translated by Alessandro Rosa - released on May 15th, 2001

PRELIMINARIES

I will need, in this Memoir, some unusual results from the functions theory: most of them are, at the present, disseminated in various papers or mathematical reviews. Therefore, in these preliminaries, I will often recall these results by referring to original works for their proofs.

Sometimes and especially in the sections 4 and 5, treating some questions about conformal mappings, I will insert some modifications or additions to known results so that they will be further more useful to me.

It is an easier way for lessening the exposition of this Memoir and for following the development of the discussed question: iterations of the rational fractions.

1. *Background from the general theory of iterations* (read KŒNIGS, *Annales of the Ėcole Normale Supérieure*, 1884, Supplement n° 2 and n° 24,25).

1° - If $\varphi(z)$ is an complex analytic function in a region R of the complex plane that includes a root ζ of the equation:

$$\varphi(\zeta) - \zeta = 0$$

satisfying $|\varphi'(\zeta)| < 1$, then one can surround ζ with a circle C_ζ, centered at ζ and with small radius ρ, so that if $|z-\zeta| < \rho$, then one will get:

$$|\varphi(z) - \zeta| < H\rho,$$

where H is positive and it ranges from 0 to 1. ζ is defined *limit point with uniform convergence*, since the sequence of images z_i of any interior point z of C

$$[z_1 = \varphi(z), \ldots, z_i = \varphi(z_{i-1}), \ldots]$$

converges to ζ.

2° Let us suppose that in the region R, where $\varphi(z)$ is analytic, one finds a system of n distinct points $\zeta, \zeta_2, \zeta_3, \ldots, \zeta_{n-1}$, generating a *periodic cycle*; that's to say

$$\zeta_1 = \varphi(\zeta), \quad \zeta_2 = \varphi(\zeta_1) = \varphi_2(\zeta), \quad \ldots, \quad \zeta_{n-1} = \varphi(\zeta_{n-2}) = \varphi_{n-1}(\zeta),$$
$$\zeta = \varphi(\zeta_{n-1}) = \varphi_n(\zeta) \quad \text{with} \quad |\varphi'(\zeta)|, |\varphi'(\zeta_1)|, \ldots, |\varphi'(\zeta_{n-1})| < 1. \quad (^1)$$

The n points can be permuted circularly by $[z,\varphi(z)]$.

One can prove that every point ζ_i of the cycle is the centre of a circle C_i, whose interior points z verify the following relation:

$$\left|\frac{\varphi_n(z) - \zeta_i}{z - \zeta_i}\right| < H_i < 1,$$

where H_i ranges from 0 to 1.

One may also surround each of the ζ_i with a circle C'_i, small enough and in the interior of C_i, so that if z belongs to C'_i, then its images $z_1, z_2, \ldots, z_{n-1}$ are respectively inside

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C_{i+1}, C_{i+2}, … C_{i-1} circularly and starting from C_{i+1}. If one starts from a point z, inside C'_i, then z_n still lies in C'_i at a distance from ζ_i that is shorter than the distance of z from ζ_i:

$$| z_n - \zeta_i | < H_i | z - \zeta_i | ;$$

likewise the points z_{n+1}, z_{n+2}, … fall as z_1, z_2, … in the circles C_{i+1}, C_{i+2}, …, so that they are nearer to ζ_{i+1}, ζ_{i+2}, … than z_1, z_2, … .

ζ_i is the limit point of the sequence z, z_n, z_{2n}, …; ζ_{i+1} is the limit point of the sequence z_{2n+1}, z_{2n+2}, … etc.

Therefore the sequence z, z_1, z_2 ... converges periodically to each one of the points ζ_i of the periodic cycle.

2. *The normal families of analytic functions*. - This notion was introduced by Montel in his thesis and in different Memoirs; two of them appeared in the *Annales de l'Ecole Normale supérieure*, in 1912 and 1916.

This notion revealed to be prolific for several questions in the functions theory and it has been treated by other authors.

In the 1st part of this Memoir, one will realize that this notion has been very useful to me. Montel says that a family of analytic functions is *normal* in a domain D, if one can extract an uniformly convergent subsequence on the interior of D from all the infinite sequences of functions of this family.

He showed (§ **2** to **6**, in his Memoir of 1916) that a *normal* family (F), of functions $f(z)$ bounded in a point, enjoys the following properties:

1. The functions are bounded in the interior of D.
2. The functions are continuous in the interior of D.
3. The functions $f(z) - a$ have, for each value of a, a finite number of zeros in the interior of D, if a is not a limit function.

A family of analytic bounded functions in the interior of a domain D generates a *normal* family.

A family of holomorphic functions in the unit disk is a normal family if it does not assume either the value 0, either the value 1 in the interior of that disk [§ **10**].

Both the exceptional values 0 and 1 can be replaced by any other two finite distinct values. In Chapter IV of his Memoir of 1916, Montel examines the normal families of meromorphic functions, which are supremely interesting for us.

([1]) One puts usually

$$\varphi[\varphi(z)] = \varphi_2(z),\ \varphi_2[\varphi(z)] = \varphi[\varphi_2(z)] = \varphi_3(z),\ \ldots,$$

and one gets

$$\varphi'_n(z) = \varphi'(z) \times \varphi'(z_1) , \ldots , \times \varphi'(z_{n-1})$$

since z_1, … , z_{n-1} are the iterated values, respectively of rank 1, 2, … , (n-1) of z.

A family (F) of meromorphic functions $f(z)$ is said *normal* in a domain D, if one may extract a uniformly convergent sequence to a limit function in the open domain D from all the infinite sequences of functions belonging to that family.
This limit function is a meromorphic function that may, in certain cases, assume an infinite or a finite constant value (§ **27** and **28** of the cited Memoir, where the uniform convergence is well defined).
In section **31**, one will find the fundamental theorem: *the meromorphic functions* $f(z)$ *in the domain* D *generate a normal family if they never assume three distinct values a, b, c in* D.
In sections **33** and **34,** one sees that, if the meromorphic functions of a normal family are bounded at a point P of a domain D, then there is a circle (c), centered at P, so that none of the functions of the family has a pole in the interior of (c).
On the other hand, if D' is a domain completely inside D and including P, then the number of poles of each function, included in D', is less than a fixed number.
Finally, I recall that in Chapter III of the Memoir of 1912, one may find some interesting suggestions about the convergence of a sequence of holomorphic functions: all of them are deduced from the fundamental theorem (§ **30**).
Consider an infinite sequence of holomorphic functions in the domain D:

$$f_1(z),\ f_2(z),\ \ldots,\ f_n(z),\ \ldots,$$

belonging to a normal family in this domain.

1. If the sequence converges in an infinite set of points in the interior of the domain D ([1]), then the sequence converges in the whole domain;
2. If the sequence converges in D, then the convergence is uniform in the interior of D.

3. *Background from the set theory*. - One will find some notions in the article (on the sets of points) included in the *Encyclopédie francaise des Science mathèmatiques*, written by Zoretti (t. II, vol.1, fasc.2) under this survey: *Structure des ensembles fermés*, (*Structure of closed sets* - § **10**, **11**, **12**, **15**, **14**).
It quotes concepts about a *point inside a set*, a ε - *chained set between two of its points or a set consisting of one piece between the same two points*, open domains (domains W), Jordan curves, etc.
A point is said to be in the interior of a set if all points of a circle (with a sufficiently small radius and centered at P) belong to the set of P. The boundary points are not interior points.

([1]) That's to say, having at least a limit point in D.

A domain (domain W) is said to be open if:

1. Every point of the set is in the *interior* of the set ;
2. Any arbitrary pair of points of the set can be always linked by a broken line whose any point still belongs to the same set.

A set is said to be ε - *chained or consisting of one piece between two of its points* if, given a value $\varepsilon > 0$, one can find out a sequence of points of the set beginning and finishing in *two given points*, so that the distance from each point to the following point is inferior than ε.
A simple Jordan line is the locus of points defined by equations so that

$$x = f(t)\ ,\quad y = g(t) \qquad\qquad (0 \le t \le 1)$$

where the following equations are never satisfied simultaneously

$$f(a) = f(b)\quad ,\quad g(a) = g(b)$$

except when $a = b$.
This is a continuous line with no double points.
Then, the line is closed when the following equalities:

$$f(a) = f(b)\ , g(a) = g(b)$$

are simultaneously satisfied for $a = 0$, $b = 1$ and for these values only.

The fundamental result, explained by Jordan, is that a closed line C splits the plane in two regions: the first one is said to be the *interior*, the second one is said to be the *exterior* region of C; one cannot trace a simple Jordan line joining an exterior point to an interior point without intersecting the curve C, at least in one point.

4. *The conformal map of simply connected domains in the interior of a circle*.
Various suggestions about the conformal maps. - In the functions theory, one has often to deal with some domains corresponding to themselves in the way of the Riemann surfaces.

In the Memoir *Sur l'uniformisation des functions analytiques* (*On conformation of analytic functions*)(Acta, t. XXXI), Poincaré defines them very accurately and progressively using the method of analytic continuation (to see § **2**, definition of domains D).

He defines that a closed line is a continuous line, traced in a domain; this closed line is the *boundary* of a continuous region whose all points belong to a domain.

A domain is said to be *simply connected* if every closed curve, traced on this domain, is the complete boundary of an continuous region whose all points belong to the domain.

In this Memoir, Poincaré establishes that one can make a conformal map of the *interior* of a domain either on the *interior* of a circle, either on the entire plane without the point at infinity, either on the *extended plane* with the point at infinity ([1]).

In a Memoir of the LXXII Volume of *Math. Annalen*, Carathéodory resumes the case where the domain is conformally mapped in the interior of a circle.

If a function $f(z)$ is holomorphic *in* and *on* a circle C, then the point $z_1 = f(z)$ describes (as the z point describes the interior of C) the interior of a simply connected domain D, bounded by only one analytic border; D belongs to the type mentioned above: it may correspond to itself, in general. The function $z_1 = f(z)$ establishes a conformal correspondence between the interior of C and the interior of the domain D.

In particular, if the domain D is bounded and it does not correspond to itself, then it is defined for $i \to \infty$ as the limit of a sequence of domains D_1, D_2, …, not corresponding to themselves; each domain is bounded by an analytic closed curve C_i, so that D_i includes D_{i-1} in its interior.

If $f_i(z)$ is the function establishing a conformal correspondence from the interior of the circle $| z | < 1$ with the interior of the domain D_i [with the condition that $f_1(0) = f_2(0) =$ … and $f'_i(0)$ retrieves a real value], then the sequence of the $f_i(z)$ converges uniformly to a limit function $f(z)$, in the interior of $| z | < 1$; $f(z)$ is analytic in $| z | < 1$ and it conformally maps the interior $| z | < 1$ in the interior of D. While following the series, one could detect some domains D including the point at infinity when $f(z)$ has some poles in the circle of mapping, since it is a rational fraction.

In this case one will move to the Riemann sphere for supporting both the variable z and the different domains: this is a familiar concept for geometry researchers.

In two remarkable Memoirs of the LXXIII Volume of *Math. Annalen*, Carathéodory uniformly maps the interior of a simply connected bounded domain G, which does not correspond to itself, in the interior of the circle $| z | < 1$; he studied the correspondence between the boundary points of $| z | = 1$ and the boundary points of the domain G.

This study was resumed more simply by Lindelof ([2]).

The border G is a continuous line ([3]), that's to say an ε - *chained* set between an arbitrary pair of its points.

A boundary point P of G is said to be *accessible* from the interior of G if one can find a simple Jordan line where all points but P are interiors in G and so that it links P to a any point inside G; this line may consist of finite number of segments of straight lines or of an infinity of segments having P as limit point.

The accessible points of the boundary are *everywhere dense on this boundary*.

([1]) If the domain has, al least, two distinct boundary points, then the map is made on a circle.

([2]) LINDELÖF, *Sur un principe general de l' Analyse et ses applications à la théorie de la representation conforme*. Helsingfors, 1915.

([3]) Read SCHÖNFLIES, *Die Lehre von den Punktmannigbaltigkeiten*, 2° Partie, p. 115.

An *accessible* point P on the boundary is said to be *simple* on the same boundary when, whatever is the simple Jordan closed line L (starting from P) whose points are *interior* to G (except P), the *interior region in* L includes only interior points of G and not boundary points of G.
This definition, less general than the one given by Carathéodory (§ **44** of the LXXIII Volume), diverges from this last one when it is applied to *accessible points* (read for examples, LINDELÖF, § 14 of the cited Memoir, Helsingfors, 1915).
A boundary, where all points are both accessible and simple, is a *simple Jordan closed curve* (read CARATHEODORY, § **48** of the Volume LXXIII, or LINDELÖF, § 11,12,13,14 in the Memoir of 1915).
Carathéodory established that one can make a conformal map of the interior of a simple Jordan closed curve in the interior of the circle $|z| < 1$; the established correspondence, between the points belonging to the interior of the two curves, extends to boundary points.
There is a biunivocal continuous correspondence between the points of a Jordan curve and the points of the circle $|z| = 1$; if an interior point, of the disk $|z| < 1$, converges to a point of the circle ($|z| = 1$), then its image converges to a determined point of the Jordan curve.
This correspondence is mutual.
There are a lot of possible works to be made on the Jordan closed curves.
It is known that this curve splits the plane in two regions so that every point of this curve is *accessible* from the both interior and exterior region.
This condition is mutual.

Every continuous line is a common boundary and it is a simple Jordan closed curve so that it splits the plane in two domains without any common points and so that every point of the continuum is accessible from the interior of each one of the two domains (to see SCHŒNFLIES, loc. cit., § 11 and 12 of the Chapter V).

5. *Some lemmas of the function theory*.
Let a simple area D of the analytic plane be bounded by a simple curve C (a circle, for example) and suppose that a function $f(z)$ is analytic in D and on C.

When z describes the area D, the point $z_1 = f(z)$ describes an area D_1 which is simply connected and bounded by an analytic closed curve C_1, the transformed from C.
In general, D_1 will correspond to itself in the same and already described manner as the Riemann surface [if $f(z)$ assumes the same value in two distinct points of the area D].

We imagine the layers of the area D_1, lying on the same plane as D, and we especially want to know the cases when the *area* D *is completely in the interior of the area* D_1, that's to say when one can find a layer of D_1 ([1]) where all points of D and its contour are *interior points* of D_1; there is not any possible misunderstanding thereabout (and if D corresponds to itself, then it would be possible to recognize that D is inside D_1 by an analytic continuation with the help of *elements*, while proceeding like Poincaré did in its Memoir, cited before).

Here the examples: in the figures 1, 2, 3, D is *inside* D_1; in the figure 4, D is *not inside* D_1. In the figure 5, D and D_1 are areas coming back into themselves and D is *completely inside* D_1.
It could happen that the function $f(z)$ has a pole in D; then the area D_1 includes the point to infinity as interior point. But of course $f(z)$ has not any singular essential point in D or on C.

Therefore, the relation $z_1 = f(z)$ establishes a biunivocal and analytic correspondence between the interior points of D and the interiors points of D_1 (it is very important: D_1 is considered as it has several layers. Then two points with the same affixe z_1 but lying on different layers are considered as two distinct points of D_1) as well as between the points of C and C_1.

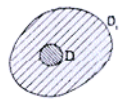
Fig.1

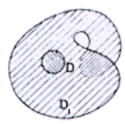
Fig.2

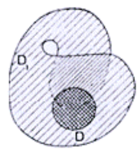
Fig.3


Fig.4

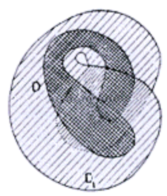
Fig.5

We can know generate a conformal mapping from the interior of D_1 ([2]) to the interior of the circle $| Z | \leq 1$ so that the points of C_1 are in a biunivocal and analytical correspondence with the points of $| Z | = 1$.
Then D will be conformally mapped on a simply connected area D with one layer and bounded by an analytic closed curve O which has no double points and it is inside the circle $| Z | = 1$, called O_1; D_1 is the area inside the circle $| Z | \leq 1$.

([1]) Therefore the considered layer D_1 has not any branch point in D when one supposes that D is simple area of analytical plan and with one layer.
([2]) If D_i includes the point at infinity, then we have only to examine the area corresponding to D_1 on the *Riemann sphere* for realizing that it is a conformal map of the point at infinity and of its neighbourhood on the neighbourhood of an interior point of D_1.

An interior point Z of D_1, corresponding to an interior point *z* of D_1, is the *image* for the conformal mapping of D_1 on D_1.
If we consider two points *z* and z_1, respectively inside D and D_1 and corresponding mutually, biunivocally and analytically in respect of the mapping $z_1 = f(z)$, then their images Z and Z_1 will be two points, respectively in the interior of O and O_1; the *images* also correspond each other in a biunivocal and analytical way in respect of a mapping $Z_1 = F(Z)$ which is called *the transformed relation of the relation* $z_1 = f(z)$ by an auxiliary conformal map.

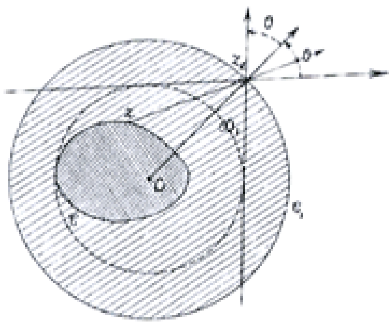

Fig. 6

F(Z) will be analytic and uniform in D and on O, and its inverse function will be analytic and uniform in D_1 and on O_1. Every interior point Z of D corresponds with one and only one interior point Z_1 of D_1, and mutually: the correspondence is analytic and it extends to the boundaries O and O_1.
The most important goal of an auxiliary conformal transformation is to bring back the areas D and D_1 to simple areas with one only layer. When Z describes the contour O in the positive direction, while the area D lies at its left, then Z_1 describes the boundary O_1 in the positive direction while the area D_1 is at its left. In these conditions, it is known that the roots number of the equation

$$F(Z) - Z = 0,$$

in the interior of the boundary O, is the same as the total variation of the argument of F(Z) - Z, when Z describes once the closed contour O in the positive direction. [In fact it is necessary to notice that the function F(Z) - Z is holomorphic on D and without any pole, while $|F(Z)| \leq 1$ in the area D].
Therefore, it is necessary to study the total variation of $\arg(Z_1 - Z)$ when Z describes O in the positive direction. The circle O_1 is centered at the origin and Z is always an *interior point of the circle* O_1; therefore

$$\arg Z_1 - \theta < \arg(Z_1 - Z) < \arg Z_1 + \theta \qquad \left(0 < \theta < \frac{\pi}{2}\right),$$

while the argument of previous complex numbers is properly defined in the initial position of Z_1 and *these arguments follow later for continuity*. The angle θ has a well determined inferior limit, because the angle 2θ of bisector OZ_1 shall include the contour O in its interior, whatever Z_1 is on O_1. Since the total variation of $\arg(Z_1)$ equals $+2\pi$, when Z describes O in the positive direction, the value of $\arg(Z_1 - Z)$ will be also the same as $+2\pi$, in the same conditions.

Therefore $F(Z) - Z = 0$ *has one and only one root in the interior of* O.

After applying a linear substitution, preserving the circle O_1, to the plane of the Z, one may suppose that this root is brought to the origin, that's to say: $F(0) = 0$.

Then the function $Z_1 = F(Z)$ is such that $Z = \Phi(Z_1)$, inverse of F(Z), is an holomorphic function in and on the circle O_1, $|Z_1| \leq 1$ and on that circle;

and one gets

$$\Phi(0) = 0$$

$$|\Phi(Z_1)| < 1 \text{ , whatever is } |Z_1| \leq 1 \text{ ,}$$

since $Z = \Phi(Z_1)$ is always in D or on O when Z_1 is in D_1, or on O_1.

If one makes a Taylor development $\Phi(Z_1)$ around the origin, then one get

$$\Phi(Z_1) = Z_1 + \Phi'(0) + \frac{Z^2}{2!} + \Phi''(0) + \ldots$$

with

$$\Phi'(0) = \frac{1}{2i\pi}\int_{O_1}\frac{\Phi(Z_1)dZ_1}{Z_1^2},$$

according to the Cauchy formula. Now, on O , one gets

$$|Z_1| = 1 \quad \text{et} \quad |\Phi(Z_1)| < 1$$

Therefore

$$\left|\int_{O_1}\frac{\Phi(Z_1)dZ_1}{Z_1^2}\right| < \int_{\varsigma_1}|dZ_1| = 2\pi$$

Therefore

$$|\Phi'(0)| < 1$$

and following

$$|F'(0)| = \left|\frac{1}{\Phi'(0)}\right| > 1$$

returning to F(Z). Resuming, we established that

$$F(0) = 0 \text{ ,} \quad |F'(0)| > 1 \text{ .}$$

On the following, there is a second demonstration of this last inequality, based on an interesting lemma about conformal maps. The domain D includes the origin. It is conformally mapped on the circle $|Z_1| \leq 1$ with the help of the function

$$Z_1 = F(Z) ,$$

where the origin is mapped to itself. The function F(Z), *vanishing at the origin*, assumes module = 1 on O. It is good to notice that $F'(0) \neq 0$, since the origin is not a branch point of O_1. The function $\log\frac{1}{F(Z)}$ is infinite at the origin as $\log\frac{1}{Z}$ and, on O ,its real part vanishes. That real part is $\log\frac{1}{|F(Z)|}$, which is an harmonic function of X,Y(Z=X+*i*Y) vanishing on O and infinite in O like $\log\frac{1}{|Z|}$. There is the Green's function of boundary O, relatively to the origin.

If one develops F(Z) around the origin, then one gets

$$F(Z) = ZF'(0) + \frac{Z^2}{2!}F''(0) + ...$$

$$|F(Z)| = |Z|\left|F'(0) + \frac{Z}{2!}F''(0) + ...\right|$$

et

$$\log\frac{1}{|F(Z)|} = \log\frac{1}{|Z|} + \log\frac{1}{\left|F'(0) + \frac{Z}{2!}F''(0) + ...\right|} = \log\frac{1}{|Z|} + \log\frac{1}{|F'(0)|} + \textit{harmonic function vanishing at}$$

the origin. Therefore $\log\frac{1}{|F'(0)|}$ is the constant term of the development of the Green's function around the origin. Since the contour O is inside the circle O_1 ($|Z_1| = 1$), it is known that, according to a lemma ([1]) of Kœbe (*Math. Annalen*, t.LXVII, p. 208), the constant term of the Green's function is < Γ, when Γ < 0; therefore $|F'(0)| > 1$.

Let us come back now to the domains D and D_1: they are biunivocally and analytically related by the mapping $z_1 = f(z)$. The point Z = 0 of D, *coinciding with its corresponding* $Z_1 = F(0) = 0$, is the image of a certain point ω of D *coinciding with its correspondent* ω_1 of D_1. It means that if *z* is the affixe of this point ω, then one gets $z_1 = f(z) = z$, and *moreover* the point ω_1 of D_1, which corresponds to the point ω of D, lies *on the same layer as* D. In other words, the biunivocal and analytic correspondence, fixed by $z_1 = f(z)$ between the points outside D and those outside D_1 has *one and only one double point* ω. Again, I do not want to say that the *f*(*z*) = *z has only one root in* D, because it may happen that a point *z* of D corresponds to a point on D_1 with the *affixe* $z_1 = f(z)$ *equalling to z* but lying on the *another layer where* D_1 has been traced over.

([1]) This lemma supposes that the contour O does not trespass O_1, but it is still verified when O shares two common parts with O_1.

For example, it may happen that *several distinct layers* of D_1, where each one of them is completely covering the area D, that's to say if on several different layers from D_1, the cylinder, projecting the curve C that bounds D, intersects some simply connected areas, consisting of one piece and projected on D and *completely inside* D_1 (*look at* figure 7). It is therefore established that the equation $f(z) - z = 0$ has at least a root ζ in D.
If furthermore one remarks that

$$\frac{dz_1}{dz} = \frac{dz_1}{dZ_1}\frac{dZ_1}{dZ}\frac{dZ}{dz} \; (^1)$$

where Z_1 is the image of z_1 and Z is the image of z ; then, one sees that one gets at the point Z = 0 which is the image of $z = \zeta = f(\zeta)$

$$\frac{dz_1}{dZ_1} = \frac{dz}{dZ}$$

and since $Z = Z_1 = 0$,

$$\left(\frac{dz_1}{dz}\right)_z = \left(\frac{dZ_1}{dZ}\right)_0 ,$$

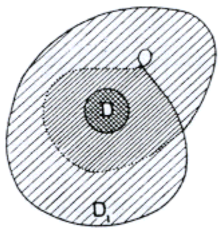

Fig. 7

and so

$$\left(\frac{dZ_1}{dZ}\right)_0 = F_1(0)$$

one gets

$$\left|\frac{dz_1}{dz}\right|_\zeta = |f'(\zeta)| > 1,$$

Therefore the equation $f(z) - z = 0$ has at least one root ζ in D and one gets in ζ

$$|f'(\zeta)| > 1,$$

Now, the following lemma will be really useful for us.

Schwarz's lemma. – (It is published in the *Œuvres* of Schwarz t. II, p.109) :
«If $f(z)$ is holomorphic for $|z| < 1$ and if, on the domain $|z| < 1$, one gets

$$|f(z)| \leq 1 , \quad \text{with} \quad f(0) = 0$$

---

(1) No difficulties for the derivates $\frac{dz_1}{dZ_1}$ and $\frac{dz}{dZ}$, since the layer of D_1, including D, has not any branch point in D. This difficulty is not very hurting: it is necessary only to substitute to the simple area D an area simply connected and built by of several layers, *the same* as D, superimposed and ramified among themselves so that the set of these layers fits well an interior connected area in D_1.

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then the following inequality holds for each point z ($0 < |z| < 1$):

$$|f(z)| < |z|$$

except in the case where $f(z)$ is a linear function $ze^{i\theta}$: in this case

$$f(z) = |z| \text{ . »}$$

In fact (1) the function $\varphi(z) = \frac{f(z)}{z}$ is uniform for $|z| < 1$ by hypothesis.

Whatever ρ is positive and < 1, the module of $\varphi(z)$ reaches its maximum for $|z| = \rho$ in the domain $|z| \leq \rho$.
One has therefore, for $|z| \leq \rho$,

$$\left|\frac{f(z)}{z}\right| \leq \frac{1}{\rho},$$

since for $|z| = \rho$, $|f(z)| \leq 1$.
Therefore, for $|z| \leq \rho$,

$$|f(z)| \leq \frac{|z|}{\rho},$$

which is satisfied for every $0 < \rho < 1$.
Therefore

$$f(z) \leq |z|$$

is retrieved in all interior points $|z| < 1$.
If $\varphi(z)$ is not a constant, then one cannot have an interior point $|f(z)| = |z|$, because there would be some interior points where $|f(z)| > |z|$, in contradiction with the previous inequality. The equality $|f(z)| = |z|$ is satisfied only when $|\varphi(z)| = 1$, since $\varphi(z)$ is constant; that's to say when $f(z) = ze^{i\theta}$. Then, the previous inequality is satisfied for all interior points of the circle $|z| < 1$. One may conclude that, at the origin, $|f'(0)| < 1$, except in the trivial case where $f(z) = ze^{i\theta}$.
This is evident if $f'(0) = 0$. Therefore, one may stop at the case where $f'(0) = 0$; but, if one examines a *small enough* circle C, centered at O and bounding an area D, then, while z describes D, the relation $z_1 = f(z)$ describes a simple area D_1 with only one layer and bounded by an analytic closed curve C_1, including the origin.
And due to $|f(z)| < |z|$, the area D_1 is *entirely included in the area* D.
The lemma, demonstrated previously, allows to conclude that one gets at the origin

$$|f'(0)| < 1$$

It is clear that starting from an interior point z of the fundamental circle $|z| < 1$, the images $z_1 = f(z)$, $z_2 = f(z_1)$, … of z have only O as their limit point [$|z_i| < |z_{i-1}|$], except in the trivial case where $f(z) = ze^{i\theta}$ (i.e: it is a rotation).

(1) Read CARATHÉODORY , Math. Annalen, t. LXXII, § 4.

The trivial case is *the only case* where the function $f(z)$ establishes a biunivocal and analytic correspondence between the interior points of the circle, when the origin is preserved by the correspondence; this is well known since Poincaré.
In all the other cases, when z describes the interior D of the circle $|z| < 1$, then $z_1 = f(z)$ describes the interior of a simply connected Riemann surface D_1, including the origin and where all points lie in the fundamental circle $|z| < 1$.
The boundary of D_1 may coincide entirely or partially with the boundary of D.

Now, we stop at the case where the boundary of D_1 is inside D, that's to say where one would get the inequality $|f(z)| < 1$ for $|z| = 1$, when it is supposed that $f(z)$ is analytic on $|z| = 1$ too.
Then it is clear that argument variation of $z_1 - z = f(z) - z$ is the same as 2π, as z describes the circle O (for $|z| = 1$); and $f(z) - z$ has only one root, which is zero, on D.
This suggests the following generalization, which will be useful in several questions about iterations when one examines the problem from both a general and a local view point.
Imagine that, as z describes a simple area Δ, bounded by an analytic curve Γ, the point $z_1 = f(z)$, since $f(z)$ is analytic in Δ and on O, describes a simply connected Riemann surface Δ_1, where all points, including the boundary Γ_1 (an analytic closed curve), are inside the area Δ.
Then the conformal mapping of Δ on the area D of the circle $|z| \leq 1$, shows that:

1. The equation $z = f(z)$ has one and only one root ζ in the interior Δ;
2. $|f'(\zeta)| < 1$;
3. The images $z_1, z_2, \ldots, z_n, \ldots$ of any point z, lying in Δ or on Γ, converge uniformly to ζ.

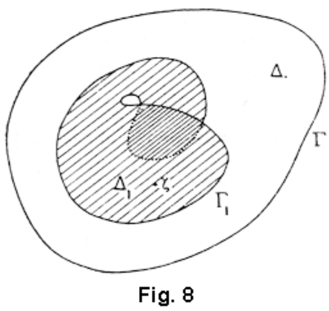

Fig. 8

On the following, there are some extensions of Schwarz's lemma: they do not need be demonstrated in details.
The Schwarz's lemma asserts that, except in the trivial case where $f(z) = ze^{i\theta}$, if z describes the interior of the circle $|z| \leq \rho < 1$, then $z_1 = f(z)$ describes a Riemann surface, which *is entirely inside the circle*: $|f(z)| < \rho$.

Now, if one supposes that one still gets $|f(z)| \le 1$ for $|z| < 1$, so that the fixed point ζ ($f(\zeta) = \zeta$) is not the origin, but another *interior* point of the circle $|\zeta| < 1$, then one will see very easily that, if ζ' is the symmetric point of ζ in respect of the circle $|z| = 1$, one will gets in the circle

$$\left|\frac{z-\zeta}{z-\zeta'}\right| \le \rho < \rho_0,$$

where ρ_0 assumes the constant value $\left|\frac{z-\zeta}{z-\zeta'}\right|$ for $|z| = 1$, $\rho_0 = |\zeta|$:

$$\left|\frac{f(z)-\zeta}{f(z)-\zeta'}\right| < \left|\frac{z-\zeta}{z-\zeta'}\right|,$$

whatever $\rho < \rho_0$, except in the case where :

$$\left|\frac{f(z)-\zeta}{f(z)-\zeta'}\right| = e^{i\vartheta}\left|\frac{z-\zeta}{z-\zeta'}\right| ,$$

that's to say the case where $f(z)$ is homographic, so that

$$\left|\frac{f(z)-\zeta}{f(z)-\zeta'}\right| = \left|\frac{z-\zeta}{z-\zeta'}\right|.$$

It means that, as z describes the interior of any circle C which is inside $|z| < 1$ and belonging to any band defined by two points-circles ζ and ζ' (limit points or Poncelet points), the point $z_1 = f(z)$ remains *in the interior of the circle* C without lying never on the circle; and it describes a simply connected Riemann surface including ζ and being *completely in the circle* C [except the case when $f(z)$ is linear].

Free from the hypothesis that an fixed point ζ may exist in $|z| < 1$, hypothesis that however may be not verified here [*read* further about fractions $\varphi(z)$ in the fundamental circle $|z| = 1$, so that $\varphi(z) = z$ has all its roots on this circle], then one will examine any point ζ and its symmetrical ζ', in respect of the circle $|z| = 1$.

Now it is clear that if $\zeta_1 = f(\zeta)$ is the value corresponding to ζ and ζ'_1 is the symmetrical point of ζ_1, in respect of the circle $|z| = 1$ (1), then the function

$$\psi(z) = \frac{f(z)-\zeta_1}{f(z)-\zeta'_1}\frac{1}{|\zeta_1|}$$

is a function in the variable $u = \frac{1}{|\zeta|}\frac{z-\zeta}{z-\zeta'}$, vanishing for $u = 0$ ($z = \zeta$) and that, for $|u| < 1$, is always ≤ 1. Therefore one has, in the circle

$$\left|\frac{z-\zeta}{z-\zeta'}\right| \le \rho < |\zeta| ,$$

who is any circle of an band (ζ, ζ') inside $|z| < 1$

$$\left|\frac{f(z)-\zeta_1}{f(z)-\zeta'_1}\right| < \left|\frac{\zeta_1}{\zeta}\right|\left|\frac{z-\zeta_1}{z-\zeta'_1}\right|,$$

(1) Supposing that ζ_1 is not on the circle $|z| = 1$.

except the case where $f(z)$ is homographic and it preserves the interior of circle $|z| < 1$; in that case the sign < shall be replaced by the sign = .
This fact is more easily enunciated if one maps the interior of $|z| < 1$ to the interior of the upper analytic half-plane [$\mathrm{Im}(z)$ = imaginary part of $z > 0$] by a proper linear substitution. Then one immediately sees that if $z_1 = f(z)$ maps every point z of the half-plane [$\mathrm{Im}(z) > 0$] to a point z_1 of the half-plane [$\mathrm{Im}(z_1) > 0$], if ζ is a point of the half-plane [$\mathrm{Im}(\zeta) > 0$] and if $\zeta_1 = f(\zeta)$ satisfying [$\mathrm{Im}(\zeta_1) > 0$], so, after defining ζ' and ζ'_1 the conjugated values of ζ and ζ_1, it is visible that

$$\left|\frac{f(z)-\zeta_1}{f(z)-\zeta_1'}\right| < \left|\frac{z-\zeta}{z-\zeta'}\right|$$

so that

$$\left|\frac{z-\zeta}{z-\zeta'}\right| \leq \rho < 1$$

whatever $\rho > 1$ is, except in the case when $f(z)$ is a linear function of the type $\frac{az+b}{cz+d}$, preserving the upper half-plane.
For such an homographic function, one always gets

$$\left|\frac{f(z)-\zeta_1}{f(z)-\zeta_1'}\right| < \left|\frac{z-\zeta}{z-\zeta'}\right|$$

and so,

$$\frac{f(z)-\zeta_1}{f(z)-\zeta_1'} = \frac{z-\zeta}{z-\zeta'} e^{i\Theta} .$$

Remarks. - If one is dealing with a function $f(z)$, which preserves the interior and the exterior one of the circle C ($|z| = 1$), that's to say when:

1. Each interior point z is mapped to the interior point $z_1 = f(z)$;
2. z', exterior and symmetric of z in respect of C, is mapped to a point z'_1, symmetric of z_1 in respect of C;
3. any point of the circle is mapped to a point of the same circle [example: the fractions in fundamental circle of Fatou (*Comptes Rendus*, May 21[th] 1917)];

and if the origin and the infinity are both double points of the mapping $z_1 = f(z)$, then one gets, except in the trivial case where $z_1 = ze^{i\Theta}$,

$$|f(z)| < |z| \qquad \text{(for all interior points)}$$

and evidently

$$|f(z)| > |z| \qquad \text{(for all exterior points)}$$

since the circle C is evidently the symmetry axis of the transformation.

New extension of the Schwarz's Lemma. – The previous study has been an analysis of the mapping $z_1 = f(z)$ inside the circle $|z| \leq 1$, when the *interior of the circle* is mapped

to itself so that every *interior point* is invariant. As I wrote above, it may happen that the mapping keeps invariant only those points lying *on the circumference* $|z| = 1$.
Now I will indicate again how one can analyse the behaviour of the transformation *in the circle*. Firstly I suppose that this circle has been already mapped in the upper analytic half-plane [$\mathrm{Im}(z) > 0$] by a linear auxiliary substitution.

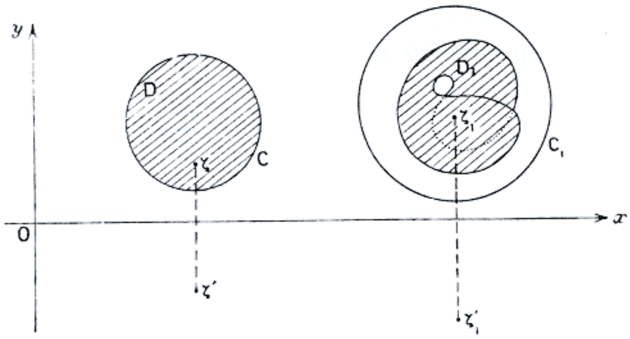

Fig.9

Then it is supposed that $z_1 = f(z)$ is an analytic function in the upper half-plane and so we will suppose that this relation *maps the real axis to itself* : each point z, lying on the real axis, is mapped to a point z_1 lying still on the real axis. Therefore we suppose this mapping being both analytic and real on the real axis; the same mapping extends to the inferior analytic half-plane by the method of the symmetries, but it is an additional question. We saw that if ζ is a point of the upper half-plane and ζ_1 is the mapped point $\zeta_1 = f(\zeta)$, as z describes the interior and the circumference of the circle C, $\left|\frac{z-\zeta_1}{z-\zeta'_1}\right| \le \rho < 1$ ([1]), one gets

$$\left|\frac{f(z)-\zeta_1}{f(z)-\zeta'_1}\right| < \left|\frac{z-\zeta}{z-\zeta'}\right|,$$

except in the trivial case where $f(z)$ is homographic (ζ' and ζ'_1 are the conjugated points of and ζ and ζ_1). So if z is in the interior of C or on its circumference, then $z_1 = f(z)$ is in the interior of the circle C_1 defined by

$$\left|\frac{z_1-\zeta_1}{z_1-\zeta'_1}\right| = \rho \ .$$

The two circles C and C_1 are homothetic in respect of a point of the real axis, so that ζ and ζ_1 are mutually mapped by the same homothethy.
The ratio between radii of C *and* C_1 *is equal to the ratio of the ordinates of* ζ *and* ζ_1.
Imagine that ζ *converges to a point* τ *of the real axis*, while the circle C preserves a finite radius R; then ζ_1 converges to a point τ_1 of the real axis ([2]) [$\tau_1 = f(\tau)$].
It is visible that, on the real axis, $f'(z)$ is real and > 0 ([3]);

([1]) ρ is any number < 1 .
([2]) It may be supposed that τ and τ_1 are at a finite distance.
([3]) Without vanishing.

therefore both ζ and ζ_1 converge respectively to τ and τ_1, concluding that the ratio

$$\frac{ordinate(\zeta_1)}{ordinate(\zeta)}$$

converges to a very determined limit which is equal to $f'(\tau)$. In these conditions, it is not difficult to conclude that, as C converges to a limit position Γ which is a circle with radius R and tangent in τ to the real axis, then C_1 converges to a limit position Γ_1 which is a circle with radius $Rf'(\tau)$ a tangent in τ_1 to the real axis.

Again, it is easy to see that, as z describes the interior of the circumference of the circle Γ, $z_1 = f(z)$ describes a domain Δ_1 *including no points outside* Γ_1 but *a priori* including points on Γ_1.

The opposite hypothesis: if Δ_1 would include *some points outside* Γ_1, then a contradiction arises; in fact, after examining a circle O infinitely near to Γ and lying above Ox, which corresponds to a O_1 circle (infinitely near to Γ_1), then O_1 should always include z_1, when z is in O; and this would not be true when z_1 is near to a point of Δ_1, outside Γ_1 and therefore outside O_1. Anyway it can be asserted that, as z describes the interior of the circumference of the circle Γ, z_1 describes a domain sharing only the point τ_1 with Γ_1, when $f(z)$ is not homographic.

As z describes the interior of the circumference of the circle Γ with radius R that, evidently, is *any* tangent circle in τ to the real axis, z_1 describes a domain whose points lie in the interior or on the circumference of Γ_1: this circumference has a radius $R_1 = Rf'(\tau)$ and it is tangent in τ_1 to OX.

It means that the imaginary part of the function $\frac{1}{f(z)-\tau_1}$ is, in the interior or on the circumference of Γ, $\leq$ to a certain negative limit value which is

$$-\frac{1}{R_1} = -\frac{1}{Rf'(\tau)}, \qquad \operatorname{Im}\left(\frac{1}{f(z)-\tau_1}\right) \leq -\frac{1}{Rf'(\tau)} \quad (^1)$$

Now when z describes the interior of Γ, one gets

$$\operatorname{Im}\left(\frac{1}{z-\tau}\right) < -\frac{1}{R},$$

the sign < can be replaced by = , when z lies on Γ.

Therefore *when z describes the circumference*, one will get

$$\operatorname{Im}\left[\frac{1}{f(z)-\tau_1} - \frac{1}{(z-\tau)f'(\tau)}\right] \leq 0 \ .$$

The function

$$\operatorname{Im}\left[\frac{1}{f(z)-\tau_1} - \frac{1}{(z-\tau)f'(\tau)}\right]$$

([1]) Because, evidently, the point $\frac{1}{z_1-\tau_1}$, as z_1 is on Γ_1, is on the half-plane $\operatorname{Im}\left(\frac{1}{z_1-\tau_1}\right) \leq -\frac{1}{R_1}$

is harmonic and uniform in all interior points of Γ and in all points of Γ, except in the point $z = \tau$.
Now, in this point, one has

$$f(z) - \tau_1 = f(z) - f(\tau) = (z-\tau)f'(\tau) + \frac{(z-\tau)^2}{2!}f''(\tau) + \ldots = (z-\tau)f'(\tau)[1 + \lambda(z-\tau) + \ldots]$$

Therefore

$$\frac{1}{f(z)-\tau_1} = \frac{1}{(z-\tau)f'(\tau)}\frac{1}{1+\lambda(z-\tau)+\ldots} = \frac{1}{(z-\tau)f'(\tau)}[1 - \lambda(z-\tau) + \ldots]$$

the quantity in parentheses are uniform around $z = \tau$.
Therefore

$$\frac{1}{f(z)-\tau_1} - \frac{1}{(z-\tau)f'(\tau)} = -\frac{\lambda}{f'(\tau)} + \ldots \ ,$$

the second member is uniform around $z = \tau$.
Therefore

$$\mathrm{Im}\left[\frac{1}{f(z)-\tau_1} - \frac{1}{(z-\tau)f'(\tau)}\right]$$

is uniform and harmonic around $z = \tau$.
If the function is not constantly = 0 on Γ, it will be therefore < 0 inside Γ, and it is ≤ 0 on Γ.
One will get therefore, in all interior points of Γ,

$$\mathrm{Im}\left[\frac{1}{f(z)-\tau_1}\right] < \mathrm{Im}\left[\frac{1}{(z-\tau)f'(\tau)}\right].$$

The last inequality cannot become an equality, because if

$$\mathrm{Im}\left[\frac{1}{f(z)-\tau_1} - \frac{1}{(z-\tau)f'(\tau)}\right]$$

is = 0 on all Γ, then this quantity vanishes in the whole Γ: evidently $f(z)$ is a homographic function

$$\frac{1}{f(z)-\tau_1} = \frac{1}{(z-\tau)f'(\tau)} + k \ ,$$

where k is any real constant.
Therefore, *except in the case when f(z) is homographic*, one gets, in all interior points of Γ,

$$\mathrm{Im}\left[\frac{1}{f(z)-\tau_1}\right] < \mathrm{Im}\left[\frac{1}{(z-\tau)f'(\tau)}\right].$$

Therefore this inequality is satisfied *in all points lying above* Ox because one may find a tangent circle Γ in τ to Ox and including the point z.
The same inequality proves (and it is necessary to be demonstrated) that the domain Δ, described by z_1 when z describes the interior of the circumference of a circle Γ with ra-

dius R and tangent in τ to Ox, lies in the *interior* of the circle Γ_1, with a radius $Rf'(\tau)$ and tangent in τ_1 to Ox, so that it shares only the point τ_1 with the circumference of Γ_1.

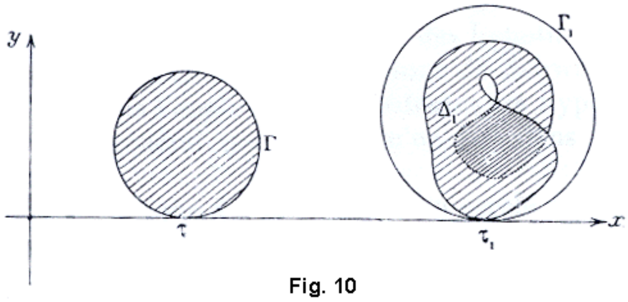

Fig. 10

Application. - This proposition is here applied to the question of the iteration: given a function $f(z)$, satisfying the previous hypotheses and being both a rational or an entire function that preserves both the two half-planes $\mathrm{Im}(z) > 0$ and $\mathrm{Im}(z) < 0$.
If τ is a double point, lying on Ox, of the transformation $z_1 = f(z)$, one will get

$$\tau_1 = f(\tau) = \tau \, .$$

Moreover, if one has $0 < f'(\tau) \leq 1$, then it is clear that every circle Γ, tangent in τ to Ox, becomes a domain Δ_1, inside Γ and *sharing only the point* τ *with* Γ.
Every point z of Γ, different from τ, becomes an interior point z_1 of Γ; z_1 generates an interior point $z_2 = f(z_1)$ of the circle drawn by z_1 and tangent in τ to Ox, ... So one sees that *the sequence* $z, z_1, z_2, \ldots$ *converges uniformly to the limit point* τ, satisfying $\tau = f(\tau)$, when $0 < f'(\tau) \leq 1$, whatever the *initial point* z is in the upper half-plane (not on Ox).

It demonstrates very simply the results, enunciated by Fatou (*Comptes Rendus*, May, 21[th] 1917) for the fractions $\varphi(z)$ on the fundamental circle, where a point $\zeta = \varphi(\zeta)$ is inside that circle, so that $|\varphi'(\zeta)| \leq 1$: every interior point or every exterior point of the circle has some images converging uniformly to ζ.
One will see further that the two cases $|\varphi'(\zeta)| < 1$ and $|\varphi'(\zeta)| = 1$ are deeply different.

Remarks. - All the hypotheses, described above about the function $f(z)$, are not equally necessary: there are more general hypotheses than can be used for extending properly the Schwarz's Lemma for the previous application; these hypotheses are so general as those made in the Schwarz's lemma:

Suppose that $f(z)$ is an analytic function *in the interior* of a circle C in the plane z (it may always be supposed the half-plane $\mathrm{Im}(z) > 0$); suppose in addition that:

1. Every interior point z of C is mapped to an point $z_1 = f(z)$, *interior of* C *or lying on the circumference of* C

$$\mathrm{Im}[F(z)] \leq 0 \qquad \text{in every point} \qquad \mathrm{Im}(z) > 0 \, .$$

2. A point O of the C circle is invariant, for example the origin $f(0) = 0$.
3. The function $f(z)$ is holomorphic in a neighbourhood of a point ([1]). (It is supposed nothing else on C but O.)

Then it is necessary that $f'(0)$ is both real and > 0 so that the first condition is satisfied around the origin. One immediately realizes that the entire discussion for demonstrating the extensions of such lemma, are still true here; that's to say, in every interior point of C, Im(z) > 0, one gets

$$\mathrm{Im}\left(\frac{1}{f(z)}\right) < \mathrm{Im}\left(\frac{1}{zf'(0)}\right),$$

the equality may be verified only if $f(z)$ is homographic.

Therefore, *as z describes the interior and the contour of any circle* T *with radius* R, *interior of* C *and tangent in* O *to* C, z_1 *describes a domain* Δ_1, *whose interior and boundary points* (*except* O) *are now in the interior of a circle* T_1, *tangent in* O *to* C *and with radius* $R_1 = f'(0)$.

Δ_1 is tangent in O to C. In particular, if $f'(0) = 1$, then Δ_1 is completely inside Γ, except for O: here Δ_1 touches Γ. As it has been enunciated before, further this lemma will be really useful for us.

6. *Reminder of notions about algebraic functions*. – Suppose that $\varphi(z)$ is a rational function; $\psi(z)$ is the algebraic inverse function of $\varphi(z)$. Suppose that $\varphi(z) = z_1$, so $z = \psi(z_1)$. Let us examine the Riemann surface R, extended on the plane z, related to $\psi(z)$; φ is a function of a degree k, then R has k layers and $z = \psi(z_1)$ is a uniform function on R so that it always retrieves *distinct* values in two *distinct* points of R, when these points has the same affixe z_1.

A value of z_1 is related to k values of z (distinct or not): its k preimages.

1. Let us imagine that z_1 describes the interior of an area Δ_1 of the analytic plane, so that Δ_1 consists of one piece and it is bounded by one or more analytic contours.

To find the areas, described by the set of preimages of z, it is enough to study the points, of the Riemann surface R, that are projected on the analytic plane to an interior point of the area Δ_1. Several cases may occur:

a) It may happen that these points generate a number k of areas S_1 in R: the areas consist of one piece, they are distinct, superimposed, each of them has one only layer being the same as Δ_1.

([1]) It is enough to suppose that $f(z)$, holomorphic on C, admits, as z converges to O in C, both first and second order finite and continuous derivatives, so that $z_1 = zf'(0) + z^2\frac{f''(0)}{2} + z^2\varepsilon(z)$, while ε(z) is holomorphic on C and converging to zero, as z converges to O on C.

This will happen only if starting from any interior point z_1 of Δ_1, with any determination of $\psi(z_1)$ and coming back there by any interior path of Δ_1, then one finds the same determination of $\psi(z_1)$ again in z_1.
Then *no branch point of* $\psi(z)$ *can be in the interior of* Δ_1, but this condition is not surely sufficient if Δ_1 is bounded by an only one contour.
Then each one of the k preimages z_1 describes an holomorphic function of z_1 in Δ_1 and it describes a simple area Δ, bounded by analytic contours and with one only layer. So, the k obtained areas are *two by two exterior* ; they cannot share anything else but boundary points: *this happens only if the boundary of* Δ_1 *includes a branch point of* $\psi(z)$.

The function $\psi(z)$ maps conformally the interior of each one of the k areas S_1 is to the interior of an area Δ_1 .

b) It may happen that these points generate less than k distinct areas on R, each one consisting of one piece, since each one of them may have many layers. It happens that, starting from an interior point z_1 of Δ_1 with a given determination of $\psi(z_1)$, one can describe *an interior path of* Δ_1 which *brings back in* z_1 *a different determination of* $\psi(z_1)$ *from the initial determination*.
It means that one may move from a point P of R, projected in the interior of Δ_1, to a point P', superimposed to P and lying on a different layer of R from P, by using a continuous path which is traced on R and whose points are projected inside Δ_1. Therefore P and P' belong to a connected area, lying on R and counting at least two layers, whose points are projected inside Δ_1.

Certainly, this occurs when the area Δ_1 includes at least a branch point of $\psi(z)$. But it is not necessary. In fact, if one examines a function of the second degree, relatively to $\psi(z)$, then there are two critical points ζ_1, ζ_2; now, consider two closed curves C_1 and C_2.

These two curves separate ζ_1 from ζ_2; they do not intersect and they define an annulus (C_1C_2) generating a double connected area. It is visible that the points of R, projected in the interior of Δ_1, generate one only area, consisting of one piece and with two superimposed layers and bounded by each one of the two curves C_1 and C_2.

If one examines p areas S_1 ($p < k$), which have been cut in the Riemann surface of $\psi(z)$, then it is visible that the set of the k preimages of z_1 describes a set of p areas Δ when z_1 describes the interior of Δ_1: every area consists of one piece, it has one layer and it conformally maps one of the areas S_1 on the plane z.
These p areas Δ are two by two exterior; they cannot share anything else but some boundary points only in the case when the boundary of Δ intersects a branch point of $\psi(z)$.

In particular it may happen that, after starting from an interior point z_1 of Δ by any determination of $\psi(z_1)$, one can trace an interior path of Δ that brings back *all other determinations* of $\psi(z_1)$ in z_1.

Then the set of points of R, projected in the interior of Δ_1, generates one only area S_1 consisting of one piece and with k superimposed layers (here the value $p = 1$). After starting from a point P of R, projected in Δ_1, one may reach to each one of the k - 1 points of R, superimposed to P, by describing a continuous path, traced on R and whose points are projected in the area Δ_1.

Then it is clear, as z_1 describes the interior of Δ_1, the set of its k preimages describes only one area Δ of the plane of the z (area with one layer): the area is a conformal map of the area S_1 with k layers. Resuming, while z_1 describes the area Δ_1, the set of k preimages describes p ($p \leq k$, p can be = 1) distinct areas Δ of the plane of z; each one of them consists of one piece, with one only layer and bounded by analytic contours [it may show edged points when the boundary intersects some critical points of $\psi(z)$].

These p areas are two by two exterior: they may share a finite number of boundary points and only when the boundary of Δ_1 intersects a critical point of $\psi(z)$.

II. Let us consider two areas Δ_1 and Δ'_1 of the analytic plane, so that Δ_1 is *inside* Δ'_1. While z_1 describes Δ_1, the set of its k preimages describes p distinct areas Δ of the plane z ($1 \leq p \leq k$); it is clear that when z_1 describes Δ_1, the set of its k preimages describes p' distinct areas Δ′ of the plane z, for $p' \leq p$. Furthermore, every area Δ is inside an area Δ′.

It may happen that p' is $< p$ and that two distinct areas Δ are inside a same area Δ′.

III. Now, if one considers two areas Δ_1 and Δ'_1 of the analytic plane and *exterior from each other*, the set of the k preimages of z_1, describing Δ_1, describes p distinct areas Δ of the plane z; and as z_1 describes Δ'_1, the set of its preimages describes p' distinct areas Δ′ of the plane z. *Each one of* Δ *is exterior from each one of the* Δ′ and, if Δ_1 and Δ'_1 do not share any common point on the boundary, then none of the Δ areas may share any common points with an area Δ′.

We will use this result later, in the definition of domains of convergence to limit points with uniform convergence $\zeta = \varphi(\zeta)$, $| \varphi'(\zeta) | < 1$. We will see immediately that the domains related to two distinct limit points *cannot share any common interior points*.