translated by Alessandro Rosa - *released* on May 15th, 2001

Memoir on iterations of rational functions [1]

Mémoire sur l'itération des fonctions rationnelles [2]

written by **Gaston Maurice Julia**

Translated in English
by
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([1]) Memoir awarded by the academy of Sciences of Paris: Grand Prix of Sciences Mathematiques, 1918.
([2]) Appearing in the *Journal de Mathématiques pures et appliquées* – 4th tome, 1918 (83th volume of the collection), published by Gauthier-Villars editor.

To Maria Elena,
thanks for your patience
owing to the time I stole you
before meeting in the evening ...

To Felice,
thanks for your technical support,
which enlightened my sight
during last review ...

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Summary

PROLOGUE

Almost all studies of fractals, coming out from iterations of rational functions in the complex domain, are based upon this fundamental memoir by Gaston Maurice Julia and upon some very important articles by Pierre Fatou. Both authors opened the entrance door to a path leading to the deeper knowledge of the behaviour of iterated rational functions in the complex domain.
As this article came out, it was claimed by contemporary mathematicians and so it allowed its author Gaston Maurice Julia, at the age of only twenty-five, to gain a very wide popularity in the academic world.
But the pour graphics possibilities, offered by the technology of that times, were not enough for this kind of studies requiring huge instruments of calculus like today computers.
Then, due to a lack of both proper and detailed methods of visualization, the interest faded away after some years of fame; we like thinking that in those times only few minds, headed by Julia himself, were conscious and able to imagine and to "see" what new was really happening over the complex domain, since the pour following of mathematicians.
Mortals should still wait a long time for gaining the necessary powers to unveil those worlds to their sight and to reach to the same high levels of abstraction where those first masters had already risen to.

So this article sank into oblivion and it was forgotten for more than 50 years until Benoit Mandelbrot brought it again to light by displaying the contents of these studies with the proper support of computer graphics.
Mandelbrot was the right man at the right time: probably he is the most famous representative of the strong interest about discrete (and continuous ... but this is another story) dynamical systems in these last 30 years.

He set the math world afire so that the enthusiasm for complex dynamical systems arose again. But, this time, it seemed to stay any longer since it could be (and still it is) excited by the fast and detailed support of modern computer graphics.
The current wide literature about both continuous and discrete dynamical systems witnesses the great squandered interest by today mathematicians.

This translation is like a tribute to the original article, written in French.
Obviously, the original expressions often looked like being someway influenced by the style of language of those times; and so the technical terms.
The translation of the whole article has been done in respect of a modern style of writing (but paying always attention to preserve original meanings) and of using current terms in topology, complex analysis and discrete dynamical systems.
As usual as any translation, the main (I suppose) purpose is to spread the subject around a wider range of people than just only those dealing with the original language that, in this case, it's French.
However, I'm conscious that some mistakes may always occur.
So, I appeal to the care of the reader about supporting further improvements of this translation: corrections, constructive suggestions, opinions or comments will be really appreciated in order " to make it work better " so that both experts and beginners may enjoy this article at best.

Hope you join us !

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INTRODUCTION

The Memoir, I submit to the judgement of the Academy, is dedicated to the study of the iteration of a rational fraction, $z_1 = \varphi(z)$, in the entire complex plane. At present this topic has not been treated, except by a local study, anywhere else but on two articles written by Fatou and appearing in the *Comptes Rendus de l' Académie de Sciences* (i.e.: *Report of the Science Academy*), published respectively on October 15th 1906 and on May 21th 1917.

Fatou showed strong interesting results but he included examples using *special* properties of the examined functions. These examples are the simplest ones that can be imagined. In this Memoir, my target is to consider any fraction $\varphi(z)$ and then to state the independent properties from the particularities of $\varphi(z)$.

I decide to descend from the general to the particular case; and I only give the examples showing the accomplishment of some possibilities that a stingy of *a priori* hypotheses and rigorous analysis has shown to me.

The Academy will estimate me if I succeed.

From my view point, this is the most qualitative study that I took up.

I try to discern what are all the circumstances that may arise. It will be shown, further in this article, that a lot of them have been ignored so far. I am not sure I have completely exhausted them all due to lack of time and knowledge too.

First of all, it is necessary to show by examples that the possibilities, revealed by analysis, are effective. These examined examples are never artful but they have always been chosen to clear up an idea. This is not an unfruitful study as, I hope, the examples, included in the 2nd, 3rd and 4th part of this Memoir, will show. It would take too long and my time is short. The rest of the subject would be treated in future.

I will indicate it whenever I will be able to do it but without being too verbose.

The main question, dominating this study, is the following: *given a point* z, *so that its images* $z_1, z_2, \dots, z_n, \dots$, *generate a set* E; *what are the properties of the set* E′, *derived from the set* E ?

The local study treats the cases where E′ consists of one point or in a finite number of points (a limit point with uniform convergence and a periodic cycle). More generally, let ζ be either a point of E′ and a limit point of points of E; I searched for what is the domain set where z may range in so that ζ persists in being an *analytic function of* z, without essential singularities.

The known examples showed that ζ may assume a constant value in a region of the z plane and different constant values in other distinct regions.

It is necessary to see what are the delimiting sets: that's to say *the set of points so that* ζ *stops being an analytic function of* z *as soon as* z *is out of this set*. It is necessary to

look for *essential singularities* of ζ which is considered as a function of *z*. This is the target of the 1st part of this Memoir.
Fatou had shown, with some examples (October 15th 1906) that the set of singularities could be *perfect discontinuous* or a *continuous line*. *In any cases*, *this set includes all the points satisfying* $z = \varphi_p(z)$ *and* $|\varphi'_p(z)| > 1$.
I decided to study, *a priori* and in the general case, the set E of the *z* points $z = \varphi_p(z)$, where $p = 1, 2, \ldots, \infty$ and $|\varphi'_p(z)| > 1$. It is a countable set.
Montel's studies, on *normal families*, gave immediately simple properties for all points P of E. *Inside a circle region*, *centered at* P, *the functions* $\varphi(z)$, $\varphi_2(z)$, *…,* $\varphi_n(z)$ [*an undefined number of iterations of* $\varphi(z)$] *assume*, *starting from a certain rank*, *all complex values*, *except two at most; the cases*, *where these exceptional values are one or two*, *are clearly defined*.
The comparison of this result with the classic Picard's theorem, on the essential singular points, induced me to think that the set E, together with its derived set E′, generates the set of the singularities that I was looking for.
I recognized that E′ surrounds E and that E′ *is a perfect set*: moreover, all points of E′ enjoy the property that I succeeded to find out for the points of E; and this property *characterizes* the E′ points. I studied *the structure of* E′ *without any* restrictive *a priori* hypotheses.
E′ could be either discontinuous, either a continuous line or a continuous area.
I easily showed the two first cases by some examples; for the third one, I showed that E′ cannot include a two dimensions plane area without including the entire plane. After comparing some common elements between the current question and the proper or improper discontinuity of the *automorphism groups* in the plane, I was induced to think that this last eventuality (i.e.: E′ is a continuous area) may occur, but I did not succeed in showing it with an example; these analogies prove at least that it is impossible to reject the case, where E′ includes the entire plane, without a severe demonstration. I did not succeed in giving it. So that I'm in doubt upon this point. I have given criteria allowing, in practice, to recognize the ordinary cases where E′ is a line or it is discontinuous; and I showed how *the structure of* E *is the same in all its parts*.
Especially, I also showed, by examples already given by Fatou, how the knowledge of particular properties of $\varphi(z)$ could be used to study the structure of the set E′.
Then, while treating the case where E′ does not include all the plane, I will show, and this is fundamentally important, that, in every region D not including any point of E′, the sequence of $\varphi_i(z)$ is normal, and *every limit point* ζ *of the images of z is an analytic complex function in* D.
Therefore the result, I was looking for, was finally obtained.

Every limit point ζ, of images of z, is an analytic complex function in z, so that z does not meet again any point of the set E′. *Therefore the set* E′ *is the set of essential singularities for the limit function of the sub-sequence of functions*, *extracted from the sequence of* $\varphi_i(z)$.

This remark agrees with the Picard's theorem (that I demonstrated) concerning the exceptional values of the sequence of $\varphi_i(z)$ around every point of the set E′. This question is theoretically solved: if one considers in the plane the different regions bounded by E′ (there are not any if E′ is everywhere discontinuous) and if one knows the behaviour of the sequence of $\varphi_i(z)$ in *an arbitrarily small part* of each one of those regions, then one will know the behaviour in each one of the regions *considered in all their extensions*: the problem refers to the *analytic continuation*, since every limit function of functions $\varphi_i(z)$ is analytic for all extension of the considered region.

This is the core of the 1st part of my Memoir. It is based upon a completely general order, as one has seen. But there is a *theoretic* solution and one knows that solutions are never satisfactory. Poincaré used to say very elegantly:

> «*We have no completely solved or unsolved problems; there are only problems which are more or less solved, etc.*»

The set E′ is theoretically well defined with the help of E, whose construction, made point by point, already requires the solution of a countable infinity of algebraic equations; it is necessary to detect clearly the geometric properties of E.

This is not completely possible, except for extremely special cases like the fractions in the *fundamental circle*. In most of the cases, it is necessary to use lower precision.

The same happens in the study of *all limit functions* of the sequence of $\varphi_i(z)$ in one of the regions delimiting the set E′.

I could not do anything else but starting from a local study, based upon the discoveries of my predecessors but with more additional hypotheses. In fact, in the 2nd part of this Memoir, I have applied general results, obtained in the 1st part, to the study of the *uniform or periodic convergence to a point or to a periodic cycle* [when $\zeta = \varphi(\zeta)$, $|\varphi'(\zeta)| < 1$ or the groups $\zeta_1, \zeta_2, \ldots, \zeta_{p-1}$, defined by $\zeta = \varphi_p(\zeta), \zeta_i = \varphi_p(\zeta_{i-1})$, $|\varphi'_p(\zeta)| < 1$].

Let us suppose the existence of such a point or of such a group (it is an additional hypothesis), so that one can easily generate fractions $\varphi(z)$ where all roots of $z = \varphi(z)$ retrieve $|\varphi'(z)| > 1$.

I did not succeed either in generating a fraction where all roots of $z = \varphi_p(z)$ retrieve $|\varphi'_p(z)| > 1$ (whatever p is) or in showing that, for any fraction $\varphi(z)$, there is necessarily a number p and one root of the equation retrieving $|\varphi'_p(z)| < 1$.

It is easy, by taking for example a *singular* fraction in the fundamental circle (wherein the only limit point for the entire plane, except E′, is a point of E′), to verify that all

roots of $z = \varphi_p(z)$ satisfy $|\varphi'_p(z)| \geq 1$, whatever p is (but, in the cited case, there's a root satisfying $|\varphi'(z)| = 1$). But, if we admit the existence of a limit point or of a periodic cycle (both of them are easy to be recognized in every special case), then one may search for the *locus* of the domain of convergence to the point or to the cycle, that's to say the set of points whose images converge *uniformly* to the limit point or, *periodically*, to the points of the periodic cycle. One can show that the limit point, or all points of the periodic cycle are included in a *connected region* R, bounded by the set E′ (1) (I mean that all its boundary points are points of E′) wherein every interior point (of R) converges to the limit point or, periodically, to the points of the periodic cycle.
R is the *immediate domain of convergence to the limit point* ; for periodic cycles, the set R of the p domains, related to p points respectively, is the *domain of convergence to that cycle*.
The great result is that every immediate domain of convergence to a point or to a periodic cycle *includes a critical point of the function* $\psi(z)$, *inverse function of* $\varphi(z)$; this critical point is the image of a point satisfying $\varphi'(z) = 0$ and it is included in the considered immediate domain.
But the *total domain of convergence*, to a point or to a periodic cycle, can be bigger than the *immediate domain*: it may consist of an *infinite number of disjoint regions, separated by* E′: *all these regions are bounded by points of* E′.
All these regions are preimages of the immediate domain; E′ is the boundary of the *total domain of convergence* to a point or to a periodic cycle.
It is easy, while studying the connectedness of the immediate domain, to prove that it may be simply connected or it may have an infinite order of connection.
Among other consequences, one argues that, when *the number of limit points is greater than 2* (counting every point of a periodic cycle), *at most one of these points has a total connected domain*, *which is the same as its immediate domain*, *and that all other points have a total domain consisting of an infinite number of regions*. The consideration of the Riemann surface R of $\varphi(z)$ is here very useful. It allows to realize the natural reasons of all these particularities. It also explains the simplicity of the results, given by Fatou in two Notes of 1906 and one of May 21th 1917.
Simple and very general examples with enough variety are examined.
Then examples of new circumstances are given:

A. The total domain has an infinite number of pieces:

1° $z_1 = \frac{-z^3 + 3z}{2}$: for this one, both the grouping of these domains and the distribution of E′ in the plane have been studied;

2° Examples of Newton's method.

(1) Here the set E′ can be only a continuous line or it can be discontinuous.

B. For the connectedness of immediate domains, the following example:

$$z_1 = A\left[\frac{z^5}{5} - 2a\frac{z^4}{4} + a^2\frac{z^3}{3}\right]$$

has been examined where, for a big enough A, the domain R_∞, with an infinite order of connectedness, is bounded by an infinity of continuous distinct lines which are exterior two by two.

C. $z_1 = z^2 - 2$: it proves that E′ can be an unclosed continuous line: the segment (-2,+2) of the real axis, for example.

It also proves that a critical point of $\psi(z)$, as well as a point satisfying $\varphi'(z) = 0$, could belong to points of E′.

In the 3rd part, using the results coming from the 2nd part, I engaged myself to clarify the nature of E′, after recognizing that it is a continuous line.

Fatou, in 1906, with the help of considerations about the functional equations, gave the following example:

$$z_1 = \frac{z + z^2}{2}$$

where E′ is not analytic; but it is necessary to have a more geometrical proof, independent from the functional equations, and to examine why E′ is not analytic.

I gave very general examples where, with the help of hypotheses (read *inequalities*) for $\varphi(z)$, I succeeded in proving directly that E′ is a *continuous closed and simple Jordan curve*. In particular, for the example:

$$z_1 = \frac{z + z^2}{2}$$

On this Jordan curve, the points of E are everywhere dense and, in general, *points where the curve itself has not any tangent line*.

[it is enough to prove that the absence of the tangent line in a point $z = \varphi_p(z)$, where $\varphi_p'(z)$ is not real; this is the general case.]

In other examples, in particular for:

$$z_1 = \frac{(-z^3 + 3z)}{2}$$

I show that E′ is a *continuous* closed curve Γ, described by

$$x = f(t)\ ,\ y = g(t)\ ,$$

where f and g are continuous in the interval $a \le t \le b$; the curve has double and everywhere dense points on itself and it consists of an infinity of curves O([1]), where each one of them *is a simple and closed Jordan curve*; every point of a curve O is the limit point of the external curves O, in respect of the firstly considered curve; the dimensions of these curves converge to zero.

([1]) and by the limit points of the curves O.

In the 2nd part, I gave a schema allowing the representation of such curve Γ.
Finally, in the 4th part, I completely studied the singular convergence to a singular limit point.

$$\zeta = \varphi(\zeta) \qquad \text{where } \varphi'(\zeta) = e^{i\theta},$$

or to an singular group, coming out from

$$\zeta = \varphi_p(\zeta) \qquad \text{where } \varphi'_p(\zeta) = e^{i\theta},$$

where θ is *real* and *commensurable* to 2π.
Such a point ζ always belongs to E′, but it is the limit of the images of the points belonging to a domain R, confining with ζ and bounded by E′.
R is called the *immediate domain of convergence* to ζ (the same happens for a singular periodic cycle). I extend the concept of the *total domain* and I give examples where this total domain consists of an *infinite number of areas*, so that E′ is a continuous curve with everywhere dense double points on the curve itself ($z_1 = z + z^3$), like in the example:

$$z_1 = \frac{(-z^3 + 3z)}{2}$$

Again, the immediate domain always includes a critical point of ψ(*z*).
I was less lucky in the study of points

$$\zeta = \varphi(\zeta) \qquad \text{where } \varphi'(\zeta) = e^{i\theta},$$

when θ is *incommensurable* with 2π: I succeeded only in reducing the number of possibilities to only two cases so that, in a given example, they are mutually excluded.

1° ζ *does not belong to* E′ ; then, there is a *centre*, in the interior of a region of the plane R ([1]), bounded by E′, traversed by analytic curves surrounding ζ and preserved by the relation $z_1 = \varphi(z)$.
On every curve, the images of a point of this curve are everywhere dense.
Then the Schrœder's equation:

$$F[\varphi(z)] = e^{i\theta} F(z)$$

has an holomorphic solution in all R. ζ *is not a limit point for the images of any point of the plane*.

2° ζ *belongs to* E′ and one can realize that there is a limit point for the images of a *critical point* of the branch of ψ(*z*) which, at the point ζ, is equal to the point ζ. I did not achieve to find out an example of a rational function φ(*z*): neither for 1°, or for 2°.

([1]) Neither R and neither some of the preimage regions of R shall include the critical point of ψ(*z*).

Therefore I am again still in doubt. If the first possibility could be verified, then one would find out an example where the sequence of $\varphi_i(z)$ admits a non constant limit function in R; for example, z would be the limit of functions $\varphi_{n_1}(z), \varphi_{n_2}(z), \ldots$, with properly chosen indexes and in a such way so that $e^{n_1 i\theta}, e^{n_2 i\theta}, \ldots$ converge to 1.

In R, one would get a development of z with a sequence of rational fractions converging uniformly in all the interior of R.

There are known results in functions theory: they are so interesting that they are mentioned here again. Other questions, as we will see it, have been left to a further treating. I have also applied here my results to functional equations which are very known since the works of Kœnigs and its successors.

The target of this Memoir is the study of iterations: maybe in future I will come back to pending questions.