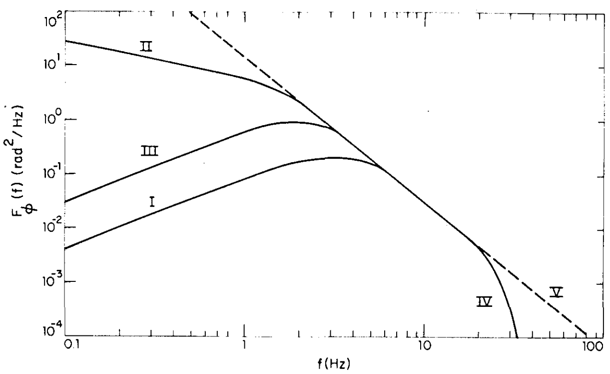

FIG. 1. Representative plots of the power spectra of segments within a phase corrector, for pistons located at the center (I) and at the edge (II and III). Curve II is for a gross piston reference and curve III for a gross tilt reference. The type of reference does not affect curve I. The high-frequency roll-off is determined by the size of the corrector, with IV representing a finite corrector segment $\frac{1}{10}$ the diameter of the aperture, and V representing a segment of size zero. The entire dashed line is the simple power spectrum given by Eq. (1).

perfectly matches the wave front, at least spatially, if not temporally. This avoids making any assumption on the nature of the corrector, and the corrector may be modal or zonal, segmented or continuous. To amplify the usefulness of the resultant formula, we carry out integrations for two cases: one representing a near-horizontal, moderately short range, and one consistent with astronomical observation.

In the more detailed analysis,[1] we considered a segmented corrector composed of any array of movable pistons which could also be tilted in order to form a least-squares fit to the wave front over the small circular region defined by the pistons. There was an option of referencing the phase at a point in the aperture to either the average phase across the aperture (gross piston reference) or the tilt across the aperture (gross tilt reference). Examples of such spectra are shown in Fig. 1 for pistons at the center and the edge of the aperture. These curves are diagrammatic in that they are not for any specific atmospheric conditions. The low frequencies in these curves are governed by the type of phase reference chosen, whereas the high frequencies are affected by the segment size. If we let the segment size go to zero, then all the curves have a common high-frequency asymptote given by a path integral of Eq. (72) in the earlier paper. The result, which applies to either plane or spherical waves, is

$$\lim_{f\to\infty} F_\phi(f) = 0.0326\, k^2 f^{-8/3} \int_0^L C_n^2(z)\, v^{5/3}(z)\, dz \quad , \tag{1}$$

where f is cyclic frequency, $k = 2\pi/\lambda$ is the wave number (λ is wavelength), L is the path length, $v(z)$ the wind speed transverse to the path, $C_n^2(z)$ the refractive-index structure parameter, and z is the incremental position along the path from $z = 0$ at the telescope (or receiver) to $z = L$ at the source. For astronomical seeing, the upper limit L is replaced by ∞.

The asymptote given by Eq. (1) is simply the spectrum of phase for Kolmogorov turbulence, where the phase is not referenced to either gross piston or gross tilt. The phase spectrum is also given by Tatarskii,[2] but only for the special case where $v(z)$ and $C_n^2(z)$ are independent of z. A tacit assumption in the derivation of Eq. (1) is that the frequencies are small compared with the characteristic frequency of amplitude scintillation, or $f \ll v/(\lambda L)^{1/2}$. Admittedly, v is actually a function of path position z, and L may be only a scale height for astronomical seeing; but rather than go into a more rigorous analysis, we note that if the inequality is reversed, such that $f \gg v/(\lambda L)^{1/2}$, then Eq. (1) is simply multiplied by $\frac{1}{2}$.

Bandwidths are determined by integrating $F_\phi(f)$ as filtered by a filter rejection response. Suppose the closed-loop servo response, the Fourier transform of the impulse response, is given by the complex function $H(f, f_c)$, where f_c represents a characteristic frequency such as a 3 dB point. The rejection response, in terms of power, is then $|1 - H(f, f_c)|^2$. Thus, the rejected, or uncorrected, power is

$$\sigma_r^2 = \int_0^\infty |1 - H(f, f_c)|^2 F_\phi(f)\, df \quad . \tag{2}$$

Typically, it is σ_r^2 which we will specify in order to determine f_c for a certain set of atmospheric conditions.

Since there are many types of servo closed-loop responses which might be implemented, we chose two extreme forms for $H(f, f_c)$ which should represent the range of possibilities. First, to represent a sharp cutoff, we used a binary filter given by

$$H(f, f_c) = \begin{cases} 1 & f \le f_c \\ 0 & f > f_c \end{cases} \tag{3}$$

Secondly, we chose an RC filter to represent a slow roll-off, and in fact, many adaptive optics servo systems have such a response in the neighborhood of the 3 dB point. (For higher frequencies, the actual response may drop off more rapidly than 20 dB/decade, but this will have little impact since the spectrum itself has a rather steep $f^{-8/3}$ dependence.) For the RC filter we have

$$H(f, f_c) = (1 + if/f_c)^{-1} \quad . \tag{4}$$

We will not concern ourselves with the phase lag associated with such a filter.

If we assume the cutoff frequency which will be eventually derived is to the right of the low-frequency breaks indicated in Fig. 1, then it is sufficient to use the asymptotic form of $F_\phi(f)$ in the integration of Eq. (2). This assumption may be alternatively stated as the rejected power σ_ϕ^2. From the parent paper,[1] we find that the aperture-averaged variance of phase referenced to gross tilt is

$$\sigma_\phi^2 = 0.141\, (D/r_0)^{5/3} \quad , \tag{5}$$

where D is the telescope diameter and r_0 is Fried's coherence length.[3] For convenience, we repeat here the definition of r_0 as

$$r_0^{-5/3} = 0.423\, k^2 \int_0^L C_n^2(z)\, Q(z)\, dz \quad , \tag{6}$$

where $Q(z)=1$ for plane waves, and $Q(z)=[(L-z)/L]^{5/3}$ for spherical waves. A typical value of σ_r might be 0.2π rad (or $\frac{1}{10}$ wave), which requires that the integrated turbulence strength be such that $D/r_0 \gg 0.74$. Fortunately, D/r_0 should be much greater than 0.74 to warrant the use of an adaptive optic in the first place. So we comfortably proceed with the integration of Eq. (2), using the asymptotic spectrum, Eq. (1). After integrating Eq. (2) with a binary filter, we invert the result to express f_c in terms of σ_r and find

$$f_c=[0.0196\ (k/\sigma_r)^2 \int_0^L C_n^2(z)\, v^{5/3}(z)\, dz]^{3/5} \quad . \tag{7}$$

For the RC-filter function, the constant 0.0196 becomes 0.102, and thus f_c is 2.70 times larger.

There are two special cases of interest. The first is consistent with many ground-based operations over near-horizontal ranges, and the second is that of astronomical observation. For the first case, we assume C_n^2 is a constant and the transverse wind speed is composed of a constant v_a plus a pseudowind ωz due to slewing at an angular rate ω. Then we find

$$f_c=\left\{7.34\times10^{-3}\left(\frac{k}{\sigma_r}\right)^2 C_n^2 \frac{v_a^{8/3}}{\omega}\left[\left(1+\frac{\omega L}{v_a}\right)^{8/3}-1\right]\right\}^{3/5} \tag{8}$$

for a binary filter, and f_c is 2.70 times higher for an RC filter. As an example, suppose $\sigma_r=0.2\pi$ rad, $\lambda=10.6$ μm, $C_n^2=10^{-13}$ m$^{-2/3}$, $v_a=4$ m/s, $\omega=0.01$ rad/s, and $L=2000$ m. For these conditions, we find f_c would be in the range of 31 Hz for a binary filter to 84 Hz for an RC filter.

For the astronomical case, the models for v and C_n^2 become more complicated. We have chosen to calculate f_c based on recently published data for an astronomical site. Miller, Zieske, and Hanson[4] report profiles of C_n^2 versus altitude for three nights at the ARPA Maui Optical Station (AMOS). Our model of their data is

$$C_N^2(z)=[2.2\times10^{-13}(z\sin\theta+10)^{-1.3}+4.3\times10^{-17}] \times\exp[-(z\sin\theta)/4000] \quad , \tag{9}$$

where θ is the elevation angle and the units of C_n^2 and z are m$^{-2/3}$ and m, respectively. For a wind velocity model, we averaged rawinsonde data[5] collected at Lihue (island of Kauai), Hawaii, for the years 1950–1970 and at Hilo, Hawaii, for the years 1950–1974. We modeled wind speed as a constant, to represent the lower altitudes, plus a Gaussian to represent the jet stream. The model, consisting of the mean wind speed plus one standard deviation, is

$$v(z)=8+30\exp\left\{-\left[\frac{(z\sin\theta-9400)}{4800}\right]^2\right\} \quad , \tag{10}$$

where v and z are in MKS units. Implicit in Eq. (10) is the knowledge that the site altitude corresponding to $z=0$ is 3048 m above MSL. We have taken the conservative assumption that the winds are entirely transverse to the path; however, if the wind is blowing predominantly downrange rather than cross-range, there would be an additional $\sin\theta$ multiplying all of Eq. (10).

For the conditions of turbulence and wind speed given in Eqs. (9) and (10), as well as for $\sigma_r=0.2\pi$ rad, $\lambda=0.5$ μm, and $\theta=90°$, the calculated cutoff frequencies are f_c (binary) = 28 Hz and $f_c(RC)=75$ Hz. For these same conditions, we may calculate r_0 and verify the assumption that $D/r_0 \gg 0.74$. Using Eq. (6) we find $r_0=0.13$ m, and thus $D \gg 0.1$ m, which is easily satisfied. This value of r_0 compares favorably with the median value of 0.10 m (at $\lambda=0.5$ μm) reported by Fried[3] for the U.S. Naval Observatory at Flagstaff, Arizona, and the Kitt Peak National Observatory.

These values of f_c and r_0 can easily be scaled to other elevation angles and wavelengths. The elevation angle scalings are $f_c \sim (\sin\theta)^{-3/5}$ assuming the winds are entirely cross-range, $f_c \sim (\sin\theta)^{2/5}$ when the winds are downrange, and $r_0 \sim (\sin\theta)^{3/5}$ independent of winds. The wavelength (or actually wave number) scalings are $f_c \sim k^{6/5}$ and $r_0 \sim k^{-6/5}$. We also suppose the reader may want to increase or decrease the entire C_n^2 profile, for which we point out that $f_c \sim (C_n^2)^{3/5}$ and $r_0 \sim (C_n^2)^{-3/5}$. Overall wind-speed scaling affects only f_c in that $f_c \sim v$. Let us now consider what may be a near-worst case, of $\theta=30°$ (winds cross-range) and a C_n^2 twice the values of the model. We shall not scale v since the model already consists of the mean plus one standard deviation. Also, at least for the visible wavelengths, $\lambda=0.5$ μm should suffice. For these near-worst-case conditions, the actual cutoff frequency may lie between f_c (binary) = 64 Hz and f_c $(RC)=172$ Hz.

In summary, we have provided simplified formulas for the bandwidth of the phase corrector and servo control of an adaptive optics system. The formulas should be used as good rules-of-thumb for perhaps all but the very final stages of the servo design. At that point, more precise power spectra[1] should be consulted. If we have used the more precise spectra with the high-frequency roll-off which results from having a finite actuator spacing, the calculated bandwidth would have been slightly lower. The specification derived in this paper is to be taken as a conservative estimate, based on an infinite number of corrector actuators. To apply our result, Eq. (7), requires knowledge of both wind speed and turbulence profiles on the optical path. To demonstrate the utility of the formula, we investigate two cases of interest: one essentially a horizontal path and one consistent with astronomical observation from a mountaintop. In both cases we found the bandwidths to be fairly low, less than 200 Hz, giving encouragement that the control system and corrector mirror need not be extremely complicated for many applications.

*This work was supported by the Advanced Research Projects Agency of the Department of Defense.

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Wave-front compensation error due to finite corrector-element size

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(Received 24 November 1975; revision received 23 August 1976)

A critical component of an active optical-compensation system is the formable mirror or other phase corrector. Because the corrector will have a finite number of controllable elements, it cannot perfectly correct a distorted wave front and the resulting error must be evaluated to model and design the system properly. This paper presents the full theory of the phase corrector in two common situations: one, where a particular known type of distortion such as focus or coma is to be corrected, and two, where the distortion is a random function of position, such as might arise from atmospheric turbulence. Results for a typical corrector are presented for both situations.

I. INTRODUCTION

Since 1973 there has been considerable effort to develop practical hardware that can compensate for phase errors in a wave front.[1–9] Such a system typically requires a telescope, a wave-front sensor, an actively deformable mirror or other phase corrector, and a set of electronics that converts the output from the wave-front sensor into control voltages for the active mirror. One of the main design questions of such a system is the number of independent actuators required to achieve a certain accuracy of fit from the active mirror. This number will depend on the response of the mirror as well as the set of wave fronts being corrected. The theory necessary to calculate this number is presented here, both for an ensemble of random wave fronts and a fixed type of distortion (i.e., focus or coma).

II. MODEL

It is assumed that the statistical structure function of the phase functions are known, and we calculate from these statistics the ensemble average error variance of the fit. If $\phi(\mathbf{x}, t)$ is the phase function being fit, then the structure function D is defined

$$D(\mathbf{x}-\mathbf{x}',\ t-t') = \langle[\phi(\mathbf{x}, t) - \phi(\mathbf{x}', t')]^2\rangle, \tag{1}$$

where the expectation value is taken over the ensemble of phase functions. This and one other expectation value are the only statistical information required for the calculation. The other information is

$$\langle\phi(\mathbf{x}, t)\rangle = 0. \tag{2}$$

The statistical assumptions in Eqs. (1) and (2) are implied whenever the ensemble of phase functions includes all translations and rotations of each phase function with equal probability. This assumption is valid in many applications such as turbulence compensation. Sometimes a particular distortion such as a focus or astigmatism correction is required instead of a more general ensemble. In these cases, Eqs. (1) and (2) may not be valid. Instead, the results of Sec. III may be applied directly for each degree of freedom and the resulting rms fitting error will be proportional to the magnitude of the corrections. The constant of proportionality can be calculated using Eqs. (7)–(9) and Eq. (11).

The active mirror or phase corrector is modeled by N discrete actuators, the jth of which causes a phase correction $R_j(\mathbf{x})$ for a unit applied signal. If $S_j(t)$ is the signal applied to the jth actuator at time t, then the total phase correction is

$$\phi_c(\mathbf{x}, t) = \sum_{j=1}^{N} S_j(t)\, R_j(\mathbf{x}). \tag{3}$$

The error is then

$$\epsilon(\mathbf{x}, t) = \phi(\mathbf{x}, t) - \sum_{j=1}^{N} S_j(t)\, R_j(\mathbf{x}), \tag{4}$$

and the ensemble average error variance E is

$$E = \langle\epsilon^2(x, t)\rangle = \frac{1}{A}\int dx \left\langle \left[\phi(\mathbf{x}, t) - \sum_{j=1}^{N} S_j(t)\, R_j(\mathbf{x})\right]^2 \right\rangle, \tag{5}$$

where A = area of the aperture.

The calculation is to choose $S_j(t)$ to give the minimum value of E and to find E as a function of the number of actuators and the response function, $R_j(\mathbf{x})$.

III. OPTIMUM $S_j(t)$

The first question addressed is how well the mirror can approximate a known function $\phi(\mathbf{x})$. The error variance is

$$e = \frac{1}{A}\int d\mathbf{x}\epsilon^2(\mathbf{x}) = \frac{1}{A}\int d\mathbf{x} \left[\phi(\mathbf{x}) - \sum_{j=1}^{N} S_j R_j(\mathbf{x})\right]^2. \tag{6}$$

The problem is to choose S_j to minimize e. The solution is standard variational analysis with the result most easily presented by defining some new quantities.

Let