"There is the possibility that not very skilled artisans, who had no access to geometers, could create complex designs that require sophisticated geometric explanations."

Elementary Constructions of Persian Mosaics

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The Medieval Islamic World was a vast area, which included North Africa, a part of Europe, a part of India and China, and the Middle East. The interchange of cultures and combinations of arts among nations in this area, who had different backgrounds and systems of beliefs, created a type of art known as Islamic Art. Recent studies have shown that mathematicians who taught practical geometry to artisans played a decisive role in the creation of those patterns and perhaps in designing the buildings themselves.

Our purpose is to not only emphasize the direct involvement of mathematicians in this art, but also to suggest the use of modularity in creations of some classes of patterns. Through comparison of extant designs and models we present the possibility of techniques of trial and error used by artisans to create fascinating designs based on sets of tiles that we call "modules."

Abul Wafa, Mohandes

For our discussion we mainly rely on a treatise written by Abul Wafa al-Buzjani in the 10th century, *On Those Parts of Geometry Needed by Craftsmen*. Buzjani was born in Buzjan, near Nishabur, a city in Khorasan, Iran, in 940 A.D. He learned mathematics from his uncles and later on moved to Baghdad, one of the world's leading cities in art and science in the medieval time, when he was in his twenties. He flourished there as a great mathematician and astronomer. He was given the title *Mohandes* by the mathematicians, scientists, and artisans of his time, which meant "the most skillful and knowledgeable professional geometer." He died in 997 or 998 A.D.

Buzjani's important contributions include geometry and trigonometry. He was the first to show the generality of the sine theorem relative to spherical triangles and developed a new method of constructing sine tables. He introduced the secant and cosecant for the first time, knew the relations between the trigonometric lines, which are now used to define them, and undertook extensive studies on conics. In geometry he solved problems about compass and straightedge constructions in plane and in sphere.

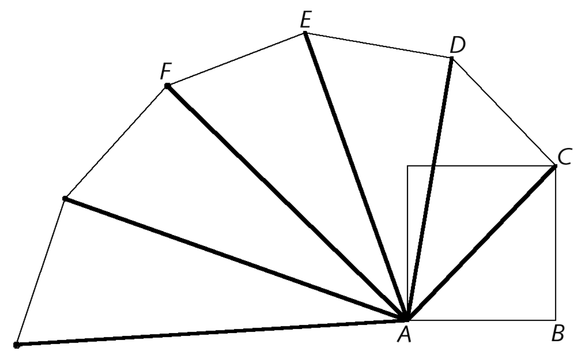


Figure 1: Consecutive constructions of $\sqrt{n}$ units line segments.

Buzjani wrote that he participated in meetings between artisans and mathematicians. "At some sessions, mathematicians gave instructions on certain principles and practices of geometry. At others, they worked on geometric constructions of two- or three- dimensional ornamental patterns or gave advice on the application of geometry to architectural construction." Therefore, it will not be far from the truth if we claim that some of the spatial properties and aesthetic elements in the structures of the Islamic art and architecture come from the direct involvement of mathematicians.

In his treatise, in a chapter titled "*On Dividing and Assembling Squares*," Buzjani presents the two different attitudes of mathematicians and artisans toward geometry:

> "A number of geometers and artisans have made errors in the matter of these squares and their assembling. The geometers made errors because they don't have practice in applied constructing, and the artisans because they lacked knowledge of reasoning and proof."

He continues:

> "I was present at a meeting in which a number of geometers and artisans participated. They were asked about the construction of a square from three squares. Geometers easily constructed a line such that the square of it is equal to the three squares but none of the artisans was satisfied. They wanted to divide those squares into pieces from which one square can be assembled."

Buzjani describes what he means by the mathematicians' approach for solving this problem. In Figure 1 $\overline{AB}$ presents one side of a square unit. Then

$$\overline{AC} = \sqrt{2},\ \overline{AD} = \sqrt{3},\ \overline{AE} = \sqrt{4},\ \overline{AF} = \sqrt{5},$$

and so on. Therefore, in each step we are able to find the side of a square with its area equal to 2, 3, 4, 5, and so on. A square with a side congruent to $\overline{AD}$ has the same area as 3 square units.

Then artisans presented several methods of cutting and assembling of these three squares. Some of these methods, based on mathematical proofs, turned out to be correct. Others were incorrect, even though they seemed correct at first glance.

We show one of these incorrect constructions here, along with Buzjani's "constructive criticism":

> "Some of the artisans locate one of these squares in the middle and divide the next one on its diagonal and divide the third square into one isosceles right triangle and two congruent trapezoids and assemble together as it is seen in the figure.

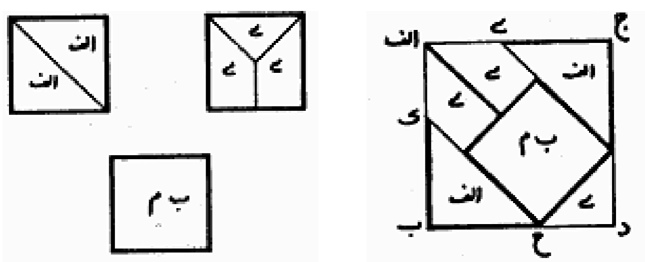


Figure 2: An incorrect construction of a square from three unit squares.

> For a layperson not familiar with the science of geometry, this solution seems correct. However, it can be shown that this is not the case. It is true that the resulting shape has four right angles. It is also true that each side of the larger shape seems to be one unit plus one half of the diagonal of the unit square. However, this construction does not result in a square because the diagonal of the unit square (which is the hypotenuse of the assembled larger triangle) is an irrational number but the measure of the line segment that this hypotenuse is located on in the larger shape is one and half a unit, which is a rational number."

Buzjani includes more information and simple approximations to reinforce his point that the construction is not correct. However, his way of using an argument based on the idea of rational and irrational numbers is undoubtedly elegant. He then, gives his solution, which is mathematically correct. At the practical level it can be performed in any medium by artisans.

> "But on the division of the squares based on reasoning we divide two squares along their diagonals. We locate each of these four triangles on one side of the third square in such a way that one vertex of the acute angle of the triangle is located on a vertex of the square. Then by means of line segments we join the vertices of the right angles of four triangles. From each larger triangle a smaller triangle will be cut using these line segments. We put each of these triangles in the congruent empty space next to it to complete the square."

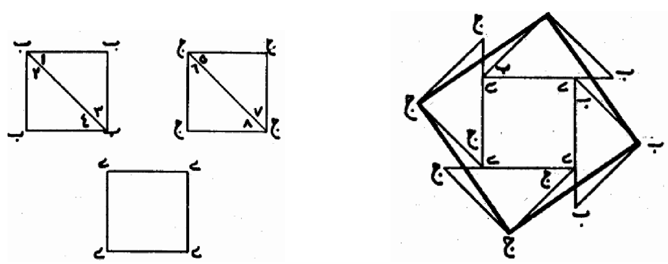


Figure 3: The correct construction of a square from three unit squares.

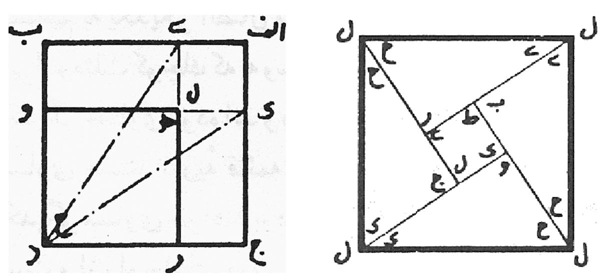
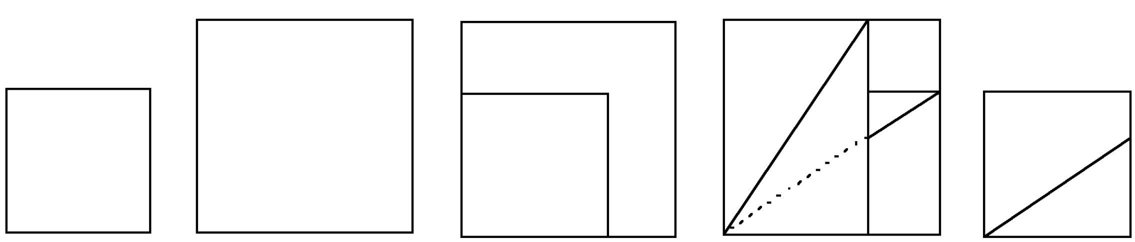


Figure 4: Generating a square from two different size squares.



(a) (d) (b) (e) (c)

Figure 5: The details of generating a square from two different squares.

Another interesting problem that Buzjani presents in his book is the composition of a single square from a finite number of different sizes of squares. For this, he solves the problem for two squares first and then comments that with the same method we are able to solve the problem for any number of squares.

The proposed solution for two squares is again based on cutting and pasting, squares and, therefore, is acceptable by artisans. His solution is elegant: We first put the small square (a) on the top of the larger square (b) as presented in (c) and then draw necessary line segments presented in (d) and (e) and finally cut the solid lines and paste them to obtain the resulting square.

For a person familiar with elementary mathematics what Buzjani is doing in this problem can be justified as follows: Let a be the size of a side of a small square and b the size of a side of the large square. Then the sum of the areas of these two squares will be $a^2 + b^2$. Now the cuts will create four right triangles with sides a and b (and hypotenuse $\sqrt{a^2+b^2}$), and a square with a side equal to $b - a$. The way that we arrange these four right triangles and the little square is in fact the visualization of the following equation:

$$a^2 + b^2 = 4\,(1/2\,ab) + (b - a)^2.$$

Modularity Art: Elementary Trial and Error, Sophisticated Symmetry

It may seem that some of the cutting and pasting activities presented by Buzjani such as mentioned above are simply mathematical challenges and lack practical value. However, if we study ornamental designs in Persian arts and architecture closely we notice the use of modular designs based on assembled cut-tiles.

So far we have seen that by using these modules one can approach ornamental qualities of patterns created by ingenious processes executed by either highly skilled artisans or mathematicians. What we wish to propose is controversial: There is the possibility that not very skilled artisans, who had no access to geometers, by the use of elementary cut-and-paste processes, based on trial and error, could create complex designs that require sophisticated geometric explanations—processes that remind us of modern fractal geometry: simple input with complicated results.

The widespread approach for constructing and arranging pattern designs in ceramic mosaic during the 10th and 11th centuries was the use of squares. The economy of energy and space for molding, casting, painting, and baking tiles forced artisans to use single-colored square tiles. The four colors available were light blue, navy blue, brown and black. The color scheme improved rapidly by increasing the number of colors made through different combinations of metals. It is natural to assume that a practical way of achieving new patterns from these squares for some artisans would be to cut them in different formats and assemble them and such a way that different colors replace one other in new arrangements. In this way the artisans could rely on the color contrast of cut-tiles to emphasize designs, rather than use a compass and straightedge. An elementary example would be to consider congruent squares in two colors of black and white. If we cut an isosceles right triangle with sides equal to one half of the side of these squares in one color and exchange it with the other triangle with opposite color, we have two two-color "modules" where one is the negative of the other. Now if we also include the two original single-color square designs, then we have four modules to work with to create patterns.

Figure 6: Four modules created based on two different colors of congruent squares.

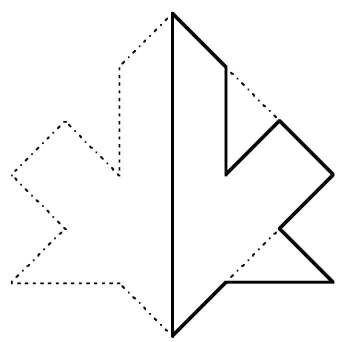


Figure 7: A "maple leaf" motif.

One interesting Islamic pattern is the "maple leaf." Pólya illustrated the seventeen wallpaper patterns in his article "Über die Analogie der Kristallsymmetrie in der Ebene" published in 1924. He illustrated a maple leaf pattern and

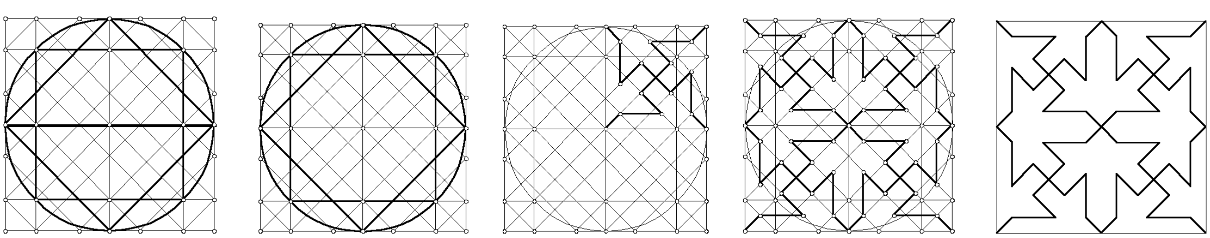


Figure 8: Traditional construction of the "maple leaf" pattern.

identified it as D°$_4$. The basic shape to construct this pattern is an isosceles right triangle. If we modify one side of this triangle and use a quarter rotation of one side to another followed by reflection of the shape under the hypotenuse, then we will have a maple leaf motif.

Figure 8 presents a traditional means of constructing the maple leaf pattern using compass and straightedge. In comparison we wish to present the modularity method and introduce the simplest possible set of modules using two single-color square tiles cut diagonally to generate the "maple leaf" pattern. If we cut a black and a white tile from their diagonals and exchange one of the generated triangles from each then we have a set of three modules of black, white, and half black-half white tile. In any other case by a diagonally shaped cut we create four modules as illustrated in Figure 6. Truchet's 1704 paper laid down a mathematical framework for studying permutations based on these tiles. Now by using 14 white, 14 black, and 8 black-white tiles we create a 36-tile grid, which is a base for construction of the maple leaf tessellation (Figure 9).

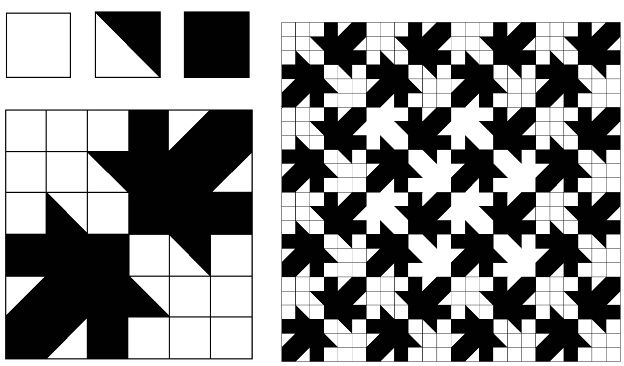


Figure 9: A set of three modules and the final result for the "maple leaf" tessellation.

Another interesting problem that Buzjani solved in assembling of squares is if we have ($m^2 + n^2$) squares, where $m \geq n$, such as 5 ($2^2 + 1^2$) or 13 ($3^2 + 2^2$), then we can make two congruent rectangles, each of whose length is equal to m and their width is equal to n. Therefore, we use $2mn$ of our squares in this way. This is possible because of the simple fact that $m^2 + n^2 \geq 2mn$. But then, the unused squares make a perfect square: $(m - n)^2$. Now if we cut these two rectangles diagonally and arrange the pieces around the square with side $m - n$ we find our desired square. In fact Buzjani mentions that the measure of the diagonals of the two rectangles is equal to the side of desired square $\left(\sqrt{m^2 + n^2}\right)$ and this is a hint for how to arrange the pieces around the square with side $m - n$. Figure 10 presents two examples of 5 and 13 squares, respectively.

We should note that the "sum of two perfect squares" problem, is in fact, a special case for the general case of "two different squares" problem, presented in Figure 4. Nevertheless the cut for the above two examples can provide us, or an artisan with a set of modules for exploring new patterns. The following figure presents a pattern, called "hat," which can be generated by only two opposite modules (without the use of original single color tiles) using the cuts from the above "5 squares" problem.

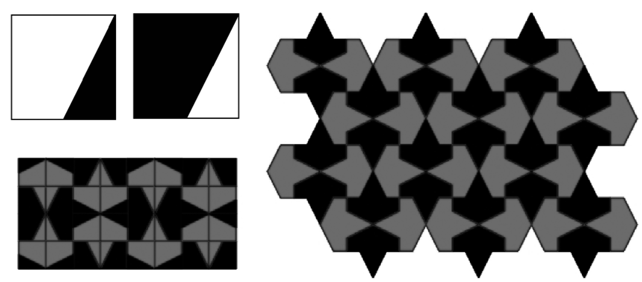


Figure 11: Modules based on the "5 squares" problem, and its generated "hat" tessellation.

Figure 12 illustrates steps taken by a geometer or a highly skilled artisan to compose the "hat" grid using compass and straightedge. You may find similar design constructions in B. L. Bodner's article, "Constructing and Classifying Designs of al-Andalus," which appeared at *2003 ISAMA/BRIDGES Conference Proceedings*, University of Granada, Spain. ■

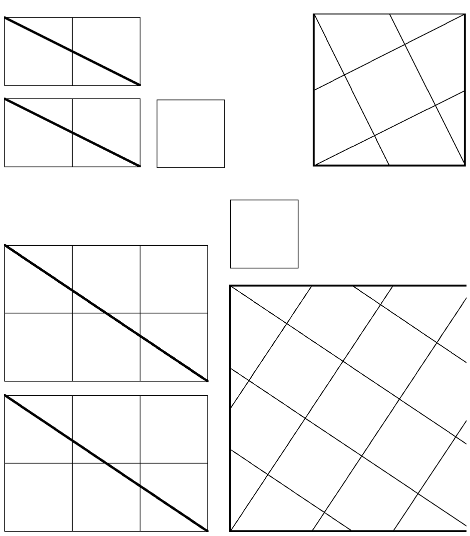


Figure 10: Cutting and assembling a square from five square units; cutting and assembling a square from thirteen square units.

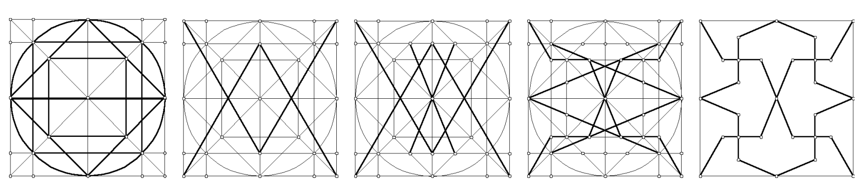


Figure 12: Polygon construction approach for generating the grid for the "hat" tiling.