

# Rauzy fractals for free group automorphisms

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CIRM Workshop, March 2006

# The classical case: setting

- ▶ Primitive substitution  $\sigma$
- ▶ Fixed or periodic point  $\sigma^n(u) = u$
- ▶ Symbolic dynamical system  $(\Omega_\sigma, S)$
- ▶  $\Omega_\sigma = \overline{\{S^n u\}}$
- ▶ Geometric representation ?

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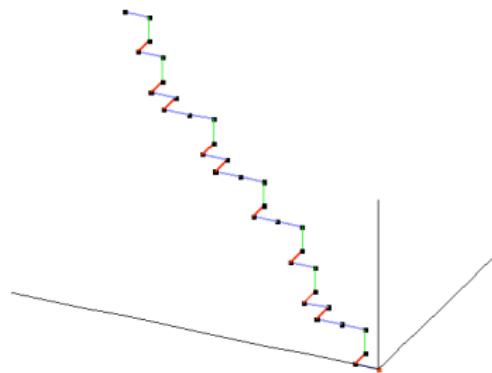
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- ▶ Geometric representation of the fixed point :
- ▶ Stepped line
- ▶ Direction of the stepped line: Perron eigenvector

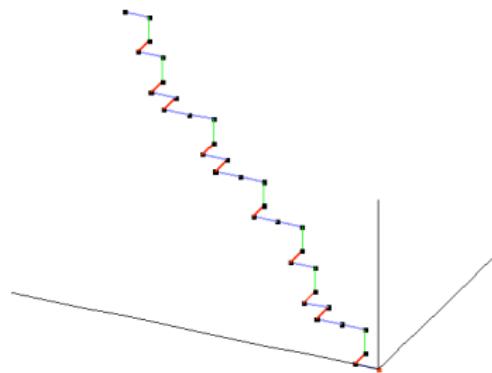
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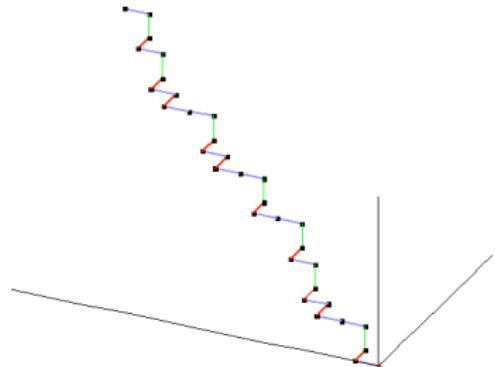


# The classical case: Pisot irreducible substitutions

- ▶ Pisot irreducible substitutions:
- ▶ all eigenvalues except 1 smaller than 1
- ▶ the stepped line stays within bounded distance of the Perron eigenline
- ▶ Rauzy fractal by projection on the contracting plane
- ▶ Geometric coincidence condition: many properties

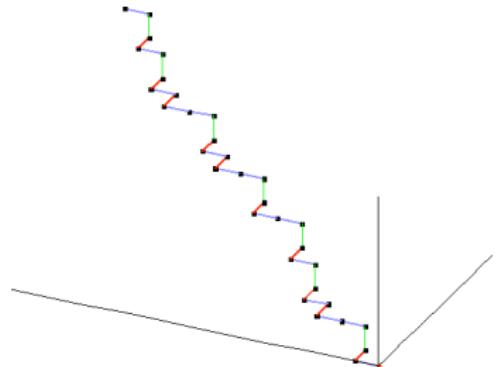
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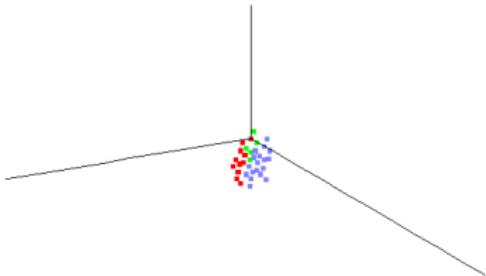
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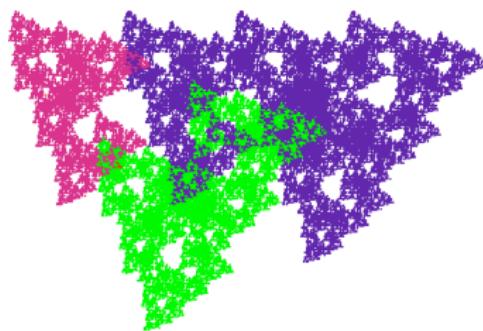
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# the reducible case

## ► Pisot reducible substitutions:

- the largest eigenvalue is a Pisot number
- $\mathbb{R}^d = H + \mathbb{R}v + E$
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- ▶ from the lecture of A. Hilion: train track
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- ▶ Pisot IWIP automorphism:
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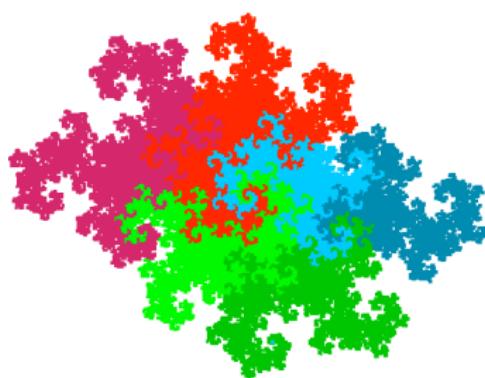
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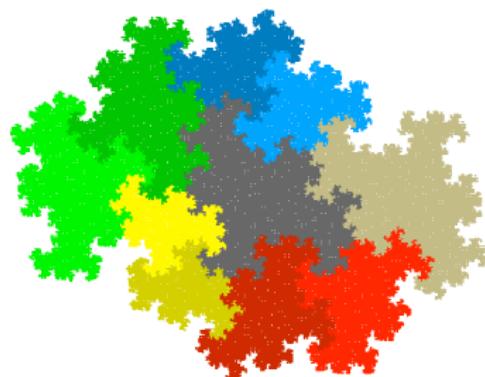
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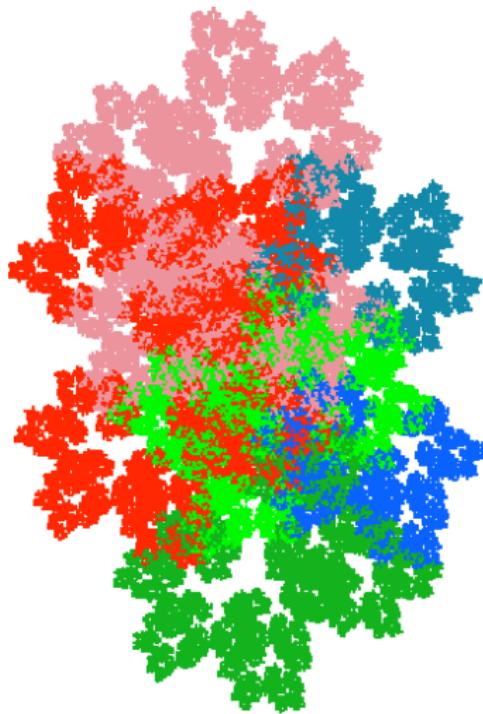
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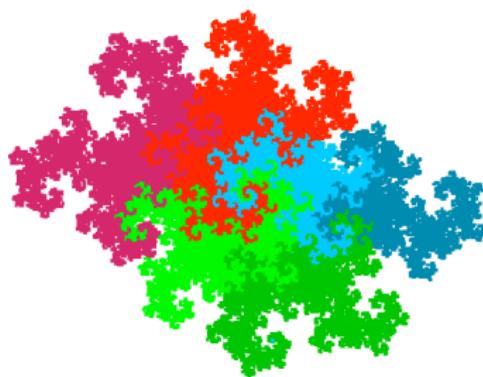
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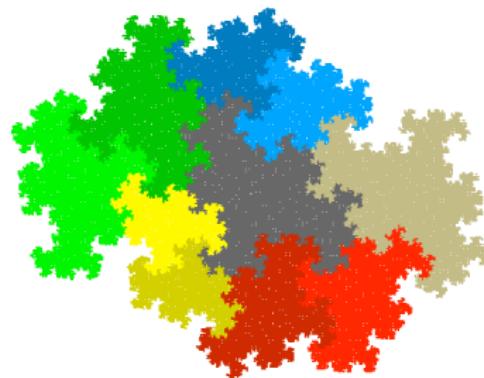
## symmetry properties

- ▶ 2d components
- ▶ the components are pairwise symmetric
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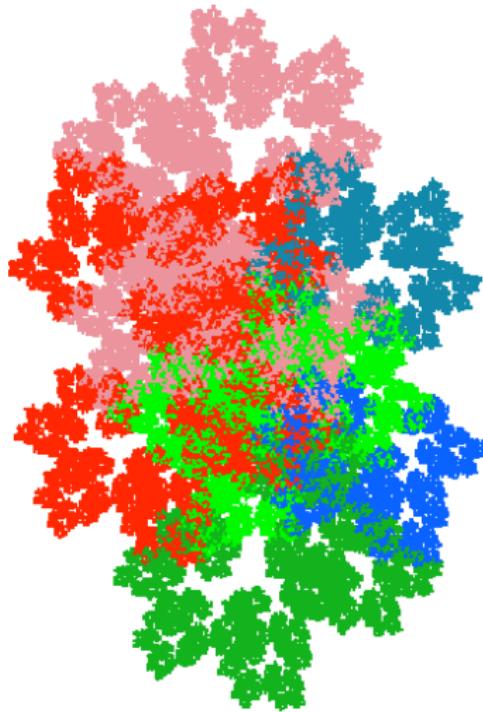
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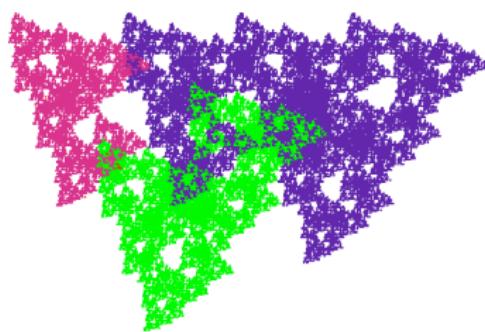
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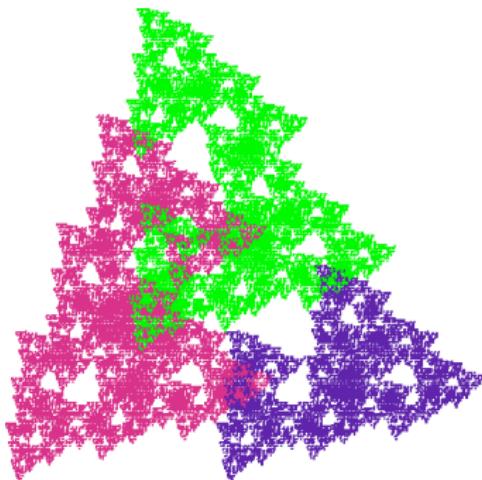
# An open question

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- ▶ these give rise to various Rauzy fractals
- ▶ they seem to have the same structure
- ▶ how can we make this precise ?



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