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Super Lie groups: structure and representations

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To my mom, my dad,
and my grandad.

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Chapter 1

Introduction

Udite, udite, o rustici
attenti non fiatate...

Dulcamara
L'elisir d'amore

Symmetry enters in theoretical physics with at least two different meanings. It can enters at a very fundamental level as a manifestation of the properties of space-time (principle of relativity) or it can appear as a property of dynamical processes themselves. To illustrate the difference, consider the fact that euclidean space is rotation invariant but this is not the case for a generic field of forces defined over it. For the sake of definitiveness we will distinguish this two kind of symmetries by calling kinematical the former and dynamical the latter. In any case, whichever meaning we give to the word *symmetry* the basic objects needed are a *group* G and a *representation* of G in a Hilbert space $\mathcal{H}$. Let us briefly illustrate how these concepts enter into the picture.

When trying to formalize the principle of relativity, two basic concepts come into play. The first is the geometry. By this we mean a space-time $\mathcal{V}_4$ whose mathematical structure depends on the theory we are considering. It is Cartan space-time if the theory is Newtonian mechanics, it is Minkowski space-time if we are in the setting of special relativity. In any case each observer relativizes $\mathcal{V}_4$ with respect to his own reference frame. Roughly speaking this establishes a correspondence $\mathcal{V}_4 \simeq \mathbb{R}^4$. On the other hand, we known from quantum mechanics that each observer describes a quantum system by the projective space $\mathcal{P}$ whose points are the one-dimensional subspaces of a separable complex Hilbert space $\mathcal{H}$. Associated to the state space $\mathcal{P}$ is the corresponding group of automorphisms, whose elements are simply the maps

$$\Sigma : P \quad \longrightarrow \quad P$$

preserving the transition probabilities: $tr\,(\Sigma(P_1)\Sigma(P_2)) = tr\,(P_1 P_2)$. The mathematical implementation of the principle of relativity consists in the assumption that each transformation of the group G relating different frames of reference is represented at the quantum level by a corresponding automorphism of $\mathcal{P}$ relating the different descriptions of the quantum systems. In other words the second ingredient we need in order to define a symmetry is simply a group homomorphism between the group G and the group $Aut(P)$. Suppose now G connected, Wigner theorem asserts that, if $\dim \mathcal{H} > 2$, then each such homomorphism is equivalent to conjugation by a unitary operator defined over $\mathcal{H}$. Such an operator is uniquely defined modulo a phase. Summarizing this discussion, a symmetry turns hence out to be a (projective) representation of the group G on the Hilbert space $\mathcal{H}$.

Dynamical symmetries are more difficult to be described. Somewhat simplifying we can think that a dynamical symmetry is a representation of a group G in the Hilbert space $\mathcal{H}$ commuting with the Hamiltonian of the system. This does not reflect, in principle at least, any particular property of the geometry of the system, nevertheless it has consequences on the spectrum of the theory.

The scheme just described is very general and completely adequate for treating non relativistic quantum mechanics, and most of the problems arising in QFT. Nevertheless, it was realized during seventies, that if it would have been possible to introduce a symmetry capable of exchanging bosons and fermions then some difficulties encountered by the standard model of elementary particles could have been solved. Among these let us mention, for example, the problems of naturalness and hierarchy. In the historical development of these ideas a great role was played by an argument due to Coleman and Mandula concerning the most general form a symmetry of the S matrix of a quantum field theory can have. Let us briefly discuss this result. A scattering process can be described by the Hilbert space of single particles, the corresponding Fock space space $\mathcal{F}$, a representation of the Poincarè group P on $\mathcal{H}$, and unitary operator S called the S-matrix of the theory . The physical content of the operator S is encoded in the "matrix elements" $\{(\phi, S\psi)\}$. More precisely $(\phi, S\psi)$ is the probability that given the entering system of particles ψ, the outgoing states ϕ is detected. A symmetry of the S-matrix is an (unbounded) operator B on $\mathcal{H}$ commuting with S, satisfying the Leibniz rule[1]. The rigorous version of Coleman Mandula argument then says that, if H is the sum of strictly positive mass representations of P and under appropriate conditions on S, the most general symmetry of S is a (representation of a) *direct product* of the Poincarè algebra and a reductive Lie algebra. This excludes the possibility of symmetries transforming bosons into fermions and *viceversa*.

The point is that in order to have a symmetry that can exchange bosons and fermions it is necessary that both concepts appear on equal footing in the geometry. This is not the case in usual geometry since everything at sight there commutes. In other words, for creating a symmetry that exchange bosons and fermions it is necessary to introduce fermions in the geometry. Once this was realized, physicists and mathematicians faced the problem of creating a new geometrical framework where usual commuting coordinates and "new" anticommuting ones can coexist. Somewhat simplifying the story, we can say that the problem was solved by Berezin and Kostant with the creation of supermathematics ([Ber87],[Kos77]). The basic facts of supergeometry are treated in chapter 3. There the notions of supermanifold and differential calculus over them are quickly reviewed. As it is pointed out in chapter 2, once the category of supermanifolds is constructed the definition of supergroup arise in a natural way. They are simply groups objectsin the category of supermanifolds. Super groups are the main device for implementing supersymmetry and their basic properties are treated in chapter 4. In that chapter, it is shown that many classical constructions carry over to the super setting. For example, the basic notion of infinitesimal symmetry (Lie algebra) is generalized in a natural way to supergroups leading to the concept of a super Lie algebra. Exactly as in ordinary Lie group theory, the super Lie algebra associated to a supergroup is defined as the vector space of left invariant vector fields. There is a large amount of literature devoted to the study of super Lie algebras. We quote at least the work of Kac ([Kac77]) on th classification of simple super Lie algebras, the classical work of Scheunert

[1]By this we mean that B can be extended to an operator on the Fock space such that $B(X \otimes Y) = B(X) \otimes Y + x \otimes B(Y)$.

[Sch79] and references therein.

Up to our knowledge the theory of representation of super Lie groups is much less developed than the corresponding theory for super Lie algebras. One of the reasons for this is surely the fact that a super Lie group is a quite difficult object to be treated. Let us now sketch where some of the difficulties arise. When one tries to introduce anticommuting variables θ_i, the first difficulty one encounters is that such θ_i are nilpotent of degree two (i.e. $\theta_i^2 = 0$). In other words, the quest for supersymmetry forces us to consider spaces on which "functions" are defined whose square is zero. In order to avoid confusion with ordinary real valued functions we will call such "functions", sections over the superspace M. If we leave for a while apart anticommutativity, we can notice that a similar situation is not new in mathematics. Grothendieck's revolution in algebraic geometry (the theory of schemes) consists exactly in this. Grothendieck realized that the geometry of a space can be enriched by allowing the existence of nilpotent elements among the sections defined over it. A supermanifold is then defined as an ordinary manifold M_0, called the *reduced manifold*, together with an algebra of sections locally isomorphic to $C^{\infty}(M_0) \otimes \Lambda(\theta^1, ..., \theta^q)$. Clearly the value of such nilpotent sections at a given point x of M is not defined. This has as a consequence that it is possible to consider spaces having the same set of points but with different geometries. For example, in Grothendieck's conception of space, the equations $y = 0$ and $y^2 = 0$ describe the same set of points but endowed with different geometric structures. This corresponds to the heuristic picture of viewing $y^2 = 0$ as a "double" line.

If we accept this point of view, we have to admit that points of a supermanifold are not important. The geometrical content is encoded in the algebra of sections defined over the supermanifold . Nevertheless such superalgebra has no simple algebraic structure (for example, due to topological subtleties, it is not a an Hopf algebra). This forces us to a deep study of the structure of supergroups. Such a study is the object of chapters 5 and 6.

In chapter 5, we review a fundamental result due to Kostant which is of fundamental importance for all subsequent developments. We prove that the category of supergroups is equivalent to the category of super Harish-Chandra pairs. Roughly speaking proving that two categories are equivalent means that each problem in one of the two categories can be translated in a completely equivalent one in the other and, moreover, that each solution to a given problem in one category gives a corresponding solution to the corresponding problem in the other category. This is a widely used approach in theoretical physics. For example whenever one substitutes a connected and simply connected Lie group with the corresponding and more manageable Lie algebra. This is similar to what happens in the case we are interested in. In fact Kostant proved that we can substitute a supergroup G with a pair consisting of the associated reduced Lie group G_0 and the super Lie algebra $\mathfrak{g}$ associated to G. Such a pair is what is called a super Harish-Chandra pair (see [DM99a]). This approach is the corner stone of our approach to representation theory of super semidirect products developed in the second part of the thesis. The first part ends with chapter 6. In such a chapter we establish a precise connection between two constructions: the functor of points approach to supergroups and an approach to supergroups that first appeared, actually in a very sketchy way, in the seminal work of Berezin [Ber87]. The main idea lying at the heart of the approach through the functor of points consists in associating to a supermanifold M a family $\{M(S)\}$ of sets, where

$$M(S) \quad := \quad \mathrm{Mor}(S, M)$$

consists of all the supermanifold morphisms between S and M, S varying over the whole category of supermanifolds. Quite remarkably, it turns out (Yoneda's lemma) that knowing the functorial family $\{M(S)\}$ is equivalent to knowing M. Even more is true. If G is a supergroup, then each $G(S)$ is a set-theoretical group and, in this case, the family of groups $\{G(S)\}$ completely determines the structure of the super Lie group G. In some sense this construction reduces the theory of super Lie groups to the theory of functorial families of ordinary set-theoretical groups. In this chapter it is shown that if $S = \mathbb{R}^{0|q}$ is a super-point then $G\left(\mathbb{R}^{0|q}\right)$ can be endowed with a Lie group structure and that the functorial family of ordinary Lie groups $\left\{G\left(\mathbb{R}^{0|q}\right)\right\}$ is rich enough for reconstructing the super Lie group G. It is also shown that

$$\mathrm{Lie}\left(G\left(\mathbb{R}^{0|q}\right)\right) \simeq (\mathfrak{g} \otimes \Lambda^q)_0$$

where $\mathfrak{g}$ is the super Lie algebra associated to G and Λ^q is the Grassmann algebra in q generators. This result is quite interesting, since it allows to establish a link between this approach and other constructions. Let us briefly give some details. Suppose $\mathfrak{g}$ a super Lie algebra. If we consider the even tensor product

$$\mathfrak{g}(\Lambda) := (\mathfrak{g} \otimes \Lambda)_0$$

with the bilinear form

$$[X \otimes \theta, Y \otimes \xi] = (-1)^{p(\xi)p(\theta)} [X, Y] \otimes \theta\xi$$

then $\mathfrak{g}(\Lambda)$ is an ordinary Lie algebra. It is well known (see [DM99a] and [Var04]) that the knowledge of $\mathfrak{g}$ is equivalent to the knowledge of the functorial family $\{\mathfrak{g}(\Lambda)\}$. Such a trick of introducing auxiliary odd parameters for making computations is commonly used by physicists and it is known under the name of *even rules principle*. We hence obtain that the group we get by exponentiating each $\mathfrak{g}(\Lambda)$ is exactly the Lie group $G\left(\mathbb{R}^{0|q}\right)$. Such result allows to replace the super Lie group G with the functorial family $G\left(\mathbb{R}^{0|q}\right)$. The advantage with respect to the functor of points approach being in the fact that we have now at disposal the power of ordinary Lie theory. How this approach can be useful is quickly explained in 9.0.11.

As it was already said, Part II of the thesis is devoted to the development of a representation theory for super Lie groups and it is based on [CCTV06]. The language adopted, as explained above, is that of super Harish-Chandra pairs. In chapter 7, we present the basic definitions and tools for such a program. In particular the concept of unitary representation of a super Harish-Chandra pair $(G_0, \mathfrak{g})$ on a super Hilbert space $\mathcal{H}$ is defined. This is, at least formally, a pair (π, ρ^π) consisting of a unitary representation π of G_0 and a representation ρ^π of $\mathfrak{g}$ satisfying appropriate compatibility relations. Nevertheless this step already presents some difficulties. It is in fact well known that if π is a unitary representation of the Lie group G_0 in an infinite dimensional Hilbert space then the differential $(\mathrm{d}\pi)(X)$, $X \in \mathfrak{g}_0$ is an unbounded operator. The fact that, given X and Y in $\mathfrak{g}_1$

$$[\rho^\pi(X), \rho^\pi(Y)] \subseteq \mathrm{d}\pi_0([X, Y])$$

implies that, in general, $\mathfrak{g}_1$ is also represented by unbounded operators. These considerations make apparent that great care is needed in defining infinite dimensional representations since the choice of an appropriate dense subspace is required. In this circle of ideas a major role is played by theorem 41. Such a theorem shows that each natural choice of a dense core for the operators $\{\rho^\pi(X)\}$ with $X \in \mathfrak{g}_1$ is actually equivalent to the choice of the subspace $C^\infty(\pi)$ of smooth vectors for the unitary representation π. In other words it is the even part of the representation that takes care of the functional analytical subtleties due to unboundedness of operators. The chapter ends proving the super version of Schur's lemma on the commutant of an irreducible representation.

In chapter 9 we address the problem of obtaining the UIRs of a class of groups large enough to cover some examples of interest for theoretical physics. In particular we consider the class of super-semidirect products. These are, by definition, super Lie groups whose reduced Lie group is an ordinary semidirect product

$$G_0 \quad \simeq \quad T_0 \rtimes L_0$$

T_0 being an abelian group, and such that

$$[\mathfrak{g}_1, \mathfrak{g}_1] \quad \subseteq \quad \mathfrak{t}_0$$

Super Poincarè groups in arbitrary dimensions are in this class. In developing the theory it is useful to keep in mind the route followed in the classical theory (see, for example,[Mac57], [Mac58] and [Mac76]) The basic ideas for determining the UIRs of a classical semi-direct product consists in associating to each UIR a corresponding transitive system of imprimitivity. Once this is done, the classical Imprimitivity theorem allows a complete geometrical classification of all transitive systems of imprimitivity and hence of all UIRs. In mathematical literature this method is called Mackey's machine while in the Physics literature the name *little group method* is often used. Like in the classical case, it can be shown that given a UIR of the SHCP $(G_0, \mathfrak{g})$ in the super Hilbert space $\mathcal{H}$, then it is induced (in a proper sense) by a representation of the stability group (or little group) that fixes some character of $\hat{T}_0$. It is from this point on that the SUSY theory acquires its own distinctive flavor. In the first place, unlike the classical situation, Theorem 7 stipulates that not all orbits are allowed, only those belonging to a suitable subset T_0^+. We shall call these orbits *admissible.* These are the orbits where the little super group admits an irreducible unitary representation which restricts to a character of T_0. These representations will be called *admissible.* These orbits satisfy a positivity condition which we interpret as the condition of positivity of energy. This condition is therefore necessary for admissibility. However it requires some effort to show that it is also sufficient for admissibility, and then to determine all the irreducible unitary representations of the little super Lie group at λ (Theorem 8).

In section 9.0.10 we discuss the case of the super Poincaré groups. We consider spacetimes of Minkowski signature and of arbitrary dimension $D \geq 4$ together with N-extended supersymmetry for arbitrary $N \geq 1$. In this case, we finally reach the conclusion that the super particles are parametrized by the admissible orbits and irreducible unitary representations of the stabilizers of the classical little groups, exactly as in the classical theory. The positive energy condition $\Phi_\lambda \geq 0$ becomes just that λ, which we replace by p to display the fact that it is a momentum vector, lies in the closure of the forward light cone. Thus the orbits of imaginary mass are excluded by supersymmetry (Theorem 10). Our approach enables us to handle super particles with *infinite spin* in the same manner as those with finite spin because of the result that the odd operators of the little group are *bounded* (Lemma 20).

Chapter 2

Abstract Nonsense

Or dunque seguitando quel discorso
Che non ho cominciato...

Dandini
La Cenerentola

The aim of this chapter is to show that it is possible to develop a quite general "theory of groups and group representations" in which all usual theories embeds. Though rather abstract this approach presents some advantages. For example whenever new geometrical structures arise (like, for example, super manifolds) it is possible to say immediately what a group and a group representation are in the new context. It can also be useful in order to point out the similarity between the theory of super Lie groups with other more known theories.

2.1 $\mathcal{C}$-groups and formal groups

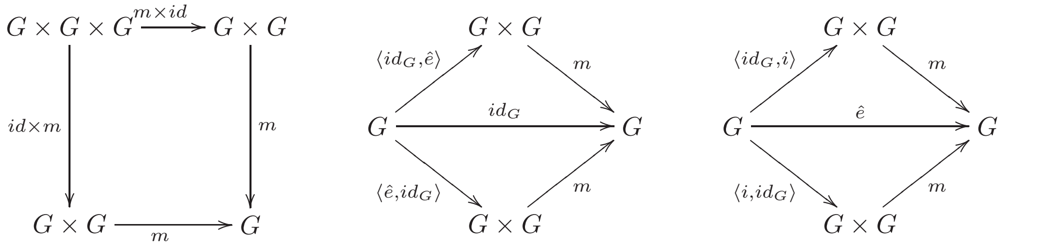
In this section we give the basic definitions in the theory of formal groups. Let $\mathcal{C}$ be a category. It is quite natural to say that $G \in \mathcal{O}bj\,(\mathcal{C})$ is a *group object* in the category $\mathcal{C}$ or briefly a $\mathcal{C}$-group if there exist morphisms μ, e and i which obey the usual rules for multiplication, unit and inverse respectively. More precisely:

Definition 1 *Let $\mathcal{C}$ be a (locally little) category with finite products and a final element $\overline{e}$. We say that (G, μ, i, e) is a* group object *in $\mathcal{C}$ if the following conditions are satisfied:*

1. *G is an object of $\mathcal{C}$*

2. *$\mu \in \mathrm{Mor}_{\mathcal{C}}\,(G \times G, G)$, $i \in \mathrm{Mor}_{\mathcal{C}}\,(G, G)$ and $e \in \mathrm{Mor}_{\mathcal{C}}\,(\overline{e}, G)$*

3. *the above morphisms satisfy the following commutative diagrams:*



where $\hat{e}$ denotes the composition of the identity $e : \overline{e} \longrightarrow G$ with the unique map $G \longrightarrow \overline{e}$. $\langle \psi, \phi \rangle$ denotes the map $\psi \times \phi \circ \operatorname{diag}$, $\operatorname{diag} : G \longrightarrow G \times G$ being the canonical diagonal map.

When no confusion is possible we will denote the $\mathcal{C}$-group (G, μ, i, e) simply by G.

Before giving some examples, we briefly comment about the hypothesis on $\mathcal{C}$ required by the above definition. All of them are satisfied in usual applications so we will be very concise. To say that a category is *locally little* means that $\operatorname{Mor}(X, Y)$ is a set for each X and Y in $\mathcal{C}$. The existence of *finite products* means essentially that given two objects X and Y there exists an object $X \times Y$ obeying the usual universal property of products. Finally, an object $\hat{e}$ in a category $\mathcal{C}$ is said *final* if for each object X in $\mathcal{C}$ there exists a unique morphism in $\operatorname{Mor}(X, \hat{e})$. It is easily shown that a final element if it exists is unique modulo isomorphisms.

Example 1 *If $\mathcal{C}$ is the category $\mathcal{S}et$ of sets, $\mathcal{T}op$ of topological spaces or $\mathcal{M}an$ of differentiable manifolds then a group object in $\mathcal{C}$ is a group, a topological group or a Lie group respectively.*

Example 2 *If $\mathcal{C}$ is the category of super manifolds $\mathcal{SM}$, then a group object G is called a super (or graded) group.*

We will of course return to example 2 in the following sections and chapters.

If $\mathcal{C}$ is a sub-category of $\mathcal{C}'$, then a $\mathcal{C}$-group G can be considered also as a $\mathcal{C}'$ group, the difference being in the morphisms. For example, if G is an algebraic group, it can also be considered as a Lie group or a topological group or a $\mathcal{S}et$ group. In a sense the structure of an object in a category is determined by morphisms. Next section shows a different approach to the theory of $\mathcal{C}$ groups which stresses the approach through morphisms.

2.2 The functor of points approach

We now show that, given a $\mathcal{C}$-group G, it is possible to construct a contravariant functor from the category $\mathcal{C}$ to the category of $\mathcal{S}et$-groups. Conversely given a functor constructed in this way it is possible to recover the $\mathcal{C}$-group G. This result is of fundamental importance in developing the theory since it allows to replace the $\mathcal{C}$-group G with the corresponding functor and this corresponds to the usual way in which computation are done in practice.

Let hence M be an object in $\mathcal{C}$ and consider the maps

$$\begin{array}{rcl} \mathcal{C} & \longrightarrow & \mathcal{S}et \\ S & \longmapsto & M(\cdot) := \operatorname{Mor}_{\mathcal{C}}(S, M) \end{array}$$

and

$$\begin{array}{rcl} \operatorname{Mor}_{\mathcal{C}}(S, T) & \longrightarrow & \operatorname{Mor}_{\mathcal{S}et}(M(T), M(S)) \\ \psi & \longmapsto & M(\psi) := (f \mapsto f \circ \psi) \end{array}$$

It is easy to see that they define a functor $M(\cdot)$ from $\mathcal{C}$ to the category of sets.

Definition 2 *Given a category $\mathcal{C}$ and an object M in $\mathcal{C}$, the functor*

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{S}et \\ S & \longmapsto & M(S) \end{array}$$

is called the **functor of points** *associated to M.*

The elements x_S of $M(S) = \mathrm{Mor}_{\mathcal{C}}(S, M)$ are called the S-points of M. If for example, we consider the category of manifolds $\mathcal{M}an$ and we take for S the manifold given by a single point pt, then the pt points of M are the usual set theoretical points of M. Clearly the knowledge of such point is not enough in order to determine the differentiable structure of M, but the following discussion will show that the knowledge of all S-points is in fact sufficient for this.

Next paragraph is devoted to discuss a very simple lemma which is nevertheless of fundamental importance for the development of the theory.

Yoneda lemma We have insofar defined an application that assigns to each object in the category $\mathcal{C}$ a functor from $\mathcal{C}$ to $\mathcal{S}et$:

$$M(\cdot) : \mathcal{C} \quad \longmapsto \quad \mathcal{S}et$$

Moreover if we define the map

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{C}}(M, N) & \longrightarrow & \mathrm{Mor}_{\mathcal{S}et}(M(S), N(S)) \\ \psi & \longmapsto & (x_S \longmapsto \psi \circ x_S) \end{array}$$

we have that each morphism in $\mathcal{C}$ induces a natural transformation between the corresponding functors of points. Since functors between two categories form by themselves a category with natural transformations as morphisms, it is easy to check that the above construction defines a covariant functor

$$\Upsilon : \mathcal{C} \quad \longrightarrow \quad [\mathcal{C}, \mathcal{S}\mathrm{et}]$$

Such functor is called **Yoneda's embedding**. In fact we have:

Lemma 1 (Yoneda) *There is a bijection between* $\mathrm{Mor}_{\mathcal{C}}(M, N)$ *and* $\mathrm{Mor}_{\mathcal{S}et}(M(S), N(S))$ *which is functorial in S.*

Proof. (**Sketch**) Let $\psi \in \mathrm{Mor}_{\mathcal{C}}(M, N)$ and define

$$\begin{array}{rcl} \psi(S) : \ M(S) & \longrightarrow & N(S) \\ x_S & \longmapsto & \psi \circ x_S \end{array}$$

It is easy to check that $\psi(\cdot)$ is a natural transformation between $M(\cdot)$ and $N(\cdot)$.

Conversely let $\mathcal{F}$ be a natural transformation between the two functors of points, consider

$$\mathcal{F}(M) : M(M) \quad \longrightarrow \quad N(M)$$

and define

$$\psi_{\mathcal{F}} \quad := \quad \mathcal{F}(M)(\mathrm{id}_M) \in \mathrm{Mor}_{\mathcal{C}}(M, N) \, .$$

Again, it is not difficult to show that

$$(\psi_{\mathcal{F}})(\cdot) = \mathcal{F}$$

■

From this we get, for example, the fact that the functor of points of an object M brings essentially the same information as the object M itself.

Since we are interested in $\mathcal{C}$-groups let us now specialize to the case in which M is in fact a $\mathcal{C}$-group and let us denote it by G. Each $G(S)$ inherits from G a group structure. Consider in fact the multiplication morphism m of G. Due to functoriality this gives a morphism

$$m(S) : (G \times G)(S) \longrightarrow G(S)$$

that, due to the identification

$$(G \times G)(S) \simeq G(S) \times G(S)$$

gives a map

$$m(S) : G(S) \times G(S) \longrightarrow G(S)$$

Similarly we get maps

$$\begin{aligned} i(S) : G(S) &\longrightarrow G(S) \\ e(S) : \hat{e} &\longrightarrow G(S) \end{aligned}$$

Due to functoriality all commutative diagrams of definition 1 are brought in analogous diagrams for $G(S)$, thus showing that $(G(S), m(S), i(S), e(S))$ is a $\mathcal{S}et$-group.

Yoneda's embedding is not, in general, surjective. A functor F from $\mathcal{C}$ to $\mathcal{S}et$ is said to be representable if it is naturally equivalent to a functor in the image of Υ or in other words we say that F is **representable** if there exists M in $\mathcal{C}$ such that $M(\cdot)$ and F are naturally equivalent.
It should be now evident that a $\mathcal{C}$-group is the same thing as a representable functor F from $\mathcal{C}$ to $\mathcal{S}et$ such that each $F(M)$ is a $\mathcal{S}et$-group in a functorial way.

2.3 The category of $\mathcal{C}$-groups

We now define (in a rather natural way) the sub-category $G\mathcal{C}$ of $\mathcal{C}$-groups for a fixed category $\mathcal{C}$. The objects of $G\mathcal{C}$ are clearly the $\mathcal{C}$-groups. Concerning morphisms, let G and G' be $\mathcal{C}$-groups. We say that ψ is in $\mathrm{Mor}_{G\mathcal{C}}(G, G')$ if $\psi \in \mathrm{Mor}_{\mathcal{C}}(G, G')$ and the following diagram commutes

$$\begin{array}{ccc} G \times G & \xrightarrow{\psi \times \psi} & G' \times G' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ G & \xrightarrow{\psi} & G' \end{array}$$

Passing to the corresponding functor of points we obtain that $\psi : G \longrightarrow G'$ is a morphism of $\mathcal{C}$-groups if and only if $\psi(S) : G(S) \longrightarrow G'(S)$ is a functorial family of morphisms of abstract groups.

Chapter 3

Supergeometry

Ecco qua: così stupendo,
sì balsamico elisire...

Dulcamara
L'elisir d'amore

In this chapter we are concerned with the description of the category $\mathcal{SM}$ of supermanifolds. In the first part we introduce the basic concepts of super differential geometry. Since the literature on the subject is rich of many comprehensive treatments no attempt is made of completeness and the first sections are only devoted to fix notations. A good knowledge of basic facts of the matter is assumed. The reader who need to know more can consult [Leĭ80], [Man97], [DM99a], [Var04].

3.1 Supermanifolds

Supermanifolds can be defined using the language of (graded) ringed spaces.

Graded ringed spaces Let M_0 be a topological space and denote by τ_{M_0} (or simply by τ, when no confusion is possible) the topology of M_0. We view τ as a category in which $\mathrm{Mor}_\tau(U, V)$ is defined to be the inclusion $i_{U,V} : U \hookrightarrow V$ if $U \subseteq V$, and empty otherwise. Let $\mathrm{g-Alg}$ denote the category of associative, graded commutative, unital and real graded algebras.

Definition 3 *Consider a topological space M_0. Let*

$$\mathcal{O}_M : \tau_{M_0} \longrightarrow \mathrm{g-Alg}$$

be a contravariant functor and let

$$\rho_{V,U} = \mathcal{O}_M(i_{U,V}) \qquad i_{U,V} \in \mathrm{Mor}_\tau(U, V)$$

$\mathcal{O}_M$ is called a **pre-sheaf** *over M_0 and $\rho_{V,U}$ are called the structural or restriction maps. The elements of $\mathcal{O}_M(U)$ are called sections over U.*
$\mathcal{O}_M$ is called a **sheaf** *over M_0 if it is a pre-sheaf and the following two conditions are satisfied*

i) *given* U *open in* M_0*, an open cover* $\{U_\alpha\}$ *of* U *and sections* s *and* s' *in* $\mathcal{O}(U)$ *then*

$$\rho_{U,U_\alpha}(s) \quad = \quad \rho_{U,U_\alpha}(s') \quad \forall \alpha$$

implies $s = s'$.

ii) *given* $\{U_\alpha\}$ *open cover of the open set* $U \subseteq M_0$ *and a family of sections* $\{s_\alpha\}$ *with* $s_\alpha \in \mathcal{O}(U_\alpha) \quad \forall\alpha$*, such that*

$$\rho_{U_\alpha, U_\beta \cap U_\alpha} s_\alpha \quad = \quad \rho_{U_\beta, U_\beta \cap U_\alpha} s_\beta \qquad \forall \alpha, \beta$$

then there exists a section s *in* $\mathcal{O}(U)$ *such that* $\rho_{U,U_\alpha} s = s_\alpha$

Remark 1 *If* s *is a section in* $\mathcal{O}_M(U)$*, and* $V \subset U$ *is open, the restriction of* s *to* V *is also denoted by* $s_{|V}$.

When no ambiguity is possible $\mathcal{O}_M(U)$ will simply be denoted $\mathcal{O}(U)$. Moreover $\mathcal{O}_M(M_0)$ will be often abbreviated in $\mathcal{O}(M)$. We will call a pair $M = (M_0, \mathcal{O}_M)$ a (graded) **ringed space**[1].

The local structure of a graded ringed space is encoded in the next construction. For each $x \in M_0$, one can consider the set $\mathcal{S}_x$ of sections defined in a neighborhood of x with the equivalence relation

$$s \sim s' \quad \Longleftrightarrow \quad \exists U \ni x \text{ such that } s_{|U} = s'_{|U}$$

The quotient space

$$\mathcal{O}(M)_x \quad := \quad \mathcal{S}_x / \sim$$

is called the **stalk** of the sheaf at x . Elements of the stalk are equivalence classes of sections coinciding in an arbitrary neighborhood of x.

Suppose that $(M_0, \mathcal{O}_M)$ and $(N_0, \mathcal{O}_N)$ are ringed space and let $\psi_0 : M_0 \longrightarrow N_0$ be a continuous mapping, then the *push-forward* of $\mathcal{O}_M$ on N_0 is the sheaf over N_0 defined by:

$$\begin{aligned} (\psi_{0*}\mathcal{O}_M) : \tau_{N_0} &\longrightarrow g - Alg \\ U &\longmapsto \mathcal{O}_M\left(\psi_0^{-1}(U)\right) \end{aligned}$$

Definition 4 *Let* $(M_0, \mathcal{O}_M)$ *and* $(N_0, \mathcal{O}_N)$ *be graded ringed spaces. A* **morphism** *of* $(M_0, \mathcal{O}_M)$ *in* $(N_0, \mathcal{O}_N)$*, with reduced morphism* ψ_0*, is a a natural transformation between* $\mathcal{O}_N$ *and* $\psi_{0*}\mathcal{O}_M$*. A morphism of (graded) ringed spaces* $\psi : M \to N$ *is a pair* $\psi = (\psi_0, \psi^*)$ *consisting of a continuous map* $\psi_0 : M_0 \to N_0$ *and a morphism* ψ^* *with reduced morphism* ψ_0.

In order to define super manifolds we need first to define local models for them.

Superdomains and their morphisms. Let U be an open subset of $\mathbb{R}^p$ with the usual relative topology. We define the *super domain* $U^{p|q}$ of dimension (p, q) as the ringed space whose topological manifold is U and whose sheaf of sections is given by

$$\mathcal{O}(V) \quad = \quad \mathcal{C}^\infty(V) \otimes \Lambda\left(\theta^1, ..., \theta^q\right)$$

[1]This terminology is misleading since the elements of the sheaf are not only rings but also algebras. Nevertheless we stick to this convention.

where V is an open set in U. If we consider cartesian coordinates $x^1, ..., x^p$ on V, then the system $x^1, ..., x^p, \theta^1, ..., \theta^q$ is called a canonical super coordinate system on V. The $\{x^i\}$ are called even coordinates while the $\{\theta^j\}$ are called odd coordinates.

A *morphism* of super domains

$$\phi : U^{p|q} \longrightarrow V^{r|s}$$

is defined as a morphism of graded ringed spaces $\phi = (\phi_0, \phi^*)$:

$$\begin{array}{rcl} \phi_0 : U^p & \longrightarrow & V^r \\ \phi^* : \mathcal{O}_V & \longrightarrow & \phi_{0*}(\mathcal{O}_U) \end{array}$$

Theorem 1 below shows that ϕ_0 is completely determined by ϕ^* and that it is automatically smooth.

Remark 2 *In the non graded setting (*i.e. $q = s = 0$*) it is well known that ϕ_0 and ϕ^* determine each other. Due to the presence of nilpotent elements this is no longer true in the super context.*

For each $x \in U^{p|q}$ one can also define the evaluation map at x as

$$\begin{array}{rcl} \delta_x : \mathcal{O}\left(U^{p|q}\right) & \longrightarrow & \mathbb{R} \\ \sum_I s_I \theta^I & \longmapsto & s_0(x) \end{array}$$

The following criterion is widely used in order to prove that a given section is zero.

Proposition 1 *Let $U^{p|q}$ be a superdomain with coordinates $x^1, ..., x^p, \theta^1, ..., \theta^q$. Let I_x be the ideal in $\mathcal{O}\left(U^{p|q}\right)$ generated by*

$$x^1 - \delta_x(x^1), ..., x^p - \delta_x(x^p), \theta^1, ..., \theta^q$$

then for every section $f \in \mathcal{O}\left(U^{p|q}\right)$ and any integer $k \geq 0$, there exists a polynomial P_k in the coordinates $x^1, ..., x^p, \theta^1, ..., \theta^q$ such that

$$f - P_k \quad \in \quad I_x^{k+1}$$

Proof. See [Leĭ80]. ■

Supermanifolds, reduced manifolds and morphisms At this point next definition is quite natural.

Definition 5 *A supermanifold M (also called graded manifold) of dimension (p, q) is a ringed space $(M_0, \mathcal{O}_M)$ such that*

i) M_0 is a Hausdorff, locally compact, second countable topological space;

ii) $\forall x \in M_0$ there exists an open neighborhood $U \ni x$ such that

$$\mathcal{O}_M(U) \simeq \mathcal{O}(V^{p|q})$$

where $V^{p|q}$ is a superdomain.

Remark 3 *Historically graded and super manifolds originated as different objects. Both were created in order to give a geometrical framework for supersymmetry. Graded manifolds were created in analogy with (commutative) algebraic geometry (see, for example, [Ber87], [Kos77], [Man97], [DM99a], [Var04]) while super manifolds were defined following the pattern of classical differential geometry (see [Bat79]). Eventually Batchelor proved (see [Bat80]) that there is an equivalence between the two categories. Actually we are exposing here the graded point of view, nevertheless we will use freely also the "super"* prefix.

Definition 6 *A* **morphism** $\phi : M \longrightarrow N$ *of super manifolds is simply a morphism* (ϕ_0, ϕ^*) *of the corresponding ringed spaces.*

Next theorem is of fundamental importance in handling morphisms, since it says that a morphism is completely determined by its action on coordinates.

Theorem 1 *Let* $U^{p|q}$ *be a super domain with graded coordinate* x^i, θ^j*, and let* $(M_0, \mathcal{O}_M)$ *be a supermanifold. If* $\psi : M \longrightarrow U^{p|q}$ *is a morphism of supermanifolds then there are sections* $\{f_i\}_{i=1}^p \in \mathcal{O}(M)_0$ *and* $\{\xi_j\}_{j=1}^q \in \mathcal{O}(M)_1$ *such that*

$$\begin{cases} \psi^*(x^i) = f^i \\ \psi^*(\theta^j) = \xi^j \end{cases} \tag{3.1}$$

Conversely for each element $\{f^i, \xi^j\}$ *in* $\mathcal{O}(M)_0^p \oplus \mathcal{O}(M)_1^q$ *there exists a unique morphism* (ψ_0, ψ^*) *of* $U^{p|q}$ *into* M *such that 3.1 holds.*

Proof. See [Leĭ80] or [Var04]. ■

Let $M = (M_0, \mathcal{O}_M)$ be a supermanifold. For each $x \in U$ define the map

$$\begin{array}{rcl} \delta_x : \mathcal{O}(M) & \longrightarrow & \mathbb{R} \\ s & \longmapsto & \tilde{s}(x) \end{array}$$

where $\tilde{s}(x)$ is defined as the unique real number such that $s - \tilde{s}(x)$ is not invertible in any neighborhood of x. It is easily shown that δ_x is a graded algebra morphism and it moreover descends to a morphism

$$\delta_x : \mathcal{O}(M)_x \longrightarrow \mathbb{R}$$

This means that

$$J_x = \ker \delta_x$$

is a graded ideal of $\mathcal{O}(M)$.

For each open set we can hence define

$$\begin{array}{rcl} \tilde{} : \mathcal{O}_M(U) & \longrightarrow & \mathcal{C}^\infty(U) \\ s & \longmapsto & (x \longmapsto \tilde{s}(x)) \end{array}$$

whose kernel $J_M(U)$ is given by the ideal of nilpotent elements in $\mathcal{O}_M(U)$. Locally such an ideal is generated by $\mathcal{O}_M(U)_1$. Hence

$$\mathcal{C}^\infty(U) \simeq \mathcal{O}_M(U)/J_M(U)$$

It can be shown that $\tilde{M} = \left(M_0, \tilde{\mathcal{O}_M}\right)$ is a purely even manifold called the **reduced manifold** associated to M. One has a natural *inclusion* of $\tilde{M}$ into M defined by

$$i_0 = \mathrm{id}_{M_0}$$

and

$$i^* : \mathcal{O}_M(U) \longrightarrow \mathcal{O}_M(U)/J_M(U) \simeq \mathcal{C}^\infty(U)$$

Suppose now ψ is a morphism of graded manifolds. Due to the fact that

$$\psi^*(J_N) \subseteq J_M$$

we have a corresponding morphism of the associated reduced manifolds defined by

$$\mathcal{C}^\infty(N) \simeq \mathcal{O}(N)/J(N) \longrightarrow \mathcal{O}(M)/\psi^*(J_N) \longrightarrow \mathcal{O}(M)/J_M \longrightarrow \mathcal{C}^\infty(M)$$

A straightforward calculation shows that

$$\begin{aligned} \widetilde{\phi \circ \psi} &= \widetilde{\phi} \circ \widetilde{\psi} \\ \widetilde{1_M} &= 1_{\widetilde{M}} \end{aligned}$$

Summarizing we have

Proposition 2 *The assignment*

$$\begin{aligned} \mathcal{SM} &\longrightarrow \mathcal{M}an \\ M &\longmapsto \widetilde{M} \\ \phi &\longmapsto \tilde{\phi} \end{aligned}$$

which assigns to a supermanifold the corresponding reduced manifold defines a functor.

3.2 Super vector spaces as supermanifolds

The reader can be confused by the fact that there does not seem to exist any relation between a super vector space and a super manifold. In particular a super vector space is a purely algebraic object with no sheaf of sections defined over it. Nevertheless we use the same notation $\mathbb{R}^{p|q}$ to indicate both a super vector space and a supermanifold. We now show how it is possible to associate in a canonical way to each super vector space a corresponding supermanifold.

Let $V = V_0 \oplus V_1$ be a super vector space and consider the contravariant functor

$$\mathcal{F}_V : \mathcal{SM} \longrightarrow \mathcal{S}et$$

defined by

$$\begin{aligned} \mathrm{Obj}(\mathcal{SM}) &\longrightarrow \mathrm{Obj}(\mathcal{S}et) \\ S &\longmapsto (\mathcal{O}(S) \otimes V)_0 \end{aligned}$$

and by

$$\begin{array}{rcl} \mathrm{Mor}\,(S,T) & \longrightarrow & \mathrm{Mor}\,(\mathcal{F}_V(T),\mathcal{F}_V(S)) \\ \psi & \longmapsto & \psi^* \otimes \mathrm{id} \end{array}$$

In the remaining part of this section, we will show that such a functor is represented by the super manifold

$$\hat{V} \quad := \quad (V_0, \mathcal{C}^\infty(V_0) \otimes \Lambda\,(V_1^*))$$

Indeed, if we fix a homogeneous basis $\left(e_i^*, f_j^*\right)$ and define the maps

$$\begin{array}{rcl} \hat{V}\,(S) := \mathrm{Mor}\left(S, \hat{V}\right) & \longrightarrow & \mathcal{F}_V\,(S) = (\mathcal{O}\,(S) \otimes V)_0 \\ \psi & \longmapsto & \sum \psi^*(e_i^*) \otimes e_i + \sum \psi^*(f_j^*) \otimes f_j \end{array}$$

it turns out, due to theorem 1, that all such maps are bijections. Suppose now that a morphism of graded vector spaces

$$f : V \quad \longrightarrow \quad W$$

is given. The map

$$\begin{array}{rcl} \mathcal{F}_V\,(S) & \longrightarrow & \mathcal{F}_W(S) \\ (\mathcal{O}\,(S) \otimes V)_0 & \overset{\mathrm{id}\otimes f}{\longmapsto} & (\mathcal{O}\,(S) \otimes W)_0 \end{array}$$

is a natural transformation that, due to Yoneda's lemma, descends to a morphism of the representing supermanifolds:

$$f : (V_0, \mathcal{C}^\infty(V_0) \otimes \Lambda\,(V_1^*)) \quad \longrightarrow \quad (W_0, \mathcal{C}^\infty(W_0) \otimes \Lambda\,(W_1^*))$$

Remark 4 *Due to this result, we will henceforth use the same letter to indicate the super vector space and the corresponding super manifold.*

Example 3 *Consider the graded vector space $\mathbb{R}^{1|1}$. We endow it with the structure of a super algebra (with identity e) defining*

$$\begin{array}{rcll} m : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} & \longmapsto & \mathbb{R}^{1|1} & (3.2) \\ (e,e) & \longmapsto & e & (3.3) \\ (f,f) & \longmapsto & -e & (3.4) \\ (e,f) & \longmapsto & f & (3.5) \end{array}$$

where (e,f) is the canonical basis of $\mathbb{R}^{1|1}$. According to our previous discussion, an easy calculation shows that the morphism induced at the super manifold level is given by:

$$\begin{array}{rcll} m^* : \mathcal{O}\left(\mathbb{R}^{1|1}\right) & \longmapsto & \mathcal{O}\left(\mathbb{R}^{1|1} \times \mathbb{R}^{1|1}\right) & (3.6) \\ e^* & \longmapsto & (e^* \otimes e^* + f^* \otimes f^*) & (3.7) \\ f^* & \longmapsto & (e^* \otimes f^* + f^* \otimes e^*) & (3.8) \\ & & & (3.9) \end{array}$$

3.3 Tangent structures

Vector fields over a supermanifold M are defined in complete analogy with the classical case.

Definition 7 *Let M be a supermanifold. A vector field X of degree $p(X)$ over M is a family of maps*

$$X_U : \mathcal{O}_M(U) \longrightarrow \mathcal{O}_M(U)$$

such that

$$X_U(ss') = X_U(s)s' + (-1)^{p(X)p(s)} s X_U(s')$$

and compatible with restrictions.

Vector fields dfine a sheaf of $\mathcal{O}_M$-modules called the tangent sheaf and denoted by $\mathcal{T}(M)$. If $U^{p|q}$ is a superdomain then

$$\frac{\partial}{\partial x^i}\left(\sum_I s_I \theta^I\right) := \sum_I \frac{\partial s_I}{\partial x^i}\theta^I$$

$$\frac{\partial}{\partial \theta^j}\left(\sum_{j\notin I} s_I \theta^I + \sum_{j\notin I} s_{jI}\theta^j\theta^I\right) := \sum_{j\notin I} \frac{\partial s_{jI}}{\partial \theta^j}\theta^I$$

are respectively even and odd vector fields. It can be proved that $\mathcal{T}(M)$ is a locally free $\mathcal{O}_M$-module locally isomorphic to $\mathcal{O}_M(U)_0^p \oplus \mathcal{O}_M(U)_1^q$ for which $\left\{\frac{\partial}{\partial x^i}\right\}$ and $\frac{\partial}{\partial \theta^j}$ form a homogeneous basis.

Definition 8 *A tangent vector v at x in M of degree $p(x)$ is a derivation of the stalk $\mathcal{O}(M)_x$ i.e. a linear map*

$$v : \mathcal{O}(M)_x \longrightarrow \mathbb{R}$$

such that

$$v([f][g]) = v([f])\,\tilde{g}(x) + (-1)^{p(v)p(f)}\tilde{f}(x)v([g])$$

Remark 5 *Notice that the minus sign in the last formula is rather useless, since if f is odd, its value at x is zero.*

If X is a vector field defined in a neighborhood of x, then

$$(X)_x = \delta_x \circ X : [f] \longrightarrow \widetilde{X(f)}(x)$$

is a tangent vector at x of the same parity as X. Clearly a vector field X is not determined by its a values at points. It can be proved that tangent vectors at x form a super vector space $T_x(M)$ of the same dimension as M and that

$$\left\{\left(\frac{\partial}{\partial x^i}\right)_x\right\}_{i=1}^p \qquad \left\{\left(\frac{\partial}{\partial \theta^j}\right)_x\right\}_{j=1}^q$$

define a homogeneous basis of $T_x(M)$.

3.4 The sheaf $\mathcal{O}_M$

In this section we want to collect some results concerning the sheaf of sections over a super manifold M. We will state and prove some technical results like the existence of *partition of unity* over M and we will also show that it is possible to endow each super algebra $\mathcal{O}(U)$ with a topology in such a way that it becomes a Fréchet algebra. We will end this section showing that it is possible to reconstruct the sheaf of M knowing the super algebra of global sections.

3.4.1 Differential operators

We define the **differential operators** $\mathcal{D}(M) \subset \mathrm{End}_{\mathbb{R}}(\mathcal{O}(M))$ over a super manifold M. Following [Kos77], the space $\mathcal{D}_k(M)$ of differential operators of degree k is defined by induction on the degree. In particular $\mathcal{D}_0(M) \simeq \mathcal{O}(M)$ is the space of multiplicative operators, while

$$\mathcal{D}_k(M) = \{D \in \mathrm{End}_{\mathbb{R}}(\mathcal{O}(M)) \mid [D,f] \in \mathcal{D}_{k-1}(M), \forall f \in \mathcal{O}(M)\}$$

where $[D,f]$ denotes the commutator in $\mathrm{End}_{\mathbb{R}}(\mathcal{O}(M))$, that is

$$[D,f] = Df - (-1)^{p(D)p(f)} fD$$

or

$$[D,f](s) = D(fs) - (-1)^{p(D)p(f)} fD(s)$$

One has

$$\mathcal{D}_{k-1}(\mathcal{O}(M)) \subseteq \mathcal{D}_k(M)$$

and by definition

$$\mathcal{D}(\mathcal{O}(M)) := \bigcup_{k=0}^{\infty} \mathcal{D}_k(M)$$

It is easily shown that $\mathcal{D}(M)$ is a sub superalgebra of $\mathrm{End}_{\mathbb{R}}(\mathcal{O}(M))$. In particular

$$\mathcal{D}_1(\mathcal{O}(M)) = \mathcal{O}(M) \oplus \mathcal{T}(M)$$

where $\mathcal{T}(M)$ are, by definition, the vector fields over M. Moreover the assignment

$$U \longmapsto \mathcal{D}(U) \quad U \subset \tilde{M} \text{ open}$$

is a sheaf of $\mathcal{O}_M$- modules. It is possible to show that, if U is an open set with coordinate (x^i, θ^j), then every $D \in \mathcal{D}(M)$ admits a unique expression of the form

$$D = \sum_{I,J} a_{I,J} \frac{\partial^{|I|}}{\partial x^I} \frac{\partial^{|J|}}{\partial \theta^J} \quad a_{I,J} \in \mathcal{O}(U)$$

where $I = (i_1, ..., i_p)$, $i_r \in \mathbb{N}$ and $J = (j_1, ..., j_q)$, $j_r \in \mathbb{Z}_2$.

3.4.2 Partitions of unity

Let us abbreviate $\mathcal{O}_M(U)$ by $\mathcal{O}(U)$. Like in classical differential geometry partitions of unity will be useful in reducing arguments to coordinate neighborhoods. Suppose s to be a section over U and consider the set of points $x \in U$ for which there exists an open neighborhood $V \subseteq U$ such that the restriction of s to V is zero. This is an open set whose complement is called the *support* of s. The proof of the following proposition is given in [Kos77].

Proposition 3 *Let $\{U_i\}_{i\in I}$ be an open covering of M_0 then there exist a locally finite refinement $\{V_j\}$ where each V_j is a coordinate open set and a family $\{\psi_j\}$ of sections such that*

i) $\psi_j \in \mathcal{O}(M)_0$

ii) $\text{supp}\psi_j \subset V_j$ *for each* j

iii) $\sum \psi_j = 1$ *and* $\tilde{\psi_j} \geq 0$ *for each* j

We will put at work partition of unity already in the next section

3.4.3 Topology of $\mathcal{O}(M)$

$\mathcal{O}(U)$ is an (infinite dimensional) super algebra over $\mathbb{R}$ and we want to define a topology over it.

Suppose M is an ordinary manifold, one of the most common choices is given by the open compact topology which is defined as follows. Let us denote by $\mathcal{K}$ the set of compact subset of M and by $\mathcal{D}$ the space of differential operators over M (see [Var84], for example). Define the map

$$\begin{array}{rcl} p_{D,K} : \mathcal{O}(M) & \longrightarrow & \mathbb{R} \\ f & \longmapsto & \sup_{x\in K} |Df| \end{array}$$

with $D \in \mathcal{D}$ and $K \in \mathcal{K}$. It is well known (see [War83]) that each $p_{D,K}$ is a seminorm and that if we endow $\mathcal{O}(M)$ with the topology defined by the family $\{p_{D,K}\}_{D\in\mathcal{D},K\in\mathcal{K}}$ then $\mathcal{O}(M)$ becomes a Fréchet algebra that is a locally convex, metrizable, complete and Hausdorff algebra. Endowed with this topology, $\mathcal{O}(M)$ acquires some pleasant property:

i) $\mathcal{C}_c^\infty(M)$ is dense in $\mathcal{O}(M)$

ii) differential operators are continuous

iii) the topological dual of $\mathcal{O}(M)$ is the space of compactly supported distributions

iv) it is well behaved with respect to Cartesian products, in the sense that (see [Gro52])

$$\mathcal{O}(M\times N) = \overline{\mathcal{O}(M)\otimes\mathcal{O}(N)}$$

The generalization of these results to the super setting presents no difficulty and we sketch it now very briefly.

Let hence M denote a super manifold, $\mathcal{K}$ the set of compact subset of M_0 and by $\mathcal{D}$ the space of differential operators over M. Similarly to (3.10), we define the maps

$$\begin{array}{rcl} p_{D,K} : \mathcal{O}(M) & \longrightarrow & \mathbb{R} \\ f & \longmapsto & \sup_{x\in K} \left|\widetilde{Df}(x)\right| \end{array}$$

where $D \in \mathcal{K}$ and $D \in \mathcal{D}$.

Lemma 2 *The family* $\{p_{D,K}\}_{D\in\mathcal{D},K\in\mathcal{K}}$ *is a separating family of seminorms*

Proof. It is immediate from the fact that $K \subseteq K'$ implies $p_{D,K} \leq p_{D,K'}$ and using local coordinates. ■

This lemma tells us that the family $\{p_{D,K}\}_{D\in\mathcal{D},K\in\mathcal{K}}$ defines a locally convex Hausdorff topological space structure on $\mathcal{O}(M)$. Metrizability follows in the usual way from paracompactness of M_0 (see [War83]).

We now show that it is possible to describe quite explicitly the topology of $\mathcal{O}(M)$.

Proposition 4 *Let* $\{U\}_\alpha$ *be an open cover of* $\tilde{M}$ *and let* $\{s_n\} \subset \mathcal{O}(M)$ *be a sequence.*

i) $\{s_n\}$ *converges to* s *in* $\mathcal{O}(M)$ *if and only if, for each* U_α, $\left\{s_{n\,|U_\alpha}\right\}$ *converges to* $s_{\,|U_\alpha}$ *in* $\mathcal{O}(U_\alpha)$.

ii) $\{s_n\}$ *is a Cauchy sequence if and only if, for each* U_α, $\left\{s_{n\,|U_\alpha}\right\}$ *is a Cauchy sequence.*

Proof. It is not restrictive to suppose that s_n converges to zero. The necessity is obvious. Let $D \in \mathcal{D}(M)$ and K compact subset of $\tilde{M}$. Let $\{V_\beta\}$ be a finite open cover of K such that each V_β is a compact closure open set with $V_\beta \subseteq U_\alpha$, for some α. If $\left\{s_{n\,|U_\alpha}\right\}$ converges to zero on each U_α, then also $p_{D,K}(s_n)$ goes to zero since

$$p_{D,K}(s_n) \quad \leq \quad \sum_\beta p_{K\cap\overline{V_\beta},D}(s_n)$$

The second part of the proposition is similar. ■

Finally we have

Proposition 5 $\mathcal{O}_M(U)$ *is complete for each* U.

Proof. Let $\{s_n\}_n$ be a Cauchy sequence in $\mathcal{O}_M(U)$ and fix a denumerable cover $\{V_\alpha\}$ of coordinate neighborhoods. Due to prop. 4, $\left\{s_{n\,|V_\alpha}\right\}$ is Cauchy for each V_α. Since $\mathcal{C}^\infty(V_\alpha) \otimes \Lambda^q$ is clearly complete in the open compact topology we have that there exists $s_\alpha \in \mathcal{O}_M(U)$ such that

$$\left\{s_{n\,|V_\alpha}\right\} \quad \longrightarrow \quad s_\alpha$$

By continuity of the restricted maps, on overlaps we have

$$\begin{aligned} \left\{s_{n\,|V_\alpha\,|V_\alpha\cap V_\beta}\right\} \quad &\longrightarrow \quad s_{\,\alpha|V_\alpha\cap V_\beta} \\ \left\{s_{n\,|V_\alpha\cap V_\beta\,|V_\alpha}\right\} \quad &\longrightarrow \quad s_{\,\beta|V_\alpha\cap V_\beta} \end{aligned}$$

But $s_{n\,|V_\alpha\,|V_\alpha\cap V_\beta} = s_{n\,|V_\beta\,|V_\alpha\cap V_\beta} = s_{n\,|V_\alpha\cap V_\beta}$, and we are done ■

It is clear that all differential operators are continuous in such a topology. The following results are proved in detail in [Bal05].

Proposition 6 *i) If $\psi : M \longrightarrow N$ is a morphism of supermanifolds then the corresponding pull-back*

$$\psi^* : \mathcal{O}(N) \longrightarrow \mathcal{O}(M)$$

is continuous.

ii) Vector fields, tangent vectors and differential operators are continuous.

Proposition 7 *$\mathcal{O}(M) \otimes \mathcal{O}(N)$ is dense in $\mathcal{O}(M \times N)$.*

We end this section with a conceptually important result. We show that the knowledge of the graded algebra $\mathcal{O}(M)$ of global sections allows to reconstruct the whole sheaf $\mathcal{O}_M$. Actually even more is true: the knowledge of $\mathcal{O}(M)$ is equivalent to the knowledge of the manifold M itself so that the well known duality between manifolds and sections also holds in the graded setting.
Suppose the global sections $\mathcal{O}(M)$ of a super manifold M are given, and let U denote an open subset of M_0. Let us define

$$\mathcal{S}_U = \{s \in \mathcal{O}(M)_0 \mid \tilde{s}(x) \neq 0 \, \forall x \in U\}$$

Consider the Cartesian product $\mathcal{O}(M) \times \mathcal{S}_U$, define the equivalence relation

$$(f, s) \sim (f', s') \iff \exists s'' \in \mathcal{S}_U \text{ such that } s''(s'f - sf') = 0$$

and denote

$$\mathcal{O}(M)\,\mathcal{S}_U^{-1} = \mathcal{O}(M) \times \mathcal{S}_U / \sim$$

Proposition 8 *The localization map*

$$\begin{aligned} \ell : \mathcal{O}(M)\,\mathcal{S}_U^{-1} &\longrightarrow \mathcal{O}(U) \\ (f, s) &\longmapsto f_{|U}\left(s_{|}\right)^{-1} \end{aligned}$$

is a graded algebra isomorphism.

Proof. We follow the proof given in [BBHR91]. Let us start proving *injectivity*. Suppose hence $f_{|U}\left(g_{|U}\right)^{-1} = 0$. This is equivalent to $f_{|U} = 0$ and we are reduced to prove that there exists $g \in \mathcal{S}_U$ such that $gf = 0$. Cover M with a family $\{V_\alpha\}$ of coordinate neighborhoods and let $\{\psi_\alpha\}$ be a subordinate partition of unity

$$f = \sum f\psi_\alpha = \sum f_\alpha$$

For each $V_\alpha \cap U$ define a section $g_\alpha \in \mathcal{C}^\infty(V_\alpha \cap U) \otimes \Lambda \simeq \mathcal{O}(V_\alpha \cap U)$ such that $g_\alpha > 0$ on $V_\alpha \cap U$ and $\text{supp} g_\alpha = \overline{V_\alpha \cap U}$. Finally define

$$g = \sum_\alpha \psi_\alpha g_\alpha$$

g is strictly positive on U by construction, and zero on $M \setminus U$, hence $gf = 0$.

We now come to *surjectivity*. Let f be a section in $\mathcal{O}_M(U)$. We want to express f as a ratio of an element $g \in \mathcal{O}(M)$ with an element $h \in \mathcal{S}_U$.

Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of U such that $\overline{U_\alpha} \subset U$ for each α. Due to Lindeloff, it is not restrictive to assume I equal to $\mathbb{N}$. Let ψ_n be even sections such that

$$\begin{cases} \mathrm{supp}\psi_n \subset \overline{U_n} \\ \psi_n > 0 \text{ on } U_n \end{cases}$$

Let us define

$$\sum_n c_n \psi_n f$$

We look for constants c_n such that

$$\begin{aligned} g &= \lim_{n\to\infty} \sum_n c_n \psi_n f \quad \in \mathcal{O}(M) \\ h &= \lim_{n\to\infty} \sum_n c_n \psi_n \quad \in \mathcal{S}_U \end{aligned}$$

Let us consider the first equation. Being $\mathcal{O}(M)$ complete, convergence can be checked using Cauchy criterion: fix an increasing family of seminorms $\{p_j\}$ defining the topology of $\mathcal{O}(M)$,

$$p_j\left(\sum_{i=k}^{k+r} c_i \psi_i f\right) \leq \sum_{i=k}^{k+r} |c_i|\, p_j(\psi_i f)$$

This suggests to define

$$c_i = \frac{1}{2^i} \cdot \frac{1}{d_i + p_i(\psi_i f)}$$

so that

$$p_j\left(\sum_{i=k}^{k+r} c_i \psi_i f\right) \leq \sum_{i=k}^{k+r} \frac{1}{2^n} \frac{p_j(\psi_i f)}{d_i + p_i(\psi_i f)}$$

If now d_i are numbers greater than 0, due to the fact that $p_j \leq p_i$ definitively, the above sequence is Cauchy and hence converges. Analogously the second equations suggests to define

$$c_i = \frac{1}{2^i} \cdot \frac{1}{d_i' + p_i(\psi_i)}$$

and hence

$$\begin{aligned} g &= \lim_{n\to\infty} \sum_n \frac{1}{2^n} \cdot \frac{1}{1 + p_n(\psi_n) + p_n(\psi_n f)} \psi_n f \\ h &= \lim_{n\to\infty} \sum_n \frac{1}{2^n} \cdot \frac{1}{1 + p_n(\psi_n) + p_n(\psi_n f)} \psi_n \end{aligned}$$

have the required properties. ∎

Next proposition says that morphisms of supermanifolds are determined by the pull-backs on global sections.

Proposition 9 *Let $\psi : \mathcal{O}(N) \longrightarrow \mathcal{O}(M)$ be a mapping of supermanifolds, then there exists a unique*

$$\phi : M \quad \longrightarrow \quad N$$

such that $\phi^* = \psi$

Proof. (**sketch**) Define the map

$$\phi_0^* : \mathcal{O}(N) / J_N(N) \quad \longrightarrow \quad \mathcal{C}^\infty(M)$$

From classical theory ϕ_0^* determines a unique $\phi_0 : \tilde{M} \longrightarrow \tilde{N}$ and

$$\begin{array}{rcl} \phi^* : \mathcal{O}_N(U) \simeq \mathcal{O}(N) \mathcal{S}^{-1}_{\phi_0^{-1}(U)} & \longrightarrow & \mathcal{O}_M(U) \simeq \mathcal{O}(M) \mathcal{S}_U^{-1} \\ (f, s) & \longmapsto & (\psi(f), \psi(s)) \end{array}$$

■

3.5 $\mathcal{O}_M^\circ$: the space of distribution with finite support

We will not need the general theory of distribution over a super manifold M. We content ourselves to characterize the distributions with finite support. More precisely we want to characterize those elements in the algebraic dual $\mathcal{O}(M)^*$ of $\mathcal{O}(M)$ whose support is finite. We recall that the *support* of an element T in $\mathcal{O}(M)^*$ is the complement of the set of those points in M_0 for which there exists an open neighborhood U_x such that

$$\langle T, \phi \rangle \quad = \quad 0$$

for each $\phi \in \mathcal{O}(M)$ with support in U_x.

Denote by

$$(\mathcal{O}_M^\circ)_x$$

the subspace of distributions with support at x and suppose that

$$T : \mathcal{O}(M) \quad \longrightarrow \quad \mathbb{R}$$

belongs to $(\mathcal{O}_M^\circ)_x$. Let U be a coordinate neighborhood of x. It is easy to check that T descends to a continuous linear functional on $\mathcal{O}(U) \simeq \mathcal{C}^\infty(U) \otimes \Lambda^q$. Consequently we have that $(\mathcal{O}_M^\circ)_x \simeq \mathcal{C}^\infty(U)_x \otimes \Lambda^{q*}$ where $\mathcal{C}^\infty(U)_x$ denotes the space of classical distributions with support at x. It is well known that the elements of $\mathcal{C}^\infty(U)'$ with support at x are exactly the derivatives of the delta distribution at x_0 (see, for example, [Hor66]). We can hence write

$$(\mathcal{O}_M^\circ)_x \quad \simeq \quad \mathcal{S}(T_x(M_0)) \otimes \Lambda^{q*}$$

and

$$\mathcal{O}_M^\circ \simeq \bigoplus_{x \in M_0} (\mathcal{O}_M^\circ)_x \quad \simeq \quad \bigoplus_{x \in M_0} \mathcal{S}(T_x(M_0)) \otimes \Lambda^{q*}$$

We now want to study in some detail the algebraic structure of $\mathcal{O}_M^\circ$. Let Δ denote the adjoint of the multiplication operator $m : \mathcal{O}(U) \otimes \mathcal{O}(U) \to \mathcal{O}(U)$, we have:

Proposition 10 $(\mathcal{O}_M^\circ, \Delta, \eta)$ *is a graded co-commutative co-algebra.*

Proof. Since $\mathcal{O}_M^\circ$ is canonically identified with a subspace of the dual of $\mathcal{O}(M)$ it is enough to prove that $\mathcal{O}_M^\circ$ is stable under the adjoint of m, that is that for each X_x in $\mathcal{O}_M^\circ$ there exists ΔX_x in $\mathcal{O}_M^\circ \otimes \mathcal{O}_M^\circ$ such that

$$\langle \Delta(X_x), f \otimes g \rangle \quad := \quad \langle X_x, m(f \otimes g) \rangle$$

Nevertheless due to Leibniz rule, $\langle X_x, m(f \otimes g) \rangle$ can be written as a (finite) linear combination of terms like

$$\left(\frac{\partial^{|P|}}{\partial x^P} f \right)_x \left(\frac{\partial^{|Q|}}{\partial x^Q} g \right)_x \quad = \quad \left\langle \left(\frac{\partial^{|P|}}{\partial x^P} \right)_x \otimes \left(\frac{\partial^{|Q|}}{\partial x^Q} \right)_x, f \otimes g \right\rangle$$

Moreover it is easily checked that

$$\eta(X_g) \quad = \quad X_g(1)$$

is a co-unit. Also graded commutativity and the appropriate commutative diagrams are easily checked. They come directly from the fact that $\mathcal{O}(M)$ is a graded algebra. ∎

The fact that $\mathcal{O}_M^\circ$ is a graded co-commutative colagebra follows also easily from next section and abstract graded co-algebra theory [2].

3.6 An algebraic approach to finite support distributions

In [Kos77] it is stated that the space of distributions with finite support over a graded manifold M identifies with the Sweedler dual $\mathcal{O}(M)^{(s)}$ of the graded algebra $\mathcal{O}(M)$. In this section we prove this result.

Definition 9 *Let A be a graded algebra and denote by A^* the corresponding algebraic dual. The Sweedler dual of A is defined as*

$$A^{(s)} \quad = \quad \{X \mid X \in A^*, \ker X \textit{ contains an ideal of finite co-dimension}\}$$

The necessity for defining $A^{(s)}$ comes from the fact that if an algebra A is not finite dimensional then the adjoint of the multiplication

$$\langle \Delta X, a \otimes b \rangle \quad := \quad \langle X, a \cdot b \rangle$$

does not turn A^* into a co-algebra since

$$(A \otimes A)^* \quad \supsetneq \quad A^* \otimes A^*$$

On the other hand, it can be proved that $A^{(s)}$ is a co-algebra with respect to the operation Δ defined above and co-unit

$$\begin{aligned} \eta : A^{(s)} &\longrightarrow \mathbb{R} \\ X &\longmapsto \langle X, 1 \rangle \end{aligned}$$

(see [Swe69]).

[2] The standard reference for results on Hopf algebras is the book of Sweedler ([Swe69]). There all results are given for non graded co-algebras, nevertheless we omit the proofs of the analogous results in the super setting whenever this is a trivial extension of the classical proofs.

Proposition 11 *The co-algebra of distribution with finite support coincides with the Sweedler dual of $\mathcal{O}(M)$:*

$$\mathcal{O}_M^\circ \quad \simeq \quad \mathcal{O}(M)^{(s)}$$

Since each point distribution has a kernel that contains an ideal of finite co-dimension we have that $\mathcal{O}_M^\circ \subseteq \mathcal{O}(M)^{(s)}$. Moreover, since it is well known that elements of $\mathcal{O}(M)^*$ that annihilates some J_x^n are finite support distributions, we are reduced to prove the following

Proposition 12 *Each ideal I of finite co-dimension of a stalk $\mathcal{O}(M)_x$ contains an ideal of the form J_x^k for some $k \in \mathbb{N}$.*

We need the following important lemma.

Lemma 3 (Nakayama) *Let A be a local (graded) commutative ring with maximal homogeneous ideal $\mathfrak{m}$. Let E be a finitely generated module for the ungraded ring A then if H is a submodule of E such that $E = \mathfrak{m}E + H$ then $E = H$.*

Proof. See [Var04]. ∎

We can now prove the proposition

Proof. First let us notice that J_x is an $\mathcal{O}(M)_x$-module finitely generated by the $p+q$ monomials $\{x^i, y^j\}$. Consider then the chain of ideals

$$\mathcal{O}(M)_x \supseteq J_x + I \supseteq J_x^2 + I \supseteq \ldots$$

Since I has finite co-dimension, we have that there exists k such that

$$J_x^k + I \quad = \quad J_x^{k+1} + I$$

Hence

$$J_x^k \quad \subseteq \quad J_x^{k+1} + I$$

Suppose U and W graded vector spaces, and V a graded vector subspace of U, then $(V + W) \cap U = U + (W \cap U)$. Then

$$J_x^k \quad = \quad \left(J_x^{k+1} + I\right) \cap J_x^k = J_x^{k+1} + I \cap J_x^{k+1}$$

Due to Nakayama lemma we have $I \cap J_x^{k+1} = J_x^k$ ∎

Chapter 4

Super groups

Questo è un gruppo rintrecciato.
Chi sviluppa più inviluppa,
Chi più sgruppa, più raggruppa;
Ed intanto la mia testa
Vola, vola e poi s'arresta;
Vo tenton per l'aria oscura,
E comincio a delirar.

La Cenerentola

In the previous chapter, we have defined the category $\mathcal{SM}$ of supermanifolds. In this chapter we examine more closely the category of $\mathcal{SM}$-groups. $\mathcal{SM}$-groups are commonly called **super groups** or **graded groups**.
All the material presented here, with the exception of section 4.2, is well known. In particular section 4.4 follows almost word by word the discussion in [Kos77] (see also [Bou98]). The result about the triviality of the sheaf of sections over a super Lie group G (sec. 4.2) is well known (see [Kos77] and [Bry87]) but, up to our knowledge, the proof presented is original, explicit and much more elementary than the others.

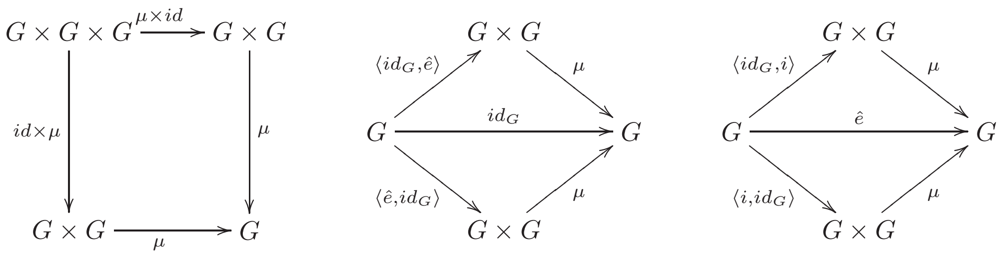
We start specializing definition 1 to our context.

Definition 10 *A real super group G is a real smooth super manifold G together with three morphisms*

$$\begin{aligned} \mu : G \times G &\longrightarrow G \\ i : G &\longrightarrow G \\ e : \mathbb{R}^{0|0} &\longrightarrow G \end{aligned}$$

called multiplication, inverse, *and* unit *respectively satisfying the following commutative di-*

agrams:



where $\hat{e}$ denotes the composition of the identity $e : \mathbb{R}^{0|0} \longrightarrow G$ with the unique map $G \longrightarrow \mathbb{R}^{0|0}$. $\langle \psi, \phi \rangle$ denotes the map $\psi \times \phi \circ \mathrm{diag}$, $diag : G \longrightarrow G \times G$ being the canonical diagonal map.

Remark 6 *As we have already discussed in the previous chapter, morphisms in the previous definition are actually graded algebras morphisms*

$$\begin{aligned} \mu^* : \mathcal{O}(G) &\longrightarrow \mathcal{O}(G \times G) \\ i^* : \mathcal{O}(G) &\longrightarrow \mathcal{O}(G) \\ e^* : \mathcal{O}(G) &\longrightarrow \mathbb{R} \end{aligned}$$

while diagrams have to be dualized

Remark 7 *Let us also notice that to each super group is associated a Lie group G. Indeed it is defined as the underlying manifold G_0 and the "reduced morphisms"*

$$\begin{aligned} \mu_0 : G_0 \times G_0 &\longrightarrow G_0 \\ i_0 : G_0 &\longrightarrow G_0 \\ e_0 : \mathbb{R}^0 &\longrightarrow G_0 \end{aligned}$$

Since the map $\phi \longmapsto \phi_0$ is functorial, it is immediate that G_0, μ, i_0, e_0 is a Lie group.

4.1 The Lie superalgebra of a super Lie group

Exactly as in the classical case one can define the notion of left-invariant vector fields on a graded manifold M. As usual care is needed since only definitions involving the algebra of sections give correct generalizations.

Definition 11 *A derivation $X : \mathcal{O}_M \longrightarrow \mathcal{O}_M$ is called* **left invariant** *if*

$$(1 \otimes X) \circ \mu^* = \mu^* \circ X$$

Clearly if X and Y are left invariant then easy calculations show that also $aX + bY$ is left invariant for all a and b in $\mathbb{R}$ as well as the graded commutator

$$[X, Y] = X \circ Y - (-1)^{p(X)p(Y)} Y \circ X$$

One can state these results as

Proposition 13 *Denote by* $\mathrm{Lie}(G)$ *the set of left invariant vector fields over G.* $\mathrm{Lie}(G)$ *is a graded Lie algebra.*

Exactly as in ordinary Lie theory one has

Proposition 14 *There is a vector space isomorphism*

$$\begin{aligned} \mathrm{Lie}\,(G) &\longrightarrow T_e\,(G) && (4.1)\\ X &\longmapsto X_e && (4.2) \end{aligned}$$

Proof. Let us first show that a left invariant vector field is uniquely determined by its value at the identity:

$$X \longmapsto X_e := \delta_e \circ X$$

Let us notice that $1 \otimes \delta_e \circ \mu^* = \mathrm{id}$. Hence

$$X = 1 \otimes \delta_e \circ 1 \otimes X \circ \mu^* = 1 \otimes X_e \circ \mu^* \qquad (4.3)$$

so that the map 4.1 is injective.

Conversely given a vector $X_e \in T_e\,(G)$, eq. 4.3 defines a left invariant vector field, thus proving surjectivity. ■

From this proposition, it follows that we can endow $T_e\,(G)$ with a Lie algebra structure in such a way that the map 4.1 become a graded Lie algebra isomorphism. $T_e\,(G)$ endowed with such structure is denoted by $\mathfrak{g}$. We thus have

Corollary 1 *There is a graded Lie algebra isomorphism*

$$\begin{aligned} \mathrm{Lie}\,(G) &\longrightarrow \mathfrak{g}\\ X &\longmapsto X_e \end{aligned}$$

whose inverse is given by

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathrm{Lie}\,(G)\\ X_e &\longmapsto 1 \otimes X_e \circ \mu^* \end{aligned}$$

$\mathrm{Lie}\,(G_0)$ is clearly a graded Lie algebra. We now want to identify its even part.

Proposition 15 *Let G and H be graded Lie groups and let $\phi : G \longrightarrow H$ be a morphism of graded Lie groups. The map*

$$(\mathrm{d}\phi)_e : \mathfrak{g} \longrightarrow \mathfrak{g}'$$

is a Lie algebra homomorphism.

Proof. It is straightforward that:

$$\begin{aligned} (\mathrm{d}\phi)_e\,[X,Y] &= [X,Y]_e \circ \phi^*\\ &= \delta_{e_G} \circ 1 \otimes [X,Y] \circ \mu_G^* \circ \phi^*\\ &= \delta_{e_H} \circ 1 \otimes ([X,Y] \circ \phi^*) \circ \mu_H^* \end{aligned}$$

But

$$\begin{aligned} X \circ Y \circ \phi^* &= 1 \otimes X_e \circ \mu^* 1 \otimes Y_e \circ \mu^* \circ \phi^*\\ &= 1 \otimes X_e \circ \mu^* \circ \phi^* \circ 1 \otimes \mathrm{d}\phi_e Y_e\\ &= \phi^* \circ 1 \otimes \mathrm{d}\phi_e X_e \circ \mu^* \circ 1 \otimes \mathrm{d}\phi_e Y_e\\ &= \phi^* \circ \left((\mathrm{d}\phi_e X_e)^L \circ (\mathrm{d}\phi_e Y_e)^L \right) \end{aligned}$$

so that the thesis easily follows. ■

From this it follows the next

Corollary 2 *The even part of the graded Lie algebra* $\mathrm{Lie}(G)$ *canonically identifies with* $\mathrm{Lie}(G_0)$.

Proof. The thesis is immediate considering the canonical inclusion

$$j : G_0 \longrightarrow G$$

and previous proposition. ■

4.2 Splitting of the sheaf

In this section we prove the following result

Proposition 16 *Let G be a SLG and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the corresponding SLA. If U is an open set in G_0 then*

$$\mathcal{O}(U) \simeq \mathcal{C}^\infty(G_0) \otimes \Lambda(\mathfrak{g}_1^*)$$

Taking in the above proposition $U = G_0$, we obtain that the super algebra of global sections over a SLG splits into $\mathcal{C}^\infty(G_0) \otimes \Lambda(\mathfrak{g}_1^*)$. We will also exhibit an explicit form of the graded isomorphism $i : \mathcal{O}(G) \to \mathcal{C}^\infty(G_0) \otimes \Lambda(\mathfrak{g}_1^*)$. As a byproduct of the above proposition we will also obtain a local diffeomorphism from the SLA $\mathfrak{g}$ to the SLG G.

The construction we will exhibit is actually a particular case of the graded version of Frobenius theorem (see, for example [DM99a]). Fix a basis $\{X_i\}_i$ of $\mathfrak{g}_1$ and consider the associated set of left invariant vector fields X_i^L. Let U be an open set in G_0 and consider also the super manifold

$$\mathfrak{g}_1 \times U$$

with the odd differential operators

$$\left\{1 \otimes X_i^L\right\}$$

defined over it. If $\{X_i^*\}$ denote the dual basis of $\{X_i\}$, we can define the even derivations $D_i = X_i^* \otimes X_i^L$. Each D_i is clearly nilpotent, and so is

$$D = \sum_i D_i$$

Notice also that D does not depend on the choice of coordinates in $\mathfrak{g}_1$.

Define the map

$$\exp(D) : \Lambda(\mathfrak{g}_1^*) \otimes \mathcal{O}(U) \longrightarrow \Lambda(\mathfrak{g}_1^*) \otimes \mathcal{O}(U) \tag{4.4}$$

$$f \longmapsto \sum_{n=0}^{\infty} \frac{1}{n!} D^n(f) \tag{4.5}$$

$$= \sum_{n=0}^{\dim \mathfrak{g}_1} \frac{1}{n!} D^n(f) \tag{4.6}$$

Lemma 4 *The map* $\exp(D)$ *is an automorphism of* $\Lambda(\mathfrak{g}_1^*) \otimes \mathcal{O}(U)$ *which defines an isomorphism of the super manifold* $\mathfrak{g}_1 \times U$ *into itself.*

Proof. Due to theorem 9, it is enough to show that $\exp(D)$ is an automorphism of the super algebra $\mathcal{O}(U)$. In commutative algebra this is a very well known fact and the proof extends immediately to the super setting. In fact:

$$\begin{aligned} \exp(D)(f)\exp(D)(g) &= \sum_k \frac{1}{k!}D^k f \sum_p \frac{1}{p!}D^p g \\ &= \sum_n \sum_{p+k=n} \frac{1}{p!k!} D^p f D^k g \\ &= \sum_n \frac{1}{n!}\sum_k \begin{pmatrix} n \\ k \end{pmatrix} D^k f D^{n-k} g \end{aligned}$$

■

The explicit form of $\exp(D)$ is easily recognized to be:

$$\begin{aligned} \exp(D) &= \sum_n \frac{1}{n!}\left(\sum_i X_i^* \otimes X_i^L\right)^n \\ &= \sum_n \frac{1}{n!}\sum_{|I|=n}\left(X_I^* \otimes X_I^L\right) \end{aligned}$$

where I is the usual multi-index notation and $|I|$ the corresponding length.
Let us now proceed heuristically. The above formula shows that $\exp(D)$ is compatible with the projection

$$pr : \mathfrak{g}_1 \times U \longrightarrow U$$

so that we obtain a map

$$\mathfrak{g}_1 \times U \overset{\exp(D)}{\longrightarrow} \mathfrak{g}_1 \times U \overset{pr}{\longrightarrow} U$$

The key observation is now that $U_0 \hookrightarrow U$ is a subsupermanifold which is transversal in each point to the distribution spanned by the left invariant vector fields X_i^L so that

$$\mathfrak{g}_1 \times U_0 \overset{i}{\hookrightarrow} \mathfrak{g}_1 \times U \longrightarrow U$$

is actually an isomorphism of super manifolds whose form is given by

$$\tau : \mathcal{O}(U) \longrightarrow \mathcal{O}(U) \tag{4.7}$$

$$f \longmapsto \sum_n \frac{1}{n!}\sum_{|I|=n}\left(X_I^* \otimes \widetilde{X_I^L(f)}\right) \tag{4.8}$$

In order to prove carefully the result we need the following proposition (see [Kos77]).

Proposition 17 *Let $\psi : \mathcal{O}(M) \longrightarrow \mathcal{O}(N)$ be an algebra morphism. ψ defines a diffeomorphism of graded manifolds if and only if*

i) ψ_0 is a diffeomorphism

ii) $(\mathrm{d}\psi)_x$ is an isomorphism for each $x \in M_0$

In this case we have

$$\psi : \mathfrak{g}_1 \times U_0 \xrightarrow{\mathrm{id}\times i} \mathfrak{g}_1 \times U \xrightarrow{(expD)} \mathfrak{g}_1 \times U \xrightarrow{\mathrm{pr}} U$$

and hence the reduced morphism is given by

$$U_0 \xrightarrow{i} \{pt\} \times U_0 \xrightarrow{(expD)_0} \{pt\} \times U_0 \xrightarrow{i} U_0$$

which is clearly a diffeomorphism. In order to determine the action of $(\mathrm{d}\psi)_x$, we first notice that, for the super manifold

$$U \quad := \quad (U_0, \mathcal{C}^\infty(U_0) \otimes \Lambda(\mathfrak{g}_1^*))$$

one has the identification:

$$T_x(U) \quad \simeq \quad T_x(U_0) \oplus \mathfrak{g}_1$$

Let $\{X_i\}$ be a basis of $T_x(U_0)$ and $\{Y_j\}$ a basis of $\mathfrak{g}_1$. We have

$$\begin{aligned} (\mathrm{d}\psi)_x X_i(f) &= X_i\left(\tilde{f}\right) \\ (\mathrm{d}\psi)_x Y_i(f) &= \widetilde{Y_i^L(f)}(x) \\ &= (l_{x^{-1}*} Y_i)(f) \end{aligned}$$

4.3 The action of G_0 on G

The reduced Lie group G_0 acts on G in a natural way. Since this action is a basic ingredient of future development of the theory we shortly describe it.

Since G is a super group, it acts on itself through group multiplication

$$G \times G \quad \longrightarrow \quad G$$

Fix an element

$$\overline{g} : \mathbb{R}^{0|0} \quad \longrightarrow \quad G_0 \longrightarrow G$$

in G_0 and define

$$l_{\overline{g}} : \mathbb{R}^{0|0} \times G \xrightarrow{\overline{g}\times\mathrm{id}} G_0 \times G \xrightarrow{i\times\mathrm{id}} G \times G \xrightarrow{m} G$$

It is easily seen that this induces an action

$$l : G_0 \times G \quad \longrightarrow \quad G$$

which we call left multiplication by G_0. At the level of sections we have

$$l_{\overline{g}}^* f \quad = \quad \delta_{\overline{g}} \otimes 1 \circ m^*(f) \tag{4.9}$$

4.4 The convolution product on $\mathcal{O}_G^o$

It is possible to use the group structure of G for defining an algebra structure on $\mathcal{O}_G^o$. Let us first examine the case in which G is an ordinary Lie group. In this case it is well known that given two distribution X_g and $Y_{\overline{g}}$ in $\mathcal{O}_G^o$, we can define the convolution product $X_g * Y_{\overline{g}}$ as

$$\begin{aligned}(X_g * Y_{\overline{g}})(f) &= \int \mathrm{d}Y_{\overline{g}}(y) \int \mathrm{d}X_g(x) f(xy) \\ &:= \langle X_g \otimes Y_{\overline{g}}, \mu^*(f)\rangle\end{aligned}$$

This product possesses the following remarkable properties:

$$\delta_{\overline{g}} * X_g = l_{\overline{g}*} X_g \tag{4.10}$$
$$X_g * \delta_{\overline{g}} = r_{\overline{g}*} X_g \tag{4.11}$$

where l_g and r_g denote the usual push-forward of left and right translations respectively. Let us now come back to the super setting, we are led to define the convolution product according to

$$(X_g * Y_{\overline{g}})(f) = \langle X_g \otimes Y_{\overline{g}}, \mu^*(f)\rangle$$

where of course now $f \in \mathcal{O}_G^o$ and μ is the super group multiplication.

Returning to the super setting, we can now use the action of G_0 on G described in the previous section. If $X_{\overline{g}}$ denotes a distribution on G concentrated in $\overline{g}$, we have

$$\begin{aligned}\langle l_{g*} X_{\overline{g}}, f\rangle &= \left\langle X_{\overline{g}}, l_g^* f\right\rangle \\ &= \langle X_{\overline{g}}, \delta_g \otimes 1 \circ m^*(f)\rangle \\ &= (\delta_g * X_{\overline{g}})(f)\end{aligned}$$

We have thus obtained the desired generalization of 4.10:

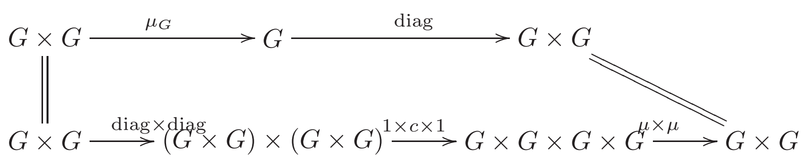
$$\begin{aligned}l_{\overline{g}*} X_g &= \delta_{\overline{g}} * X_g \\ r_{\overline{g}*} X_g &= X_g * \delta_{\overline{g}}\end{aligned}$$

It is quite interesting now to examine more closely the algebraic structure of $\mathcal{O}_G^o$. In fact we know that $\mathcal{O}_G^o$ is a graded co-algebra from prop. 10 and a graded algebra with identity δ_e from the previous discussion. It is natural to ask if the two structure are compatible.

Proposition 18 *If we endow $\mathcal{O}_G^o$ with the algebraic structures of the previous discussion we have that both co-multiplication and co-unit are algebra homomorphisms.*

We need the following calculus lemma.

Lemma 5 *Let $c_{M,N} : M \times N \longrightarrow N \times M$ the flip map. The following diagram commutes*



that is

$$\mu_G^*(st) = \mu_G^* \circ \mathrm{diag}_G^*(s \otimes t) = \mathrm{diag}_G^* \otimes \mathrm{diag}_G^* \circ (1 \otimes c^* \otimes 1) \circ \mu^* \otimes \mu^*(s \otimes t) \quad \forall s,t \in \mathcal{O}(G)$$

Proof. Straightforward. ■

Proof. We have to show that co-multiplication and co-unit are graded algebra morphisms. Let us start with co-multiplication. Consider

$$\begin{aligned}
\langle \Delta\,(X_g * Y_h), s \otimes t\rangle &= \langle X_g \otimes Y_h, \mu_G^* \circ (s \cdot t)\rangle \\
\text{(by lemma 5)} &= \langle X_g \otimes Y_h, \mathrm{diag}_G^* \otimes \mathrm{diag}_G^* \circ (1 \otimes c^* \otimes 1) \circ \mu^* \otimes \mu^*(s \otimes t)\rangle \\
&= \langle \Delta\,(X_g) * \Delta\,(Y_h), (1 \otimes c^* \otimes 1) \circ \mu^* \otimes \mu^*(s \otimes t)\rangle \\
&= \langle \Delta X_g * \Delta U_h, s \otimes t\rangle
\end{aligned}$$

and

$$\langle \Delta\delta_e, s \otimes t\rangle = \langle \delta_e \otimes \delta_e, s \otimes t\rangle$$

Consider now the co-unit. We have:

$$\begin{aligned}
\langle \eta\,(X_g * Y_g), s\rangle &= \langle X_g \otimes Y_h, \mu^* \circ \delta_e(s)\rangle \\
&= \delta_e(s) X_g(1) Y_h(1) \\
&= \delta_e(s)\eta\,(X_g)\,\eta\,(Y_h)
\end{aligned}$$

■

This proves (see, for example, [Swe69]) that $\mathcal{O}_G^o$ is a *bi-algebra.* Actually even more is true.

Proposition 19 *$\mathcal{O}_G^o$ is a graded Hopf algebra.*

Proof. It simply follows from the fact that δ_e is a unit for the convolution product. ■

We end this section pointing out an interesting structure of $\mathcal{O}_G^o$ which parallels a general structure theorem on Hopf algebras (see again [Swe69]).

4.4.1 $\mathcal{O}_G^o$ as a smash product

Let us examine more closely the product law in $\mathcal{O}_G^o$. Fix hence a distribution X_g in $\mathcal{O}_G^o$. Each such element can be canonically written as the convolution product of a distribution concentrated on the identity e of the supergroup with the "element" g of G_0 on which it is concentrated:

$$X_g = \delta_g * \left(\delta_{g^{-1}} * X_g\right) = \delta_g * l_{g^{-1}*} X_g \tag{4.12}$$

Thus each element in $\mathcal{O}_G^o$ can be written as a product $\delta_g * X$, with $X \in \mathcal{O}_G^o$ concentrated at e.

In order to simplify notations we introduce the symbol X_e as a shortcut for $l_{g^{-1}*}X_g$ so that the previous relation becomes $X_g = \delta_g * X_e$. Next proposition gives a caracterization of the distributions with support in e.

Proposition 20 *The space of distributions concentrated at the identity e of the supergroup G is isomorphic, as an algebra, to the enveloping algebra $E\,(\mathfrak{g})$ of the super Lie algebra $\mathfrak{g}$ associated with the group.*

Proof. It goes exactly as in the classical case. ■

Due to decomposition 4.12, we can now define a map

$$\begin{array}{rcl} \psi : \mathcal{O}_G^o & \longrightarrow & \mathbb{R}\left(G_0\right) \otimes E\left(\mathfrak{g}\right) \\ \sum_i X_{g_i} & \longmapsto & \sum_i g_i \otimes \left(X_e\right)_i \end{array} \tag{4.13}$$

where, of course, $\left(X_e\right)_i = l_{g_i^{-1}*} X_{g_i}$. By proposition 20, the above map is a bijection.

We can hence transport all the algebraic structures defined over $\mathcal{O}_G^o$ to corresponding structures over $\mathbb{R}\left(G_0\right) \otimes E\left(\mathfrak{g}\right)$. In this way $\mathbb{R}\left(G_0\right) \otimes E\left(\mathfrak{g}\right)$ becomes a graded Hopf algebra. The coalgebra structure is easily seen to be the one induced by the coalgebra structures of $\mathbb{R}\left(G_0\right)$ and $E\left(\mathfrak{g}\right)$, namely

$$\begin{array}{rcl} \Delta_{\mathbb{R}(G_0)} : \delta_g & \longmapsto & \delta_g \otimes \delta_g \\ \Delta_{\mathfrak{g}} : X_e & \longmapsto & X_e \otimes \delta_e + \delta_e \otimes X_e \quad \forall X_e \in \mathfrak{g} \end{array}$$

More interesting is the algebra structure induced through the map 4.13. We now want to spell out this product.

$$\left(g \otimes X\right)\left(\overline{g} \otimes Y\right) := \psi \circ \left[\psi^{-1}\left(g \otimes X\right)\psi^{-1}\left(\overline{g} \otimes Y\right)\right] \tag{4.14}$$

$$= \psi\left(X_g * Y_{\overline{g}}\right) \tag{4.15}$$

$$= \psi\left(\delta_{g\overline{g}} * \left(X^{\overline{g}^{-1}} * Y\right)\right) \tag{4.16}$$

that we can rewrite as

$$\left(g \otimes X\right)\left(\overline{g} \otimes Y\right) = \left(g\overline{g} \otimes X^{\overline{g}} Y\right) \tag{4.17}$$

where $X^{\overline{g}^{-1}}$ clearly denotes the *adjoint action* of G_0 on $\mathfrak{g}$.

Since this formula resembles the one for semi-direct products, it is costumary in the literature to denote $\mathbb{R}\left(G_0\right) \otimes E\left(\mathfrak{g}\right)$, endowed with the hopf algebra structure just described, with

$$\mathbb{R}\left(G_0\right) \rtimes E\left(\mathfrak{g}\right)$$

This is called the **smash product** of $\mathbb{R}\left(G_0\right)$ with $E\left(\mathfrak{g}\right)$ (see [Swe69]).

Chapter 5

Harish-Chandra pairs

Questa o quella per me pari sono...
Duca di Mantova
Rigoletto

In the previous chapter we have associated to each super group G the graded Hopf algebra $\mathcal{O}_G^o$ of finite support distributions over G. We have also shown that it can be written as a smash product

$$\mathcal{O}_G^o \quad = \quad \mathbb{R}(G_0) \rtimes E(\mathfrak{g})$$

In this chapter, following [Kos77], we define the Lie-Hopf algebra (LHA) $E\,(G_0, \mathfrak{g})$ associated to a super group G as the graded Hopf-algebra $\mathcal{O}_G^o$ in which G_0 is endowed with its Lie group structure. Clearly the Lie-Hopf algebra associated to a super Lie group G is completely determined once the triple

$$(G_0, \mathfrak{g}, \alpha)$$

consisting of the reduced Lie group G_0, the super Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and the adjoint action α of G_0 on $\mathfrak{g}$ is given. According to [DM99a], each triple $(G_0, \mathfrak{g}, \alpha)$ with

i) G_0 Lie group,

ii) $\mathfrak{g}$ super Lie algebra such that $\mathfrak{g}_0 \simeq \mathrm{Lie}\,(G_0)$

iii) α action of G_0 on $\mathfrak{g}$ such that $\alpha_{|\mathfrak{g}_0} = \mathrm{Ad}$

is called a **super Harish-Chandra pair** (SHCP). This also gives the general definition of a Lie-Hopf algebra, as the smash product constructed starting from a generic SHCP. We show that, given such a LHA $E\,(G_0, \mathfrak{g})$, a supergroup G can be constructed in terms of the algebraic dual $E\,(G_0, \mathfrak{g})'$. The LHA of G turns out to be exactly the given one. Once defined morphisms between SHCP we have the category $\mathcal{SHC}$ of super Harish-Chandra pairs. The category $\mathcal{LH}$ of Lie-Hopf algebras is then automatically defined.

The aim of this chapter is to show that the functor

$$\mathcal{SG}rp \quad \longrightarrow \quad \mathcal{LH} \simeq \mathcal{SHC}$$

is an equivalence of categories. This is of central importance for our approach to the representation theory of super Lie groups. The detailed proof of the equivalence we will give follows [BT06].

5.1 Lie-Hopf algebras and super Harish-Chandra pairs

Let G be a super group. As we have seen in the previous chapter, the space of finite-support distributions $\mathcal{O}_G^o$ is canonically isomorphic to the smash-product $\mathbb{R}(G_0) \rtimes E(\mathfrak{g})$. Clearly, one has

$$\mathcal{O}_G^o \hookrightarrow \mathcal{O}(G)'$$

where $\mathcal{O}(G)'$ denotes the algebraic dual of $\mathcal{O}(G)$.

Lemma 6 *$\mathcal{O}_G^o$ separates the points of $\mathcal{O}(G)$, that is*

$$\langle f-h, X_g\rangle = 0 \quad \forall X_g \in \mathcal{O}_G^o \quad \Longrightarrow \quad f=h$$

Proof. Let $f \in \mathcal{O}(G)$ be a non null section over G, and let U be an open coordinate neighborhood such that

$$f_{|U} \;=\; \sum_I f_I \theta^I \neq 0$$

Clearly there exist f_J and $x \in U$ such that $f_J(x) \neq 0$. Consider the element of $\mathcal{O}_G^o$ given by $\left(\frac{\partial}{\partial\theta^J}\right)_x$, the thesis easily follows. ■

Proposition 21 *Each morphism of supergroups $\psi : G \to H$ uniquely determines a morphism of graded Hopf algebras $\phi^* : \mathcal{O}_G^o \to \mathcal{O}_H^{(o)}$*

Proof. Let $\psi \in Mor(G,H)$ with $\psi = (\psi_0, \psi^*)$ and denote ψ^* with ϕ. We now show that the adjoint of ϕ defines a morphism of Lie-Hopf algebras. We have

$$\langle f, \phi^*(X_g)\rangle \;=\; \langle \phi(f), X_g\rangle$$

where $\phi^*(X_g)$ belongs to the algebraic dual $\mathcal{O}(H)'$ of $\mathcal{O}(H)$. Nevertheless due to proposition 6 we have

$$\langle \phi^*(f_n), X_g\rangle \;\longrightarrow\; 0 \quad \text{if } f_n \stackrel{n\to\infty}{\longrightarrow} 0$$

and moreover it is easy to see that if

$$\text{supp } f \cap \psi_0(g) \;=\; \emptyset$$

then $\langle f, \phi^*(X_g)\rangle = 0$. The injectivity of the map $\psi \mapsto \psi^*$ is an easy consequence of the fact that $\mathcal{O}_G^o$ separates sections. ■

The above proposition hence establish that the map

$$Mor(G,H) \;\stackrel{*}{\longrightarrow}\; Mor(\mathcal{O}_G^o, \mathcal{O}_H^o)$$

is injective. Nevertheless, in general, the above map is not bijective since if $\psi \in Mor(\mathcal{O}_G^o, \mathcal{O}_H^o)$ then $\psi^*(\mathcal{O}(H)) \subseteq (\mathcal{O}(G))'$. In order to obtain a bijection we now give more structure to $\mathcal{O}_G^o$ so to restrict the class of morphisms.

Definition 12 *We say that $(G_0, \mathfrak{g})$ is a* super Harish-Chandra pair *(SHCP) if*

- *G_0 is a classical Lie group*

- $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *is a super Lie algebra such that* $\mathfrak{g}_0$ *is isomoprphic to the Lie algebra of* G_0*;*
- *the adjoint representation of* $\mathfrak{g}_0$ *on* $\mathfrak{g}$ *exponentiates to a representation of* G_0*.*

If G is a supergroup, the pair $(G_0, \mathfrak{g})$ given by the reduced Lie group G and the super Lie algebra $\mathfrak{g}$ is a super Harish-Chandra pair. A morphism of SHCPs is simply a couple of morphisms $\psi = \left(\psi_0, \rho^\psi\right)$ preserving the SHCP structure.

Definition 13 *Let* $(G_0, \mathfrak{g})$ *and* $(H_0, \mathfrak{h})$ *be SHCP. A morphism between the two is defined as a couple* (ψ_0, ρ^ψ) *such that*

i) $\psi_0 : G_0 \to H_0$ *is a Lie groups homomorphism*

ii) $\rho^\psi : \mathfrak{g} \to \mathfrak{h}$ *is a super Lie algebra homomorphism*

iii) ψ_0 *and* ρ^ψ *are compatible in the sense that:*

$$\rho^\psi_{|\mathfrak{g}_0} = \mathrm{d}\psi_0$$

Given a morphism $\phi : G \longrightarrow H$ of super Lie groups, then it determines the morphism of the corresponding super Harish-Chandra pairs

$$\left(\phi_0, (\mathrm{d}\phi)_e\right)$$

It is now natural to give the following

Definition 14 *A Lie-Hopf algebra* $E\,(G_0, \mathfrak{g})$ *is defined as the smash product*

$$\mathbb{R}\,(G_0) \rtimes_\alpha E\,(\mathfrak{g})$$

where $(G_0, \mathfrak{g}, \alpha)$ *is a super Harish-Chandra pair.*

Of course one can give the following natural definition.

Definition 15 *Let* $E\,(G_0, \mathfrak{g})$ *and* $E\,(H_0, \mathfrak{h})$ *be Lie-Hopf algebras. We say that* $\psi :\ E\,(G_0, \mathfrak{g}) \longrightarrow E\,(H_0, \mathfrak{h})$ *is a morphism of Lie-Hopf algebras if*

i) ϕ *is a morphism of graded Hopf algebras*

ii) $\phi_{|G_0}$ *is a Lie group morphism*

iii) $\phi_{|\mathfrak{g}_0} = \mathrm{d}\phi_0$

We have thus defined two strictly related categories. The category $\mathcal{SHC}$ of super Harish-Chandra pairs and the category $\mathcal{LH}$ of Lie-Hopf algebras. As it may be expected, we have:

Proposition 22 $\mathcal{SHC}$ *and* $\mathcal{LH}$ *are equivalent categories.*

Proof. Straightforward. ■

These simple considerations can be summarized saying that we have defined a functor F from the category of super Lie groups to the category of super Harish Chandra pairs. This chapter will be devoted to prove the following

Theorem 2 *The category of super Lie groups is equivalent to the category of super Harish Chandra pairs.*

Roughly speaking this theorem says that each problem in the category of SLG can be reformulated in an equivalent problem in the language of SHCP. We will put at work this approach when we will study representation theory of SLGs.
Before embarking in the proof of the theorem let us outline the road we will follow. Proving categorical equivalence means showing that the functor F defining the equivalence is

i) bijective on the objects

ii) bijective on morphisms

A moment of thought makes clear that in our case this means showing that

i) given a $(G_0, \mathfrak{g})$ there exists a SLG G having $(G_0, \mathfrak{g})$ as corresponding SHCP

ii) given a morphism $f : (G_0, \mathfrak{g}) \to (H_0, \mathfrak{h})$ there exists a unique morphism ψ of corresponding SLG from which f arises.

5.2 Reconstruction of the Lie super group from the Harish-Chandra pair

Let $(G_0, \mathfrak{g})$ be a super Harish-Chandra pair and consider the corresponding Lie-Hopf algebra as defined in section 5.1. The algebraic dual $E(G_0, \mathfrak{g})'$ is naturally endowed with the structure of a left $E(\mathfrak{g})$-module by defining $Z.\phi$ for $Z \in E(\mathfrak{g})$, $\phi \in E(G_0, \mathfrak{g})'$ as

$$\langle xX, Z.\phi \rangle = \langle xXZ, \phi \rangle \quad \forall x \in G_0,\ X \in E(\mathfrak{g})$$

Clearly if $E(G_0, \mathfrak{g})$ comes from a super Lie group G then $E(G_0, \mathfrak{g}) \subset \mathcal{O}(G)'$ and we have a map

$$i : \mathcal{O}(G) \longrightarrow E(G_0, \mathfrak{g})'$$

Moreover, since $E(G_0, \mathfrak{g})$ is non singularly paired with $\mathcal{O}(G)$ (prop. 6), the above map is injective and the action of $E(\mathfrak{g})$ on $i(\mathcal{O}(G))$ is the usual action through left invariant differential operators. The evaluation map is defined by

$$\begin{aligned} \tilde{}: E(G_0, \mathfrak{g})' &\longrightarrow \mathbb{R}^{G_0} \\ \phi &\longmapsto \tilde{\phi} : (x \mapsto \langle x, \phi \rangle) \end{aligned}$$

Define hence the super vector space

$$A(G) = \left\{ \phi \in E(G_0, \mathfrak{g})' \mid \widetilde{(Z.\phi)} \in \mathcal{C}^\infty(G_0) \text{ and } \widetilde{(ZX.\phi)} = X\,\widetilde{(Z.\phi)}\ \forall Z \in E(\mathfrak{g}),\ X \in \mathfrak{g}_0 \right\}$$

We have

Proposition 23 *If $(G_0, \mathfrak{g})$ is a super Harish-Chandra pair, then $(G_0, A(G))$ is a super Lie group with operations*

$$\begin{aligned} \mu^* &= m^*_{E(G_0,\mathfrak{g})}\big|_{A(G)} \\ i^* &= s^*_{|_{A(G)}} \\ e^* &= \eta^*_{|_{A(G)}} \end{aligned}$$

where m, s amd η are multiplication, antipode and co-unit in $E(G_0, \mathfrak{g})$. Moreover the super Harish-Chandra pair associated to $(G_0, A(G))$ is precisely $(G_0, \mathfrak{g})$.

Proof. We divide the proof into the following steps:

i) $A(G)$ is a graded algebra

ii) the maps μ^*, i^* and e^* are graded algebra morphisms with respect to which $A(G)$ is a Hopf algebra.

iii) $(G_0, A(G))$ is a super Lie group

iv) the Harish-Chandra pair associated to $(G_0, A(G))$ is $(G_0, \mathfrak{g})$.

Notice that *iii*) follows at once from *i*), *ii*), and prop. 9.

First step We start showing that $A(G)$ is a graded algebra and in order to establish this we have to prove that it is stable by the adjoint of the co-multiplication map. In fact the identity is given by the adjoint of the counit, and the fact that the appropriate commutative diagrams hold follows dualizing the co-algebra diagrams of $E(G_0, \mathfrak{g})$. In other words we have to prove that if ϕ, $\psi \in A(G)$, then $(Z.(\phi \cdot \psi))\tilde{} \in \mathcal{C}^\infty(G_0)$ for all $Z \in E(\mathfrak{g})$ and $(XZ.(\phi\psi))\tilde{} = X(Z.(\phi\psi))\tilde{}$ for each $X \in \mathfrak{g}_0$. Let hence be $Z \in E(\mathfrak{g})$

$$\begin{aligned} Z.(\phi \cdot \psi)\tilde{}(x) &= \langle x, Z.(\phi \cdot \psi)\rangle \\ &= \langle (x \otimes x)\Delta(Z), \phi \otimes \psi\rangle \\ &= (-1)^{Z_{(2)}\phi}\left\langle xZ_{(1)}, \phi\right\rangle \left\langle xZ_{(2)}, \psi\right\rangle \\ &= (-1)^{Z_{(2)}\phi} Z_{(1)}.\tilde{\phi}(x) \cdot Z_{(2)}.\tilde{\psi}(x) \end{aligned}$$

then $(Z.(\phi \cdot \psi))\tilde{} \in \mathcal{C}^\infty(G_0)$.

Let now $\overline{X} \in \mathfrak{g}_0$.

$$\begin{aligned} \left(\overline{X}Z.(\phi \cdot \psi)\right)\tilde{}(x) &= \left\langle \Delta\left(x\overline{X}Z\right), \phi \otimes \psi\right\rangle \\ &= \sum \left\langle x\overline{X}Z_{(1)} \otimes xZ_{(2)} + xZ_{(1)} \otimes x\overline{X}Z_{(2)}, \phi \otimes \psi\right\rangle \\ &= \sum (-1)^{p(\phi)p(Z_{(2)})}\left[\left\langle x\overline{X}Z_{(1)}, \phi\right\rangle \left\langle xZ_{(2)}, \psi\right\rangle + \left\langle xZ_{(1)}, \phi\right\rangle \left\langle x\overline{X}Z_{(2)}, \psi\right\rangle\right] \\ &= \sum (-1)^{p(\phi)p(Z_{(2)})}\overline{X}\left(\left(Z_{(1)}.\phi\right)\tilde{}\left(Z_{(2)}.\phi\right)\tilde{}\right)(x) \\ &= \overline{X}(Z.(\phi \cdot \psi))\tilde{}(x) \end{aligned}$$

Second step We have to prove that the maps defined in the proposition preserve $A(G)$. The necessary commutative diagrams then automatically follow from the analogous diagrams for $E(G_0, \mathfrak{g})$. We begin with the inverse:

$$\langle xX, i^*(\phi)\rangle = \langle s(xX), \phi\rangle \, \forall xX \in E(G_0, \mathfrak{g})$$

If $Z \in E(\mathfrak{g})$ then

$$\begin{aligned} (Z.i^*(\phi))\tilde{}(x) &= \langle xZ, i^*(\phi)\rangle \\ &= \left\langle x^{-1}s(Z)^x, \phi\right\rangle \\ &= \left(s(Z)^x.\phi\right)\tilde{}\left(x^{-1}\right) \in \mathcal{C}^\infty(G_0) \end{aligned}$$

Suppose now $\overline{X} \in \mathfrak{g}_0$, then

$$\begin{aligned}
\left(\overline{X}Z.i^*(\phi)\right)^{\sim}(x) &= \left\langle x\overline{X}Z, i^*(\phi)\right\rangle \\
&= \left\langle x^{-1}\left[\overline{X}, s\left(Z\right)\right]^{x}, \phi\right\rangle - \left\langle x^{-1}\overline{X}^{x} s\left(Z\right)^{x}, \phi\right\rangle \\
&= \frac{d}{dt}\left[\left(s\left(Z\right)^{x\exp\left(t\overline{X}\right)}\phi\right)^{\sim}\left(x^{-1}\right)\right]\Big|_{t=0} - \frac{d}{dt}\left[\left(s\left(Z\right)^{x}.\phi\right)^{\sim}\left(\exp\left(t\overline{X}\right)x^{-1}\right)\right]\Big|_{t=0} \\
&= \frac{d}{dt}\left[\left(s\left(Z\right)^{x\exp\left(t\overline{X}\right)}\phi\right)^{\sim}\left(x\exp\left(t\overline{X}\right)\right)\right]\Big|_{t=0} \\
&= \overline{X}\left(Z.i^*(\phi)\right)^{\sim}(x)
\end{aligned}$$

We now consider the adjoint of the multiplication in $E(G_0, \mathfrak{g})$. Notice that $A\left(G \times G\right)$ is defined as a a subspace of $E\left(G_0 \times G_0, \mathfrak{g} \oplus \mathfrak{g}\right)'$ and that $E\left(G_0 \times G_0, \mathfrak{g} \oplus \mathfrak{g}\right) \simeq E\left(G_0, \mathfrak{g}\right) \otimes E\left(G_0, \mathfrak{g}\right)$. We have to prove that μ^* defines a map from $A(G)$ to $A(G \times G)$. Let hence $\phi \in A(G)$, x and x' in G_0 and $Z \otimes Z' \in E\left(\mathfrak{g} \oplus \mathfrak{g}\right) \simeq E(\mathfrak{g}) \otimes E(\mathfrak{g})$, then

$$\begin{aligned}
\left\langle x \otimes x', \left(Z \otimes Z'\right).\mu^*(\phi)\right\rangle &= \left\langle xx'Z^{x'^{-1}}Z', \phi\right\rangle \\
&= \left(Z^{x'^{-1}}Z'.\phi\right)^{\sim}\left(xx'\right)
\end{aligned}$$

From this it follows that $\left[\left(Z \otimes Z'\right)\mu^*(\phi)\right]^{\sim} \in \mathcal{C}^{\infty}(G_0 \times G_0)$.
Let now $\overline{X} \otimes 1 \in \mathfrak{g}_0 \oplus \mathfrak{g}_0 \subset E\left(\mathfrak{g}_0 \oplus \mathfrak{g}_0\right) \simeq E\left(\mathfrak{g}_0\right) \otimes E\left(\mathfrak{g}_0\right)$. We have:

$$\begin{aligned}
\left[\left(\overline{X} \otimes 1\right)\left(Z \otimes Z'\right)\mu^*(\phi)\right]^{\sim}(x, x') &= \left\langle xx'\overline{X}^{x'^{-1}}Z^{x'^{-1}}Z', \phi\right\rangle \\
&= \overline{X}^{x'^{-1}}\left(Z^{x'^{-1}}Z'.\phi\right)^{\sim}\left(xx'\right) \\
&= \frac{d}{dt}\left[\left(Z^{x'^{-1}}Z'.\phi\right)^{\sim}\left(x\exp\left(t\overline{X}\right)x'\right)\right]\Big|_{t=0} \\
&= \left(\overline{X} \otimes 1\right)\left[\left(Z \otimes Z'\right)\mu^*(\phi)\right]^{\sim}(x, x')
\end{aligned}$$

Finally, if $1 \otimes \overline{X} \in \mathfrak{g}_0 \oplus \mathfrak{g}_0$, then

$$\begin{aligned}
\left[\left(1 \otimes \overline{X}\right)\left(Z \otimes Z'\right)\mu^*(\phi)\right]^{\sim}(x, x') &= \left\langle xx'Z^{x'^{-1}}\overline{X}Z', \phi\right\rangle \\
&= \left\langle xx'\left[\overline{X}, Z^{x'^{-1}}\right]Z', \phi\right\rangle + \left\langle xx'\overline{X}Z^{x'^{-1}}Z', \phi\right\rangle \\
&= \frac{d}{dt}\left[\left(Z^{x'\exp\left(t\overline{X}\right)^{-1}}Z'.\phi\right)^{\sim}\left(xx'\right)\right]\Big|_{t=0} \\
&\quad + \frac{d}{dt}\left[\left(Z^{x'^{-1}}Z'.\phi\right)^{\sim}\left(xx'\exp\left(t\overline{X}\right)\right)\right]\Big|_{t=0} \\
&= \left(1 \otimes \overline{X}\right)\left[\left(Z \otimes Z'\right)\mu^*(\phi)\right]^{\sim}(x, x')
\end{aligned}$$

We end this part of the proof showing that each Y in $\mathfrak{g}$ acts as a superderivation of the same parity as Y. Let X denote a generic element of $E\left(\mathfrak{g}\right)$, then

$$\begin{aligned}
\left\langle xX, Y.\left(\phi \cdot \psi\right)\right\rangle &= \left\langle \Delta\left(xXY\right), \phi \otimes \psi\right\rangle \\
&= \left\langle xX, \left[\left(Y.\phi\right) \cdot \psi + (-1)^{p(\phi)p(Y)}\phi \cdot \left(Y.\psi\right)\right]\right\rangle
\end{aligned}$$

Last step Let us consider linear map

$$\Phi : \mathcal{C}^{\infty}(G_0) \otimes \Lambda\left(\mathfrak{g}_1^*\right) \quad \longrightarrow \quad A\left(G\right)$$

defined over decomposable elements according to

$$\Phi\left(f \otimes \omega^*\right) \quad = \quad \phi_{f\omega^*}$$

with $\phi_{f\omega^*} \in E\left(G_0, \mathfrak{g}\right)'$ defined by

$$\left\langle xX\lambda(\omega), \phi_{f\omega^*}\right\rangle \quad := \quad Xf(x)\left\langle \omega, \omega^*\right\rangle$$

where λ is the isomorphism $E\left(\mathfrak{g}_0\right) \otimes \Lambda\left(\mathfrak{g}_1\right) \ni X \otimes \omega \longmapsto X\lambda(\omega) \in E\left(\mathfrak{g}\right)$ defined in appendix A. According to such isomorphism we will express each element in $E(\mathfrak{g})$ as a product $X\lambda(\omega)$ with $X \in E(\mathfrak{g}_0)$ and $\omega \in \Lambda(\mathfrak{g}_1)$. Let us first verify that $\phi_{f\omega^*}$ is in $A(G)$. Given $Y = X\lambda(\omega) \in E\left(\mathfrak{g}\right)$ and $\overline{X} \in \mathfrak{g}_0$, we have:

$$\begin{aligned} \left(Y.\phi_{f\omega^*}\right)^{\sim}(x) &= \left\langle xX\lambda(\omega), \phi_{f\omega^*}\right\rangle \\ &= Xf(x)\left\langle \omega, \omega^*\right\rangle \end{aligned}$$

and

$$\begin{aligned} \left(\overline{X}Y.\phi_{f\omega^*}\right)^{\sim}(x) &= \left\langle x\overline{X}X\lambda(\omega), \phi_{f\omega^*}\right\rangle \\ &= \overline{X}Xf(x)\left\langle \omega, \omega^*\right\rangle \\ &= \overline{X}\left(Y.\phi_{f\omega^*}\right)^{\sim}(x) \end{aligned}$$

showing that Φ is well defined taking values in $A(G)$. We now pass to show that $\Phi : \mathcal{C}^{\infty}(G_0) \otimes \Lambda\left(\mathfrak{g}_1^*\right)^{\mathrm{opp}} \longrightarrow A\left(G\right)$ is a super-algebra morphism, where $\Lambda\left(\mathfrak{g}_1^*\right)^{\mathrm{opp}}$is $\Lambda\left(\mathfrak{g}_1^*\right)$ with multiplication law $\vee$ defined by

$$\omega \vee \tau \quad := \quad \tau \wedge \omega \quad \forall \omega, \tau \in \Lambda\left(\mathfrak{g}_1^*\right)$$

Clearly Φ is an even linear map, hence it is sufficient to prove that

i) $\Phi\left(fg \otimes 1\right) = \Phi\left(f \otimes 1\right) \cdot \Phi\left(g \otimes 1\right)$ for each $f, g \in \mathcal{C}^{\infty}(G_0)$

ii) $\Phi\left(f \otimes \omega^*\right) = \Phi\left(f \otimes 1\right) \cdot \Phi\left(1 \otimes \omega^*\right)$ for each $f \in \mathcal{C}^{\infty}(G_0)$ and $\omega^* \in \Lambda\left(\mathfrak{g}_1^*\right)$

iii) $\Phi\left(1 \otimes Z^* \vee \omega^*\right) = \Phi\left(1 \otimes Z^*\right) \cdot \Phi\left(1 \otimes \omega^*\right)$, for all $Z^* \in \mathfrak{g}_1^*, \omega^* \in \Lambda\left(\mathfrak{g}_1^*\right)$

In fact, if the above properties hold true, then

$$\begin{aligned} \Phi\left(\left(f \otimes Z_1^* \vee ... \vee Z_k^*\right) \cdot \left(g \otimes Z_{k+1}^* \vee ... \vee Z_n^*\right)\right) &= \Phi\left(fg \otimes 1\right) \cdot \Phi\left(Z_1^* \vee ... \vee Z_n^*\right) \\ &= \Phi(f) \cdot \Phi(g) \cdot \Phi\left(Z_1^* \vee ... \vee Z_k^*\right) \cdot \Phi\left(Z_{k+1}^* \vee ... \vee Z_n^*\right) \\ &= \Phi\left(f \otimes Z_1^* \vee ... \vee Z_k^*\right) \cdot \Phi\left(g \otimes Z_{k+1}^* \vee ... \vee Z_n^*\right) \end{aligned}$$

We now verify i). We will use formula A.1 in appendix A for evaluating $\Delta\lambda$; moreover, in order to simplify notations, we will write $\Delta(X) = X_{(1)} \otimes X_{(2)}$.

$$\begin{aligned} \left\langle xX\lambda(Z_1 \wedge ... \wedge Z_k), \phi_f \cdot \phi_g\right\rangle &= \left\langle (x \otimes x)\,\Delta\left(X\right)\Delta\left(\lambda\left(Z_1 \wedge ... \wedge Z_k\right)\right), \phi_f \otimes \phi_g\right\rangle \\ &= \sum_{r=0}^{k} \sum_{\substack{J \subset \{1...k\} \\ \text{increasing order} \\ |J| = r}} (-1)^{p\left((1,\dots,k)\to\left(J,\overline{J}\right)\right)} \left\langle xX_{(1)}\lambda\left(Z_J\right), \phi_f\right\rangle \times \\ &\quad \times \left\langle xX_{(2)}\lambda\left(Z_{\overline{J}}\right), \phi_g\right\rangle \\ &= \left\langle xX\lambda\left(Z_1 \wedge ... \wedge Z_k\right), \phi_{fg}\right\rangle \end{aligned}$$

moreover

$$\begin{aligned}\langle xX\lambda(1), \phi_f \cdot \phi_g\rangle &= \langle (x\otimes x)\,\Delta(X), \phi_f\otimes\phi_g\rangle\\ &= X_{(1)}f(x)X_{(2)}f(x)\\ &= X\,(fg)\,(x)\\ &= \langle xX\lambda(1), \phi_{fg}\rangle\end{aligned}$$

Consider now ii). It is not restrictive to suppose $\omega^* \in \Lambda^n(\mathfrak{g}_1^*)$ with $n > 0$.

$$\begin{aligned}\langle xX\lambda(Z_1\wedge\ldots\wedge Z_k), \phi_f\cdot\phi_{\omega^*}\rangle &= \sum_{r=0}^{k}\sum_{\substack{J\subset\{1\ldots k\}\\ \text{increasing order}\\ |J|=r}} (-1)^{p\left((1,\ldots,k)\to(J,\overline{J})\right)}\left\langle xX_{(1)}\lambda(Z_J), \phi_f\right\rangle\left\langle xX_{(2)}\lambda(Z_{\overline{J}}), \phi_{\omega^*}\right\rangle\\ &= \left\langle xX_{(1)}, \phi_f\right\rangle\left\langle xX_{(2)}\lambda(Z_1\wedge\ldots\wedge Z_k), \phi_{\omega^*}\right\rangle\\ &= Xf(x)\,\langle Z_1\wedge\ldots\wedge Z_k, \omega^*\rangle\\ &= \langle xX\lambda(Z_1\wedge\ldots\wedge Z_k), \phi_{f\omega^*}\rangle\end{aligned}$$

and

$$\langle xX, \phi_f\cdot\phi_{\omega^*}\rangle \ = 0 = \ \langle xX, \phi_{f\omega^*}\rangle$$

We finally come to iii). With the same notations as above:

$$\begin{aligned}\langle xX\lambda(Z_1\wedge\ldots\wedge Z_k), \phi_{Z^*}\cdot\phi_{\omega^*}\rangle &= \sum_{r=0}^{k}\sum_{\substack{J\subset\{1\ldots k\}\\ \text{increasing order}\\ |J|=r}} (-1)^{p\left((1,\ldots,k)\to(J,\overline{J})\right)}\times\\ &\quad\times\left\langle xX_{(1)}\lambda(Z_J)\otimes xX_{(2)}\lambda(Z_{\overline{J}}), \phi_{Z^*}\otimes\phi_{\omega^*}\right\rangle\\ &= \sum_{j=1}^{k}(-1)^{p((1,\ldots,k)\to(j,1,2,\ldots,j-1,j+1,\ldots,k))}(-1)^{k-1}\times\\ &\quad\times\left\langle xX_{(1)}\lambda(Z_J), \phi_{Z^*}\right\rangle\left\langle xX_{(2)}\lambda\left(Z_{(1,2,\ldots,j-1,j+1,\ldots,k)}\right), \phi_{\omega^*}\right\rangle\end{aligned}$$

Now, if X belongs to $\mathfrak{g}_0E(\mathfrak{g}_0)$, then by the last formula

$$\langle xX\lambda(Z_1\wedge\ldots\wedge Z_k), \phi_{Z^*}\cdot\phi_{\omega^*}\rangle \ = 0 = \ \langle xX\lambda(Z_1\wedge\ldots\wedge Z_k), \phi_{Z^*\vee\omega^*}\rangle$$

If $X = 1$, then

$$\begin{aligned}&= (-1)^{k-1}\sum_{j=1}^{k}(-1)^{j+1}\langle Z_j, Z^*\rangle\left\langle Z_{(1,2,\ldots,j-1,j+1,\ldots,k)}, \omega^*\right\rangle\\ &= (-1)^{k-1}\langle Z^*\lrcorner(Z_1\wedge\ldots\wedge Z_k), \omega^*\rangle\\ &= \langle (Z_1\wedge\ldots\wedge Z_k), \omega^*\wedge Z^*\rangle\\ &= \langle x\lambda(Z_1\wedge\ldots\wedge Z_k), \phi_{\omega^*\wedge Z^*}\rangle\end{aligned}$$

We have to prove that Φ is an isomorphism. Φ is clearly injective. Suppose hence that ϕ is an element in $A(G)$. Define $\varphi : G_0 \longrightarrow \Lambda(\mathfrak{g}_1^*)$ as

$$\langle \omega, \varphi(x)\rangle \ := \ \langle x\lambda(\omega), \phi\rangle \quad \forall\omega\in\Lambda(\mathfrak{g}_1)$$

φ can be identified with an element $\varphi \in \mathcal{C}^{\infty}(G_0) \otimes \Lambda(\mathfrak{g}_1^*)$ since

$$\langle \omega, \varphi(x) \rangle = (\lambda(\omega).\phi)\tilde{}\,(x)$$

Moreover $\Phi(\varphi) = \phi$. In fact:

$$\begin{aligned} \langle xX\lambda(\omega), \Phi(\varphi) \rangle &= \langle \omega, X\varphi(x) \rangle \\ &= X(\lambda(\omega).\phi)\tilde{}\,(x) \\ &= (X\lambda(\omega).\phi)\tilde{}\,(x) \\ &= \langle xX\lambda(\omega), \phi \rangle \end{aligned}$$

To end the proof, we show that each $X \in \mathfrak{g}$ is a left invariant vector field. From this follows that $\mathfrak{g}$ is the Lie algebra of G. Let $Y \in E(\mathfrak{g})$, then, given $\phi \in A(G)$,

$$\begin{aligned} &\langle xY, [(\mathrm{id} \otimes \delta_e) \circ (\mathrm{id} \otimes Z) \circ m^*](\phi) \rangle \\ &\qquad = \langle xY \otimes eZ, m^*(\phi) \rangle \\ &\qquad = \langle xY, Z.\phi \rangle = \end{aligned}$$

so that

$$(\mathrm{id} \otimes \delta_e) \circ (\mathrm{id} \otimes Z) \circ m^* = Z$$

■

Chapter 6

The Weil-Berezin functor

...Tu sei quell'albero
D'onde germoglieranno
Per forza matematica
Sei germi in men d'un anno
Con gran felicità.
Si sa, si sa, si sa.

Buralicchio
L'equivoco stravagante

In this chapter we analyze the "fine" structure of a super manifold by means of a nice device invented by A. Weil ([Wei53]) in the purely even context (for a complete and beautiful treatment of Weil's ideas, we remand to [KMS93]). Eventually we will establish a link between this approach and a construction that first appeared in [Ber87].

The idea is quite simple. Since, as we have seen, the geometrical structure of a super manifold M is completely encoded in in the graded algebra of sections, the *local* differential structure near a point x has to be enclosed in the stalk $\mathcal{O}(M)_x$ at x.

This is an object with a peculiar algebraic structure and it is natural that in order to study it one has to use objects of the "same kind". This naturally leads to introduce the category $\mathcal{L}oc$ of local algebras (section 6.1), and to consider the space

$$M^A \;\; := \;\; \bigcup_{x \in M} \mathrm{g-Alg}\left(\mathcal{O}(M)_x, A\right) \; = \; \mathrm{g-Alg}\left(\mathcal{O}(M), A\right)$$

of superalgebra morphisms from $\mathcal{O}(M)$ to the local algebra A. The elements of M^A are called A-near points of M. It is interesting to note that the super algebra of sections over a super-point $\mathbb{R}^{0|s}$ forms a local algebra isomorphic to a Grassman algebra Λ^s in s generators . $M^{\mathbb{R}^{\Lambda^s}}$ can hence be interpreted as the *analysis* of the supermanifold M through the super-points $\mathbb{R}^{0|s}$. In section 6.2, we show that M^A can be endowed with the structure of a a purely even manifold fibered over the reduced manifold $\tilde{M}$. As one may expect the above construction exhibits a nice functorial behavior in the sense that it defines a bi-functor

$$\begin{array}{rcl} \mathcal{SM} \times \mathcal{L}oc & \longrightarrow & \mathcal{M}an \\ (M, A) & \longmapsto & M^A \end{array}$$

This is proved in section 6.3.

One of the more beautiful aspects of the A-near points theory is that it allows to develop the differential calculus on manifolds using algebraic techniques that resemble very closely the original method of "infinitesimal numbers" due to Fermat. Consider in fact the (even) local algebra $\mathbb{R}^\epsilon = \mathbb{R} \oplus \mathbb{R}^\epsilon$ where $\epsilon^2 = 0$, and let M be a classical manifold. As noticed by Weil, the set of $\mathbb{R}^\epsilon$-point of M canonically identifies with the tangent bundle over M. Such construction extends without difficulty to the super context (section 6.4). This approach allows to compute all the differential structures over M^A we are interested in. In doing this, the transitivity theorem 30 plays a crucial role. The section ends showing that also vector fields over M can be prolonged to vector fields over M^A.

Finally, in section 6.5, we apply the above results to the case in which M is a supergroup G. In such a case, each G^A acquires a Lie group structure in a functorial way so that we get a functor

$$\begin{array}{rcl} \mathcal{L}oc & \longrightarrow & \mathcal{LG} \\ A & \longmapsto & G^A \end{array}$$

from the category of local algebras to the category of Lie groups that can be "restricted" to a functor

$$\begin{array}{rcl} \mathfrak{B} : \mathcal{PL}oc & \longrightarrow & \mathcal{LG} \\ \Lambda & \longmapsto & \mathrm{g-Alg}\left(\mathcal{O}\left(G\right), \Lambda\right) \end{array}$$

where $\mathcal{PL}oc$ denotes the full subcategory of $\mathcal{L}oc$ whose objects are the algebras of sections over some super-point. $\mathcal{PL}oc$ clearly identifies with the category of Grassman algebras. We call $\mathfrak{B}$, the Weil-Berezin functor. Due to prop. 9, we have that there is a bijection

$$\mathrm{g-Alg}\left(\mathcal{O}\left(G\right), \Lambda\right) \quad \longleftrightarrow \quad \mathrm{Mor}_{\mathcal{SM}}\left(\mathbb{R}^{0|s}, G\right)$$

The Weil-Berezin functor is hence a relative of the functor of points $G(\cdot)$, but it differs from it in the following two aspects:

i) it is defined only on the subcategory of super-points;

ii) it takes values in the category of Lie groups.

Such a functor determines the supergroup G uniquely. With respect to this point, notice that if $\mathfrak{B}$ were only the restriction of $G(\cdot)$ to super-points then this fact would not have been true. In fact there would have been no possibility of recovering the smooth structure of G. Nevertheless, in our case, being true point ii), we have that $G^{\mathbb{R}^{0|0}}$ is a Lie group that actually identifies with the reduced group G_0. Moreover using the results of section 6.4, we can "differentiate" the Weil-Berezin functor and obtain a functor

$$\mathcal{PL}oc \longrightarrow \mathcal{LA} \tag{6.1}$$

$$\Lambda \longmapsto T_e\left(G^\Lambda\right) = \left(\mathfrak{g} \otimes \Lambda\right)_0 \tag{6.2}$$

This actually was the starting point of Berezin construction. It is well known (see [DM99a]) that the functor 6.1 determines uniquely the super Lie algebra $\mathfrak{g}$, so that $\mathfrak{B}$ brings complete information about the SHCP $(G_0, \mathfrak{g})$ and hence about the super group G.

6.1 Local algebras

In this section we extend the theory of A near points over a manifold (see [Wei53]) to the graded setting. In what follows we will call **local (graded) algebra** a graded algebra A which is also finite-dimensional, commutative, unital and with a graded ideal $J = J_0 \oplus J_1$ such that

i) $\dim A/J = 1$

ii) J is nilpotent, i.e. there exists $n \in \mathbb{N}$ such that $J^n = 0$

Example 4 *Let $(M, \mathcal{O}(M))$ be a graded manifold. Fix a point $x \in M_0$. If I_x denotes the ideal of sections f vanishing at x we have that for each $p = 0, ...$*

$$\mathcal{O}(M) / I_x^{p+1}$$

is a local algebra with nilpotent ideal I_x / I_x^{p+1}.

Example 5 *Another fundamental example of local algebra is given by the sections over a point pt. Clearly it identifies with some exterior algebra in N generators $\Lambda^N = \mathbb{R} \oplus \Lambda_+^N$.*

Example 6 *Consider the graded vector space $\mathbb{R}^{\epsilon,\theta} = (\mathbb{R} + \mathbb{R}\epsilon) \oplus \mathbb{R}\theta$ with $\epsilon \in \left(\mathbb{R}^{\epsilon,\theta}\right)_0$, $\theta \in \left(\mathbb{R}^{\epsilon,\theta}\right)_1$ and relations $\epsilon^2 = \theta^2 = \epsilon\theta = 0$. $\mathbb{R}^{\epsilon,\theta}$ is clearly a local algebra and $J = \mathbb{R}\epsilon \oplus \mathbb{R}\theta$. We will also denote by $\mathbb{R}^\epsilon$ the even algebra $\left(\mathbb{R}^{\epsilon,\theta}\right)_0$.*

Notice that last example shows that for a local algebra, J_0 need not to be the ideal generated by J_1.

The smallest integer n such that $J^n = 0$ is called the *height* of J. Since A is unital, $\mathbb{R}$ canonically embeds in A and moreover it splits into the direct sum

$$A = \mathbb{R} \oplus J \tag{6.3}$$

Hence we define the evaluation map

$$\begin{aligned} \tilde{}: A &\longrightarrow \mathbb{R} \\ a &\longmapsto \tilde{a} := \mathrm{pr}_{\mathbb{R}}(a) = a \mathrm{mod} J \end{aligned}$$

$\tilde{a}$ is also called the *finite part* of a.

Consider the algebra of "graded polynomials" in $r|p$ variables.

$$P(r|p) = \mathbb{R}\left[x^1, ..., x^r\right] \otimes \Lambda\left(\theta^1, ..., \theta^p\right)$$

For each choice of a $r|p$-tuple

$$\begin{aligned} i_1, ..., i_r &\in J_0 \\ j_1, ..., j_p &\in J_1 \end{aligned}$$

the map

$$\psi : \mathbb{R}^{r|p} \ni P \longmapsto P\left[i_1 ... i_r, j_1 ... j_p\right] \tag{6.4}$$

sends $\mathbb{R}^{r|p}$ into A and it is a mapping of graded algebras.

Lemma 7 *The map 6.4 is onto if and only if* $i_1, ..., i_r, j_1, ..., j_p$ *span* J/J^2.

The lemma shows that local algebras are finite dimensional quotient of the algebra of formal power series over $\mathbb{R}$. In particular are local algebras all algebras of the form

$$\mathbb{R}^{n|m} \big/ J^{p+1}$$

where J is the ideal of those element which evaluated are zero. This is a local algebra of height p. We omit the proof since it goes exactly as in the classical case and we will not need such a result.

6.2 A-near points

Definition 16 *Let* A *be a local algebra,* M *be a graded manifold and* $x \in M_0$. *We call* A*-point near to* x *in* M_0 *every homomorphism*

$$x^A \quad \in \quad g - Alg\left(\mathcal{O}\left(M\right), A\right)$$

such that

$$\widetilde{x^A\left(f\right)} = \tilde{f}(x) \qquad \forall f \in \mathcal{O}\left(M\right)$$

Notice that if x^A is a point near to x then

$$\tilde{x^A} := pr_{\mathbb{R}} \circ x^A \quad \in \quad g - Alg\left(\mathcal{O}\left(M\right), \mathbb{R}\right)$$

and such morphism corresponds to the evaluation at x. This allows to construct a projection

$$\pi : M^A \quad \longrightarrow \quad M \tag{6.5}$$

$$x^A \quad \longmapsto \quad \tilde{x^A} \tag{6.6}$$

Remark 8 *If* A *is an exterior algebra* Λ *then it can also be interpreted as the sheaf of sections of a super manifold of the form* $\mathbb{R}^{0|s}$. *In this case the map* $\mathbb{R}^{0|s} \longrightarrow M$ *given by*

$$\begin{cases} \psi_0\left(pt\right) = \tilde{x^A} \\ \psi^* = x^A \end{cases}$$

is a morphism of super manifolds.

Definition 17 *We define the set of* A *near points to a super manifold* M *as*

$$M^A \quad := \quad g - Alg\left(\mathcal{O}\left(M\right), A\right)$$

Next proposition says that an A-point x^A is local in the sense that it depends only on germs of sections near $\tilde{x}_A$.

Lemma 8 *If* $f \in \mathcal{O}\left(M\right)$ *vanishes in a neighborhood of* $\tilde{x}_A$ *then* $x^A(f) = 0$

Proof. Let U be the neighborhood of $\tilde{x}_A$ on which f vanishes, and V be a neighborhood of $\tilde{x^A}$ such that $\overline{V} \subset U$. Let $\phi \in \mathcal{O}\left(M\right)_0$ be such that $\phi_V = 1$ and supp$\phi \subset U$. Since $\phi f = 0$

$$0 = x^A(f\phi) = x^A(\phi)x^A(f)$$

and, due to the fact that $x^A(\phi)$ is invertible in A, we are done. ■

From this, next results follow easily:

Proposition 24 *$x^A(f)$ depends only on the germ of f at $\tilde{x}_A$.*

This proposition says that an A-point x^A can actually be viewed as an element of

$$\mathcal{O}(M)^A_x \quad := \quad g - Alg\left(\mathcal{O}(M)_x, A\right) \tag{6.7}$$

Proposition 25 *Let x^A be an A-point near $\tilde{x^A}$, then x^A is uniquely determined by the elements*

$$\begin{aligned} x^A(x^i) &\in x^i(\tilde{x}_A) + J_0 \\ x^A(y^j) &\in J_1 \end{aligned}$$

$\left\{x^i, y^j\right\}$ being graded coordinates near $\tilde{x^A}$. Moreover to each assignment of this kind there corresponds a unique A-point near $\tilde{x^A}$.

Proof. Fix x^A and define

$$\begin{aligned} x^A(x^i) &= f_i = x^i(\tilde{x^A}) + \overline{f_i} \in A_0 \\ x^A(y^j) &= F_j \in A_1 \end{aligned}$$

Consider the morphism x_f which is defined as x^A on coordinate sections, and extend it to a generic section using formal Taylor development (this is meaningful due to the nilpotency of the sections). Define also the morphism

$$x^A - x_f \quad \in \quad \mathrm{Hom}\left(\mathcal{O}(M), A\right)_0$$

and notice that

$$\left(x^A - x_f\right)(x^i) \quad = \quad \left(x^A - x_f\right)(y^j) = 0 \qquad \forall i, j$$

Due to proposition 1, given $s \in \mathcal{O}(M)$ and an integer k, there exists a polynomial $P^k_{\tilde{x^A}}$ such that

$$\left(s - P^k_{\tilde{x^A}}\right) \quad \in \quad J^{k+1}_{\tilde{x^A}}$$

From this the first part follows easily. The second is clear. ■

Manifold structure We will devote this section to show that M^A can be endowed with a manifold structure. As a preliminary remark let us note that every section in $\mathcal{O}(M)$ determines a map

$$f^A : M^A \quad \longrightarrow \quad A \tag{6.8}$$

defined as

$$f^A\left(x^A\right) \quad = \quad x^A(f)$$

If we fix a basis in A we can write

$$\begin{aligned} f^A(x^A) &= \sum \left\langle x^A(f), a_k^* \right\rangle a_k \\ &= \sum f_k^A(x^A) a_k \end{aligned}$$

where $f_k : M^A \to \mathbb{R}$. We endow M^A with a manifold structure such that the maps f_k become smooth.

Proposition 26 *Let M be a $n|m$-dimensional graded manifold and A a local algebra endowed with its natural vector space topology. The set M^A can be given a smooth structure canonically determined by the condition that each $f \in \mathcal{O}(M)$ is smooth when considered as a map from M^A to A. The triple $\left(M^A, \pi, M\right)$ is a fiber bundle.*

Proof. We use theorem 1 in Appendix. The first step consists in defining a denumerable family of subsets U_α^A of M^A with corresponding maps

$$h_\alpha : U_\alpha^A \longrightarrow \mathbb{R}^n$$

To this end consider a denumerable open cover of M through coordinate neighborhoods (U_α). Using the projection π defined by eq. 6.5 we can lift each open set U_α to the set

$$U_\alpha^A = \pi^{-1}(U_\alpha)$$

Clearly $\left\{U_\alpha^A\right\}$ is a cover of M^A. Let $x^A \in U_\alpha^A$ and let x^i, y^j be a graded coordinate system on U_α. Due to prop. 25, x^A is completely determined by its action on x^i, y^j, We have

$$\begin{aligned}
x^A(x^i) &= \sum_{k=0}^{r} \left\langle x^A(x^i), a_k^* \right\rangle a_k \\
&= \sum_{k=0}^{r} x_k^i(x^A) a_k \\
&= x^i(\tilde{x^A}) + n^i \\
x^A(y^j) &= \sum_{k=1}^{r} \left\langle x^A(y^j), a_k^* \right\rangle a_k \\
&= \sum_{k=1}^{r} y_k^j(x^A) a_k = N^j
\end{aligned}$$

hence each x^A in U_α^A is completely determined by the value of the functions

$$\left\{x_k^i, y_k^i\right\}$$

Define the map

$$\begin{aligned}
h_\alpha : U_\alpha^A &\longrightarrow U_\alpha \times \mathbb{R}^{(\dim A_0 - 1)\cdot n} \times \mathbb{R}^{\dim A_1 \cdot m} \\
x^A &\longmapsto \left(x_0^1(x^A), ..., y_{\dim A}^m(x^A)\right)
\end{aligned}$$

Using again prop. 25, it is straightforward that the above map is a bijection.
Let now U_β be another chart and let

$$\begin{aligned}
\overline{x}^i &= \sum_P \overline{x}_P^i(x^1, ..., x^n) y^P \qquad i = 1, ..., n \\
\overline{y}^j &= \sum_Q \overline{y}_Q^j(x^1, ..., x^n) y^Q \qquad j = 1, ..., m
\end{aligned}$$

be the transition functions on $U_\alpha \cap V_\beta$. We have

$$\overline{x}_k^i = \left\langle x^A(\overline{x}^i), a_k^* \right\rangle = \sum_P \left\langle x^A \left(\overline{x}_P^i(x^1, ..., x^n) y^P\right) x^A \left(y^P\right), a_k^* \right\rangle$$

and hence

$$\left\langle \sum_P \overline{x}^i_P \left(x^1 + n^1, ..., x^n + n^n\right) N^P, a^*_k \right\rangle = \sum_{P,F} \frac{\partial^{|F|} \overline{x}^i_P}{\partial x^F} (x^1, ..., x^n) \left\langle n^F \cdot N^P, a^*_k \right\rangle$$

where

$$\begin{aligned} \left\langle n^F \cdot N^P, a^*_k \right\rangle &= \left\langle n_1^{F_1} ... n_n^{F_n} N_1^{P_1} ... N_m^{P_m}, a^*_k \right\rangle \\ &= \left\langle \left(\sum x^1_k a_k\right)^{F_1} ... \left(\sum x^n_k a_k\right)^{F_n} \cdot \left(\sum y^1_k a_k\right)^{P_1} ... \left(\sum y^m_k a_k\right)^{P_m}, a^*_k \right\rangle \end{aligned}$$

Since this last expression is polynomial in x^i_k and y^j_k, we obtain that the transition functions are smooth. All other conditions in Proposition B are easily seen to be satisfied, so that M^A is endowed with a smooth structure. Such a structure can in principle depend on the choice of the atlas $\{U_\alpha\}$ of M and on the choice of the basis of A. Nevertheless calculations completely similar to that already done clarify that this is not the case. ■

Remark 9 *The bundle structure described by the above proposition can be seen, using the identification given by eq.6.7, as corresponding to the decomposition*

$$M^A \simeq \bigcup_{x \in M} \mathcal{O}(M)^A_x$$

Using previous proposition and remembering the discussion in section 3.2, one easily proves the following

Corollary 3 $\left(\mathbb{R}^{1|1}\right)^A$ *is diffeomorphic to* A *with the canonical vector space topology through the map*

$$\begin{array}{ccccc} \left(\mathbb{R}^{1|1}\right)^A & \longrightarrow & \left(\mathbb{R}^{1|1} \otimes A\right)_0 & \longrightarrow & A \\ x^A & \longmapsto & e \otimes x^A(x) + f \otimes x^A(\theta) & \longmapsto & x^A(x) + x^A(\theta) \end{array}$$

where e, f *denote the canonical basis of* $\mathbb{R}^{1|1}$*. More generally if* V *is a graded vector space one has that* V^A *and* $(V \otimes A)_0$ *are diffeomorphic through*

$$\begin{array}{ccc} (V)^A & \longrightarrow & (V \otimes A)_0 \\ x^A & \longmapsto & \sum e_i \otimes x^A(e^*_i) + \sum f_j \otimes x^A(f^*_j) \end{array}$$

Proof. As a set we have:

$$\begin{aligned} \left(\mathbb{R}^{1|1}\right)^A &\simeq g - Alg\left(\mathcal{O}\left(\mathbb{R}^{1|1}\right), A\right) \\ &\simeq A_0 \oplus A_1 \end{aligned}$$

where $x^A \in \left(\mathbb{R}^{1|1}\right)^A$ is identified with $(a'_0, a'_1) = \left(x^A(x), x^A(\theta)\right) \in A_0 \oplus A_1$. The smooth functions on A are easily identified through

$$\begin{aligned} x_k\left(x^A\right) &= \left\langle x^A(x), a^*_k \right\rangle \\ &= \left\langle a'_0, a^*_k \right\rangle \\ \theta_k\left(x^A\right) &= \left\langle x^A(\theta), a^*_k \right\rangle \\ &= \left\langle a'_1, a^*_k \right\rangle \end{aligned}$$

with the functions generated by the Cartesian coordinates on A. The proof of the last part is similar. ∎

Remark 10 *Notice that* $\left(\mathbb{R}^{1|1}\right)^A$ *and* A *are identified as classical manifolds.*

Remark 11 *Due to the previous proposition we have that each element of* $\mathcal{O}(M) \otimes A^*$ *identifies with the smooth function on* M^A *defined by*

$$f \otimes a^*(x^A) \quad = \quad \left\langle x^A(f), a^* \right\rangle$$

Moreover such functions determines the smooth structure on M^A *since they contain an atlas over* M^A*.*

6.3 The functor of A-near points

Extension of morphisms Let M and N be a graded manifold and let $\psi : M \longrightarrow N$ be a morphism of graded manifold. For each local algebra A ψ induces a morphism

$$\begin{aligned} \psi^A : M^A &\longrightarrow N^A \\ x^A &\longmapsto \psi^A(x^A) := x^A \circ \psi^* \end{aligned}$$

We call ψ^A the **extension** or the **prolongation** of ψ.

Proposition 27 *Let* M, N *and* ψ *be as above, then* $\psi^A : M^A \longrightarrow N^A$ *is smooth. Moreover*

$$(\psi \circ \phi)^A \quad = \quad \psi^A \circ \phi^A$$

Proof. We show that $\left(\psi^A\right)^* (f \otimes a^*)$ is a smooth function on M^A. But this follows immediately from

$$\begin{aligned} \left[\left(\psi^A\right)^* (f \otimes a^*)\right](x^A) &= (f \otimes a^*)\left(\psi^A(x^A)\right) \\ &= (f \otimes a^*)\left(x^A \circ \psi^*\right) \\ &= x^A \circ \psi^*(f) \otimes a^* \\ &= \left((\psi^*)(f) \otimes a^*\right)(x^A) \end{aligned}$$

which is smooth.
For the last part, we note that

$$\begin{aligned} (\psi \circ \phi)^A (x^A) &= x^A \circ (\psi \circ \phi)^* \\ &= x^A \circ \phi^* \circ \psi^* \\ &= \phi^A(x^A) \circ \psi^* \\ &= \psi^A \circ \phi^A(x^A) \end{aligned}$$

∎

Suppose now that a morphism of local algebras $\sigma : A \longrightarrow B$ is given and define

$$\begin{aligned} \sigma^M : M^A &\longrightarrow M^B \\ x^A &\longmapsto \sigma \circ x^A \end{aligned}$$

Proposition 28 *Let M, A, B and σ be as above, then $\sigma^M : M^A \longrightarrow M^B$ is a smooth mapping and*

$$\sigma^M \circ \tau^M = (\sigma \circ \tau)^M$$

Proof. The proof goes as the previous one. Let $f^B \otimes b^*$ be a smooth function on M^B then

$$\begin{aligned} \sigma^{M*}(f \otimes b^*)(x^A) &= (f \otimes b^*) \circ \sigma^M(x^A) \\ &= f \otimes b^*(\sigma \circ x^A) \\ &= x^A(f) \otimes \sigma^*(b^*) \\ &= [f \otimes \sigma^*(b^*)]\,(x^A) \end{aligned}$$

which is clearly smooth. ■

We can summarize the content of propositions from 26 to 28 in the following theorem.

Theorem 3 *The assignments*

$$\begin{aligned} \mathcal{O}bj\,(\mathcal{SM}) \times \mathcal{O}bj\,(\mathcal{L}oc) &\longrightarrow \mathcal{O}bj\,(\mathcal{M}an) \\ (M, A) &\longmapsto M^A \end{aligned}$$

and

$$\begin{aligned} \mathrm{Mor}_{\mathcal{SM}} \times \mathrm{Mor}_{\mathcal{L}oc} &\longrightarrow \mathrm{Mor}_{\mathcal{M}an} \\ (\psi, \sigma) &\longmapsto \sigma^M \circ \psi^A \end{aligned}$$

defines a bi-covariant bi-functor

$$\mathcal{SM} \times \mathcal{L}oc \longrightarrow \mathcal{M}an$$

Moreover we have

Proposition 29 *The functor*

$$\begin{aligned} \mathcal{SM} &\longrightarrow \mathcal{M}an \\ M &\longmapsto M^A \end{aligned}$$

is product preserving in the sense that

$$(M \times N)^A \simeq M^A \times N^A$$

where the identification is settled by

$$\begin{aligned} \psi :\ M^A \times N^A &\longrightarrow (M \times N)^A \\ \left(x^A, y^A\right) &\longmapsto m^A \circ \left(x^A \otimes y^A\right) \end{aligned}$$

Proof. Let us first of all notice that both $M^A \times N^A$ and $(M \times N)^A$ are fibered over the cartesian product $M \times N$ and that ψ is fibered over the identity. Consider $\left(\tilde{x^A}, \tilde{y^A}\right) \in M \times N$, let $U \ni \tilde{x^A}$ be a coordinate neighborhood with coordinate x^i, θ^j and let $V \ni \tilde{y^A}$ be a coordinate neighborhood with coordinate y^r, η^s. The bijectivity and the smoothness of ψ follows from

$$
\begin{aligned}
m^A \circ \left(x^A \otimes y^A\right)\left(x^i \otimes 1\right) &= x^A(x^i)\\
m^A \circ \left(x^A \otimes y^A\right)\left(1 \otimes y^r\right) &= y^A(y^r)\\
m^A \circ \left(x^A \otimes y^A\right)\left(\theta^j \otimes 1\right) &= x^A(\theta^j)\\
m^A \circ \left(x^A \otimes y^A\right)\left(1 \otimes \eta^s\right) &= y^A(\eta^s)
\end{aligned}
$$

■

Next lemma shows that the A-near point functor is well behaved with respect to algebra tensor products. From now on we use freely the results in section 3.2.

Lemma 9 *Let A be a graded algebra and B a local algebra. According to corollary 3, there exists a canonical diffeomorphism*

$$
i : (A)^B \longrightarrow (A \otimes B)_0
$$

Moreover i preserves multiplication in the sense that if m denotes the multiplication in A and m^B the corresponding prolongation, then

$$
i\left(m^B\left(x^B, y^B\right)\right) = i\left(x^B\right) \cdot i\left(y^B\right)
$$

where $\cdot$ is the canonical product law in $(A \otimes B)_0$. Moreover if $\mathbb{R}^{1|1}$ is endowed with the super algebra structure given in section 3.2, then also the identification $\left(\mathbb{R}^{1|1} \otimes A\right)_0 \simeq A$, as ungraded algebras, preserves the product law.

Proof. The diffeomorphism follows from corollary 3. In order to show that multiplication is preserved let us write explicitly the identification

$$
\begin{aligned}
i : A^B &\longrightarrow (A \otimes B)_0\\
x^B &\longmapsto \sum a^k \otimes x^B\left(a^{*k}\right)
\end{aligned}
$$

Moreover multiplication in A and co-multiplication in A^* are related according to

$$
\begin{aligned}
m^A\left(a_i \otimes a_j\right) &= \sum_p c_{ij}^p a_p\\
m^{A*}\left(a^{*k}\right) &= \sum_{ij} c_{ij}^k a^{*i} \otimes a^{*j}
\end{aligned}
$$

Hence, we have:

$$
\begin{aligned}
\sum_{kij} a_k \otimes c_{ij}^k m^B\left(x^B\left(a^{*i}\right) \otimes x^B\left(a^{*j}\right)\right) &= \sum_{ij}\left(\sum_k c_{ij}^k a_k\right) \otimes x^B\left(a^{*i}\right) \cdot x^B\left(a^{*j}\right)\\
&= \sum_{ij} a^i \cdot a^j \otimes x^B\left(a^{*i}\right) \cdot x^B\left(a^{*j}\right)
\end{aligned}
$$

■

Remark 12 *Great care has to be paid to the following fact . In what follows A appears with two roles: as a super vector space that can also be viewed as a super manifold (see 3.2) or as the result of an "exponentiation" (like in the case $\left(\mathbb{R}^{1|1}\right)^A$) In this case A is regarded as an even manifold since the Weil functor takes values in the category of classical manifolds. In order to simplify notation we have avoided to explicitly indicate this fact hoping that the context will make it clear.*

We briefly stop in order to check the consistency of the notations[1]. If f is a section in $\mathcal{O}(M)$ then we can canonically define a morphism of graded manifolds $\phi_f : M \longrightarrow \mathbb{R}^{1|1}$ through

$$\begin{aligned} \phi^*(x) &= f_0 \\ \phi^*(\theta) &= f_1 \end{aligned}$$

Viceversa, to each morphism $\psi : M \longrightarrow \mathbb{R}^{1|1}$ we can associate the section $f_\psi = \psi^*(x)+\psi^*(\theta)$. Due to prop. 1, this is a bijection. Hence, it is natural to denote f and ϕ_f with the same letter. According to our previous proposition, given a local algebra A we can define a map

$$f^A : M^A \longrightarrow \left(\mathbb{R}^{1|1}\right)^A$$

Nevertheless the same symbol f^A was already used in eq 6.8 to denote the map

$$f^A(x^A) = x^A(f)$$

but, using lemma 9, we have

$$\begin{aligned} f^A : M^A &\longrightarrow \left(\mathbb{R}^{1|1}\right)^A \longrightarrow A \\ x^A &\longmapsto x^A \circ f^* \longmapsto x^A\left(f^*(x)\right) + x^A\left(f^*(\theta)\right) = x^A(f) \end{aligned}$$

Transitivity Given a graded manifold M we can form the manifold M^A of A-points of M. We can further iterate the process constructing the manifold of B-points $\left(M^A\right)^B$, nevertheless since M^A is a purely even manifold it is clear that

$$\left(M^A\right)^B \simeq \left(M^A\right)^{B_0}$$

In this paragraph we prove the following

Proposition 30 $\left(M^A\right)^B = \left(M^A\right)^{B_0}$ *is diffeomorphic to* $M^{A\otimes B_0}$ *through*

$$\begin{aligned} \tau : \left(M^A\right)^{B_0} &\longrightarrow M^{A\otimes B_0} \\ \left(x^A\right)^B &\longmapsto x^{A\otimes B_0} : \left(f \longmapsto i\left[\left(f^A\right)^B\left(x^A\right)^B\right]\right) \end{aligned}$$

where i is the identification $\left[\left(\mathbb{R}^{1|1}\right)^A\right]^B \longrightarrow A\otimes B_0$. *$\tau$ has the following properties:*

[1] We can quote here what Godement wrote about classical differential geometry ([God04]). " La théorie des variétés est remplie d'identifications canoniques et de construction functorielles qui, théoriquement, soulèvent à chaque pas des questions de compatibilité destinées à rassurer le lecteur sur le caractère non contradictoire des notions et abus de langage introduits. ... On n'est pas sur la Terre pour s'amuser. "

i) $\tau\left(\left(x^A\right)^B \cdot \left(y^A\right)^B\right) = \tau\left(\left(x^A\right)^B\right) \cdot \tau\left(\left(y^A\right)^B\right)$

ii) $\left(\tau\left(\left(x^A\right)^B\right)\right)(f) = \left(x^A\right)(f) \otimes 1_B + \xi$ *where* $\xi \in J_A \otimes B_0$

iii) $\tau \circ \left(\psi^A\right)^B = \psi^{A\otimes B_0} \circ \tau$

Remark 13 *Before proving the proposition notice that, in order to avoid heavy notations, we have adopted the convention that the image of* $\left(x^A\right)^B$ *under the identification is obtained "lowering B".*

Proof. Let f be a section in $\mathcal{O}(M)$. As already discussed f can be identified with a morphism $f : M \longrightarrow \mathbb{R}^{1|1}$ and hence extended to $f^A : M^A \longrightarrow A$ and again to $\left(f^A\right)^B$: $\left(M^A\right)^{B_0} \longrightarrow A^{B_0} \simeq A \otimes B_0$ Let us now consider the correspondence which assigns to each $\left(x^A\right)^{B_0}$ the mapping

$$\begin{array}{rcl} \mathcal{O}(M) & \longmapsto & A \otimes B_0 \\ f & \longmapsto & \left(f^A\right)^{B_0}\left(x^A\right)^{B_0} \end{array}$$

Using lemma 9, such a map is easily seen to be an $A \otimes B_0$-point near to $M^{A\otimes B_0}$, so that we have

$$\left(M^A\right)^{B_0} \longrightarrow M^{A\otimes B_0}$$

We now prove that the above correspondence is a diffeomorphism. Being a result local in M we can suppose $M = \mathbb{R}^{p|q}$:

$$\begin{aligned} \left(\left(\mathbb{R}^{p|q}\right)^A\right)^B &= (A_0^p \oplus A_1^q) \otimes B_0 \\ &= \mathbb{R}^p \otimes A_0 \otimes B_0 \oplus \mathbb{R}^q \otimes A_1 \otimes B_0 \\ &= \left(\mathbb{R}^{p|q}\right)^{A\otimes B_0} \end{aligned}$$

There remain points ii) and iii). Point ii) follows immediately noticing that

$$\begin{aligned} x^{A\otimes B_0} &= \sum_k a_k \otimes \left(\left(f^A\right)^B \left(x^A\right)^B\right)(a_k^*) \\ &= x^a(f) + \sum_k a_k \otimes b_k \end{aligned}$$

Point *ii)* is a consequence of

$$\begin{aligned} \psi^{A\otimes B_0}\tau\left[\left(x^A\right)^B\right](f) &= \tau\left[\left(x^A\right)^B\right]\psi^*(f) \\ &= \left[\psi^*(f)^A\right]^B \left(x^A\right)^B \\ &= \left[\tau\left(\psi^A\right)^B\left(x^A\right)^B\right](f) \end{aligned}$$

■

From the above propostion it is easy to show that

$$\left(x^A\right)^{B_0}(f \otimes a^*) \quad = \quad a^* \lrcorner x^{A\otimes B_0}(f) \tag{6.9}$$

Let us briefly sketch how[2]. Previous proposition gives the equality

$$x^{A\otimes B_0}(f) \quad = \quad \sum_k a_k \otimes \left(\left(f^A\right)^B (x^A)^B\right)(a_k^*)$$

so that

$$a^* \lrcorner x^{A\otimes B_0}(f) \quad = \quad \left(\left(f^A\right)^B (x^A)^B\right)(a^*)$$

Nevertheless

$$\begin{aligned}\left[\left(f^A\right)^B (x^A)^B\right](a^*) &= (x^A)^B \left[\langle f^A(\cdot), a^*\rangle\right] \\ &= (x^A)^B \left[\langle(\cdot)(f), a^*\rangle\right] \\ &= (x^A)^B (f \otimes a^*)\end{aligned}$$

and the equality follows.

Due to 6.3, every element x^A of M^A can be written as

$$f(x^A) \quad = \quad f(x) + L(f)$$

where $f(x) \in \mathbb{R}$ and $L(f) \in J$. Writing down the condition that x' is a graded algebra morphism we get easily

$$L(fg) \quad = \quad f(x)L(g) + L(f)g(x) + L(f)L(g)$$

suppose now that J has height 1 (that is $J^2 = 0$) then

$$L(fg) \quad = \quad L(f)g(x) + f(x)L(g)$$

where $L(\cdot)$ belongs to J. This suggests a way to construct algebraically the differential structures (tangent vectors, jets,...) over a manifold. We devote next paragraph to this topic.

6.4 Differential calculus with A-near points

Tangent structure Let $x^A \in g - Alg(\mathcal{O}(M), A)$. Due to the decomposition of A into $\mathbb{R} \oplus J$ we can write x^A as

$$x^A \quad = \quad \tilde{x^A} + L_{x^A}$$

where $\tilde{x^A} \in g - Alg(\mathcal{O}(M), \mathbb{R})$. It is easy to check that

$$L_{x^A}(fg) \quad = \quad L_{x^A} f g(x) + f(x) L_{x^A} g + L_{x^A}(f) L_{x^A}(g) \tag{6.10}$$

[2] We do not keep track of all the identifications made.

Suppose now that $A=\mathbb{R}^{\epsilon,\theta}$ and write

$$L_{x^A}(f) \quad = \quad L_\epsilon(f)\epsilon + L_\theta(f)\theta$$

Notice also that clearly L_ϵ (resp. L_θ) vanishes on odd (resp. even) sections. 6.10 is hence equivalent to

$$\begin{aligned} L_\epsilon(fg) &= L_\epsilon f g(x) + f(x) L_\epsilon g \\ L_\theta(fg) &= L_\theta f g(x) + f(x) L_\theta g \end{aligned}$$

From this it follows that each $x_{\mathbb{R}^{\epsilon,\theta}}$ canonically identifies a tangent vector at $x = \widetilde{x_{\mathbb{R}^{\epsilon,\theta}}}$. Conversely given a tangent vector $v = v_0 + v_1$ at x one easily verify that

$$\delta_x + v_0\epsilon + v_1\theta$$

is an element in M^A. Clearly if M is a manifold then $M^{\mathbb{R}^{\epsilon,\theta}} = M^{\mathbb{R}^\epsilon}$. As one may expect

Proposition 31 *Let M be a (purely even) manifold. $M^{\mathbb{R}^\epsilon}$ endowed with the smooth structure defined in prop. 26 is isomorphic to the tangent bundle over M*

Right now we have seen that if N is a manifold then $N^{\mathbb{R}^\epsilon}$ is the tangent bundle on N. Suppose now N has the form M^A (which is by no means restrictive) then, due to prop. 30, we have an isomorphism

$$\tau : T(M^A) \simeq (M^A)^{\mathbb{R}^\epsilon} \quad \longrightarrow \quad M^{A\otimes\mathbb{R}^\epsilon} \tag{6.11}$$

Due to point ii) of prop. 30 one has that if $\left(x^A\right)^{\mathbb{R}^\epsilon} = \delta_{x^A} + \epsilon V_{x^A}$ with $V_{x^A} \in T_{x^A}\left(M^A\right)$ hence $x^{A\otimes\mathbb{R}^\epsilon} = \tau\left(x^A\right)^{\mathbb{R}^\epsilon} = \delta_{x^A} + \epsilon v_{x^A}$

where $v_{x^A} : \mathcal{O}(M) \to A$ satisfies the following rule:

$$v_{x^A}(fg) \quad = \quad v_{x^A}(f)\cdot x^A(g) + x^A(f)\cdot v_{x^A}(g)$$

In general, let M be a supermanifold, A a superalgebra and let

$$\psi : \mathcal{O}(M) \quad \longrightarrow \quad A$$

be a morphism of graded algebras. One defines the space of ψ-linear derivations of M (ψ derivations for short) as the A-module

$$\mathrm{Der}_\psi\left(\mathcal{O}(M), A\right) := \left\{X | X \in \mathrm{Hom}\left(\mathcal{O}(M), A\right), X(fg) = X(f)\psi(g) + (-1)^{p(f)p(X)}\psi(f)X(g)\right\}$$

Hence we have that each tangent vector at x^A canonically identifies a x^A-derivation. Clearly such derivation sends even sections (resp. odd) into the even (resp. odd) subspace of A. Conversely each such derivation canonically identifies a tangent vector at x^A. Hence we have

Proposition 32 *The tangent space at x^A in M^A canonically identifies with*

$$\mathrm{Der}_{x^A}\left(\mathcal{O}(M), A\right)_0$$

We now need some insight into the structure of $\mathrm{Der}_{x^A}\left(\mathcal{O}\left(M\right),A\right)_0$. The following proposition describes it explicitly.

Let M be a right A-module and let N be a left B-module for some algebras A and B. Suppose moreover that an algebra morphism $\psi : A \longrightarrow B$ is given. One defines the tensor product $M \otimes_\psi N$ as the vector space $M \otimes N$ with the equivalence relation

$$m \cdot a \otimes n \quad \sim \quad m \otimes \psi(a) n$$

Proposition 33 *Let M be a supermanifold and let $x \in M$. Denote with $\mathcal{T}_x\left(M\right)$ the germs of vector fields at x. One has the identification of left A-modules*

$$A \otimes_{x^A} \mathcal{T}_{\tilde{x^A}}\left(M\right) \quad \simeq \quad \mathrm{Der}_{x^A}\left(\mathcal{O}\left(M\right),A\right) \tag{6.12}$$

The theorem is clearly local so that it is enough to prove it in the case M is a superdomain. Next lemma do this.

Proposition 34 *Let M be a superdomain of dimension (p,q) with coordinate system $\left\{x^i, y^j\right\}$ and let A a local algebra. To any set of elements*

$$\underline{f} = \{f_i, F_j\} \qquad i = 1, ...p \quad j = 1, ..., q; \quad f_i, F_j \in A$$

there corresponds a ψ-derivation

$$X_{\underline{f}} : \mathcal{O}\left(M\right) \quad \longrightarrow \quad A$$

given by

$$X_{\underline{f}}(h) \quad = \quad \sum_i f_i \psi\left(\frac{\partial h}{\partial x^i}\right) + \sum_j F_j \psi\left(\frac{\partial h}{\partial y^j}\right)$$

$X_{\underline{f}}$ is even (resp. odd) if and only if $\{f_i\}$ are even (resp. odd) and $\{F_j\}$ are odd (resp. even). Moreover any ψ-derivation is of this form for a uniquely determined set of functions $\underline{f}$.

Proof. That $X_{\underline{f}}$ is a ψ-derivation is clear. That the family $\underline{f}$ is uniquely determined is also immediate from the fact that they are the value of $X_{\underline{f}}$ on the coordinate functions.
Let now X be a generic ψ-derivation, Define

$$f^i \quad = \quad X\left(x^i\right) \tag{6.13}$$

$$F^j \quad = \quad X\left(y^j\right) \tag{6.14}$$

and

$$X_{\underline{f}} \quad = \quad f^i \psi_*\left(\frac{\partial}{\partial x^i}\right) + F^j \psi_*\left(\frac{\partial}{\partial y^j}\right)$$

Let

$$D \quad = \quad X - X_f$$

Clearly

$$D\left(x^i\right) \quad = \quad D\left(y^j\right) = 0$$

We now show that this implies $D = 0$. Let $h \in \mathcal{O}(M)$. Due to proposition 1 for each $m \in M$ and for each integer $k \in \mathbb{N}$ there exists a polynomial $P \in \mathcal{P}_{m,k}$ such that

$$h - P \quad \in \quad I_m^{k+1}$$

Due to Leibniz rule

$$D(h - P) \quad \in \quad J^k$$

where J is the nilpotent ideal in A, while due to 6.13

$$D(P) \quad = \quad 0$$

hence $D(h)$ is in J^k for arbitrary k and we are done. ■

Collecting the above results we get the following important result.

Theorem 4 *We have the identification*

$$T_{x^A} M^A \quad \simeq \quad \left(A \otimes_{x^A} \mathcal{T}_{\tilde{x^A}}(M)\right)_0$$

Push-forwards

Proposition 35 *Let M and N be super manifolds, A and B local algebras. Consider morphisms $\psi : M \longrightarrow N$ and $\sigma : A \longrightarrow B$. The push-forward of the map*

$$\begin{aligned} \sigma^N \circ \psi^A : M^A \quad &\longrightarrow \quad N^B \\ x^A \quad &\longmapsto \quad \sigma \circ x^A \circ \psi^* \end{aligned}$$

is given by

$$\begin{aligned} \left(\sigma^N \circ \psi^A\right)_{*x^A} : T_{x^A}\left(M^A\right) \quad &\longrightarrow \quad T_{\psi^A(x^A)}\left(N^B\right) \\ v_{x^A} \quad &\longmapsto \quad \sigma \circ (v_{x^A}) \circ \psi^* \end{aligned}$$

where the identification given by prop. 32 is assumed.

Proof. We calculate the push-forward of ψ^A, the calculation for σ^M being completely similar. The thesis follows by composition.

Keeping in mind point iii) of proposition 30 we have:

$$\begin{aligned} \tau\left[\left(\psi^A\right)_*(V_{x^A})\right] \quad &= \quad \tau\left[\left(\psi^A\right)_*\left[\left(x^A\right)^{\mathbb{R}^\epsilon}\right] - x^A\right] \\ &= \quad \tau\left[\left(\left(x^A\right)^{\mathbb{R}^\epsilon} - x^A\right) \circ \left(\psi^A\right)^*\right] \\ &= \quad \tau\left[\left(\psi^A\right)^{\mathbb{R}^\epsilon}\left(\left(x^A\right)^{\mathbb{R}^\epsilon} - x^A\right)\right] \\ &= \quad \psi^{A\otimes\mathbb{R}^\epsilon}\left[\tau\left(\left(x^A\right)^{\mathbb{R}^\epsilon} - x^A\right)\right] \\ &= \quad \tau\left(\left(x^A\right)^{\mathbb{R}^\epsilon} - x^A\right) \circ \psi^* \end{aligned}$$

■

Extension of vector fields It is possible to use prop. 33 in order to extend vector fields on M to vector fields on M^A. Indeed if X is a vector field on M of parity $p(X)$, then for each $a \in A_{p(X)}$ one can define the vector field on M^A

$$aX^A : x^A \longmapsto a \otimes_{x^A} X$$

For example if $a = 1$ and X is an even vector field on M, then X^A is the vector field defined by

$$X^A_{x^A}(f) := x^A (X(f)) \quad \forall f \in \mathcal{O}(M)$$

and, more generally,

$$\left(aX^A\right)_{x^A}(f) := a \cdot \left(x^A X(f)\right)$$

Moreover $A \otimes_{x^A} \mathcal{T}_{\tilde{x^A}}(M)$ can also be seen, in a canonical way, as a right A module by defining

$$a \otimes_{x^A} X \cdot a' := (-1)^{p(a')(p(a)+p(X))} a' \cdot a \otimes_{x^A} X$$

So that we can also define the vector field

$$X^A a : x^A \longmapsto (1 \otimes_{x^A} X) \cdot a(f) = (-1)^{p(f)p(X)} x^A (X(f)) \cdot a$$

$X^A a$ is called the *left prolongation* of X while aX^A is called the *right prolongation*. Due to eq. 6.9 we have that the corresponding vector fields are defined through[3]

$$\begin{aligned} \left(\left(X^A a\right)(f \otimes c^*)\right)(x^A) &= c^* \lrcorner \left(X^A a\right)_{x^A}(f) \\ &= (-1)^{p(X)p(f)} \left(X(f) \otimes a_{\llcorner} c^*\right)(x^A) \end{aligned}$$

We now pass to consider the commutator of two vector fields. It is easily calculated as follows

$$\begin{aligned} \left(X^A a \circ Y^B b\right)(f \otimes c^*) &= X^A a \left(Y^A b (f \otimes c^*)\right) \\ &= (-1)^{p(Y)p(f)+(p(Y)+p(f))p(X)} (XY)(f) \otimes a_{\llcorner} b_{\llcorner} c^* \end{aligned}$$

where $a_{\llcorner}$ denotes the adjoint of multiplication $(\cdot) \cdot a$. Hence we have:

$$\left[X^A a, Y^A b\right] = (-1)^{p(X)p(Y)} \left([X, Y]^A ab\right)$$

We now use the machinery developed in this section for investigating a construction that first appeared, in the super context, in Berezin book [Ber87].

[3] In what follows all checks and all calculations will be done for the left prolongation only, since it is the one we will be most concerned with.

6.5 The Weil-Berezin functor

Let (G, μ) be a super group. As we have described in the previous section we can associate to G the functor of near points:

$$\begin{array}{rcl} \mathcal{L}oc & \longrightarrow & \mathcal{M}an \\ A & \longmapsto & G^A \end{array}$$

As one may expect, we have

Proposition 36 *If G is a super group, the corresponding functor of near points take values in the category of Lie groups.*

Proof. In order to prove the theorem it is enough to show that each G^A can be endowed with a Lie group structure but this is an easy consequence of prop. 3. In fact each map μ, i and e can be transported to a smooth map

$$\begin{array}{rcl} \mu^A : (G \times G)^A \simeq G^A \times G^A & \longrightarrow & G^A \\ i^A : G^A & \longrightarrow & G^A \\ e^A : \mathbb{R}^0 & \longrightarrow & G^A \end{array}$$

and each of them obeys the necessary commutative diagrams. ∎

The explicit form of the maps μ^A, i^A, e^A is easily recognized to be

$$\begin{array}{rcl} m^A(x^A, y^A) = x^A \cdot y^A & = & m \circ x^A \otimes y^A \circ \mu^* \\ i^A(x^A) & = & x^A \circ i^* \\ e^A(\mathrm{pt}) & = & j \circ \delta_e \end{array}$$

where here $j : \mathbb{R} \longrightarrow A$ denotes the canonical inclusion of the scalars in the superalgebra A.

Lie algebra of G^A In this section we propose to investigate the structure of the Lie algebra of G^A. We have already seen that the tangent space at the identity of each G^A can be identified with

$$(A \otimes \mathfrak{g})_0$$

Actually, given a super Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, it is well known (see [DM99a], for example) that the correspondence

$$A \longrightarrow \mathfrak{g}^A := (A \otimes \mathfrak{g})_0$$

defines in a natural way a functor from the category of local algebras (for example) to the category of (ordinary) Lie algebras. Indeed on each $\mathfrak{g}^A$ one can define the commutator

$$\begin{array}{rcl} \left[\sum a_i X_i + \sum b_j Y_j, \sum c_p X_p + \sum d_q Y_q\right] & = & \sum a_i c_p \left[X_i, X_p\right] + \sum a_i d_q \left[X_i, Y_q\right] \\ & = & \sum b_j, c_p \left[Y_j, X_p\right] - \sum b_j d_q \left[Y_j, Y_q\right] \end{array}$$

where $a_i, c_p \in A_0$, $b_j, d_q \in A_1$ and $\{X_i, Y_j\}$ is a homogeneous basis of $\mathfrak{g}$. Moreover such functor determines uniquely the graded Lie algebra $\mathfrak{g}$. All such considerations are commonly used by physicists for making computations and fall under the name of *even rules principle* (see, for a complete account, [DM99a] and [Var04]).

It is not at all obvious that such Lie algebra structure is the one induced by the group G^A. We now want to prove that this is the case.
Define the map[4]

$$\begin{array}{rcl} \epsilon : (A \otimes \mathfrak{g})_0 & \longrightarrow & \mathrm{Vect} G^A \\ a \otimes X & \longmapsto & X^A a \end{array}$$

Proposition 37 *The map ϵ is a Lie algebra isomorphism between $(A \otimes \mathfrak{g}_0)$ and* $\mathrm{Lie}\left(G^A\right)$.

We need the following lemma.

Lemma 10

$$\left(\mathrm{id}_{\mathcal{O}(G)} \otimes \left(X^A a\right)_{e^a}\right) \circ \mu^*(f) \quad = \quad (-1)^{p(X)p(f)} \left[\left(\mathrm{id}_{\mathcal{O}(G)} \otimes X\right) \circ \mu^*(f)\right] \cdot a$$

Proof. First of all let us notice that $\left(X^A a\right)_{e^A} \in \mathrm{Der}_{x^A}\left(\mathcal{O}(G), A\right)_0$ is a continuous operator on $\mathcal{O}(G)$. This follows noticing that

$$\begin{array}{rcccccc} \left(X^A a\right)_{e^A} : \mathcal{O}(G) & \longrightarrow & \mathbb{R} & \hookrightarrow & A \\ f & \longmapsto & \delta_e\left(X(f)\right) & \longmapsto & a \cdot \delta_e\left(X(f)\right) \end{array}$$

and using prop. 6.

We are now ready for the calculation. We consider two cases. First we suppose that f is an even section, hence we have

$$\mu^*(f) \quad = \quad \sum f_{(1)} \otimes f_{(2)} + \sum g_{(1)} \otimes g_{(2)}$$

with $f_{(i)} \in \mathcal{O}(G)_0$, $g_{(i)} \in \mathcal{O}(G)_1$ and where the sum is actually a convergent series. Due to the continuity of the operator we have that

$$\begin{aligned} \left(\mathrm{id}_{\mathcal{O}(G)} \otimes \left(X^A a\right)_{e^a}\right) \circ \mu^*(f) &= \sum f_{(1)} \otimes X_e\left(f_{(2)}\right) \cdot a + (-1)^{p(X)} \sum g_{(1)} \otimes X_e\left(g_{(2)}\right) \cdot a \\ &= \left[(1 \otimes X_e) \circ \mu^*\right] \cdot 1 \otimes a \end{aligned}$$

In the case f is an odd section, then

$$\mu^*(f) \quad = \quad \sum f_{(1)} \otimes f_{(2)} + \sum g_{(1)} \otimes g_{(2)}$$

with $f_{(1)}$ and $g_{(2)}$ even, and $f_{(2)}, g_{(1)} \in \mathcal{O}(G)_1$. A completely similar calculation yields the desired result. ■

Proof. (of prop. 37) Due to simple dimension counting, the problem is reduced to prove that

i each $X^A a$ is actually left invariant

ii $\epsilon([aX, bY]) = \left[X^A a, Y^B b\right]$

[4]It is defined on decomposable elements and then extended by linearity.

Since point ii) follows immediately from eq. 6.15, let us show point i). Define the left multiplication

$$\begin{array}{rcl} L_{x^A} : G^A & \longrightarrow & G^A \\ y^A & \longmapsto & x^A y^A = m \circ x^A \otimes y^A \circ \mu^* \end{array}$$

We have

$$\begin{array}{rcl} \left(L_{x^A *} \left(X^A a\right)_{e^A}\right)(f) & = & m \circ x^A \otimes \left(X^A a\right)_{e^A} \circ \mu^*(f) \\ \text{(using lemma 10)} & = & \left(x^A \left(1 \otimes X_e \circ \mu^*(f)\right)\right) \cdot a \\ & = & \left(Xa\right)_{x^A}(f) \end{array}$$

and we are done. ■

We can heuristically summarize the result saying that

$$A \longrightarrow \mathfrak{g}^A$$

is the Lie algebra of the group

$$A \longrightarrow G^A$$

Structure of G^A In this paragraph we say some word about the structure of $\mathfrak{g}^A$ and G^A. First notice that if we denote by A^+ the nilpotent maximal ideal J of A, then $\mathfrak{n}(A) := \left(\mathfrak{g} \otimes A^+\right)_0$ is a nilpotent Lie subalgebra of $\mathfrak{g}^A$. Even more is true since $\mathfrak{g}^A$ decomposes into the semidirect product

$$\mathfrak{g}^A = \mathfrak{g}_0 \oplus' \left(\mathfrak{g} \otimes A^+\right)_0 = \mathfrak{g}_0 \oplus' \mathfrak{n}(A)$$

On the group level we have

Proposition 38 *Each G^A has the structure of semi-direct product*

$$G^A = G_0 \times' N^A$$

where G_0 is the Lie group underlying G and N^A is the simply connected nilpotent Lie group corresponding to $\mathfrak{n}(A)$.

Proof. (**sketch**) Let $x^A \in G^A$ and define

$$x^{A0} = x^A \circ r^*_{\tilde{x^A}^{-1}}$$

so that

$$\begin{array}{rcl} \tilde{x^{A0}} & = & \delta_e \\ x^A & = & x^{A0} \circ r^*_{\tilde{x^A}} \end{array}$$

Hence given a point x^A in G^A we have defined a point x^{A0} in

$$G_e^A := \left\{y^A \mid y^A \in \mathrm{g}-\mathrm{Alg}\left(\mathcal{O}\left(G\right), A\right) \text{ and } \tilde{y_a} = \delta_e\right\}$$

G_e^A is easily proved to be a subgroup of G^A.
Derfine the map

$$\begin{array}{rcl} I : G^A & \longrightarrow & G_0 \times G_e^A \\ x^A & \longmapsto & \left(\tilde{x}_A, x^{A0}\right) \end{array}$$

then we have:

i I is a bijection with inverse

$$\left(g, x^{A0}\right) \longmapsto x^{A0} \circ r^*_{\tilde{A}}$$

ii the group law on G^A can be transported on $G_0 \times G^A_e$ and gives

$$\begin{aligned} \left(\tilde{x^A}, x^{A0}\right) \cdot \left(\tilde{y}_A, y^{A0}\right) &= I\left(x^{A0} \otimes y^{A0} 1 \otimes Ad\left(\tilde{x^A}\right)^* \circ \mu^* \circ r_{\tilde{x^A}\tilde{y}^A}\right) \\ &= \left(\tilde{x^A}\tilde{y^A}, x^{A0} y^{A0\tilde{x^A}}\right) \end{aligned}$$

The fact that G^A_e is simply connected follows easily from the fact that it is isomorphic to A^+. ∎

Proposition 39 *The Weil-Berezin functor uniquely determines yhe super Lie group G.*

Proof.

i) G_0 is given by

$$G^{\mathbb{R}} = \mathrm{g} - \mathrm{Alg}\left(\mathcal{O}\left(G\right), \mathbb{R}\right)$$

ii) $\mathfrak{g}$ is determined by

$$\Lambda \longmapsto \left(\mathfrak{g} \otimes \Lambda\right)_0$$

iii) The adjoint action is determined as follows. Each G^Λ acts through the corresponding adjoint representation Ad^Λ on the corresponding $\left(\mathfrak{g} \otimes \Lambda\right)_0$:

$$\begin{aligned} G^\Lambda &\longrightarrow \mathrm{End}\left(\left(\mathfrak{g} \otimes \Lambda\right)_0\right) \\ x^\Lambda &\longmapsto \left(X \otimes \theta \mapsto m^\Lambda \left[\left(m^\Lambda \circ \left(x^\Lambda \otimes \left(\tau\left(\theta X^\Lambda\right)_{e^\Lambda}\right)\right) \circ \mu^*\right) \otimes \left(x^\Lambda\right)^{-1}\right] \circ \mu^*\right) \end{aligned}$$

From this, it follows that $\left\{\mathrm{Ad}^\Lambda\right\}$ is a functorial family of Λ_0-multilinear maps. Since $G_0 \hookrightarrow G^\Lambda = G_0 \rtimes N\left(\Lambda\right)$, we have, by the even rule principle, a map

$$\mathrm{Ad} : G_0 \longrightarrow \mathrm{End}\left(\mathfrak{g}\right)$$

Due to the fact that each Ad^Λ is a representation, also Ad is. The continuity is easily established.

∎

Chapter 7

Representation theory

Già sapea che la commedia
Si cangiava al second'atto

Dandini
La Cenerentola

7.1 Finite-dimensional representation theory

In the classical setting, a representation of a group G on a complex vector space V is nothing else that a group morphism $\pi : G \longrightarrow \mathrm{Aut}(V)$. Analogously one can define the concept of representation of a super Lie group G on a graded vector space $V = V_0 \oplus V_1$. In fact, if V has (finite) dimension (p, q) then one can consider the group of automorphism of V

$$\mathrm{Gl}(V) = \mathrm{Aut}(V) \quad \subset \quad \underline{\mathrm{End}}(V)$$

and endow it with a super Lie group structure. A representation of the super Lie group G on V is simply a morphism of super groups

$$\Pi : G \quad \longrightarrow \quad \mathrm{Gl}(V)$$

We can translate this considerations in the language of SHCP. Indeed the SHCP associated to $\mathrm{Aut}(V)$ is

$$\begin{aligned} \mathrm{Gl}(V)_0 &= \mathrm{Gl}(V_0) \times \mathrm{Gl}(V_1) \\ \mathfrak{gl}(V) &= \underline{\mathrm{End}}(V) \end{aligned}$$

with the natural adjoint action

$$\mathrm{Ad}(M)\, X \quad = \quad M X M^{-1}$$

Keeping in mind definition 13, the following is completely natural[1].

[1]With respect to the conventions adopted insofar, here we change notations. Now a morphism is denoted with $\Pi = (\pi, rho^{\pi})$ rather than $\pi = (\pi_0, \rho^{\pi})$. This is due to the fact that the symbol π_0 is reserved for the action of π on the even part of the super vector space V.

Definition 18 *A* **representation** *of the super Lie group $G = (G_0, \mathfrak{g})$ on the graded vector space $V = V_0 \oplus V_1$ is a morphism of corresponding SHCPs. Explicitly this means that a pair $\Pi = (\pi, \rho^\pi)$ is given such that:*

i) π is the direct sum of two representations $\pi_0 : \tilde{G} \longrightarrow \mathrm{Gl}(V_0)$ and $\pi_1 : \tilde{G} \longrightarrow \mathrm{Gl}(V_1)$:

$$\pi = \begin{pmatrix} \pi_0 & \\ & \pi_1 \end{pmatrix}$$

ii) ρ^π is a graded Lie algebra morphism

$$\rho^\pi : \mathfrak{g} \longrightarrow \underline{\mathrm{End}}(V)$$

iii) ρ^π intertwines the adjoint actions, in the sense that:

$$\rho^\pi(X^g) = \pi(g)\rho^\pi(X)\pi(g)^{-1}$$

The remaining part of the chapter is devoted to the extension of the theory to the infinite-dimensional setting.

7.2 Super Lie groups and their unitary representations

7.2.1 Super Hilbert spaces

All sesquilinear forms are linear in the first argument and conjugate linear in the second. We use the usual terminology of super geometry as in [DM99b],[Var04]. A *super Hilbert space* (SHS) is a super vector space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ over $\mathbb{C}$ with a scalar product $(\cdot\ ,\ \cdot)$ such that $\mathcal{H}$ is a Hilbert space under $(\cdot\ ,\ \cdot)$, and $\mathcal{H}_i (i = 0, 1)$ are mutually orthogonal closed linear subspaces. If we define

$$\langle x, y \rangle = \begin{cases} 0 & \text{if } x \text{ and } y \text{ are of opposite parity} \\ (x, y) & \text{if } x \text{ and } y \text{ are even} \\ i(x, y) & \text{if } x \text{ and } y \text{ are odd} \end{cases}$$

then $\langle x, y \rangle$ is an even super Hermitian form with

$$\langle y, x \rangle = (-1)^{p(x)p(y)}\overline{\langle x, y \rangle},\ \langle x, x \rangle > 0 (x \neq 0 \text{ even}),\ i^{-1}\langle x, x \rangle > 0 (x \neq 0 \text{ odd}).$$

If $T(\mathcal{H} \to \mathcal{H})$ is a bounded linear operator, we denote by T^* its Hilbert space adjoint and by $T^\dagger$ its super adjoint given by $\langle Tx, y \rangle = (-1)^{p(T)p(x)}\langle x, T^\dagger y \rangle$. Clearly $T^\dagger$ is bounded, $p(T) = p(T^\dagger)$, and $T^\dagger = T^*$ or $-iT^*$ according as T is even or odd. For unbounded T we define $T^\dagger$ in terms of T^* by the above formula. These definitions are equally consistent if we use $-i$ in place of i. But our convention is as above.

7.2.2 SUSY quantum mechanics

In SUSY quantum mechanics in a SHS $\mathcal{H}$, it is usual to stipulate that the Hamiltonian $H = X^2$ where X is an odd operator [Wit82]; it is customary to argue that this implies that $H \geq 0$ (positivity of energy); but this is true only if we know that H is essentially self adjoint on the domain of X^2. We shall now prove two lemmas which clarify this situation and will play a crucial role when we consider systems with a super Lie group of symmetries.

If A is a linear operator on $\mathcal{H}$, we denote by $D(A)$ its domain. We always assume that $D(A)$ is dense in $\mathcal{H}$, and then refer to it as *densely defined*. We write $A \prec B$ if $D(A) \subset D(B)$ and B restricts to A on $D(A)$; A is symmetric iff $A \prec A^*$, and then A has a closure $\overline{A}$. $A \prec B \Longrightarrow B^* \prec A^*$. If A is symmetric we say that it is essentially self adjoint if $\overline{A}$ is self adjoint; in this case $A^* = \overline{A}$. If A is symmetric and B is a symmetric extension of A, then $A \prec B \prec A^*$; in fact $A \prec A^*$ and $A \prec B \prec B^*$, and so $B^* \prec A^*$ and $A \prec B \prec B^* \prec A^*$. If A is self adjoint and $L \subset D(A)$, we say that L is a *core* for A if A is the closure of its restriction to L. A vector $\psi \in \mathcal{H}$ is analytic for a symmetric operator H if $\psi \in D(H^n)$ for all n and the series $\sum_n t^n (n!)^{-1} ||H^n\psi|| < \infty$ for some $t > 0$. It is a well known result of Nelson [Nel59] that if $D \subset D(H)$ and contains a dense set of analytic vectors, then H is essentially self adjoint on D. In this case $\psi \in D(H)$ is analytic for H if and only if $t \longmapsto e^{itH}\psi$ is analytic in $t \in \mathbb{R}$. If A is self adjoint, then A^2, defined on the domain $D(A^2) = \{\psi \mid \psi, A\psi \in D(A)\}$, is self adjoint; this is well known and follows easily from the spectral theorem.

Lemma 11 *Let H be a self adjoint operator on $\mathcal{H}$ and $U(t) = e^{itH}$ the corresponding one parameter unitary group. Let $\mathcal{B} \subset D(H)$ be a dense U-invariant linear subspace. We then have the following.*

(*i*) *$\mathcal{B}$ is a core for H.*

(*ii*) *Let X be a symmetric operator with $\mathcal{B} \subset D(X)$ such that $X\mathcal{B} \subset D(X)$ and $X^2\big|_{\mathcal{B}} = H\big|_{\mathcal{B}}$. Then $X\big|_{\mathcal{B}}$ is essentially self adjoint, $\overline{X\big|_{\mathcal{B}}} = \overline{X}$ and $\overline{X}^2 = H$.*

In particular, $H \geq 0$, $D(H) \subset D(\overline{X})$. Finally, these results are valid if we only assume that $\mathcal{B}$ is invariant under H and contains a dense set of analytic vectors.

Proof. Let $H_1 = H\big|_{\mathcal{B}}$. We must show that if $L(\lambda)(\lambda \in \mathbb{C})$ is the subspace of ψ such that $H_1^*\psi = \lambda\psi$, then $L(\lambda) = 0$ if $\Im(\lambda) \neq 0$. Now $L(\lambda)$ is a closed subspace. Moreover, as H_1 is invariant under U, so is H_1^* and so $L(\lambda)$ is invariant under U also. So the vectors in $L(\lambda)$ that are C^∞ for U are dense in $L(\lambda)$ and so it is enough to prove that 0 is the only C^∞ vector in $L(\lambda)$. But $H = H^* \prec H_1^*$ while the C^∞ vectors for U are all in $D(H)$, and so if ψ is a C^∞ vector in $L(\lambda)$, $H\psi = H_1^*\psi = \lambda\psi$. This is a contradiction since H is self adjoint and so all its eigenvalues are real. This proves (i).

Let $X_1 = X\big|_{\mathcal{B}}$. Clearly, X_1 is symmetric on $\mathcal{B}$. It is enough to show that X_1 is essentially self adjoint and $\overline{X_1}^2 = H$, since in this case $X_1 \prec X \prec X_1^* = \overline{X_1}$ and hence $\overline{X} = \overline{X_1}$. We have $X_1^2 = H_1$. So $H \geq 0$ on $\mathcal{B}$ and hence $H \geq 0$ by (i). Again it is a question of showing that for $\lambda \in \mathbb{C}$ with $\Im(\lambda) \neq 0$, we must have $M(\lambda) = 0$ where $M(\lambda)$ is the eigenspace for X_1^* for the eigenvalue λ. Let $\psi \in M(\lambda)$. Now, for $\varphi \in \mathcal{B}$,

$$(X_1^2\varphi, \psi) = (X_1\varphi, X_1^*\psi) = \overline{\lambda}(X_1\varphi, \psi) = \overline{\lambda}^2(\varphi, \psi) = (\varphi, \lambda^2\psi).$$

Hence $\psi \in D((X_1^2)^*)$ and $(X_1^2)^*\psi = \lambda^2\psi$. But $X_1^2 = H_1$ and so $(X_1^2)^* = H_1^* = H$ by (i). So $H\psi = \lambda^2\psi$. Hence λ^2 is real and ≥ 0. This contradicts the fact that $\Im(\lambda) \neq 0$. Furthermore, $X_1^2 \prec \overline{X_1}^2$ and so $\overline{X_1}^2 = (\overline{X_1}^2)^* \prec (X_1^2)^* = H_1^* = H$. On the other hand, as $\overline{X_1}^2$ is closed, $H = \overline{H}_1 = \overline{X_1^2} \prec \overline{X_1}^2$. So $H = \overline{X_1}^2 = \overline{X}^2$. This means that $D(H) \subset D(\overline{X})$.

Finally, let us assume that $H\mathcal{B} \subset \mathcal{B}$ and that $\mathcal{B}$ contains a dense set of analytic vectors for H. Clearly $\mathcal{B}$ is a core for H. If $\psi \in \mathcal{B}$ is analytic for H we have $X^{2n}\psi = H^n\psi \in \mathcal{B}$ and $X^{2n+1}\psi \in D(X)$ by assumption, and

$$||X^n\psi||^2 = |(H^n\psi, \psi)| \leq ||\psi||||H^n\psi|| \leq M^n n!$$

for some $M > 0$ and all n. Thus ψ is analytic for X and so its essential self adjointness is a consequence of the theorem of Nelson. The rest of the argument is unchanged. ■

Lemma 12 *Let A be a self adjoint operator in $\mathcal{H}$. Let M be a smooth (resp. analytic) manifold and $f(M \longrightarrow \mathcal{H})$ a smooth (resp. analytic) map. We assume that (i) $f(M) \subset D(A^2)$ and (ii) $A^2 f : m \longmapsto A^2 f(m)$ is a smooth (resp. analytic) map of M into $\mathcal{H}$. Then $Af : m \longmapsto Af(m)$ is a smooth (resp. analytic) map of M into $\mathcal{H}$. Moreover, if E is any smooth differential operator on M, $(Ef)(m) \in D(A^2)$ for all $m \in M$, and*

$$E(A^2 f) = A^2 Ef, \qquad E(Af) = AEf.$$

Proof. It is standard that if $g(M \longrightarrow \mathcal{H})$ is smooth (resp. analytic) and L is a bounded linear operator on $\mathcal{H}$, then Lg is a smooth (resp. analytic) map. We have

$$A\psi = A(I + A^2)^{-1}(I + A^2)\psi, \qquad \psi \in D(A^2).$$

Moreover $A(I + A^2)^{-1}$ is a bounded operator. Now $(I + A^2)f$ is smooth (resp. analytic) and so it is immediate from the above that Af is smooth (resp. analytic).

For the last part we assume that M is an open set in $\mathbb{R}^n$ since the result is clearly local. Let $x^i (1 \leq i \leq n)$ be the coordinates and let $\partial_j = \partial/\partial x^j, \partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$. It is enough to prove that

$$(\partial^\alpha f)(m) \in D(A^2), \qquad A^2\partial^\alpha f = \partial^\alpha(A^2 f), \qquad A\partial^\alpha f = \partial^\alpha(Af).$$

We begin with a simple observation. Since $(A^2\psi, \psi) = ||A\psi||^2$ for all $\psi \in D(A^2)$, it follows that whenever $\psi_n \in D(A^2)$ and (ψ_n) and $(A^2\psi_n)$ are Cauchy sequences, then $(A\psi_n)$ is also a Cauchy sequence; moreover, if $\psi = \lim_n \psi_n$, then $\psi \in D(A^2)$ and $A^2\psi = \lim_n A^2\psi_n$, $A\psi = \lim_n A\psi_n$. This said, we shall prove the above formulae by induction on $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Assume them for $|\alpha| \leq r$ and fix $j, 1 \leq j \leq n$. Let $g = \partial^\alpha f, |\alpha| = r$. Let

$$g_h(x) = \frac{1}{h}\left(g(x^1, \ldots, x^j + h, \ldots, x^n) - g(x^1, \ldots, x^n)\right) \qquad (h \text{ is in } j^{\text{th}} \text{ place}).$$

Then, as $h \to 0$, $g_h(x) \to \partial_j\partial^\alpha f(x)$ while $A^2 g_h(x) = (\partial^\alpha A^2 f)_h(x) \to \partial_j\partial^\alpha A^2 f(x)$, and $Ag_h(x) = (\partial^\alpha Af)_h(x) \to \partial_j\partial^\alpha Af(x)$. From the observation made above we have $\partial_j\partial^\alpha f(x) \in D(A^2)$ and A^2 and A map it respectively into $\partial_j\partial^\alpha A^2 f(x)$ and $\partial_j\partial^\alpha Af(x)$. ■

Definition 19 *For self adjoint operators L, X with L bounded, we write $L \leftrightarrow X$ to mean that L commutes with the spectral projections of X.*

Lemma 13 *Let X be a self adjoint operator and $\mathcal{B}$ a dense subspace of $\mathcal{H}$ which is a core for X such that $X\mathcal{B} \subset \mathcal{B}$. If L is a bounded self adjoint operator such that $L\mathcal{B} \subset \mathcal{B}$, then the following are equivalent: (i) $LX = XL$ on $\mathcal{B}$ (ii) $L\overline{X} = \overline{X}L$ on $D(\overline{X})$ (this carries with it the inclusion $LD(\overline{X}) \subset D(\overline{X})$) (iii) $L \leftrightarrow \overline{X}$. In this case $e^{itL}\overline{X} = \overline{X}e^{itL}$ for all $t \in \mathbb{R}$.*

Proof. (i) $\iff$ (ii). Let $b \in D(\overline{X})$. Then there is a sequence (b_n) in $\mathcal{B}$ such that $b_n \to b, Xb_n \to \overline{X}b$. Then $XLb_n = LXb_n \to L\overline{X}b$. Since $Lb_n \to Lb$ we infer that $Lb \in D(\overline{X})$ and $\overline{X}Lb = L\overline{X}b$. This proves (i) $\implies$ (ii). The reverse implication is trivial.

(ii) $\implies$ (iii). We have $L^n\overline{X}b = \overline{X}L^nb$ for all $b \in D(\overline{X}), n \geq 1$. So $e^{itL}\overline{X}b = \sum_n((it)^n/n!)\overline{X}L^nb$. If $s_N = \sum_{n \leq N}((it)^n/n!)L^nb$, then $s_N \to e^{itL}b, \overline{X}s_N \to e^{itL}\overline{X}b$. So $e^{itL}b \in D(\overline{X})$ and $\overline{X}e^{itL}b = e^{itL}\overline{X}b$. If $U(t) = e^{itL}$, this means that $U(t)\overline{X}U(t)^{-1} = \overline{X}$ and so, by the uniqueness of the spectral resolution of X, $U(t)$ commutes with the spectral projections of $\overline{X}$. But then $L \leftrightarrow \overline{X}$.

(iii) $\implies$ (i). Under (iii) we have $U(t)\overline{X}U(t)^{-1} = \overline{X}$ or $\overline{X}U(t)b = U(t)\overline{X}b$ for $b \in D(\overline{X})$, $U(t)b$ being in $D(\overline{X})$ for all t. Let $b_t = (it)^{-1}(U(t)b - b)$. Then $\overline{X}b_t = (it)^{-1}(U(t)\overline{X}b - \overline{X}b)$. Hence, as $t \to 0$, $b_t \to Lb$ while $\overline{X}b_t \to L\overline{X}b$. Hence $Lb \in D(\overline{X})$ and $\overline{X}Lb = L\overline{X}b$. Thus we have (ii), hence (i). ■

7.2.3 Unitary representations of super Lie groups

We take the point of view [DM99b] that a super Lie group (SLG) is a *super Harish-Chandra pair* $(G_0, \mathfrak{g})$ that is a pair $(G_0, \mathfrak{g})$ where G_0 is a classical Lie group, $\mathfrak{g}$ is a super Lie algebra which is a G_0-module, $\mathrm{Lie}(G_0) = \mathfrak{g}_0$, and the action of $\mathfrak{g}_0$ on $\mathfrak{g}$ is the differential of the action of G_0. The notion of morphisms between two super Lie groups in the above sense is the obvious one from which it is easy to see what is meant by a finite dimensional representation of a SLG $(G_0, \mathfrak{g})$: it is a triple (π_0, π, V) where π_0 is an even representation of G_0 in a super vector space V of finite dimension over $\mathbb{C}$, i.e., a representation such that $\pi_0(g)$ is even for all $g \in G_0$; π is a representation of the super Lie algebra $\mathfrak{g}$ in V such that $\pi\big|_{\mathfrak{g}_0} = d\pi_0$; and

$$\pi(gX) = \pi_0(g)\pi(X)\pi_0(g)^{-1}, \qquad g \in G_0, X \in \mathfrak{g}_1.$$

If V is a SHS and $\pi(X)^\dagger = -\pi(X)$ for all $X \in \mathfrak{g}$, we say that (π_0, π, V) is a unitary representation (UR) of the SLG $(G_0, \mathfrak{g})$. The condition on π is equivalent to saying that π_0 is a unitary representation of G_0 in the usual sense and $\pi(X)^* = -i\pi(X)$ for all $X \in \mathfrak{g}_1$. It is then clear that a finite dimensional UR of $(G_0, \mathfrak{g})$ is a triple (π_0, π, V), where

(a) π_0 is an even unitary representation of G_0 is a SHS V;

(b) π is a linear map of $\mathfrak{g}_1$ into the space $\mathfrak{gl}(V)_1$ of odd endomorphisms of V with $\pi(X)^* = -i\pi(X)$ for all $X \in \mathfrak{g}_1$;

(c) $d\pi_0([X, Y]) = \pi(X)\pi(Y) + \pi(Y)\pi(X)$ for $X, Y \in \mathfrak{g}_1$;

(d) $\pi(g_0X) = \pi_0(g_0)\pi(X)\pi_0(g_0)^{-1}$ for $X \in \mathfrak{g}_1, g_0 \in G_0$.

Let

$$\zeta = e^{-i\pi/4}, \qquad \rho(X) = \zeta\pi(X).$$

Then, we may replace $\pi(X)$ by $\rho(X)$ for $X \in \mathfrak{g}_1$; the condition (b) becomes the requirement that $\rho(X)$ is self adjoint for all $X \in \mathfrak{g}_1$, while the commutation rule in condition (c) becomes

$$-id\pi_0([X, Y]) = \rho(X)\rho(Y) + \rho(Y)\rho(X), \qquad X, Y \in \mathfrak{g}_1.$$

If we want to extend this definition to the infinite dimensional context it is necessary to take into account the fact that the $\pi(X)$ for $X \in \mathfrak{g}_1$ will in general be unbounded; indeed, from (c) above we find that $d\pi_0([X,X]) = 2\pi(X)^2$, and as $d\pi_0$ typically takes elements of $\mathfrak{g}_0$ into unbounded operators, the $\pi(X)$ cannot be bounded. So the concept of a UR of a SLG in the infinite dimensional case must be formulated with greater care to take into account the domains of definition of the $\pi(X)$ for $X \in \mathfrak{g}_1$. In the physics literature this aspect is generally ignored. We shall prove below that contrary to what one may expect, the domain restrictions can be formulated with great freedom, and the formal and rigorous pictures are essentially the same. In particular, the concept of a UR of a super Lie group is a very stable one and allows great flexibility of handling.

If V is a super vector space (not necessarily finite dimensional), we write $\mathbf{End}(V)$ for the super algebra of all endomorphisms of V. If π_0 is a unitary representation of G_0 in a Hilbert space $\mathcal{H}$, we write $C^\infty(\pi_0)$ for the subspace of differentiable vectors in $\mathcal{H}$ for π_0. We denote by $C^\omega(\pi_0)$ the subspace of analytic vectors of π_0. Here we recall that a vector $v \in \mathcal{H}$ is called a *differentiable vector* (resp. *analytic vector*) for π_0 if the map $g \mapsto \pi_0(g)v$ is smooth (resp. analytic). If $\mathcal{H}$ is a SHS and π_0 is even, then $C^\infty(\pi_0)$ and $C^\omega(\pi_0)$ are π_0-invariant dense linear super subspaces. We also need the following fact which is standard but we shall give a partial proof because the argument will be used again later.

Lemma 14 *$C^\infty(\pi_0)$ and $C^\omega(\pi_0)$ are stable under $d\pi_0(\mathfrak{g}_0)$. For any $Z \in \mathfrak{g}_0$, $id\pi_0(Z)$ is essentially self adjoint both on $C^\infty(\pi_0)$ and on $C^\omega(\pi_0)$; moreover, for any $Z_1, \dots, Z_r \in \mathfrak{g}_0$ and $\psi \in C^\infty(\pi_0)$, (resp. $\psi \in C^\omega(\pi_0)$) the map $g \longmapsto d\pi_0(Z_1)\dots d\pi_0(Z_r)\pi_0(g)\psi$ is C^∞ (resp. analytic).*

Proof. We prove only the second statement. That $id\pi_0(Z)$ is essentially self adjoint on $C^\infty(\pi_0)$ and on $C^\omega(\pi_0)$ is immediate from Lemma 11. Using the adjoint representation we have, for any $Z \in \mathfrak{g}_0$, $gZ = \sum_i c_i(g)W_i$ for $g \in G_0$ where the c_i are analytic functions on G_0 and $W_i \in \mathfrak{g}_0$. Hence, as

$$d\pi_0(Z)\pi_0(g) = \pi_0(g)d\pi_0(g^{-1}Z),$$

we can write $d\pi_0(Z_1)\dots d\pi_0(Z_r)\pi_0(g)\psi$ as a linear combination with analytic coefficients of $\pi_0(g)d\pi_0(R_1)\dots d\pi_0(R_r)\psi$ for suitable $R_j \in \mathfrak{g}_0$. The result is then immediate. ∎

Definition 20 *A* unitary representation *(UR) of a SLG $(G_0, \mathfrak{g})$ is a triple $(\pi_0, \rho, \mathcal{H})$, $\mathcal{H}$ a SHS, with the following properties.*

(a) π_0 is an even UR of G_0 in $\mathcal{H}$;

(b) $\rho(X \longmapsto \rho(X))$ is a linear map of $\mathfrak{g}_1$ into $\mathbf{End}(C^\infty(\pi_0))_1$ such that

(i) $\rho(g_0X) = \pi_0(g_0)\rho(X)\pi_0(g_0)^{-1}$ $(X \in \mathfrak{g}_1, g_0 \in G_0)$,

(ii) $\rho(X)$ with domain $C^\infty(\pi_0)$ is symmetric for all $X \in \mathfrak{g}_1$,

(iii) $-id\pi_0([X,Y]) = \rho(X)\rho(Y) + \rho(Y)\rho(X)$ $(X, Y \in \mathfrak{g}_1)$ on $C^\infty(\pi_0)$.

Proposition 40 *If $(\pi_0, \rho, \mathcal{H})$ is a* UR *of the* SLG *$(G_0, \mathfrak{g})$, then $\rho(X)$ with domain $C^\infty(\pi_0)$ is essentially self adjoint for all $X \in \mathfrak{g}_1$. Moreover*

$$\pi : X_0 + X_1 \longmapsto d\pi_0(X_0) + \zeta^{-1}\rho(X_1) \qquad (X_i \in \mathfrak{g}_i)$$

is a representation of $\mathfrak{g}$ in $C^\infty(\pi_0)$.

Proof. Let $Z = (1/2)[X, X]$. We apply Lemma 11 with $U(t) = \pi_0(\exp tZ) = e^{itH}, \mathcal{B} = C^\infty(\pi_0)$. Then $H = -id\pi_0(Z) = \rho(X)^2$ on $C^\infty(\pi_0)$. We conclude that H and $\rho(X)$ are essentially self adjoint on $C^\infty(\pi_0)$ and that $H = \overline{\rho(X)}^2$; in particular, $H \geq 0$. For the second assertion the only non obvious statement is that for $Z \in \mathfrak{g}_0, X \in \mathfrak{g}_1, \psi \in C^\infty(\pi_0)$,

$$\rho([Z, X])\psi = d\pi_0(Z)\rho(X)\psi - \rho(X)d\pi_0(Z)\psi.$$

Let $g_t = \exp(tZ)$ and let (X_k) be a basis for $\mathfrak{g}_1$. Then $gX = \sum_k c_k(g)X_k$ where the c_k are smooth functions on G_0. So

$$\pi_0(g_t)\rho(X)\psi = \rho(g_t X)\pi_0(g_t)\psi = \sum_k c_k(g_t)\rho(X_k)\pi_0(g_t)\psi.$$

Now $g \longmapsto \pi_0(g)\psi$ is a smooth map into $C^\infty(\pi_0)$. On the other hand, if $H_k = -(1/2)[X_k, X_k]$, we have $id\pi_0(H_k) = \rho(X_k)^2$ on $C^\infty(\pi_0)$, so $\pi_0(g)\psi \in D(\rho(X_k)^2)$, and by Lemma 14, $\rho(X_k)^2\pi_0(g)\psi = id\pi_0(H_k)\pi_0(g)\psi$ is smooth in g. Lemma 12 now applies and shows that $\rho(X_k)\pi_0(g_t)\psi$ is smooth in t and

$$\left(\frac{d}{dt}\right)_{t=0} \rho(X_k)\pi_0(g_t)\psi = \rho(X_k)d\pi_0(Z)\psi.$$

Hence

$$d\pi_0(Z)\rho(X)\psi = \sum_k c_k(1)\rho(X_k)d\pi_0(Z)\psi + \sum_k (Zc_k)(1)\rho(X_k)\psi.$$

Since

$$[Z, X] = \sum_k (Zc_k)(1)X_k, \qquad X = \sum_k c_k(1)X_k$$

the right side is equal to

$$\rho(X)d\pi_0(Z)\psi + \rho([Z, X])\psi.$$

■

Remark 14 *For Z such that $\exp tZ$ represents time translation, H is the energy operator, and so is positive in the supersymmetric model.*

We shall now show that one may replace $C^\infty(\pi_0)$ by a more or less arbitrary domain without changing the content of the definition. This shows that the concept of a UR of a SLG is a viable one even in the infinite dimensional context.

Let us consider a system $(\pi_0, \rho, \mathcal{B}, \mathcal{H})$ with the following properties.

(a) $\mathcal{B}$ is a dense super linear subspace of $\mathcal{H}$ invariant under π_0 and $\mathcal{B} \subset D(d\pi_0(Z))$ for all $Z \in [\mathfrak{g}_1, \mathfrak{g}_1]$;

(b) $(\rho(X))_{X \in \mathfrak{g}_1}$ is a set of linear operators such that:

  (i) $\rho(X)$ is symmetric for all $X \in \mathfrak{g}_1$,

  (ii) $\mathcal{B} \subset D(\rho(X))$ for all $X \in \mathfrak{g}_1$,

  (iii) $\rho(X)\mathcal{B}_i \subset \mathcal{H}_{i+1\,(\mathrm{mod}\,2)}$ for all $X \in \mathfrak{g}_1$,

  (iv) $\rho(aX + bY) = a\rho(X) + b\rho(Y)$ on $\mathcal{B}$ for $X, Y \in \mathfrak{g}_1$ and a, b scalars,

(v) $\pi_0(g)\rho(X)\pi_0(g)^{-1} = \rho(gX)$ on $\mathcal{B}$ for all $g \in G_0, X \in \mathfrak{g}_1$,

(vi) $\rho(X)\mathcal{B} \subset D(\rho(Y))$ for all $X, Y \in \mathfrak{g}_1$, and $-id\pi_0([X,Y]) = \rho(X)\rho(Y) + \rho(Y)\rho(X)$ on $\mathcal{B}$.

Proposition 41 *Let $(\pi_0, \rho, \mathcal{B}, \mathcal{H})$ be as above. We then have the following.*

(a) For any $X \in \mathfrak{g}_1$, $\rho(X)$ is essentially self adjoint and $C^\infty(\pi_0) \subset D(\overline{\rho(X)})$.

(b) Let $\overline{\rho}(X) = \overline{\rho(X)}\big|_{C^\infty(\pi_0)}$ for $X \in \mathfrak{g}_1$. Then $(\pi_0, \overline{\rho}, \mathcal{H})$ is a UR *of the* SLG *$(G_0, \mathfrak{g})$.*

If $(\pi_0, \rho', \mathcal{H})$ is a UR *of the* SLG *$(G_0, \mathfrak{g})$, such that $\mathcal{B} \subset D(\overline{\rho'(X)})$ and $\overline{\rho'(X)}$ restricts to $\rho(X)$ on $\mathcal{B}$ for all $X \in \mathfrak{g}_1$, then $\rho' = \overline{\rho}$.*

Proof. Let $X \in \mathfrak{g}_1$. By assumption $\mathcal{B}$ is invariant under the one parameter unitary group generated by $H = -(1/2)id\pi_0([X,X])$ while $H = \rho(X)^2$ on $\mathcal{B}$. So, by Lemma 11, $\rho(X)$ is essentially self adjoint, $\overline{\rho(X)} = \overline{\rho(X)|_{\mathcal{B}}}$, $H = (\overline{\rho(X)})^2$, and $D(H) \subset D(\overline{\rho(X)})$. Since $C^\infty(\pi_0) \subset D(H)$, we have proved (a).

Let us now prove (b). If a is scalar and $X \in \mathfrak{g}$, $\overline{\rho}(aX) = a\overline{\rho}(X)$ follows from item (iv) and the fact that $\overline{\rho(X)} = \overline{\rho(X)|_{\mathcal{B}}}$. For the additivity of $\overline{\rho}$, let $X, Y \in \mathfrak{g}_1$. Then $\rho(X+Y)$ is essentially self adjoint and its closure restricts to $\overline{\rho}(X+Y)$ on $C^\infty(\pi_0)$. Then, viewing $\overline{\rho(X)} + \overline{\rho(Y)}$ as a symmetric operator defined on the intersection of the domains of the two operators (which includes $C^\infty(\pi_0)$), we see that $\overline{\rho(X)} + \overline{\rho(Y)}$ is a symmetric extension of $\rho(X)|_{\mathcal{B}} + \rho(Y)|_{\mathcal{B}} = \rho(X+Y)|_{\mathcal{B}}$. But as $\rho(X+Y)|_{\mathcal{B}}$ is essentially self adjoint, we have, by the remark made earlier,

$$\overline{\rho(X)} + \overline{\rho(Y)} \prec \overline{\rho(X+Y)|_{\mathcal{B}}} = \overline{\rho(X+Y)}.$$

Restricting both of these operators to $C^\infty(\pi_0)$ we find that $\overline{\rho}(X+Y) = \overline{\rho}(X) + \overline{\rho}(Y)$. From the relation $\pi_0(g)\rho(X)\pi_0(g)^{-1} = \rho(gX)$ on $\mathcal{B}$ follows $\pi_0(g)\overline{\rho}(X)\pi_0(g)^{-1} = \overline{\rho}(gX)$.

The key step is now to prove that for any $X \in \mathfrak{g}_1$, $\overline{\rho(X)}$ maps $C^\infty(\pi_0)$ into itself. Fix $X \in \mathfrak{g}_1, \psi \in C^\infty(\pi_0)$. Now

$$\pi_0(g)\overline{\rho(X)}\psi = \overline{\rho(gX)}\pi_0(g)\psi.$$

So, writing $gX = \sum_k c_k(g)X_k$ where the c_k are smooth functions on G_0 and the (X_k) a basis for $\mathfrak{g}_1$, we have, remembering the linearity of $\overline{\rho}$ on $C^\infty(\pi_0)$,

$$\pi_0(g)\overline{\rho(X)}\psi = \sum_k c_k(g)\overline{\rho(X_k)}\pi_0(g)\psi.$$

It is thus enough to show that $g \longmapsto \overline{\rho(X_k)}\pi_0(g)\psi$ is smooth. If $H_k = -[X_k, X_k]/2$, we know from Lemma 14 that $\pi_0(g)\psi$ lies in $D(H_k)$ and $id\pi_0(H_k)\pi_0(g)\psi = \overline{\rho(X_k)}^2\pi_0(g)\psi$ is smooth in g. Lemma 12 now shows that $\overline{\rho(X_k)}\pi_0(g)\psi$ is smooth in g.

It remains only to show that for $X, Y \in \mathfrak{g}_1$ we have

$$-id\pi_0([X,Y]) = \overline{\rho}(X)\overline{\rho}(Y) + \overline{\rho}(Y)\overline{\rho}(X)$$

on $C^\infty(\pi_0)$. But, the right side is $\overline{\rho}(X+Y)^2 - \overline{\rho}(X)^2 - \overline{\rho}(Y)^2$ while the left side is the restriction of $(-i/2)d\pi_0([X+Y, X+Y]) + (i/2)d\pi_0([X,X]) + (i/2)d\pi_0([Y,Y])$ to $C^\infty(\pi_0)$, and so we are done.

We must show the uniqueness of $\overline{\rho}$. Let ρ' have the required properties also. Then $\rho'(X)$ is essentially self adjoint on $C^\infty(\pi_0)$ and $\mathcal{B}$ is a core for its closure, by Lemma 11. Hence $\rho'(X) = \overline{\rho}(X)$. The proof is complete. ■

We shall now prove a variant of the above result involving analytic vectors.

Proposition 42 *(i) If $(\pi_0, \rho, \mathcal{H})$ is a* UR *of the* SLG $(G_0, \mathfrak{g})$*, then $\rho(X)$ maps $C^\omega(\pi_0)$ into itself for all $X \in \mathfrak{g}_1$, so that π, as in Proposition (40), is a representation of $\mathfrak{g}$ in $C^\infty(\pi_0)$. (ii) Let G_0 be connected. Let π_0 be an even unitary representation of G_0 and $\mathcal{B} \subset C^\omega(\pi_0)$ a dense linear super subspace. Let π be a representation of $\mathfrak{g}$ in $\mathcal{B}$ such that $\pi(Z) \prec d\pi_0(Z)$ for $Z \in \mathfrak{g}_0$ and $\rho(X) = \zeta\pi(X)$ is symmetric for $X \in \mathfrak{g}_1$. Then, for each $X \in \mathfrak{g}_1$, $\overline{\rho(X)}$ is essentially self adjoint on $\mathcal{B}$ and $C^\infty(\pi_0) \subset D(\overline{\rho(X)})$. If $\overline{\rho}(X)$ is the restriction of $\overline{\rho(X)}$ to $C^\infty(\pi_0)$, then $(\pi_0, \overline{\rho}, \mathcal{H})$ is a* UR *of the* SLG $(G_0, \mathfrak{g})$ *and is the unique one in the following sense: if $(\pi_0, \rho', \mathcal{H})$ is a* UR *with $\mathcal{B} \subset D(\overline{\rho'(X)})$ and $\rho'(X)\mid_{\mathcal{B}} = \rho(X)$ for all $X \in \mathfrak{g}_1$, then $\rho' = \overline{\rho}$.*

Proof. (i) This is proved as its C^∞ analogue in the proof of Proposition 41, using the analytic parts of Lemmas 12 and 14.

(ii) The proof that $\rho(X)$ for $X \in \mathfrak{g}_1$ is essentially self adjoint with $D(\overline{\rho(X)}) \supset C^\infty(\pi_0)$ follows as before from (the analytic part of) Lemma 11. The same goes for the linearity of $\overline{\rho}$.

We shall now show that for $X \in \mathfrak{g}_1, g \in G_0$,

$$\pi_0(g^{-1})\overline{\rho(X)}\pi_0(g) = \overline{\rho(g^{-1}X)}. \tag{7.1}$$

Write $g^{-1}X = \sum_k c_k(g)X_k$ where (X_k) is a basis for $\mathfrak{g}_1$ and the c_k are analytic functions on G_0. We begin by showing that for all $\psi \in \mathcal{B}$

$$\overline{\rho(X)}\pi_0(g)\psi = \pi_0(g)\rho(g^{-1}X)\psi. \tag{7.2}$$

Now

$$\pi_0(g)\rho(g^{-1}X)\psi = \sum_k c_k(g)\pi_0(g)\rho(X_k)\psi.$$

We argue as in Proposition 41 to conclude, using Lemmas 12 and 14, that the function $\overline{\rho(X)}\pi_0(g)\psi$ is analytic in g and its derivatives can be calculated explicitly. It is also clear that the right side is analytic in g since $\rho(X_k)\psi \in \mathcal{B}$ for all k. So, as G_0 is connected, it is enough to prove that the two sides in (7.2) have all derivatives equal at $g = 1$. This comes down to showing that for any integer $n \geq 0$ and any $Z \in \mathfrak{g}_0$,

$$\rho(X)d\pi_0(Z)^n\psi = \sum_{k,r}\binom{n}{r}(Z^r c_k)(1)d\pi_0(Z)^{n-r}\rho(X_k)\psi. \tag{7.3}$$

Let λ be the representation of G_0 on $\mathfrak{g}_1$ and write λ again for $d\lambda$. Then, taking $g_t = \exp(tZ)$,

$$\lambda(g_t^{-1})(X) = \sum_k c_k(g_t)X_k,$$

from which we get, on differentiating n times with respect to t at $t = 0$,

$$\lambda(-Z)^r(X) = \sum_k (Z^r c_k)(1)X_k.$$

Hence the right side of (7.3) becomes

$$\sum_r \binom{n}{r} d\pi_0(Z)^{n-r}\rho(\lambda(-Z)^r(X))\psi.$$

On the other hand, from the fact that π is a representation of $\mathfrak{g}$ in $\mathcal{B}$ we get

$$\rho(X)d\pi_0(Z) = d\pi_0(Z)\rho(X) + \rho(\lambda(-Z)(X))$$

on $\mathcal{B}$. Iterating this we get, on $\mathcal{B}$,

$$\rho(X)d\pi_0(Z)^n = \sum_r \binom{n}{r} d\pi_0(Z)^{n-r}\rho(\lambda(-Z)^r(X))$$

which gives (7.2). But then (7.1) follows from (7.2) by a simple closure argument.

Using (7.1), the proof that $\overline{\rho}(X)$ maps $C^\infty(\pi_0)$ into itself is the same of Proposition 41. The proof of the relation $-id\pi_0([X,Y]) = \overline{\rho}(X)\overline{\rho}(Y) + \overline{\rho}(Y)\overline{\rho}(X)$ for $X, Y \in \mathfrak{g}_1$ is also the same. The proof is complete. ■

7.2.4 The category of unitary representations of a super Lie group

If $\Pi = (\pi_0, \rho, \mathcal{H})$ and $\Pi' = (\pi_0', \rho', \mathcal{H}')$ are two UR's of a SLG $(G_0, \mathfrak{g})$, a *morphism* $A : \Pi \longrightarrow \Pi'$ is an even bounded linear operator from $\mathcal{H}$ to $\mathcal{H}'$ such that A intertwines π_0, ρ and π_0', ρ'; notice that as soon as A intertwines π_0 and π_0', it maps $C^\infty(\pi_0)$ into $C^\infty(\pi_0')$, and so the requirement that it intertwine ρ and ρ' makes sense. An *isomorphism* is then a morphism A such that A^{-1} is a bounded operator; in this case A is a linear isomorphism of $C^\infty(\pi_0)$ with $C^\infty(\pi_0')$ intertwining ρ and ρ'. If A is unitary we then speak of unitary equivalence of Π and Π'. It is easily checked that equivalence implies unitary equivalence, just as in the classical case. Π' is a *subrepresentation* of Π if $\mathcal{H}'$ is a closed graded subspace of $\mathcal{H}$ invariant under π_0 and ρ, and π_0' (resp. ρ') is the restriction of π_0 (resp. ρ) to $\mathcal{H}'$ (resp. $C^\infty(\pi_0) \cap \mathcal{H}'$). The UR Π is said to be *irreducible* if there is no proper nonzero closed graded subspace $\mathcal{H}'$ that defines a subrepresentation. If Π' is a nonzero proper subrepresentation of Π, and $\mathcal{H}''$ is $\mathcal{H}'^{\perp}$, it follows from the self adjointness of $\rho(X)$ for $X \in \mathfrak{g}_1$ that $\mathcal{H}'' \cap C^\infty(\pi_0)$ is invariant under all $\rho(X)(X \in \mathfrak{g}_1)$; then the restrictions of π_0, ρ to $\mathcal{H}''$ define a subrepresentation Π'' such that $\Pi = \Pi' \oplus \Pi''$ in an obvious manner.

Lemma 15 Π *is irreducible if and only if* $\mathrm{Hom}(\Pi, \Pi) = \mathbb{C}$.

Proof. If Π splits as above, then the orthogonal projection $\mathcal{H} \longrightarrow \mathcal{H}'$ is a nonscalar element of $\mathrm{Hom}(\Pi, \Pi)$. Conversely, suppose that Π is irreducible and $A \in \mathrm{Hom}(\Pi, \Pi)$. Then $A^* \in \mathrm{Hom}(\Pi, \Pi)$ also and so, to prove that A is a scalar we may suppose that A is self adjoint. Let P be the spectral measure of A. Clearly all the $P(E)$ are even. Then P commutes with π_0 and so $P(E)$ leaves $C^\infty(\pi_0)$ invariant for all Borel sets E. Moreover, by Lemma 13, the relation $A\rho(X) = \rho(X)A$ on $C^\infty(\pi_0)$ implies that $P(E) \leftrightarrow \overline{\rho(X)}$ for all E and $x \in \mathfrak{g}_1$, and hence that $P(E)\rho(X) = \rho(X)P(E)$ on $C^\infty(\pi_0)$ for all E, X. The range of $P(E)$ thus defines a subrepresentation and so $P(E) = 0$ or I. Since this is true for all E, A must be a scalar. ■

Lemma 16 *Let* $(R_0, \mathfrak{r})$ *be a* SLG *and* (θ, ρ^θ) *a* UR *of it, in a Hilbert space* $\mathcal{L}$*. Let* P^X *be the spectral measure of* $\overline{\rho^\theta(X)}, X \in \mathfrak{r}_1$*. Then the following properties of a closed linear subspace* $\mathcal{M}$ *of* $\mathcal{L}$ *are equivalent:* (i) $\mathcal{M}$ *is stable under* θ *and* $\mathcal{M}^\infty = C^\infty(\theta) \cap \mathcal{M}$ *is stable under all* $\rho^\theta(X), (X \in \mathfrak{r}_1)$ (ii) $\mathcal{M}$ *is stable under* θ *and all the spectral projections* P_F^X *(Borel* $F \subset \mathbb{R}$*). In particular,* (θ, ρ^θ) *is irreducible if and only if* $\mathcal{L}$ *is irreducible under* θ *and all* P_F^X*.*

Proof. Follows from Lemma 13 applied to the orthogonal projection $L : \mathcal{L} \longrightarrow \mathcal{M}$. Indeed, suppose that $\mathcal{M}$ is a closed linear subspace of $\mathcal{L}$ stable under θ. Then L maps $\mathcal{L}^\infty$ onto $\mathcal{M}^\infty$. By Lemma 13 L commutes with $\rho^\theta(X)$ on $\mathcal{L}^\infty$ if and only if $L \leftrightarrow \overline{\rho^\theta(X)}$; this is the same as saying that P^X stabilizes $\mathcal{M}$. ■

Chapter 8

Induced representations of super Lie groups

Die Frist ist um...

L'Olandese volante

8.0.5 Smooth structure of the classical induced representation and its system of imprimitivity

Let G_0 be a unimodular Lie group and H_0 a closed Lie subgroup. We write $\Omega = G_0/H_0$ and assume that Ω has a G_0-invariant measure; one can easily modify the treatment below to avoid these assumptions. We write $x \mapsto \overline{x}$ for the natural map from G_0 to Ω and $d\overline{x}$ for a choice of the invariant measure on Ω. For any UR σ of H_0 in a Hilbert space $\mathcal{K}$ one has the representation π of G_0 induced by σ. One may take π as acting in the Hilbert space $\mathcal{H}$ of (equivalence classes) of Borel functions f from G_0 to $\mathcal{K}$ such that (i) $f(x\xi) = \sigma(\xi)^{-1}f(x)$ for all $x \in G_0, \xi \in H_0$, and (ii) $||f||^2_{\mathcal{H}} := \int_\Omega |f(x)|^2_{\mathcal{K}} d\overline{x} < \infty$. Here $|f(x)|_{\mathcal{K}}$ is the norm of $f(x)$ in $\mathcal{K}$, and the function $x \mapsto |f(x)|^2_{\mathcal{K}}$ is defined on Ω so that it makes sense to integrate it on Ω. Let P be the natural projection valued measure on $\mathcal{H}$ defined as follows: for any Borel set $E \subset \Omega$ the projection $P(E)$ is the operator $f \mapsto \chi_E f$ where χ_E is the characteristic function of E. Then $(\pi, \mathcal{H}, P)$ is the classical system of imprimitivity (SI) associated to the UR σ of H_0. In our case G_0 is a Lie group and it is better to work with a smooth version of π; its structure is determined by a well known theorem of Dixmier-Malliavin in a manner that will be explained below.

We begin with a standard but technical lemma that says that certain integrals containing a parameter are smooth.

Lemma 17 *Let M, N be smooth manifolds, dn a smooth measure on N, and B a separable Banach space with norm $|\cdot|$. Let $F : M \times N \longrightarrow B$ be a map with the following properties: (i) For each $n \in N$, $m \mapsto F(m,n)$ is smooth (ii) If $A \subset M$ is an open set with compact closure, and G is any derivative of F with respect to m, there is a $g_A \in L^1(N, dn)$ such that $|G(m,n)| \leq g_A(n)$ for all $m \in A, n \in N$. Then*

$$f(m) = \int_N F(m,n)dn$$

exists for all m and f is a smooth map of M into B.

Proof. It is a question of proving that the integrals

$$\int_N |G(m,n)|dn$$

converge uniformly when m varies in an open subset A of M with compact closure. But the integrand is majorized by g_A which is integrable on N and so the uniform convergence is clear. ■

We also observe that any $f \in \mathcal{H}$ lies in $L^{p,\mathrm{loc}}(G_0)$ for $p = 1, 2$, i.e., $\theta(x)|f(x)|^2_{\mathcal{K}}$ is integrable on G_0 for any continuous compactly supported scalar function $\theta \geq 0$. In fact

$$\int_{G_0} \theta(x)|f(x)|^2_{\mathcal{K}} dx = \int_\Omega \left(\int_{H_0} \theta(x\xi)|f(x\xi)|^2_{\mathcal{K}} d\xi \right) d\overline{x} = \int_\Omega \overline{\theta}(x)|f(x)|^2_{\mathcal{K}} d\overline{x} < \infty$$

where $\overline{\theta}(x) = \int_{G_0} \theta(x\xi)d\xi$.

In $\mathcal{H}$ we have the space $C^\infty(\pi)$ of smooth vectors for π. We also have its Garding subspace, the subspace spanned by all vectors $\pi(\alpha)h$ where $\alpha \in C_c^\infty(G_0)$ and $h \in \mathcal{H}$. We have

$$(\pi(\alpha)h)(z) = \int_{G_0} \alpha(x)h(x^{-1}z)dx = \int_{G_0} \alpha(zt^{-1})h(t)dt \qquad (z \in G_0).$$

The integrals exist because h is locally L^2 on G_0 as mentioned above. Since $h \in L^{1,\mathrm{loc}}(G_0), \alpha \in C_c^\infty(G_0)$, the conditions of Lemma 17 are met and so $\pi(\alpha)h$ is smooth. Thus all elements of the Garding space are smooth functions. But the Dixmier-Malliavin theorem asserts that the Garding space is exactly the same as $C^\infty(\pi)$ [DM78]. Thus all elements of $C^\infty(\pi)$ are smooth functions from G_0 to $\mathcal{K}$. This is the key point that leads to the smooth versions of the induced representation and the SI at the classical level.

Let us define $\mathcal{B}$ as the space of all functions f from G_0 to $\mathcal{K}$ such that (i) f is smooth and $f(x\xi) = \sigma(\xi)^{-1}f(x)$ for all $x \in G_0, \xi \in H_0$ (ii) f has compact support mod H_0. Let $C_c^\infty(\pi)$ be the subspace of all elements of $C^\infty(\pi)$ with compact support mod H_0.

Proposition 43 *$\mathcal{B}$ has the following properties: (i) $\mathcal{B} = C_c^\infty(\pi)$ (ii) $\mathcal{B}$ is dense in $\mathcal{H}$ (iii) $f(x) \in C^\infty(\sigma)$ for all $x \in G_0$ (iv) $\mathcal{B}$ is stable under $d\pi$.*

Proof. (i) Let $f \in \mathcal{B}$. To show that $f \in C^\infty(\pi)$ it is enough to show that for any $u \in \mathcal{H}$ the map $x \mapsto (\pi(x^{-1})f, u)_{\mathcal{H}}$ is smooth in x. Now

$$(\pi(x^{-1})f, u)_{\mathcal{H}} = \int_\Omega (f(xy), u(y))_{\mathcal{K}} d\overline{y}.$$

Since $|u|_{\mathcal{K}}$ is locally L^1 on X and f is smooth, the conditions of Lemma 17 are met. We have $\mathcal{B} \subset C_c^\infty(\pi)$. The reverse inclusion is immediate from the Dixmier-Malliavin theorem, as remarked above.

(ii) It is enough to prove that any $h \in \mathcal{H}$ with compact support mod H_0 is in the closure of $\mathcal{B}$. We know that $\pi(\alpha)h \to h$ as $\alpha \in C_c^\infty(G_0)$ goes suitably to the delta function at the identity of G_0. But $\pi(\alpha)h$ is smooth and has compact support mod H_0 because h has the same property, so that $\pi(\alpha)h \in \mathcal{B}$.

(iii) Fix $x \in G_0$. Since $\sigma(\xi)f(x) = f(x\xi^{-1})$ for $\xi \in H_0$ it is clear that $f(x) \in C^\infty(\sigma)$.

(iv) Let $f \in \mathcal{B}, Z \in \mathfrak{g}_0$. Then

$$(d\pi(Z)f)(x) = (d/dt)_{t=0} f(\exp(-tZ)x)$$

is smooth and we are done. ■

We refer to $(\pi, \mathcal{B})$ as the *smooth representation induced by* σ. We shall also define the smooth version of the SI. For any $u \in C_c^\infty(\Omega)$ let $M(u)$ be the bounded operator on $\mathcal{H}$ which is multiplication by u. Then $M(u)$ leaves $\mathcal{B}$ invariant and $M : u \mapsto M(u)$ is a $*$-representation of the $*$-algebra $C_c^\infty(\Omega)$ in $\mathcal{H}$. It is natural to refer to $(\pi, \mathcal{B}, M)$ as the *smooth system of imprimitivity* associated to σ. Observe that $f \in C^\infty(\pi)$ has compact support mod H_0 if and only if there is some $u \in C_c^\infty(\Omega)$ such that $f = M(u)f$. Proposition 43 shows that $\mathcal{B}$ is thus determined intrinsically by the SI associated to σ. The passage from $(\pi, \mathcal{H}, P)$ to $(\pi, \mathcal{B}, M)$ is thus functorial and is a categorical equivalence. Thus we are justified in working just with smooth SI's.

It is easy to see that the assignment that takes σ to the associated smooth SI is functorial. Indeed, let R be a morphism from σ to σ', i.e., R is a bounded operator from $\mathcal{K}$ to $\mathcal{K}'$ intertwining σ and σ'. We then define $T_R = T(\mathcal{H} \longrightarrow \mathcal{H}')$ by $(T_R f)(x) = Rf(x)(x \in G_0)$. It is then immediate that T_R is a morphism from the (smooth) SI associated to σ to the (smooth) SI associated to σ'. This functor is an equivalence of categories. To verify this one must show that every morphism between the two SI's is of this form. This is of course classical but we sketch the argument depending on the following lemma which will be essentially used in the super context also.

Lemma 18 *Suppose $f \in \mathcal{B}$ and $f(1) = 0$. Then we can find $u_i \in C_c^\infty(\Omega), g_i \in \mathcal{B}$ such that (i) $u_i(1) = 0$ for all i (ii) we have $f = \sum_i u_i g_i$.*

Proof. If f vanishes in a neighborhood of 1, we can choose $u \in C_c^\infty(\Omega)$ such that $u = 0$ in a neighborhood of 1 and $f = uf$. The result is thus true for f. Let $f \in \mathcal{B}$ be arbitrary but vanishing at 1. Let $\mathfrak{z}$ be a linear subspace of $\mathfrak{g}_0 = \text{Lie}\ (G_0)$ complementary to $\mathfrak{h}_0 = \text{Lie}\ (H_0)$. Then there is a sufficiently small $r > 0$ such that if $\mathfrak{z}_r = \{Z \in \mathfrak{z} \mid |Z| < r\}$, $|\cdot|$ being a norm on $\mathfrak{z}$, the map

$$\mathfrak{z}_r \times H_0 \longrightarrow G_0, \qquad (Z, \xi) \longmapsto \exp Z \cdot \xi$$

is a diffeomorphism onto an open set $G_1 = G_1 H_0$ of G_0. We transfer f from G_1 to a function, denoted by φ on $\mathfrak{z}_r \times H_0$. We have $\varphi(0, \xi) = 0$, and $\varphi(Z, \xi\xi') = \sigma(\xi')^{-1}\varphi(Z, \xi)$ for $\xi' \in H_0$. If $t_i (1 \le i \le k)$ are the linear coordinates on $\mathfrak{z}$,

$$\varphi(Z, \xi) = \sum_i t_i(Z) \int_0^1 (\partial\varphi/\partial t_i)(sZ, \xi) ds.$$

The functions $\psi_i(Z, \xi) = \int_0^1 (\partial\varphi/\partial t_i)(sZ, \xi) ds$ are smooth by Lemma 17 while $\psi_i(Z, \xi\xi') = \sigma(\xi')^{-1}\psi_i(Z, \xi)$ for $\xi' \in H_0$. So, going back to G_1 we can write $f = \sum_i t_i h_i$ where t_i are now in $C^\infty(G_1)$, right invariant under H_0 and vanishing at 1, while the h_i are smooth and satisfy $h_i(x\xi) = \sigma(\xi)^{-1} h_i(x)$ for $x \in G_1, \xi \in H_0$. If $u \in C_c^\infty(\Omega)$ is such that u is 1 in a neighborhood of 1 and supp $(u) \subset G_1$, then $u^2 f = \sum_i u_i g_i$ where $u_i = ut_i \in C_c^\infty(\Omega), u_i(1) = 0$, and $g_i = uh_i \in \mathcal{B}$. Since $f = u^2 f + (1 - u^2) f$ and $(1 - u^2) f = 0$ in a neighborhood of 1, we are done. ■

We can now determine all the morphisms from $\mathcal{H}$ to $\mathcal{H}'$. Let T be a morphism $\mathcal{H} \longrightarrow \mathcal{H}'$. Then, as T commutes with multiplications by elements of $C_c^\infty(\Omega)$, it maps $\mathcal{B}$ to $\mathcal{B}'$. Moreover, for the same reason, the above lemma shows that if $f \in \mathcal{B}$ and $f(1) = 0$, then $(Tf)(1) = 0$. So the map

$$R : f(1) \longmapsto (Tf)(1) \qquad (f \in \mathcal{B})$$

is well defined. From the fact that T intertwines π and π' we obtain that $(Tf)(x) = Rf(x)$ for all $x \in G_0$. To complete the proof we must show two things: (1) R is defined on all of $C^\infty(\sigma)$ and (2) R is bounded. For (1), let $v \in C^\infty(\sigma)$. In the earlier notation, if $u \in C_c^\infty(G_0)$ is 1 in 1 and has support contained in G_1, then $h : (\exp Z, \xi) \mapsto u(\exp Z)\sigma(\xi)^{-1}v$ is in $\mathcal{B}$ and $h(1) = v$. For proving (2), let the constant $C > 0$ be such that

$$||Tg||_{\mathcal{H}'} \leq C||g||_{\mathcal{H}} \qquad (g \in \mathcal{H}').$$

Then, taking $g = u^{1/2}f$ for $f \in \mathcal{B}$ and $u \geq 0$ in $C_c^\infty(\Omega)$, we get

$$\int_\Omega u(x)|Rf(x)|_{\mathcal{K}}^2 d\overline{x} \leq C \int_\Omega u(x)|f(x)|_{\mathcal{K}'}^2 d\overline{x}$$

for all $f \in \mathcal{B}$ and $u \geq 0$ in $C_c^\infty(\Omega)$. So $|Rf(x)|_{\mathcal{K}} \leq C|f(x)|_{\mathcal{K}}$ for almost all x. As f and $Rf = Tf$ are continuous this inequality is valid for all x, in particular for $x = 1$, proving that R is bounded.

8.0.6 Representations induced from a special sub super Lie group

It is now our purpose to extend this smooth classical theory to the super context. A SLG $(H_0, \mathfrak{h})$ is a *sub super Lie group* of the SLG $(G_0, \mathfrak{g})$ if $H_0 \subset G_0, \mathfrak{h} \subset \mathfrak{g}$, and the action of H_0 on $\mathfrak{h}$ is the restriction of the action of H_0 (as a subgroup of G_0) on $\mathfrak{g}$. We shall always suppose that H_0 is closed in G_0. The sub SLG $(H_0, \mathfrak{h})$ is called *special* if $\mathfrak{h}$ has the same odd part as $\mathfrak{g}$, i.e., $\mathfrak{h}_1 = \mathfrak{g}_1$. In this case the super homogeneous space associated is purely even and coincides with $\Omega = G_0/H_0$. As in §8.0.5 we shall assume that Ω admits an invariant measure although it is not difficult to modify the treatment to avoid this assumption. Both conditions are satisfied in the case of the super Poincaré groups and their variants.

We start with a UR $(\sigma, \rho^\sigma, \mathcal{K})$ of $(H_0, \mathfrak{h})$ and associate to it the smooth induced representation $(\pi, \mathcal{B})$ of the classical group G_0. In our case $\mathcal{K}$ is a SHS and so $\mathcal{H}$ becomes a SHS in a natural manner, the parity subspaces being the subspaces where f takes its values in the corresponding parity subspace of $\mathcal{K}$. π is an even UR.

We shall now define the operators $\rho^\pi(X)$ for $X \in \mathfrak{g}_1$ as follows:

$$(\rho^\pi(X)f)(x) = \rho^\sigma(x^{-1}X)f(x) \qquad (f \in \mathcal{B}).$$

Since the values of f are in $C^\infty(\sigma)$ the right side is well defined. In order to prove that the definition gives us an odd operator on $\mathcal{B}$ we need a lemma.

Lemma 19 $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{h}_0$ *and is stable under* G_0. *In particular it is an ideal in* $\mathfrak{g}_0$.

Proof. For $g \in G_0, Y, Y' \in \mathfrak{g}_1$, we have $g[Y,Y'] = [gY, gY'] \in [\mathfrak{g}_1, \mathfrak{g}_1]$. Since $\mathfrak{h}_0 \oplus \mathfrak{g}_1$ is a super Lie algebra, $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{h}_0$. ■

Proposition 44 *$\rho^\pi(X)$ is an odd linear map $\mathcal{B} \longrightarrow \mathcal{B}$ for all $X \in \mathfrak{g}_1$. Moreover $\rho^\pi(X)$ is local, i.e.,* $\mathrm{supp}(\rho^\pi(X)f) \subset \mathrm{supp}(f)$ *for $f \in \mathcal{B}$. Finally, if $Z \in \mathfrak{h}_0$, we have $-d\sigma(Z)f(g) = (Zf)(g)$.*

Proof. The support relation is trivial. Further, for $x \in G_0, \xi \in H_0$,

$$\begin{aligned}(\rho^\pi(X)f)(x\xi) &= \rho^\sigma(\xi^{-1}x^{-1}X)f(x\xi)\\ &= \sigma(\xi)^{-1}\rho^\sigma(x^{-1}X)\sigma(\xi)f(x\xi)\\ &= \sigma(\xi)^{-1}(\rho^\pi(X)f)(x).\end{aligned}$$

It is thus a question of proving that $g \mapsto \rho^\sigma(g^{-1}X)f(g)$ is smooth. If (X_k) is a basis for $\mathfrak{g}_1$, $g^{-1}X = \sum_k c_k(g)X_k$ where the c_k are smooth functions and so it is enough to prove that $g \mapsto \rho^\sigma(Y)f(g)$ is smooth for any $Y \in \mathfrak{g}, f \in \mathcal{B}$. We use Lemma 12. If $Z = (1/2)[Y,Y]$, we have $\rho^\sigma(Y)^2 f(g) = -id\sigma(Z)f(g)$, and we need only show that $-d\sigma(Z)f(g)$ is smooth in g. But $Z \in \mathfrak{h}_0$ and $f(g\exp tZ) = \sigma(\exp(-tZ))f(g)$ so that $-d\sigma(Z)f(g) = (Zf)(g)$ is clearly smooth in g. Note that this argument applies to any $Z \in \mathfrak{h}_0$, giving the last assertion. ■

Proposition 45 *$(\pi, \rho^\sigma, \mathcal{B})$ is a* UR *of the* SLG *$(G_0, \mathfrak{g})$.*

Proof. The symmetry of $\rho^\pi(X)$ and the relations $\rho^\pi(yX) = \pi(y)\rho^\pi(X)\pi(y)^{-1}$ follow immediately from the corresponding relations for ρ^σ. Suppose now that $X, Y \in \mathfrak{g}_1$. Then $(\rho^\pi(X)\rho^\pi(Y)f)(x) = \rho^\sigma(x^{-1}X)\rho^\sigma(x^{-1}Y)f(x)$. Hence

$$\begin{aligned}((\rho^\pi(X)\rho^\pi(Y) + \rho^\pi(Y)\rho^\pi(X))f)(x) &= -id\sigma(x^{-1}[X,Y])f(x)\\ &= i(x^{-1}[X,Y]f)(x)(\text{ Proposition 44 })\\ &= i(d/dt)_{t=0}f(x(x^{-1}\exp t[X,Y]x))\\ &= i(d/dt)_{t=0}f(\exp t[X,Y]x)\\ &= i(d/dt)_{t=0}(\pi(\exp(-t[X,Y])f)(x)\\ &= -i(d\pi([X,Y])f)(x).\end{aligned}$$

This proves the proposition. ■

We refer to $(\pi, \rho^\pi, \mathcal{H})$ as the UR of the SLG $(G_0, \mathfrak{g})$ *induced* by the UR $(\sigma, \rho^\sigma, \mathcal{K})$ of $(H_0, \mathfrak{h})$, and to $(\pi, \rho^\pi, \mathcal{B})$ as the corresponding smooth induced UR.

Write P for the natural projection valued measure in $\mathcal{H}$ based on Ω: for any Borel $E \subset \Omega$, $P(E)$ is the operator in $\mathcal{H}$ of multiplication by χ_E, the characteristic function of E.

Recall the definition of $\leftrightarrow$ before Lemma 13.

Proposition 46 *For $X \in \mathfrak{g}_1$, and $u \in C_c^\infty(\Omega)$, we have $M(u) \leftrightarrow \overline{\rho^\pi(X)}$. Furthermore $P(E) \leftrightarrow \overline{\rho^\pi(X)}$ for Borel $E \subset \Omega$.*

Proof. It is standard that a bounded operator commutes with all $P(E)$ if and only if it commutes with all $M(u)$ for $u \in C_c^\infty(\Omega)$. It is thus enough to prove that $M(u) \leftrightarrow \overline{\rho^\pi(X)}$ for all u, X. On $\mathcal{B}$ we have $M(u)\rho^\pi(X) = \rho^\pi(X)M(u)$ trivially from the definitions, and so we are done in view of Lemma 13. ■

Theorem 5 *The assignment that takes* (σ, ρ^σ) *to* $(\pi, \rho^\pi, \mathcal{B}, M)$ *is a fully faithful functor.*

Proof. Let R be a morphism intertwining (σ, ρ^σ) and $(\sigma', \rho^{\sigma'})$, and let $T : \mathcal{B} \longrightarrow \mathcal{B}'$ be associated to R such that $(Tf)(x) = Rf(x)$. It is then immediate that T intertwines ρ^π and $\rho^{\pi'}$. Conversely, if T is a morphism between the induced systems, from the classical discussion following Lemma 18 we know that $(Tf)(x) = Rf(x)$ for a bounded even operator R intertwining σ and σ'. Since T intertwines ρ^π and $\rho^{\pi'}$ we conclude that R must intertwine ρ^σ and $\rho^{\sigma'}$. ■

8.0.7 Super systems of imprimitivity and the super imprimitivity theorem

A *super system of imprimitivity* (SSI) based on Ω is a collection $(\pi, \rho^\pi, \mathcal{H}, P)$ where $(\pi, \rho^\pi, \mathcal{H})$ is a UR of the SLG $(G_0, \mathfrak{g})$, $(\pi, \mathcal{H}, P)$ is a classical system of imprimitivity, π, P are both even, and $\overline{\rho^\pi(X)} \leftrightarrow P(E)$ for all $X \in \mathfrak{g}_1$ and Borel $E \subset \Omega$.

Let $(\pi, \rho^\pi, \mathcal{H})$ be the induced representation defined in §8.0.6 and let P be the projection valued measure introduced above. Proposition 46 shows that $(\pi, \rho^\pi, \mathcal{H}, P)$ is a SSI based on Ω. We call this the SSI *induced by* (σ, ρ^σ).

Theorem 6 (super imprimitivity theorem) *The assignment that takes* (σ, ρ^σ) *to* $(\pi, \rho^\pi, \mathcal{H}, P)$ *is an equivalence of categories from the category of* UR *'s of the special sub* SLG $(H_0, \mathfrak{h})$ *to the category of* SSI *'s based on* Ω.

Proof. Let us first prove that any SSI of the SLG $(G_0, \mathfrak{g})$ is induced from a UR of the SLG $(H_0, \mathfrak{h})$. We may assume, in view of the classical imprimitivity theorem that π is the representation induced by a UR σ of H_0 in $\mathcal{K}$ and that π acts by left translations on $\mathcal{H}$. By assumption $\rho^\pi(X)$ leaves $C^\infty(\pi)$ invariant. We claim that it leaves $C_c^\infty(\pi)$ also invariant. Indeed, let $f \in C_c^\infty(\pi)$; then there is $u \in C_c^\infty(\Omega)$ such that $f = uf$. On the other hand, by Lemma 13, $\overline{\rho^\pi(X)}M(u) = M(u)\overline{\rho^\pi(X)}$ so that $uf \in D(\overline{\rho^\pi(X)})$ and $\overline{\rho^\pi(X)}(uf) = u\overline{\rho^\pi(X)}f$. Since $uf = f$ this comes to $\rho^\pi(X)f = u\rho^\pi(X)f$, showing that $\rho^\pi(X)f \in C_c^\infty(\pi)$. Thus the $\rho^\pi(X)$ leave $\mathcal{B}$ invariant and commute with all $M(u)$ there. In other words we may work with the smooth SSI.

By Lemma 18 the map $f(1) \longmapsto (\rho^\pi(X)f)(1)$ is well defined and so, as in §8.0.5 we can define a map

$$\rho^\sigma(X) : C^\infty(\sigma) \longrightarrow C^\infty(\sigma)$$

by

$$\rho^\sigma(X)v = (\rho^\pi(X)f)(1), \qquad f(1) = v, \quad f \in \mathcal{B}.$$

Then, for $f \in \mathcal{B}, x \in G_0$,

$$\begin{aligned}(\rho^\pi(X)f)(x) &= (\pi(x^{-1})\rho^\pi(X)f)(1)\\ &= (\rho^\pi(x^{-1}X)\pi(x^{-1})f)(1)\\ &= \rho^\sigma(x^{-1}X)(\pi(x^{-1})f)(1)\\ &= \rho^\sigma(x^{-1}X)f(x).\end{aligned}$$

If we now prove that $(\sigma, \rho^\sigma, \mathcal{K})$ is a UR of the SLG $(H_0, \mathfrak{h})$, we are done. This is completely formal.

Covariance with respect to H_0: For $f \in \mathcal{B}, \xi \in H_0$,

$$\begin{aligned}\rho^\sigma(\xi X)f(1) &= (\rho^\pi(\xi X)f)(1)\\ &= (\pi(\xi)\rho^\pi(X)\pi(\xi^{-1})f)(1)\\ &= (\rho^\pi(X)\pi(\xi^{-1})f)(\xi^{-1})\\ &= \sigma(\xi)(\rho^\pi(X)\pi(\xi^{-1})f)(1)\\ &= \sigma(\xi)\rho^\sigma(X)(\pi(\xi^{-1})f)(1)\\ &= \sigma(\xi)\rho^\sigma(X)\sigma(\xi)^{-1}f(1).\end{aligned}$$

Odd commutators: Let $X, Y \in \mathfrak{g}_1 = \mathfrak{h}_1$ so that $Z = [X, Y] \in \mathfrak{h}_0$. We have

$$[\rho^\pi(X), \rho^\pi(Y)]f = -id\pi([X.Y])f \tag{$*$}$$

for all $f \in \mathcal{B}$. Now,

$$\begin{aligned}i(-d\pi(Z)f)(1) &= i(d/dt)_{t=0}f(\exp tZ)\\ &= i(d/dt)_{t=0}\sigma(\exp(-tZ))f(1)\\ &= -id\sigma(Z)f(1).\end{aligned}$$

On the other hand,

$$(\rho^\pi(X)\rho^\pi(Y)f)(1) = \rho^\sigma(X)\rho^\sigma(Y)f(1)$$

so that the left side of $(*)$, evaluated at 1, becomes

$$[\rho^\sigma(X), \rho^\sigma(Y)]f(1).$$

Thus

$$[\rho^\sigma(X), \rho^\sigma(Y)]f(1) = -id\sigma(Z)f(1).$$

Symmetry: From the symmetry of the $\rho^\pi(X)$ we have, for all $f, g \in \mathcal{B}, a, b \in C_c^\infty(\Omega)$,

$$(\rho^\pi(X)(af), bg)_\mathcal{H} = (af, \rho^\pi(X)(bg))_\mathcal{H}.$$

This means that

$$\int (\rho^\sigma(x^{-1}X)f(x), g(x))_\mathcal{K} a(x)\overline{b(x)}d\overline{x} = \int (f(x), \rho^\sigma(x^{-1}X)g(x))_\mathcal{K} a(x)\overline{b(x)}d\overline{x}.$$

Since a and b are arbitrary we conclude that

$$(\rho^\sigma(x^{-1}X)f(x), g(x))_\mathcal{K} = (f(x), \rho^\sigma(x^{-1}X)g(x))_\mathcal{K}$$

for almost all x. All functions in sight are continuous and so this relation is true for *all* x. The evaluation at 1 gives the symmetry of $\rho^\sigma(X)$ on $C^\infty(\sigma)$.

This proves that $(\sigma, \rho^\sigma, \mathcal{K})$ is a UR of the SLG $(H_0, \mathfrak{h})$ and that the corresponding induced SSI is the one we started with.

To complete the proof we must show that the set of morphisms of the induced SSI's is in canonical bijection with the set of morphisms of the inducing UR's of the sub SLG in question. Let $(\pi, \rho^\pi, \mathcal{H}, P)$ and $(\pi', \rho^{\pi'}, \mathcal{H}', P')$ be the SSI's induced by (σ, ρ^σ) and $(\sigma', \rho^{\sigma'})$ respectively. For any morphism R from (σ, ρ^σ) to $(\sigma', \rho^{\sigma'})$ let T be as in Theorem 5. Then T extends uniquely to a bounded even operator from $\mathcal{H}$ to $\mathcal{H}'$, and the relations $TM(u) = M'(u)T$ for all $u \in C_c^\infty(\Omega)$ imply that $TP(E) = P'(E)T$ for all Borel $E \subset \Omega$. Hence T is a morphism from $(\pi, \rho^\pi, \mathcal{H}, P)$ to $(\pi', \rho^{\pi'}, \mathcal{H}', P')$. It is clear that the assignment $R \longmapsto T$ is functorial. To complete the proof we must show that any morphism T from $(\pi, \rho^\pi, \mathcal{H}, P)$ to $(\pi', \rho^{\pi'}, \mathcal{H}', P')$ is of this form for a unique R. But T must take $\mathcal{B} = C_c^\infty(\pi_0)$ to $\mathcal{B}' = C_c^\infty(\pi_0')$ and commute with the actions of $C_c^\infty(\Omega)$. Hence T is a morphism from $(\pi, \rho^\pi, \mathcal{B}, M)$ to $(\pi', \rho^{\pi'}, \mathcal{B}', M')$. Theorem 5 now implies that T arises from a unique morphism of (σ, ρ^σ) to $(\sigma', \rho^{\sigma'})$.

This finishes the proof of Theorem 6. ∎

Chapter 9

Representations of super semidirect products

Vieni, cara Susanna,
finiscimi l'istoria!

La contessa
Le Nozze di Figaro

9.0.8 Super semidirect products and their irreducible unitary representations

We start with a classical semidirect product $G_0 = T_0 \times' L_0$ where T_0 is a vector space of finite dimension over $\mathbb{R}$, the translation group, and L_0 is a closed unimodular subgroup of $\mathrm{GL}(T_0)$ acting on T_0 naturally. For any Lie group the corresponding gothic letter denotes its Lie algebra. In applications L_0 is usually an orthogonal group of Minkowskian signature, or its 2-fold cover, the corresponding spin group. By a *super semidirect product* (SSDP) we mean a SLG $(G_0, \mathfrak{g})$ where T_0 acts trivially on $\mathfrak{g}_1$ and $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{t}_0$. Clearly $\mathfrak{t} := \mathfrak{t}_0 \oplus \mathfrak{g}_1$ is also a super Lie algebra, and $(T_0, \mathfrak{t})$ is a SLG called the *super translation group.* For any closed subgroup $S_0 \subset L_0$, $H_0 = T_0 S_0$ is a closed subgroup of G_0, $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{g}_1$ is a super Lie algebra where $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{s}_0$ is the Lie algebra of H_0. Notice that $(H_0, \mathfrak{h})$ is a *special* sub SLG of $(G_0, \mathfrak{g})$. We begin by showing that the irreducible UR's of $(G_0, \mathfrak{g})$ are in natural bijection with the irreducible UR's of suitable special sub SLG's of the form $(H_0, \mathfrak{h})$ with the property that the translations act as scalars. For brevity we shall write $S = (G_0, \mathfrak{g}), T = (T_0, \mathfrak{t})$.

The action of L_0 on T_0 induces an action on the dual T_0^* of T_0. We assume that this action is regular, i.e., the orbits are all locally closed. By the well known theorem of Effros this implies that if Q is any projection valued measure on T_0^* such that $Q_E = 0$ or the identity operator I for any invariant Borel subset E of T_0^*, then Q is necessarily concentrated on a single orbit. This is precisely the condition under which the classical method of little groups of Frobenius-Mackey-Wigner works. For any $\lambda \in T_0^*$ let L_0^λ be the stabilizer of λ in L_0 and let $\mathfrak{g}^\lambda = \mathfrak{t}_0 \oplus \mathfrak{l}_0^\lambda \oplus \mathfrak{g}_1$. The SLG $(T_0 L_0^\lambda, \mathfrak{g}^\lambda)$ will be denoted by S^λ. We shall call it the *little super group* at λ. It is a special sub SLG of $(G_0, \mathfrak{g})$. Two λ's are called equivalent if they are in the same L_0-orbit. If θ is a UR of the classical group $T_0 L_0$ and O is an orbit in

T_0^*, its spectrum is said to be *in* O if the spectral measure (via the SNAG theorem) of the restriction of θ to T_0 is supported by O.

Given $\lambda \in T_0^*$, a UR (σ, ρ^σ) of S^λ is λ-*admissible* if $\sigma(t) = e^{i\lambda(t)}I$ for $t \in T_0$. λ itself is called *admissible* if there is an irreducible UR which is λ-admissible. It is obvious that the property of being admissible is preserved under the action of L_0. Let

$$T_0^+ = \left\{ \lambda \in T_0^* \;\middle|\; \lambda \text{ admissible} \right\}.$$

Then T_0^+ is an invariant subset of T_0^*.

Theorem 7 *The spectrum of every irreducible* UR *of the* SLG $(G_0, \mathfrak{g})$ *is in some orbit in* T_0^+. *For each orbit in* T_0^+ *and choice of* λ *in that orbit, the assignment that takes a* λ-*admissible* UR $\gamma := (\sigma, \rho^\sigma)$ *of* S^λ *into the* UR U^γ *of* $(G_0, \mathfrak{g})$ *induced by it, is a functor which is an equivalence of categories between the category of the* λ-*admissible* UR*'s of* S^λ *and the category of* UR*'s of* $(G_0, \mathfrak{g})$ *with their spectra in that orbit. Varying* λ *in that orbit changes the functor into an equivalent one. In particular this functor gives a bijection between the respective sets of equivalence classes of irreducible* UR*'s.*

Proof. Notice first of all that since T_0 acts trivially on $\mathfrak{g}_1$, $\pi_0(t)$ commutes with $\rho^\pi(X)$ on $C^\infty(\pi_0)$ for all $t \in T_0$, $X \in \mathfrak{g}_1$. Hence $P_E \leftrightarrow \rho^\pi(X)$ for all Borel $E \subset \Omega$, $X \in \mathfrak{g}_1$. For the first statement, let E be an *invariant* Borel subset of T_0^*. Let P be the spectral measure of the restriction of π to T_0. Then P_E commutes with $\pi, \rho^\pi(X)$. So, if (π, ρ^π) is irreducible, $P_E = 0$ or I. Hence P is concentrated in some orbit O, i.e., $P_O = I$. The system (π, ρ^π) is clearly equivalent to (π, ρ^π, P) since P and the restriction of π to T_0 generate the same algebra. If $\lambda \in O$ and L_0^λ is the stabilizer of λ in L_0, we can transfer P from O to a projection valued measure P^* on $L_0/L_0^\lambda = T_0L_0/T_0L_0^\lambda$. So (π, ρ^π) is equivalent to the SSI (π, ρ^π, P^*). The rest of the theorem is an immediate consequence of Theorem 6. The fact that $\sigma(t) = e^{i\lambda(t)}I$ for $t \in T_0$ is classical. Indeed, in the smooth model for π treated in §8.0.6, the fact that the spectrum of π is contained in the orbit of λ implies that $(\pi(t)f)(x) = e^{i\lambda(x^{-1}tx)}f(x)$ for all $f \in \mathcal{B}, t \in T_0, x \in G_0$. Hence $f(t^{-1}) = e^{i\lambda(t)}f(1)$ while $f(t^{-1}) = \sigma(t)f(1)$. So $\sigma(t) = e^{i\lambda(t)}I$. ■

Remark 15 *In the classical theory all orbits of* L_0 *are allowed and an additional argument of the positivity of energy is needed to single out the physically occurring representations. In SUSY theories as exemplified by Theorem 7, a restriction is already present: only orbits in* T_0^+ *are permitted. We shall prove in the next section that* T_0^+ *may be interpreted precisely as the set of all positive energy representations.*

9.0.9 Determination of the admissible orbits. Product structure of the representations of the little super groups

We fix $\lambda \in T_0^+$ and let (σ, ρ^σ) be a λ-admissible irreducible UR of S^λ. Clearly

$$-id\sigma(Z) = \lambda(Z)I \qquad (Z \in \mathfrak{t}_0).$$

Define

$$\Phi_\lambda(X_1, X_2) = (1/2)\lambda([X_1, X_2]) \qquad (X_1, X_2 \in \mathfrak{g}_1).$$

Then, on $C^\infty(\sigma)$,

$$[\rho^\sigma(X_1), \rho^\sigma(X_2)] = \lambda([X_1, X_2])I = 2\Phi_\lambda(X_1, X_2)I.$$

Clearly Φ_λ is a symmetric bilinear form on $\mathfrak{g}_1 \times \mathfrak{g}_1$. Let

$$Q_\lambda(X) = \Phi_\lambda(X, X) = (1/2)\lambda([X, X]).$$

Then Q_λ is invariant under L_0^λ because for $X_1, X_2 \in \mathfrak{g}_1, h \in L_1$,

$$[\rho^\sigma(hX_1), \rho^\sigma(hX_2)] = \sigma(h)[\rho^\sigma(X_1), \rho^\sigma(X_2)]\sigma(h)^{-1} = 2\Phi_\lambda(X_1, X_2).$$

Now

$$\rho^\sigma(X)^2 = Q_\lambda(X)I \qquad (X \in \mathfrak{g}_1).$$

Since $\rho^\sigma(X)$ is essentially self adjoint on $C^\infty(\sigma)$, it is immediate that $Q_\lambda(X) \geq 0$. We thus obtain the necessary condition for admissibility:

$$Q_\lambda(X) = \Phi_\lambda(X, X) \geq 0 \qquad (X \in \mathfrak{g}_1).$$

In the remainder of this subsection we shall show that the condition that $\Phi_\lambda \geq 0$, which we refer to as the *positive energy condition*, is also sufficient to ensure that λ is admissible. We will then find all the λ-admissible irreducible UR's of S^λ.

It will follow in the next section that if the super Lie group $(G_0, \mathfrak{g}_1)$ is a super Poincaré group, the condition $\Phi_\lambda \geq 0$ expresses precisely the positivity of the energy. This is the reason for our describing this condition in the general case also as the positive energy condition.

From now on we fix λ such that $\Phi_\lambda \geq 0$.

Lemma 20 *For any admissible* UR (σ, ρ^σ) *of* S^λ, $\rho^\sigma(X)$ *is a bounded self adjoint operator for* $X \in \mathfrak{g}_1$, *and* $\rho^\sigma(X)^2 = Q_\lambda(X)I$. *Moreover,* $Q_\lambda \geq 0$ *and is invariant under* L_0^λ.

Proof. We have, for $X \in \mathfrak{g}_1, \psi \in C^\infty(\sigma)$,

$$|\rho^\sigma(X)\psi|^2_{\mathcal{K}} = (\rho^\sigma(X)^2\psi, \psi) = Q_\lambda(X)^2|\psi|^2_{\mathcal{K}}$$

which proves the lemma. ■

This lemma suggests we study the following situation. Let W be a finite dimensional *real* vector space and let q be a nonnegative quadratic form on W, i.e., $q(w) \geq 0$ for $w \in W$. Let φ be the corresponding symmetric bilinear form ($q(w) = \varphi(w, w)$). Let $\mathcal{C}$ be the real algebra generated by W with the relations $w^2 = q(w)1 (w \in W)$. If q is nondegenerate, i.e., positive definite, this is the Clifford algebra associated to the quadratic vector space (W, q). If $q = 0$ it is just the exterior algebra over W. If $(w_i)_{1 \leq i \leq n}$ is a basis for W such that $\varphi(w_i, w_j) = \varepsilon_i \delta_{ij}$ with $\varepsilon_i = 0$ or 1 according as $i \leq a$ or $> a$, then $\mathcal{C}$ is the algebra generated by the w_i with the relations $w_i w_j + w_j w_i = 2\varepsilon_i \delta_{ij}$. Let W_0 be the radical of q, i.e., $W_0 = \{w_0 | \varphi(w_0, w) = 0 \text{ for all } w \in W\}$; in the above notation W_0 is spanned by the w_i for $i \leq a$. If $W^\sim = W/W_0$ and $q^\sim, \varphi^\sim$ are the corresponding objects induced on $W^\sim$, $q^\sim$ is positive definite, and so we have the usual Clifford algebra $\mathcal{C}^\sim$ generated by $(W^\sim, q^\sim)$ with $W^\sim \subset \mathcal{C}^\sim$. The natural map $W \longrightarrow W^\sim$ extends uniquely to a morphism $\mathcal{C} \longrightarrow \mathcal{C}^\sim$ which is clearly surjective. We claim that its kernel is the ideal $\mathcal{C}_0$ in $\mathcal{C}$ generated by W_0.

Indeed, let I be this kernel. If $s \in I$, s is a linear combination of elements $w_I w_J$ where w_I is a product $w_{i_1} \ldots w_{i_r} (i_1 < \cdots < i_r \leq a)$ and w_J is a product $w_{j_1} \ldots w_{j_s} (a < j_1 < \cdots < j_s)$; hence, $s \equiv \sum_J c_J w_J \bmod \mathcal{C}_0$, and as the image of this element in $\mathcal{C}^\sim$ is 0, $c_J = 0$ for all J because the images of the w_J are linearly independent in $\mathcal{C}^\sim$. Hence $s \in \mathcal{C}_0$, proving our claim.

A representation θ of $\mathcal{C}$ by *bounded* operators in a SHS $\mathcal{K}$ is called *self adjoint* (SA) if $\theta(w)$ is odd and self adjoint for all $w \in W$. θ can be viewed as a representation of the complexification $\mathbb{C} \otimes \mathcal{C}$ of $\mathcal{C}$; a representation of $\mathbb{C} \otimes \mathcal{C}$ arises in this manner from a SA representation of $\mathcal{C}$ if and only if it maps elements of W into odd operators and takes complex conjugates to adjoints. Also we wish to stress that irreducibility is in the graded sense.

Lemma 21 (*i*) *If τ is a* SA *representation of $\mathcal{C}$ in $\mathcal{K}$, then $\tau = 0$ on $\mathcal{C}_0$ and so it is the lift of a* SA *representation $\tau^\sim$ of $\mathcal{C}^\sim$.* (*ii*) *There exist irreducible* SA *representations τ of $\mathcal{C}$; these are finite dimensional, unique if* $\dim(W^\sim)$ *is odd, and unique up to parity reversal if* $\dim(W^\sim)$ *is even.* (*iii*) *Let τ be an irreducible* SA *representation of $\mathcal{C}$ in a* SHS $\mathcal{L}$ *and let θ be any* SA *representation of $\mathcal{C}$ in a* SHS $\mathcal{R}$. *Then $\mathcal{R} \simeq \mathcal{K} \otimes \mathcal{L}$ where $\mathcal{K}$ is a* SHS *and $\theta(a) = 1 \otimes \tau(a)$ for all $a \in \mathcal{C}$; moreover, if* $\dim(W^\sim)$ *is odd, we can choose $\mathcal{K}$ to be purely even.*

Proof. (i) If $w \in W_0$, then $\tau(w_0)^2 = q(w_0)I = 0$ and so, $\tau(w_0)$ itself must be 0 since it is self adjoint.

(ii) In view of (i) we may assume that $W_0 = 0$ so that q is positive definite.

Case I : $\dim(W) = 2m$. Select an ON basis $a_1, b_1, \ldots, a_m, b_m$ for W. Let $e_j = (1/2)(a_j + ib_j), f_j = (1/2)(a_j - ib_j)$. Then $\varphi(e_j, e_k) = \varphi(f_j, f_k) = 0$ while $\varphi(e_j, f_k) = (1/2)\delta_{jk}$. Then $\mathbb{C} \otimes \mathcal{C}$ is generated by the e_j, f_k with the relations

$$e_j e_k + e_k e_j = f_j f_k + f_k f_j = 0, \quad e_j f_k + f_k e_j = \delta_{jk}.$$

We now set up the standard "Schrödinger" representation of $\mathbb{C} \otimes \mathcal{C}$. The representation acts on the SHS $\mathcal{L} = \Lambda(U)$ where U is a Hilbert space of dimension m and the grading on $\mathcal{L}$ is the $\mathbb{Z}_2$-grading induced by the usual $\mathbb{Z}$-grading of $\Lambda(U)$. Let $(u_j)_{1 \leq j \leq m}$ be an ON basis for U. We define

$$\tau(e_j)f = u_j \wedge f, \qquad \tau(f_j)f = \partial(u_j)(f) \qquad (f \in \Lambda(U)),$$

$\partial(u)$ for any $u \in U$ being the odd derivation on $\Lambda(U)$ such that $\partial(u)v = 2(v, u)$ (here $(\cdot, \cdot)$ is the scalar product in $\Lambda(U)$ extending the scalar product of U). It is standard that τ is an irreducible representation of $\mathbb{C} \otimes \mathcal{C}$. The vector 1 is called the *Clifford vacuum.* We shall now verify that τ is SA Since $a_j = e_j + f_j, b_j = -i(e_j - f_j)$, we need to verify that $\tau(f_j) = \tau(e_j)^*$ for all j, $*$ denoting adjoints. For any subset $K = \{k_1 < \cdots < k_r\} \subset \{1, 2, \ldots, m\}$ we write $u_K = u_{k_1} \wedge \cdots \wedge u_{k_r}$. Then we should verify that

$$(u_j \wedge u_K, u_L) = (u_K, \partial(u_j)u_L) \qquad (K, L \subset \{1, 2, \ldots, m\}).$$

Write $K = \{k_1, \ldots, k_r\}, L = \{\ell_1, \ldots, \ell_s\}$ where $k_1 < \cdots < k_r, \ell_1 < \cdots < \ell_s$. We assume that $j = \ell_a$ for some a and $K = L \setminus \{\ell_a\}$, as otherwise both sides are 0. Then $K = \{\ell_1, \ldots, \ell_{a-1}, \ell_{a+1}, \ldots, \ell_s\}$ (note that $r = s - 1$). But then both sides are equal to $(-1)^{a-1}$.

From the general theory of Clifford algebras we know that if τ' is another irreducible SA representation of $\mathcal{C}$, then either $\tau \approx \tau'$ or else $\tau \approx \Pi\tau'\Pi$ where Π is the parity revarsal map and we write $\approx$ for linear (not necessarily unitary) equivalence. So it remains to show that $\approx$ implies unitary equivalence which we write $\simeq$. This is standard since the linear equivalence preserves self adjointness. Indeed, if $R : \tau_1 \longrightarrow \tau_2$ is an even linear isomorphism, then R^*R is an even automorphism of τ_1 and so $R^*R = a^2I$ where a is a scalar which is > 0. Then $U = a^{-1}R$ is an even unitary isomorphism $\tau_1 \simeq \tau_2$. Also for use in the odd case to be treated next, we note that τ is irreducible in the ungraded sense since its image is the full endomorphism algebra of $\mathcal{L}$.

Case II : $\dim(W) = 2m + 1$. It is enough to construct an irreducible SA representation as it will be unique up to linear, and hence unitary, equivalence.

Let $a_0, a_1, \ldots, a_{2m}$ be an ON basis for W. Write $x_j = ia_0a_j (1 \le j \le 2m)$, $\quad x_0 = i^m a_0a_1 \ldots a_{2m}$. Then $x_0^2 = 1, x_jx_k + x_kx_j = 2\delta_{jk}(j, k = 1, 2, \ldots, 2m)$. Moreover x_0 commutes with all a_j and hence with all x_j. The x_j generate a Clifford algebra over $\mathbb{R}$ corresponding to a positive definite quadratic form and so there is an irreducible ungraded representation τ^+ of it in an ungraded Hilbert space $\mathcal{L}^+$ such that $\tau^+(x_j)$ is self adjoint for all $j = 1, 2, \ldots, 2m$ (cf. remark above). Within $\mathbb{C} \otimes \mathcal{C}$ the x_j generate $\mathbb{C} \otimes \mathcal{C}^+$ so that τ^+ is a representation of $\mathbb{C} \otimes \mathcal{C}^+$ in $\mathcal{L}^+$ such that $i\tau^+(a_0a_j)$ is self adjoint for all j. We now take

$$\mathcal{L} = \mathcal{L}^+ \oplus \mathcal{L}^+, \quad \tau = \tau^+ \oplus \tau^+, \quad \tau(x_0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here $\mathcal{L}$ is given the $\mathbb{Z}_2$-grading such that the first and second copies of $\mathcal{L}^+$ are the even and odd parts. It is clear that τ is an irreducible representation of $\mathbb{C} \otimes \mathcal{C}$. We wish to show that $\tau(a_r)$ is odd and self adjoint for $0 \le r \le 2m$. But this follows from the fact that the $\tau(x_r)$ are self adjoint, $\tau(x_j)$ are even, and $\tau(x_0)$ is odd, in view of the formulae

$$a_0 = i^m x_0x_1 \ldots x_{2m}, \quad a_j = -ia_0x_j.$$

This finishes the proof of (ii).

(iii) Let now θ be a SA representation of $\mathcal{C}$ in a SHS $\mathcal{R}$ of *possibly infinite* dimension. For any homogeneous $\psi \in \mathcal{R}$ the cyclic subspace $\theta(\mathcal{C})\psi$ is finite dimensional, hence closed, graded and is θ-stable; moreover by the SA nature of θ, for any graded invariant subspace its orthogonal complement is also graded and invariant. Hence we can write $\mathcal{R} = \oplus_\alpha \mathcal{R}_\alpha$ where the sum is direct and each $\mathcal{R}_\alpha$ is graded, invariant, and irreducible. Let $\mathcal{L}$ be a SHS on which we have an irreducible SA representation of $\mathcal{C}$. If $\dim(W)^\sim$ is even we can thus write $\mathcal{R} = (\mathcal{M}_0 \otimes \mathcal{L}) \oplus (\mathcal{M}_1 \otimes \Pi\mathcal{L})$ where the $\mathcal{M}_j$ are even Hilbert spaces; if $\dim(W^\sim)$ is odd we can write $\mathcal{R} = \mathcal{K} \otimes \mathcal{L}$ where $\mathcal{K}$ is an even Hilbert space. In the first case, since $\mathcal{M}_1 \otimes \Pi\mathcal{L} = \Pi\mathcal{M}_1 \otimes \mathcal{L}$, we have $\mathcal{R} = \mathcal{K} \otimes \mathcal{L}$ where $\mathcal{K}$ is a SHS with $\mathcal{K}_0 = \mathcal{M}_0, \mathcal{K}_1 = \Pi\mathcal{M}_1$. ■

For studying the question of admissibility of λ we need a second ingredient. Let H be a *not necessarily connected* Lie group and let us be given a morphism

$$j : H \longrightarrow \mathrm{O}(W^\sim)$$

so that H acts on $W^\sim$ preserving the quadratic form on $W^\sim$. We wish to find out when there is a UR κ of H, *possibly projective, and preferably, but not necessarily, even*, in the space of the irreducible SA representation $\tau^\sim$, such that

$$\kappa(t)\tau^\sim(w)\kappa(t)^{-1} = \tau^\sim(tw) \qquad (t \in H, w \in W^\sim) \qquad (*).$$

For $h \in \mathrm{O}(W^{\sim})$, let

$$\tau_h^{\sim}(w) = \tau^{\sim}(hw) \qquad (w \in W^{\sim}).$$

Then $\tau_h^{\sim}$ is also an irreducible SA representation of $\mathcal{C}^{\sim}$ and so we can find a unitary operator $K(h)$ such that

$$\tau_h^{\sim}(w) = K(h)\tau^{\sim}(w)K(h)^{-1} \qquad (w \in W^{\sim}).$$

If $\dim W^{\sim}$ is even, $\tau^{\sim}$ is irreducible even as an ungraded representation, and so $K(h)$ will be unique up to a phase; it will be even or odd according as $\tau_h^{\sim} \simeq \tau^{\sim}$ or $\tau_h^{\sim} \simeq \Pi\tau^{\sim}\Pi$ where Π is parity reversal. If $\dim W^{\sim}$ is odd, $\tau^{\sim}$ is irreducible only as a graded representation and so we also need to require $K(h)$ to be an *even* operator in order that it is uniquely determined up to a phase. With this additional requirement in the odd dimensional case, we then see that in both cases the class of κ as a projective UR of H is uniquely determined, i.e., the class of its multiplier μ in $H^2(H, \mathbb{T})$ is fixed. In the following, we shall show that μ can be chosen to be ± 1-valued, and examine the structure of κ more closely.

We begin with some preparation (see [Del99],[Var04]). Let $\mathcal{C}^{\sim\times}$ be the group of invertible elements in $\mathcal{C}^{\sim}$. Define the *full Clifford group* as follows:

$$\Gamma = \left\{ x \in \mathcal{C}^{\sim\times} \cap (\mathcal{C}^{\sim+} \cup \mathcal{C}^{\sim-}) \mid xW^{\sim}x^{-1} \subset W^{\sim} \right\}.$$

We have a homomorphism $\alpha : \Gamma \longrightarrow \mathrm{O(W^{\sim})}$ given by

$$\alpha(x)w = (-1)^{p(x)} x w x^{-1}$$

for all $w \in W^{\sim}$, $p(x)$ being 0 or 1 according as $x \in \mathcal{C}^{\sim+}$ or $x \in \mathcal{C}^{\sim-}$. Let β be the principal antiautomorphism of $\mathcal{C}^{\sim+}$; then $x\beta(x) \in \mathbb{R}^{\times}$ for all $x \in \Gamma$, and we write G for the kernel of the homomorphism $x \mapsto x\beta(x)$ of Γ into $\mathbb{R}^{\times}$. Since $W^{\sim}$ is a positive definite quadratic space, we have an exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow G \xrightarrow{\alpha} \mathrm{O}(W^{\sim}) \longrightarrow 1.$$

For $\dim W^{\sim} \geq 2$, the connected component G^0 of G is contained in $\mathcal{C}^{\sim+}$ and coincides with $\mathrm{Spin}(W^{\sim})$.

Lemma 22 *Let $\tau^{\sim}$ be a SA irreducible representation of $\mathcal{C}^{\sim}$. We then have the following.* (i) *$\tau^{\sim}$ restricts to a unitary representation of G.* (ii) *The operator $\tau^{\sim}(x)$ is even or odd according as $x \in G^0$ or $x \in G \setminus G^0$.* (iii) *$\tau^{\sim}(x)\tau^{\sim}(w)\tau^{\sim}(x)^{-1} = (-1)^{p(x)}\tau^{\sim}(\alpha(x)(w))$ for $x \in G, w \in W^{\sim}$.*

Proof. Each $x \in G$ is expressible in the form $x = cv_1 \ldots v_r$, where v_i are unit vectors in $W^{\sim}$ and $c \in \{\pm 1\}$. Since $\tau^{\sim}(v_i)$ is odd, the parity of $\tau^{\sim}(x)$ is the same of x. Moreover, since the $\tau^{\sim}(v_i)$ are self adjoint,

$$\tau^{\sim}(x)\tau^{\sim}(x)^{*} = c^2\tau^{\sim}(v_1)\ldots\tau^{\sim}(v_r)\tau^{\sim}(v_r)\ldots\tau^{\sim}(v_1) = I.$$

This proves (i). (ii) and (iii) are obvious. ■

We now consider two cases.

Case I. $j(H) \subset \mathrm{SO}(W^{\sim})$.

Let ζ be a Borel map of $\mathrm{SO}(W^{\sim})$ into $\mathrm{Spin}(W^{\sim})$ which is a right inverse of $\alpha(\mathrm{Spin}(W^{\sim}) \longrightarrow \mathrm{SO}(W^{\sim}))$ with $\zeta(1) = 1$. Then $\zeta(xy) = \pm\zeta(x)\zeta(y)$ for $x, y \in \mathrm{SO}(W^{\sim})$, and so

$$\kappa_H = \tau^{\sim} \circ \zeta \circ j$$

is an even projective UR of H satisfying $(*)$ with a $\pm$-valued multiplier μ_H. Since $\zeta(1) = 1$ it follows that μ_H is *normalized*, i.e.,

$$\mu_H(h, 1) = \mu_H(1, h) = 1 \qquad (h \in H).$$

Clearly, the class of μ_H in $H^2(H, \mathbb{Z}_2)$ is trivial if and only if $j : H \to \mathrm{SO}(W^\sim)$ can be lifted to a morphism $\widehat{j} : H \to \mathrm{Spin}(W^\sim)$. In particular, this happens if H is connected and simply connected.

Suppose now H is connected but $\widehat{j}$ does not exist. We can then find a two-fold cover $H^\sim$ of H with a covering map $p(H^\sim \longrightarrow H)$ such that $j(H \to \mathrm{SO}(W^\sim))$ lifts to a morphism $j^\sim(H^\sim \to \mathrm{Spin}(W^\sim))$, and if ξ is the nontrivial element in $\ker p$, then $j^\sim(\xi) = -1$.

Lemma 23 *If j maps H into* $\mathrm{SO}(W^\sim)$*, there is a projectively unique even projective* UR *κ of H satisfying $(*)$, with a normalized ±1-valued multiplier μ. If H is connected, for κ to be an ordinary even representation (which will be unique up to multiplication by a character of H) it is necessary and sufficient that either (i) $j(H \to \mathrm{SO}(W^\sim))$ can be lifted to* $\mathrm{Spin}(W^\sim)$ *or (ii) there exists a character χ of $H^\sim$ such that $\chi(\xi) = -1$. In particular, if $H = A \times' T$, where A is simply connected and T is a torus, then κ is an ordinary even unitary representation.*

Proof. The first statement has already been proved.

We next prove the sufficiency part of the second statement. Sufficiency of (i) has already been observed. To see that (ii) is sufficient, note that $\kappa^\sim = \tau^\sim \circ j^\sim$ is an even UR of $H^\sim$ satisfying $(*)$; one can clearly replace $\kappa^\sim$ by $\kappa^\sim\chi$ without destroying $(*)$. As $\kappa^\sim(\xi) = -1$, we have $(\kappa^\sim\chi)(\xi) = 1$, and so it is immediate that $\kappa^\sim\chi$ descends to H.

We leave the necessity part to the reader; it will not be used in the sequel.

The statement for $H = A \times' T$ will follow if we show that $H^\sim$ has a character χ as in condition (ii). We have $H^\sim = A \times' T^\sim$, $T^\sim$ being the double cover of T, and $\xi = (1, t)$, with $t \neq 1, t^2 = 1$. There exists a character χ of $T^\sim$ such that $\chi(t) = -1$, and such a character can be extended to $H^\sim$ by making it trivial on A. ■

Case II. $j(H) \not\subset \mathrm{SO}(W^\sim)$.
Let

$$H_0 = j^{-1}(\mathrm{SO}(W^\sim)).$$

Then H_0 is a normal subgroup of H of index 2. We must distinguish two subcases.

Case II.a. $\dim(W^\sim)$ is even.
Let ζ_0 be a Borel right inverse of $\alpha(G^0 \longrightarrow \mathrm{SO}(W^\sim))$ with $\zeta_0(1) = 1$. Fix a unit vector $v_0 \in W^\sim$ and let $r_0 = -\alpha(v_0)$. Since $\alpha(v_0)$ is the reflection in the hyperplane orthogonal to v_0 and $\dim(W^\sim)$ is even, we see that $r_0 \in \mathrm{O}(W^\sim) \setminus \mathrm{SO}(W^\sim)$. We then define a map $\zeta(\mathrm{O}(W^\sim) \to G)$ by

$$\zeta(h) = \begin{cases} \zeta_0(h) & \text{if } h \in \mathrm{SO}(W^\sim) \\ \zeta_0(h_0)v_0 & \text{if } h = h_0 r_0, h_0 \in \mathrm{SO}(W^\sim). \end{cases}$$

Once again we have $\zeta(h_1h_2) = \pm\zeta(h_1)\zeta(h_2)$. Define

$$\kappa_H = \tau^\sim \circ \zeta \circ j.$$

Then κ_H is a projective UR of H satisfying $(*)$ with a $\pm$-valued normalized multiplier μ_H. But κ_H is *not* an even representation; elements of $H \setminus H_0$ map into *odd* unitary operators in the space of $\tau^\sim$. We shall call such a representation of H *graded with respect to* H_0, or simply *graded.* We have thus proved the following.

Lemma 24 *If* $j(H) \not\subset \mathrm{SO}(W^\sim)$, *and* $\dim(W^\sim)$ *is even, and if we define* $H_0 = j^{-1}(\mathrm{SO}(W^\sim))$, *then there is a projective* UR κ_H, *graded with respect to* H_0, *and satisfying* $(*)$ *with a* $\pm$-*valued normalized multiplier* μ_H.

Before we take up the case when $\dim(W^\sim)$ is odd, we shall describe how the projective graded representations of H are constructed. This is a very general situation and so we shall work with a locally compact second countable group A and a closed subgroup A_0 of index 2; A_0 is automatically normal and we write $A_1 = A \setminus A_0$. Gradedness is with respect to A_0. We fix a $\pm$-valued multiplier μ for A which is normalized. For brevity a representation will mean a unitary μ-representation. Moreover, with a slight abuse of language a μ-representation of A_0 will mean a $\mu|_{A_0 \times A_0}$-representation of A_0. If R_g is a graded representation in a SHS $\mathcal{H}$, R the corresponding ungraded representation, and P_j is the orthogonal projection $\mathcal{H} \longrightarrow \mathcal{H}_j$, we associate to R the projection valued measure P on A/A_0 where $P_{A_0} = P_0$ and $P_{A_1} = P_1$. Then the condition that R_g is graded is exactly the same as saying that (R, P) is a system of imprimitivity for A based on A/A_0. Conversely, given a system of imprimitivity (R, P) for A based on A/A_0, let us define the grading for $\mathcal{H} = \mathcal{H}(R)$ by $\mathcal{H}_j =$ range of $P_{A_j} (j = 0, 1)$; then R becomes a graded representation. Moreover for graded representations R_g, R'_g, we have $\mathrm{Hom}(R_g, R'_g) = \mathrm{Hom}((R, P), (R', P'))$. In other words, the category of systems of imprimitivity for A based on A/A_0 and the category of representations of A graded with respect to A_0 are equivalent naturally.

For any μ-representation r of A_0 in a purely even Hilbert space $\mathcal{H}(r)$, let

$$R_r := \mathrm{Ind}_{A_0}^{A}\, r$$

be the representation of A induced by r. We recall that R_r acts in the Hilbert space $\mathcal{H}(R_r)$ of all (equivalence classes of Borel) functions $f(A \to \mathcal{H}(r))$ such that for each $\alpha \in A_0$,

$$f(\alpha a) = \mu(\alpha, a) r(\alpha) f(a)$$

for almost all $a \in A$; and

$$(R_r(a)f)(y) = \mu(y, a) f(ya) \qquad (a, y \in A).$$

The space $\mathcal{H}(R_r)$ is naturally graded by defining

$$\mathcal{H}(R_r)_j = \{f \in \mathcal{H}(R_r) \mid \mathrm{supp}(f) \subset A_j\} \qquad (j = 0, 1).$$

It is then obvious that R_r is a graded μ-representation. We write $\widehat{R_r}$ for R_r treated as a graded representation. These remarks suggest the following lemma.

Lemma 25 *For any unitary* μ-*representation* r *of* A_0 *let* $R_r = \mathrm{Ind}(r)$ *and let* $\widehat{R_r}$ *be the graded* μ-*representation defined by* R_r. *Then the assignment* $r \mapsto \widehat{R_r}$ *is an equivalence from the category of unitary* μ-*representations of* A_0 *to the category of unitary graded* μ-*representations of* A.

Proof. Let us first assume that $\mu = 1$. Then we are dealing with UR's and the above remarks imply the Lemma in view of the classical imprimitivity theorem.

When μ is not 1 we go to the central extension $A^{\sim}$ of A by $\mathbb{Z}_2$ defined by μ. Recall that $A^{\sim} = A \times_{\mu} \mathbb{Z}_2$ with multiplication defined by

$$(a, \xi)(a', \xi') = (aa', \xi\xi'\mu(a, a')) \qquad (a, a' \in A, \xi, \xi' \in \mathbb{Z}_2).$$

(We must give to $A^{\sim}$ the Weil topology). Then $A_0^{\sim} = A_0 \times_{\mu} \mathbb{Z}_2$ and $A^{\sim}/A_0^{\sim} = A/A_0$. The μ-representations R of A are in natural bijection with UR's $R^{\sim}$ of $A^{\sim}$ such that $R^{\sim}$ is nontrivial on $\mathbb{Z}_2$ by the correspondence

$$R^{\sim}(a, \xi) = \xi R(a), \quad R(a) = R^{\sim}(a, 1).$$

The assignment $R \mapsto R^{\sim}$ is an equivalence of categories. Analogous considerations hold for μ-representations r of A_0 and UR's $r^{\sim}$ of $A_0^{\sim}$ which are nontrivial on $\mathbb{Z}_2$. The Lemma would now follow if we establish two things: (a) For any unitary μ-representation r of A_0, and $R_r = \mathrm{Ind}(r)$, we have

$$R_r^{\sim} \simeq \mathrm{Ind}(r^{\sim})$$

and (b) If ρ is a UR of $A_0^{\sim}$ and $\mathrm{Ind}(\rho) = R^{\sim}$ for some μ-representation R of A, then $\rho = r^{\sim}$ for some μ-representation r of A_0. To prove (a) we set up the map $f \longmapsto f^{\sim}$ from $\mathcal{H}(R_r^{\sim})$ to $\mathcal{H}(\mathrm{Ind}(r^{\sim}))$ by

$$f^{\sim}(a, \xi) = f(a)\xi.$$

It is an easy calculation that this is an isomorphism of $R_r^{\sim}$ with $\mathcal{H}(\mathrm{Ind}(r^{\sim}))$ that intertwines the two projection valued measures on A/A_0 and $A^{\sim}/A_0^{\sim} \approx A/A_0$. To prove (b) we have only to check that $\rho(1, \xi) = \xi$; this however is a straightforward calculation. ■

Remark 16 *Given a graded μ-representation R of A, let r be the μ-representation of A_0 defined by*

$$r(\alpha) = R(\alpha)\big|_{\mathcal{H}(R)_0} \qquad (\alpha \in A_0).$$

It is then easy to show that $R \simeq R_r$. In fact it is enough to verify this (as before) when $\mu = 1$. In this simple situation this is well known.

We now resume our discussion and treat the odd dimensional case.

Case II.b. $\dim(W^{\sim})$ is odd.

We shall exhibit a projective even UR κ of H satisfying $(*)$. We refer back to the construction of $\tau^{\sim}$ in Lemma 21. Then

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is an odd unitary operator such that $S^2 = -1$ and $\tau^{\sim}(x)S = (-1)^{p(x)} S\tau^{\sim}(x)$ for all $x \in \mathcal{C}^{\sim}$. Let $\gamma(\mathrm{O}(W^{\sim}) \to G)$ be a Borel right inverse of α. Then, it is easily checked that

$$\kappa_H(h) = \begin{cases} (\tau^{\sim} \circ \gamma \circ j)(h) & \text{if } h \in H_0 \\ (\tau^{\sim} \circ \gamma \circ j)(h)\, S & \text{if } h \in H \setminus H_0 \end{cases}$$

has the required properties.

We now return to our original setting. First the graded representations of H are obtained by taking $H = A, H_0 = A_0$ in the foregoing discussion. Let $\mathfrak{g}_{1\lambda} = \mathfrak{g}_1/\mathrm{rad}\ \Phi_\lambda$. We write $\mathcal{C}_\lambda$

for the algebra generated by $\mathfrak{g}_1$ with the relations $X^2 = Q_\lambda(X)1$ for all $X \in \mathfrak{g}_1$. We have a map

$$j_\lambda : L_0^\lambda \longrightarrow \mathrm{O}(\mathfrak{g}_{1\lambda}).$$

We take in the preceding theory

$$q = Q_\lambda, \quad \varphi = \Phi_\lambda, \quad H = L_0^\lambda, \quad W^\sim = \mathfrak{g}_{1\lambda}, \quad j = j_\lambda, \quad \mathcal{C}^\sim = \mathcal{C}_\lambda^\sim.$$

Furthermore let

$$\kappa_\lambda = \kappa, \quad \mu_\lambda = \mu, \quad \tau_\lambda^\sim = \tau^\sim, \quad \tau_\lambda = \text{ lift of } \tau^\sim \text{ to } \mathcal{C}_\lambda.$$

Then μ_λ is a normalized multiplier for L_0^λ which we can choose to be $\pm$-valued, κ_λ is a μ_λ-representation (unitary) of L_0^λ in the space of τ_λ, and

$$\kappa_\lambda(t)\tau_\lambda(X)\kappa_\lambda(t)^{-1} = \tau_\lambda(tX) \qquad (t \in L_0^\lambda).$$

Moreover, κ_λ is graded if and only if $j_\lambda(L_0^\lambda) \not\subset \mathrm{SO}(\mathfrak{g}_{1\lambda})$ and $\dim(\mathfrak{g}_{1\lambda})$ is even, otherwise κ_λ is even. Finally, let

$$L_{00}^\lambda = \begin{cases} j_\lambda^{-1}(\mathrm{SO}(\mathfrak{g}_{1\lambda})) & \text{if } j_\lambda(L_0^\lambda) \not\subset \mathrm{SO}(\mathfrak{g}_{1\lambda}) \text{ and } \dim(\mathfrak{g}_{1\lambda}) \text{ is even} \\ L_0^\lambda & \text{otherwise.} \end{cases}$$

For any unitary μ_λ-representation r of L_{00}^λ in an even Hilbert space $\mathcal{K}_\lambda$, let $\widehat{R_r}$ be the unitary μ_λ- representation of L_0^λ induced by r, which is graded if $j_\lambda(L_0^\lambda) \not\subset \mathrm{SO}(\mathfrak{g}_{1\lambda})$ and $\dim(\mathfrak{g}_{1\lambda})$ is even, and is just r in all other cases.

Theorem 8 *Let λ be such that $\Phi_\lambda \geq 0$. Then λ is admissible, i.e., $\lambda \in T_0^+$. For a fixed such λ let τ_λ be an irreducible* SA *representation of $\mathcal{C}_\lambda$ in a* SHS *$\mathcal{L}_\lambda$ and κ_λ the unitary μ_λ-representation of L_0^λ in $\mathcal{L}_\lambda$ associated to τ_λ as above. For any unitary μ_λ-representation r of L_{00}^λ in an even Hilbert space $\mathcal{K}_\lambda$, let $\widehat{R_r}$ be the unitary μ_λ- representation of L_0^λ defined as above, and let*

$$\theta_{r\lambda} = (\sigma_{r\lambda}, \rho_\lambda^\sigma)$$

be the UR *of the little* SLG *S^λ, where, for $X \in \mathfrak{g}_1, h \in L_0^\lambda, t \in T_0$,*

$$\sigma_{r\lambda}(th) = e^{i\lambda(t)}\sigma'_{r\lambda}(h)$$

and

$$\sigma'_{r\lambda}(h) = \widehat{R_r}(h) \otimes \kappa_\lambda(h), \quad \rho_\lambda^\sigma(X) = 1 \otimes \tau_\lambda(X) \qquad (X \in \mathfrak{g}_1).$$

Then $\theta_{r\lambda}$ is an admissible UR *of S^λ. The assignment $r \longmapsto \theta_{r\lambda}$ is functorial, commutes with direct sums, and is an equivalence of categories from the category of unitary μ_λ-representations of L_{00}^λ to the category of admissible* UR*'s of the little super group S^λ. If L_0^λ is connected and satisfies either of the conditions of Lemma 23, then $r \longmapsto \theta_{r\lambda}$ is an equivalence from the category of even* UR*'s of L_0^λ into the category of admissible* UR*'s of S^λ.*

Proof. Once κ_λ is fixed, the assignment

$$r \longmapsto \theta_{r\lambda}$$

is clearly functorial (although it depends on κ_λ). If $\dim(\mathfrak{g}_{1\lambda})$ is even, a morphism $M : r_1 \longrightarrow r_2$ obviously gives rise to the morphism $\widehat{M} : R_{r_1} \longrightarrow R_{r_2}$ and hence to the morphism $\widehat{M} \otimes 1$

from $\theta_{r_1\lambda}$ to $\theta_{r_2\lambda}$. Conversely, if T is a bounded even operator commuting with $1 \otimes \tau_\lambda$, it is immediate (since the $\tau_\lambda(X)$ generate the full super algebra of endomorphisms of $\mathcal{L}_\lambda$) that T must be of the form $M' \otimes 1$ where $M' : \mathcal{H}(\widehat{R}_{r_1}) \to \mathcal{H}(\widehat{R}_{r_2})$ is a bounded even operator. If now T intertwines $\widehat{R}_{r_1} \otimes \kappa_\lambda$ and $\widehat{R}_{r_2} \otimes \kappa_\lambda$, then M' must belong to $\mathrm{Hom}(\widehat{R}_{r_1}, \widehat{R}_{r_2}) \approx \mathrm{Hom}(r_1, r_2)$. Thus $r \longmapsto \theta_{r\lambda}$ is a fully faithful functor. If $\dim(\mathfrak{g}_{1\lambda})$ is odd, we can choose $\mathcal{K}_\lambda$ to be purely even (see Lemma 21). If T is a bounded even operator commuting with $1 \otimes \tau_\lambda$, we use the fact that it commutes with

$$1 \otimes \begin{pmatrix} \tau^+(a) & 0 \\ 0 & \tau^+(a) \end{pmatrix} \text{ and } 1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(in the notation of Lemma 21) to conclude, via an argument similar to the one used in the even dimensional case, that T is of the form $M \otimes 1$. Arguing as before we conclude that $M \in \mathrm{Hom}(r_1, r_2)$. Thus $r \mapsto \theta_{r\lambda}$ is a fully faithful functor in this case also. It remains to show that every admissible UR of S^λ is of the form $\theta_{r\lambda}$.

Let θ be an admissible UR of S^λ in $\mathcal{H}$. Then $\theta = (\xi, \tau)$ where ξ is an even UR of $T_0 L_0^\lambda$ which restricts to $e^{i\lambda} I$ on T_0, τ is a SA representation of $\mathcal{C}_\lambda$ related to ξ as usual. We may then assume by Lemma 21 that $\mathcal{H} = \mathcal{K} \otimes \mathcal{L}_\lambda$ and $\tau = 1 \otimes \tau_\lambda$. If $\dim(\mathfrak{g}_{1\lambda})$ is odd, we choose $\mathcal{K}$ purely even. Then $1 \otimes \tau_\lambda(hX) = \xi(h)[1 \otimes \tau_\lambda(X)]\xi(h)^{-1}$. But the same relation is true if we replace ξ by $1 \otimes \kappa_\lambda$. So if $\xi_1 = [1 \otimes \kappa_\lambda]^{-1}\xi$, then $\xi_1(h)$ is even or odd according to the grading of $\kappa_\lambda(h)$, and commutes with $1 \otimes \tau_\lambda$. Hence ξ_1 is of the form $R' \otimes 1$ for a Borel map R' from L_0^λ into the unitary group of $\mathcal{K}$. Thus

$$\xi(h) = [1 \otimes \kappa_\lambda(h)][R'(h) \otimes 1].$$

The two factors on the right side of this equation commute; the left side is an even UR and the first factor on the right is a unitary μ_λ-representation of L_0^λ, which is graded or even according to $\dim(\mathfrak{g}_{1\lambda})$ even or $\dim(\mathfrak{g}_{1\lambda})$ odd. So R' is a $\mu_\lambda^{-1} = \mu_\lambda$-representation of L_0^λ in $\mathcal{K}$, which is graded or even according to $\dim(\mathfrak{g}_{1\lambda})$ even or $\dim(\mathfrak{g}_{1\lambda})$ odd. This finishes the proof. ■

Remark 17 *If $\lambda = 0$, then $\Phi_\lambda = 0, \mathcal{L}_\lambda = 0$, and $\theta_{r0} = (r, 0)$.*

Remark 18 *For the super Poincaré groups we shall see in the next subsection that the situation is much simpler and L_0^λ is always connected and satisfies the conditions of Lemma 23.*

Combining Theorems 7 and 8 we obtain the following theorem. Let $\Theta_{r\lambda}$ be the UR of $(G_0, \mathfrak{g})$ induced by $\theta_{r\lambda}$ as described in Theorem 8.

Theorem 9 *Let λ be such that $\Phi_\lambda \geq 0$. The assignment that takes r to the* UR *$\Theta_{r\lambda}$ is an equivalence of categories from the category of unitary μ_λ-representations of L_{00}^λ to the category of* UR*'s of $(G_0, \mathfrak{g})$ whose spectra are contained in the orbit of λ. In particular, for r irreducible, $\Theta_{r\lambda}$ is irreducible, and every irreducible* UR *of $(G_0, \mathfrak{g})$ is obtained in this way. If the conditions of Lemma 23 are satisfied, then the r's come from the category of* UR*'s of L_0^λ.*

In the case of super Poincaré groups (see Remark 18 above), $\Theta_{r\lambda}$ induced by $\theta_{r\lambda}$ represents a *superparticle*. In general the UR $\pi_{r\lambda}$ of T_0L_0 contained in $\Theta_{r\lambda}$ will *not* be an irreducible UR of G_0. Its decomposition into irreducibles gives the *multiplet* that the UR of S determines. This is of course the set of irreducible UR's $U_{r\lambda j}$ of G_0 induced by the $r_{\lambda j}$ where the $r_{\lambda j}$ are the irreducible UR's of L_0^λ contained in $r \otimes \kappa_\lambda$:

$$r \otimes \kappa_\lambda = \bigoplus r_{\lambda j}, \qquad \pi_{r\lambda} = \bigoplus U_{r\lambda j}.$$

The set $(r_{\lambda j})$ thus defines the *multiplet*. For r trivial the corresponding multiplet is called *fundamental.*

9.0.10 The case of the super Poincaré groups

We shall now specialize the entire theory to the case when $(G_0, \mathfrak{g})$ is a *super Poincaré group* (SPG). This means that the following conditions are satisfied.

(a) $T_0 = \mathbb{R}^{1,D-1}$ is the D-dimensional Minkowski space of signature $(1, D-1)$ with $D \geq 4$; the Minkowski bilinear form is $\langle x, x' \rangle = x_0 x_0' - \sum_j x_j x_j'$.

(b) $L_0 = \mathrm{Spin}(1, D-1)$.

(c) $\mathfrak{g}_1$ is a real spinorial module for L_0, i.e., is a direct sum of spin representations over $\mathbb{C}$.

(d) For any $0 \neq X \in \mathfrak{g}_1$, and any $x \in T_0$ lying in the interior Γ^+ of the forward light cone Γ, we have

$$\langle [X, X], x \rangle > 0.$$

If in (c) $\mathfrak{g}_1$ is the sum of N real irreducible spin modules of L_0, we say we are in the context of *N-extended supersymmetry.* Sometimes N refers to the number of irreducible components over $\mathbb{C}$. In (d)

$$\Gamma = \{x \mid \langle x, x \rangle \geq 0, x_0 \geq 0\}, \qquad \Gamma^+ = \{x \mid \langle x, x \rangle > 0, x_0 > 0\}.$$

In the case when $D = 4$ and $\mathfrak{g}_1$ is the Majorana spinor, the condition (d) is *automatic* (one may have to change the sign of the odd commutators to achieve this); in the general case, as we shall see below, it ensures that only positive energy representations are allowed.

We identify T_0^* with $\mathbb{R}^{1,D-1}$ by the pairing $\langle x, p \rangle = x_0 p_0 - \sum_j x_j p_j$. The dual action of L_0 is then the original action. The orbit structure of T_0^* is classical.

Lemma 26 (*i*) *Let V be a finite dimensional real vector space with a nondegenerate quadratic form and let V_1 be a subspace of V on which the quadratic form remains nondegenerate. Then the spin representations of* $\mathrm{Spin}(V)$ *restrict on* $\mathrm{Spin}(V_1)$ *to direct sums of spin representations of* $\mathrm{Spin}(V_1)$. (*ii*) *Suppose $V = \mathbb{R}^{1,D-1}$. Let $p \in V$ be such that $\langle p, p \rangle = \pm m^2 \neq 0$ and $V_1 = p^\perp$. Then V_1 is a quadratic subspace, the stabilizer L_0^p of p in* $\mathrm{Spin}(V)$ *is precisely* $\mathrm{Spin}(V_1)$*, and it is* $\simeq \mathrm{Spin}(D-1)$ *for $\langle p, p \rangle = m^2$ and* $\simeq \mathrm{Spin}(1, D-2)$ *for $\langle p, p \rangle = -m^2 \neq 0$.*

Proof. (i) Let $\mathcal{C}, \mathcal{C}_1$ be the Clifford algebras of V and V_1. Then $\mathcal{C}_1^+ \subset \mathcal{C}^+$ and hence, as the spin groups are imbedded in the even parts of the Clifford algebras, we have $\mathrm{Spin}(V_1) \subset \mathrm{Spin}(V)$. Now the spin modules are precisely the modules for the even parts of the corresponding Clifford algebras and so, as these algebras are semisimple, the decomposition of the spin module of $\mathrm{Spin}(V)$, viewed as an irreducible module for $\mathcal{C}^+$, into irreducible modules for $\mathcal{C}_1^+$ under restriction to $\mathcal{C}_1^+$, gives the decomposition of the restriction of the original spin module to $\mathrm{Spin}(V_1)$. See [Del99],[Var04].

(ii) Choose an orthogonal basis $(e_\alpha)_{0\leq\alpha\leq D-1}$ such that $\langle e_0, e_0\rangle = -\langle e_j, e_j\rangle = 1$ for $1 \leq j \leq D-1$. It is easy to see that we can move p to either $(m, 0, \dots, 0)$ or $(0, m, 0, \dots, 0)$ by L_0 and so we may assume that p is in one of these two positions. For $u \in C(V)^+$ it is then a straightforward matter to verify that $up = pu$ if and only if $u \in C(V_1)^+$. From the characterization of the spin group ([Del99],[Var04]) it is now clear that $L_0^p = \mathrm{Spin}(V_1)$. ■

Lemma 27 *Let M be a connected real semisimple Lie group whose universal cover does not have a compact factor, i.e., the Lie algebra of M does not have a factor Lie algebra whose group is compact. Then M has no nontrivial morphisms into any compact Lie group, and hence no nontrivial finite dimensional* UR*'s.*

Proof. We may assume that M is simply connected. If such a morphism exists we have a nontrivial morphism $\mathfrak{m} \longrightarrow \mathfrak{k}$ where $\mathfrak{m}$ is the Lie algebra of M and $\mathfrak{k}$ is the Lie algebra of a compact Lie group. Let $\mathfrak{a}$ be the kernel of this Lie algebra morphism. Then $\mathfrak{a}$ is an ideal of $\mathfrak{m}$ different from $\mathfrak{m}$, and so we can write $\mathfrak{m}$ as $\mathfrak{a} \times \mathfrak{k}'$ where $\mathfrak{k}'$ is also an ideal and is non zero; moreover, the map from $\mathfrak{k}'$ to $\mathfrak{k}$ is injective. $\mathfrak{k}'$ is semisimple and admits an invariant negative definite form (the restriction from the Cartan- Killing form of $\mathfrak{k}$), and so its associated simply connected group K' is compact. If A is the simply connected group for $\mathfrak{a}$, we have $M = A \times K'$, showing that M admits a compact factor, contrary to hypothesis. ■

Corollary 4 *If V is a quadratic vector space of signature $(p, q), p, q > 0, p + q \geq 3$, then* $\mathrm{Spin}(V)$ *does not have any nontrivial map into a compact Lie group.*

Proof. The Lie algebra is semisimple and the simple factors are not compact. ■

Lemma 28 *We have $\Gamma = \{p \mid \Phi_p \geq 0\}$, i.e., for any $p \in \mathbb{R}^{1,D-1}$,*

$$\Phi_p \geq 0 \Longleftrightarrow p_0 \geq 0, \langle p, p\rangle \geq 0.$$

Moreover,

$$p_0 > 0, \langle p, p\rangle > 0 \Longrightarrow \Phi_p > 0.$$

Proof. For $0 \neq X \in \mathfrak{g}_1$ we have $\langle [X, X], x\rangle > 0$ for all $x \in \Gamma^+$ and hence the inequality is true with ≥ 0 replacing > 0 for $x \in \Gamma$. Hence $2\Phi_p(X, X) = \langle [X, X], p\rangle \geq 0$ if $p \in \Gamma$. So

$$\Gamma \subset \{p \mid \Phi_p \geq 0\}.$$

We shall show next that

$$\{p \mid \Phi_p \geq 0\} \subset \{p \mid \langle p, p\rangle \geq 0\}.$$

Suppose on the contrary that $\Phi_p \geq 0$ but $\langle p, p\rangle < 0$. Since Φ_p is invariant under L_0^p which is connected, we have a map $L_0^p \longrightarrow \mathrm{SO}(\mathfrak{g}_{1p})$. Then $L_0^p = \mathrm{Spin}(V_1) = \mathrm{Spin}(1, D-2)$ by Lemma 26, and Corollary 4 shows that L_0^p has no nontrivial morphisms into any compact Lie group. Hence L_0^p acts trivially on $\mathfrak{g}_{1p}$. Since L_0^p is a semisimple group, $\mathfrak{g}_{1p}$ can be lifted to an L_0^p-invariant subspace of $\mathfrak{g}_1$. Hence, if $\mathfrak{g}_{1p} \neq 0$, the action of L_0^p on $\mathfrak{g}_1$ must have non zero trivial submodules. However, by Lemma 26, the spin modules of $\mathrm{Spin}(V)$ restrict on $\mathrm{Spin}(V_1)$ to direct sums of spin modules of the smaller group and there is no trivial module in this decomposition. Hence $\mathfrak{g}_{1p} = 0$, i.e., $\Phi_p = 0$. Hence p vanishes on $[\mathfrak{g}_1, \mathfrak{g}_1]$. Now $[\mathfrak{g}_1, \mathfrak{g}_1]$ is stable under L_0 and non zero, and so must be the whole of $\mathfrak{t}_0$. So $p = 0$, a contradiction.

To finish the proof we should prove that if $\Phi_p \geq 0$ then $p_0 \geq 0$. Otherwise $p_0 < 0$ and so $-p \in \Gamma$ and so from what we have already proved, we have $\Phi_{-p} = -\Phi_p \geq 0$. Hence $\Phi_p = 0$. But then as before $p = 0$, a contradiction.

Finally, if $p_0 > 0$ and $\langle p, p \rangle > 0$, then $\Phi_p > 0$ by definition of the SPG structure. This completes the proof. ■

Theorem 10 *Let $S = (G_0, \mathfrak{g})$ be a* SPG*. Then all stabilizers are connected and*

$$T_0^+ = \{p \mid \Phi_p \geq 0\} = \Gamma.$$

Moreover, κ_p is an even UR *of L_0^p, and the irreducible* UR*'s of S whose spectra are in the orbit of p are in natural bijection with the irreducible* UR*'s of L_0^p. The corresponding multiplet is then the set of irreducible* UR*'s parametrized by the irreducibles of L_0^p occurring in the decomposition of $\alpha \otimes \kappa_p$ as a* UR *of L_0^p.*

Proof. In view of Theorem 8 and Lemma 28 we have $T_0^+ = \Gamma$. For $p \in \Gamma$, the stabilizers are all known classically. If $\langle p, p \rangle > 0$, $L_0^p = \mathrm{Spin}(D-1)$; if $\langle p, p \rangle = 0$ but $p_0 > 0$, then $L_0^p = \mathbb{R}^{D-2} \times' \mathrm{Spin}(D-2)$; and for $p = 0$, $L_0^p = L_0$. So, except when $D = 4$ and p is non zero and is in the zero mass orbit, the stabilizer is connected and simply connected, thus κ_p is an even UR of L_0^p by Lemma 23. But in the exceptional case, $L_0^p = \mathbb{R}^2 \times' S^1$ where S^1 is the circle, and Lemma 23 is again applicable. This finishes the proof. ■

9.0.11 Determination of κ_p and the structure of the multiplets. Examples

We have seen that the multiplet defined by the super particle $\Theta_{\alpha p}$ is parametrized by the set of irreducible UR's of L_0^p that occur in the decomposition of $\alpha \otimes \kappa_p$. Clearly it is desirable to determine κ_p as explicitly as possible. We shall do this in what follows.

To determine κ_p the following lemma is useful. (W, q) is a positive definite quadratic vector space and φ is the bilinear form of q. $C(W)$ is the Clifford algebra of W and H is a connected Lie group with a morphism $H \longrightarrow \mathrm{SO}(W)$. τ is an irreducible SA representation of $C(W)$ and κ is a UR such that $\kappa(t)\tau(u)\kappa(t)^{-1} = \tau(tu)$ for all $u \in W, t \in H$. *We write $\approx$ for equivalence after multiplying by a suitable character.*

Lemma 29 *Suppose that $\dim(W) = 2m$ is even and $W_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} W$ has an isotropic subspace E of dimension m stable under H. Let η be the action of H on $\Lambda(E)$ extending its action on E. Then*

$$\kappa \approx \Lambda(E) \approx \Lambda(E^*) \qquad (E^* \textit{ is the complex conjugate of } E).$$

Proof. Clearly E^* is also isotropic and H-stable. $E\cap E^*=0$ as otherwise $E\cap E^*\cap W$ will be a non zero isotropic subspace of W. So $W_{\mathbb{C}}=E\oplus E^*$. We write τ' for the representation of $C(W)$ in $\Lambda(E)$ where

$$\tau'(u)(x)=u\wedge x,\qquad \tau'(v)(x)=\partial(v)(x)\qquad (u\in E, v\in E^*, x\in\Lambda(E)).$$

Here $\partial(v)$ is the odd derivation taking $x\in E$ to $2\varphi(x,v)$. It is then routine to show that

$$\eta(t)\tau'(u)\eta(t)^{-1}=\tau'(tu)\qquad (u\in E\cup E^*).$$

Now τ' is equivalent to τ and so we can transfer η to an action, written again as η, of H in the space of τ satisfying the above relation with respect to τ. It is not necessary that η be unitary. But we can normalize it to be a UR, namely $\kappa(t)=|\det(\eta(t))|^{-1/\dim(\tau)}\eta(t)$. ■

Remark 19 *It is easy to give an independent argument that $\Lambda(E)\approx\Lambda(E^*)$. For the unitary group $\mathrm{U}(E)$ of E let Λ_r be the representation on $\Lambda^r(E)$, and let Λ be their direct sum; then a simple calculation of the characters on the diagonal group shows that ${\Lambda_r}^*\simeq\det^{-1}\otimes\Lambda_{n-r}$. Hence $\Lambda^*\simeq\det^{-1}\otimes\Lambda$, showing that $\Lambda^*\approx\Lambda$. It is then immediate that this result remains true for any group which acts unitarily on E.*

Corollary 5 *The conditions of the above lemma are met if $W=A\oplus B$ where A,B are orthogonal submodules for H which are equivalent. Moreover*

$$\kappa\approx\Lambda(E)\simeq\Lambda(E^*)\simeq\Lambda(A)\simeq\Lambda(B).$$

Proof. Take ON bases $(a_j),(b_j)$ for A and B respectively so that the map $a_j\mapsto b_j$ is an isomorphism of H-modules. If E is the span of the $e_j=a_j+ib_j$, it is easy to check that E is isotropic, and is a module for H which is equivalent to A and B. ■

We now assume that for some $r\geq 3$ we have a map

$$H\longrightarrow \mathrm{Spin}(r)\longrightarrow\mathrm{Spin}(W)$$

where the first map is surjective, and H acts on W through $\mathrm{Spin}(W)$. Further let the representation of $\mathrm{Spin}(r)$ on W be spinorial. We write σ_r for the (complex) spin representation of $\mathrm{Spin}(r)$ if r is odd and $\sigma_r^{\pm}$ for the (complex) spin representations of $\mathrm{Spin}(r)$ if r is even. Likewise we write $s_r, s_r^{\pm}$ for the real irreducible spin modules. Note that $\dim(W)$ must be even.

Lemma 30 *Let the representation of* $\mathrm{Spin}(r)$ *on W be spinorial. Let n be the number of real irreducible constituents of W as a module for* $\mathrm{Spin}(r)$*, and, when r is even, let $n^{\pm}$ be the number of irreducible constituents of real or quaternionic type. We then have the following determination of κ.*

r mod 8	κ
$0(n^{\pm}$ *even*$)$	$\Lambda\big(((n^+/2)\sigma_r^+\oplus(n^-/2)\sigma_r^-)\big)$
$1,7(n$ *even*$)$	$\Lambda\big((n/2)\sigma_r\big)$
$2,6$	$\Lambda\big(n\sigma_r^+\big)\approx\Lambda\big(n\sigma_r^-\big)$
$3,5$	$\Lambda\big(n\sigma_r\big)$
4	$\Lambda\big(n^+\sigma_r^+\oplus n^-\sigma_r^-\big)$

Proof. This is a routine application of the the Lemma and Corollary above if we note the following facts.

$r \equiv 0$: Here $\sigma_r^{\pm} = s_r^{\pm}$ and $W = n^+ s_r^+ + n^- s_r^-$.

$r \equiv 1, 7$: Here $\sigma_r = s_r, W = ns_r$.

$r \equiv 2, 6$: Over $\mathbb{C}$, s_r becomes $\sigma_r^+ \oplus \sigma_r^-$ while $\sigma_r^{\pm}$ do not admit a non zero invariant form. So $W_{\mathbb{C}} = E \oplus E^*$ where $E = n\sigma_r^+, E^* = n\sigma_r^-$, and q is zero on E.

$r \equiv 3, 5$: s_r is quaternionic, $W = ns_r$, $W_{\mathbb{C}} = 2n\sigma_r$ and σ_r does not admit an invariant symmetric form.

$r \equiv 4$: $s_r^{\pm}$ are quaternionic and $\sigma_r^{\pm}$ do not admit a non zero invariant symmetric form; $W_{\mathbb{C}} = E \oplus E^*$, where $E = n^+\sigma_r^+ + n^-\sigma_r^-$ and q is zero on E.

In deriving these the reader should use the results in [Del99] and [Var04] on the reality of the complex spin modules and the theory of invariant forms for them. ■

Super Poincaré group associated to $\mathbb{R}^{1,3}$: N=1 supersymmetry

Here $T_0 = \mathbb{R}^{1,3}, L_0 = \mathrm{SL}(2, \mathbb{C})_{\mathbb{R}}$ where the suffix $\mathbb{R}$ means that the complex group is viewed as a real Lie group. Let $\mathfrak{s} = \mathbf{2} \oplus \overline{\mathbf{2}}$, $\mathbf{2}$ being the holomorphic representation of L_0 in $\mathbb{C}^2$ and $\overline{\mathbf{2}}$ its complex conjugate. Thus we identify $\mathfrak{s}$ with $\mathbb{C}^2 \oplus \mathbb{C}^2$ and introduce the conjugation on $\mathfrak{s}$ given by $\overline{(u, v)} = (\overline{v}, \overline{u})$. The action $(u, v) \mapsto (gu, \overline{g}v)$ of L_0 ($\overline{g}$ is the complex conjugate of g) commutes with the conjugation and so defines the real form $\mathfrak{s}_{\mathbb{R}}$ invariant under L_0 (*Majorana spinor*). We take $\mathfrak{t}_0$ to be the space of 2×2 Hermitian matrices and the action of L_0 on it as $g, A \mapsto gA\overline{g}^{\mathrm{T}}$. For $(u_i, \overline{u_i}) \in \mathfrak{s}_{\mathbb{R}} (i = 1, 2)$ we put

$$[(u_1, \overline{u_1}), (u_2, \overline{u_2})] = \frac{1}{2}(u_1 \overline{u_2}^{\mathrm{T}} + u_2 \overline{u_1}^{\mathrm{T}}).$$

Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{l}_0$, $\mathfrak{g}_1 = \mathfrak{s}_{\mathbb{R}}$ is a super Lie algebra and $(T_0 L_0, \mathfrak{g})$ is the SLG with which we are concerned.

Here $\mathbb{R}^{1,3} \simeq \mathfrak{t}_0$ by the map $a \mapsto h_a = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}$; $\mathfrak{t}_0 \simeq \mathfrak{t}_0^*$ with $p \in \mathfrak{t}_0$ viewed as the linear form $a \mapsto \langle a, p \rangle = a_0 p_0 - a_1 p_1 - a_2 p_2 - a_3 p_3$. Then

$$Q_p((u, \overline{u})) = \frac{1}{4} \overline{u}^{\mathrm{T}} h_{\check{p}} u, \qquad \check{p} = (p_0, -p_1, -p_2, -p_3).$$

I : $p_0 > 0$, $m^2 = \langle p, p \rangle > 0$. We take $p = mI$ so that $L_0^p = \mathrm{SU}(2)$. Take $E = \{(u, 0)\}, E^* = \{(0, \overline{u})\}$. Then we are in the set up of Lemma 29. Then

$$\kappa_p = \Lambda(E) \simeq 2D^0 \oplus D^{1/2}, \qquad D^j \otimes \Lambda(E) = \begin{cases} 2D^j \oplus D^{j+1/2} \oplus D^{j-1/2} & (j \geq 1/2) \\ 2D^0 \oplus D^{1/2} & (j = 0). \end{cases}$$

Thus the multiplet with mass m has the same mass m and spins

$$\begin{cases} \{j, j, j + 1/2, j - 1/2\} (j > 0) \\ \{0, 0, 1/2\} (j = 0) \end{cases}$$

II : $p_0 > 0$, $\langle p, p \rangle = 0$. Here we take $p = (1, 0, 0, -1), h_p = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. Then $L_0^p = \begin{pmatrix} a & 0 \\ c & \overline{a} \end{pmatrix}$. The characters $\chi_{n/2} : a \mapsto a^n$ ($n \in \mathbb{Z}$) are viewed as characters of L_0^p. Here $Q_p((u, \overline{u})) =$

$\frac{1}{4}\overline{u}^{\mathrm{T}} h_{\tilde{p}} u = |u_1|^2$. The radical of Φ_p is the span of (e_2, e_2) and $(ie_2, -ie_2)$, e_1, e_2 being the standard basis of $\mathbb{C}^2$. We identify $\mathfrak{g}_{1p}$ with the span of (e_1, e_1) and $(ie_1, -ie_1)$. We now apply Lemma 29 with $E = \mathbb{C}(e_1, 0)$ which carries the character defined by $\chi_{1/2}$; then

$$\Lambda(E) = \chi_0 \oplus \chi_{1/2}, \qquad \chi_{n/2} \otimes \Lambda(E) = \chi_{n/2} \oplus \chi_{(n+1)/2}.$$

The multiplet is $\{n/2, (n+1)/2\}$. These results go back to [SS74].

Extended supersymmetry

Here the SLG has still the Poincaré group as its even part but $\mathfrak{g}_1$ is the sum of $N > 1$ copies of $\mathfrak{s}_{\mathbb{R}}$. It is known ([Del99],[Var04])that one can identify $\mathfrak{g}_1$ with the direct sum $\mathfrak{s}_{\mathbb{R}}^N$ of N copies of $\mathfrak{s}_{\mathbb{R}}$ in such a way that for the odd commutators we have

$$[(s_1, s_2, \ldots, s_N), (s'_1, s'_2, \ldots, s'_N)] = \sum_{1 \le i \le N} [s_i, s'_i]^1,$$

so that

$$Q_\lambda((s_1, \ldots, s_N)) = \sum_{1 \le i \le N} Q^1_\lambda((s_i, s_i)).$$

Here the index 1 means the [,] and Q for the case $N = 1$ discussed above. Let $E^N = NE^1$.

I : $p_0 > 0, m^2 = \langle p, p\rangle > 0$. Then we apply Lemma 29 with $E = E^N$ so that $\kappa_p = \Lambda(ND^{1/2})$. The decomposition of the exterior algebra of $ND^{1/2}$ is tedious but there is no difficulty in principle. We have

$$\kappa_p = \sum_{0 \le r \le N} c_{Nr} D^{r/2} \quad c_{Nr} > 0, \quad c_{NN} = 1.$$

Then $j + N/2$ is the maximum value of r for which D^r occurs in $D^j \otimes \Lambda(ND^{1/2})$. The multiplet defined by the super particle of mass m is thus

$$\begin{cases} \{j - N/2, j - N/2 + 1/2, \ldots, j + N/2 - 1/2, j + N/2\} \ (j \ge N/2) \\ \{0, 1/2, \ldots, j + N/2 - 1/2, j + N/2\} \ (0 \le j < N/2) \end{cases}$$

II : $p_0 > 0, m = 0$. Here

$$\kappa_\lambda = \Lambda(N\chi_{1/2}) = \sum_{0 \le r \le N} \binom{N}{r} \chi_{r/2}.$$

The multiplet of the super particle has the helicity content

$$\{r/2, (r+1)/2, \ldots, (r+N)/2\}.$$

Super particles of infinite spin

The little groups for zero mass have irreducible UR's which are infinite dimensional. Since L^p_0 is also a semidirect product its irreducible UR's can be determined by the usual method. The orbits of S^1 in $\mathbb{C}$ (which is identified with its dual) are the circles $\{|a| = r\}$ for $r > 0$ and the stabilizers of the points are all the same, the group $\{\pm 1\}$. The irreducible UR's of infinite dimension can then be parametrized as $\{\alpha_{r,\pm}\}$. Now $\Lambda(E) = \chi_0 \oplus \chi_{1/2}$ and an easy calculation gives

$$\alpha_{r,\pm} \otimes \Lambda(E) = \alpha_{r,+} \oplus \alpha_{r,-}.$$

The particles in the multiplet with mass 0 corresponding to spin $(r, \pm)$ consist of both types of infinite spin with the same r.

Super Poincaré groups of Minkowski super spacetimes of arbitrary dimension

Let $T_0 = \mathbb{R}^{1,D-1}$. We first determine κ_p in the massive case. Here $L_0^p = \mathrm{Spin}(D-1)$ and the form Φ_p is strictly positive definite. So $\mathfrak{g}_{1p} = \mathfrak{g}_1$ and L_0^p acts on it by restriction, hence spinorially by Lemma 26. So Lemma 30 applies at once. It only remains to determine $n, n^\pm$ in terms of the corresponding $N, N^\pm$ for $\mathfrak{g}_1$ viewed as a module for L_0. Notation is as in Lemma 30, and res is restriction to L_0^p; $r = D-1$. This is done by writing $\mathfrak{g}_1$ as a sum of the s_D and determining the restrictions of the s_D to L_0^p by dimension counting. We again omit the details but refer the reader to [Del99],[Var04].

Proposition 47 *When $p_0 > 0, m^2 = \langle p,p\rangle > 0$, κ_p, the fundamental multiplet of the super particle of mass m, is given according to the following table:*

D *mod* 8	res (s_D)	κ_p
0	$2s_{D-1}$	$\Lambda\big(N\sigma_{D-1}\big)$
$1(N=2k)$	$s_{D-1}^+ + s_{D-1}^-$	$\Lambda\big(k\sigma_{D-1}^+ \oplus k\sigma_{D-1}^-\big)$
$2(N=2k)$	s_{D-1}	$\Lambda\big(k\sigma_{D-1}\big)$
3	s_{D-1}	$\Lambda\big(N\sigma_{D-1}^\pm\big)$
4	s_{D-1}	$\Lambda\big(N\sigma_{D-1}\big)$
5	$s_{D-1}^+ + s_{D-1}^-$	$\Lambda\big(N\sigma_{D-1}^+ \oplus N\sigma_{D-1}^-\big)$
6	s_{D-1}	$\Lambda\big(N\sigma_{D-1}\big)$
7	$2s_{D-1}$	$\Lambda\big((2N)\sigma_{D-1}^\pm\big)$

We now extend these results to the case when p has zero mass. Let $V = \mathbb{R}^{1,D-1}(D \geq 4)$ and $p \neq 0$ a null vector in V. Let e_j be the standard basis vectors for V so that $(e_0, e_0) = -(e_j, e_j) = 1$ for $j = 1, \ldots, D-1$. We may assume that $p = e_0 + e_1$. Let V_1' be the span of $e_j (2 \leq j \leq D-1)$. The signature of V_1' is $(0, D-2)$. Then we have the flag $0 \subset \mathbb{R}p \subset p^\perp \subset V$ left stable by the stabilizer L_0^p of p in L_0. The quadratic form on $p^\perp$ has $\mathbb{R}p$ as its radical and so induces a nondegenerate form on $V_1 := p^\perp/\mathbb{R}p$. Write $L_1 = \mathrm{Spin}(V_1)$. Note that $V_1' \simeq V_1$.

We have a map $x \mapsto x'$ from L_0^p to L_1 where, for $v \in p^\perp$ with image $v' \in V_1$, $x'v' = (xv)'$. It is known that this is surjective and its kernel $T_1 := T_0^p$ is isomorphic to V_1 canonically: for $x \in T_0^p$, the vector $xe_0 - e_0 \in p^\perp$, and the map that sends x to the image $t(x)$ of $xe_0 - e_0$ in V_1 is well defined and is an isomorphism of T_0^p with V_1. The map $x \mapsto (t(x), x')$ is an isomorphism of L_0^p with the semidirect product $V_1 \times' L_1$. The Lie algebra of the big spin group L_0 has the $e_r e_s (r < s)$ as basis and it is a simple calculation that the Lie algebra of L_0^p has as basis $t_j = (e_0 + e_1)e_j (2 \leq j \leq D-1), e_r e_s (2 \leq r < s \leq D-1)$ with the t_j forming a basis of the Lie algebra of T_1. The $e_r e_s (2 \leq r < s \leq D-1)$ span a Lie subalgebra of the Lie algebra of L_0 and the corresponding subgroup $H \subset L_0^p$ is such that $L_0^p \simeq T_1 \times' H$. For all of this see [Var04], pp. 36-37.

We shall now determine the structure of the restriction to L_0^p of the irreducible spin representation(s) of L, *over* $\mathbb{R}$ *as well as over* $\mathbb{C}$. Since this may not be known widely we give some details. We begin with some preliminary remarks.

Let U be any finite dimensional complex L_0^p-module. Write $U = \oplus_\chi U_\chi$ where U_χ, for any character (not necessarily unitary) χ of T_1, is the subspace of all elements $u \in U$ such that $(t - \chi(t))^m u = 0$ for sufficiently large m. The action of L_1 permutes the U_χ, and so, since L_1 has no finite nontrivial orbit in the space of characters of T_1, it follows that the spectrum of T_1 consists only of the trivial character, i.e., T_1 acts *unipotently*. In particular

U_1, the subspace of T_1-invariant elements of U, is $\neq 0$, an assertion which is then valid for real modules also. It follows that we have a strictly increasing filtration $(U_i)_{i\geq 1}$ where U_{i+1} is the preimage in U of $(U/U_i)_1$. In particular, if U is semisimple, $U = U_1$.

Lemma 31 *Let W be an irreducible real or complex spin module for L_0. Let W_1 be the subspace of all elements of W fixed by T_1 and $W^1 := W/W_1$. We then have the following.*

(i) $0 \neq W_1 \neq W$, W_1 is the unique proper non zero L_0^p-submodule of W, and T_1 acts trivially also on W^1.

(ii) W_1 and W^1 are both irreducible L_0^p-modules on which $L_0^p/T_1 \simeq L_1$ acts as a spin module.

(iii) The exact sequence

$$0 \longrightarrow W_1 \longrightarrow W \longrightarrow W/W_1 \longrightarrow 0$$

does not split.

(iv) Over $\mathbb{R}$, W, W_1, W^1 are all of the same type. If $\dim(V)$ is odd, $W_1 \simeq W^1$. Let $\dim(V)$ be even; then, over $\mathbb{C}$, W_1, W^1 are the two irreducible spin modules for L_1; over $\mathbb{R}$, $W_1 \simeq W^1$ when W is of complex type, namely, when $D \equiv 0, 4 \bmod 8$; otherwise, the modules W_1, W^1 are the two irreducible modules of L_1 (which are either real or quaternionic).

Proof. We first work over $\mathbb{C}$. Let C be the Clifford algebra of V. The key point is that $W_1 \neq W$. Suppose $W_1 = W$. Then $t_j = 0$ on W for all j. If D is odd, C^+ is a full matrix algebra and so all of its modules are faithful, giving a contradiction. Let D be even and W one of the spin modules for L_0. We know that inner automorphism by the invertible odd element e_2 changes W to the other spin module. But as $e_2 t_j e_2^{-1} = t_j$ for $j > 2$ and $-t_2$ for $j = 2$, it follows that $t_j = 0$ on the other spin module also. Hence $t_j = 0$ in the irreducible module for the full Clifford algebra C. Now C is isomorphic to a full matrix algebra and so its modules are faithful, giving again $t_j \neq 0$, a contradiction.

Let (W_i) be the strictly increasing flag of L_0^p-modules, with T_1 acting trivially on each W_{i+1}/W_i, defined by the previous discussion. Let m be such that $W_m = W$. Clearly $m \geq 2$. On the other hand, the element -1 of L_0 lies in L_0^p and as it acts as -1 on W, it acts as -1 on all the W_{i+1}/W_i. Hence $\dim(W_{i+1}/W_i) \geq \dim(\sigma_{D-2})$ (see Lemma 6.8.1 of [Var04]), and there is equality if and only if $W_{i+1}/W_i \simeq \sigma_{D-2}$. Since $\dim(\sigma_D) = 2\dim(\sigma_{D-2})$, we see at once that $m = 2$ and that both W_1 and W/W_1 are irreducible L_0^p-modules which are spin modules for L_1. The exact sequence in (iii) cannot split, as otherwise T_1 will be trivial on all of W. Suppose now U is a non zero proper L_0^p-submodule of W. Then $\dim(U) = \dim(W_1)$ for the same dimensional argument as above, and so U is irreducible, thus T_1 is trivial on it, showing that $U = W_1$. We have thus proved (i)-(iii).

We now prove (iv). There is nothing to prove when D is odd since there is only one spin module. Suppose D is even. Let us again write $\sigma_{D-2}^{\pm}$ for the L_0^p-modules obtained by lifting the irreducible spin modules of L_1 to L_0^p. We consider two cases.

$D \equiv 0, 4 \bmod 8$: In this case $\sigma_D^{\pm}$ are self dual while $\sigma_{D-2}^{\pm}$ are dual to each other. It is not restrictive to assume $W = \sigma_D^+$ and $W_1 = \sigma_{D-2}^+$. We have the quotient map $W = \sigma_D^+ \longrightarrow W^1 = \sigma$; and σ is to be determined. Writing σ' for the dual of σ, we get $\sigma' \subset (\sigma_D^+)' \simeq \sigma_D^+$, so that, by the uniqueness of the submodule proved above, $\sigma' = \sigma_{D-2}^+$. Hence $\sigma = \sigma_{D-2}^-$.

$D \equiv 2, 6 \bmod 8$: Now $\sigma_D^{\pm}$ are dual to each other while $\sigma_{D-2}^{\pm}$ are self dual. Again, suppose $W = \sigma_D^+$ and $W_1 = \sigma_{D-2}^+$. The above argument then gives $\sigma' \subset \sigma_D^-$. On the other hand,

the inner automorphism by e_2 transforms σ_D^+ into σ_D^-, and the subspace of σ_D^+ fixed by T_1 into the corresponding subspace of σ_D^-, while at the same time changing σ_{D-2}^+ into σ_{D-2}^-. Hence it changes the inclusion $\sigma_{D-2}^+ \subset \sigma_D^+$ into the inclusion $\sigma_{D-2}^- \subset \sigma_D^-$. Hence we have $\sigma' = \sigma_{D-2}^-$. Dualizing, this gives $\sigma = \sigma_{D-2}^-$ once again. This finishes the proof of the lemma over $\mathbb{C}$.

We now work over $\mathbb{R}$. Since both V and V_1 have the same signature $D-2$, it is immediate that the real spin modules for L_0 and L_1 are of the same type. As an L_0-module, the complexification $W_\mathbb{C}$ of W is either irreducible or is a direct sum $U \oplus \overline{U}$ where U is a complex spin module for L_0. In the first case W is of real type and the lemma follows from the lemma for the complex spin modules. In the second case W is of quaternionic or complex type according as U and $\overline{U}$ are equivalent or not.

Complex type : We have $W_\mathbb{C} = U \oplus Z$ where $Z = \overline{U}$ is the complex conjugate of U. We have $U \simeq \sigma_D^+, Z \simeq \sigma_D^-$. Since $\mathbb{C}W_1 = U_1 \oplus Z_1$ it is clear that $0 \neq W_1 \neq W$. The real irreducible spin modules of L_1 have dimension $2^{D/2-1}$ and so we find that $\dim(W_1) = 2^{D/2-1}$ and W_1, W^1 are both irreducible; they are equivalent as they are of complex type. A non zero proper submodule R of W then has dimension $2^{D/2-1}$ and so must be irreducible. Hence either $R = W_1$ or $W = W_1 \oplus R$. But then $W = W_1$, a contradiction. The same argument shows that the exact sequence in the lemma does not split.

Quaternionic type : For proving (i)-(iii) of the lemma the argument is the same as in the complex type, except that $Z \simeq U$. We now check (iv). The case of odd dimension is obvious. So let D be even and $W_1 \simeq s_{D-2}^+$. Then $\mathbb{C}W_1 \simeq 2\sigma_{D-2}^+$ so that $U_1 \simeq Z_1 \simeq \sigma_{D-2}^+$. Then $U^1 \simeq Z^1 \simeq \sigma_{D-2}^-$ and so $\mathbb{C}W^1 \simeq 2\sigma_{D-2}^-$. Hence $W^1 \simeq s_{D-2}^-$.

The lemma is completely proved. ■

Remark 20 *Since we are interested in the quotient W^1 rather than W_1 below, we change our convention slightly; for $W = s_D^\pm$ we write $W^1 = s_{D-2}^\pm$ and $W_1 = s_{D-2}^\mp$.*

We now come to the discussion of the structure of κ_p when p is in a massless orbit.

Proposition 48 *Let $p_0 > 0, \langle p, p\rangle = 0$. Then* rad $(\Phi_p) = \mathfrak{g}_1^{T_1}$, *the subspace of elements of $\mathfrak{g}_1$ fixed by T_1. Moreover T_1 acts trivially on $\mathfrak{g}_{1p}$,* rad $(\Phi_p) \simeq \mathfrak{g}_{1p}$ *except in the cases $D \equiv 2, 6 \bmod 8$ when* rad (Φ_p) *and $\mathfrak{g}_{1p}$ are dual to each other, and $L_0^p/T_1 \simeq L_1$ acts spinorially on* rad Φ_p *and $\mathfrak{g}_{1p}$. In all cases* $\dim(\text{rad }(\Phi_p)) = \dim(\mathfrak{g}_{1p}) = (1/2)\dim(\mathfrak{g}_1)$. *For $\mathfrak{g}_{1p}$ as well as the associated κ_p the results are as in the following table.*

D mod 8	$\mathfrak{g}_{1p}$	κ_p
$0, 4$	Ns_{D-2}	$\Lambda(N\sigma_{D-2}^\pm)$
$2 (N^\pm = 2n^\pm)$	$N^+s_{D-2}^+ + N^-s_{D-2}^-$	$\Lambda(n^+\sigma_{D-2}^+ + n^-\sigma_{D-2}^-)$
6	$N^+s_{D-2}^+ + N^-s_{D-2}^-$	$\Lambda(N^+\sigma_{D-2}^+ + N^-\sigma_{D-2}^-)$
$1, 3 (N = 2n)$	Ns_{D-2}	$\Lambda(n\sigma_{D-2})$
$5, 7$	Ns_{D-2}	$\Lambda(N\sigma_{D-2})$

Proof. We have $\mathfrak{g}_1 = \oplus_{1\le i\le N}\mathfrak{h}_i$ where the $\mathfrak{h}_i$ are real irreducible spin modules and $[\mathfrak{h}_i, \mathfrak{h}_j] = 0$ for $i \neq j$ while $\langle [X, X], q\rangle > 0$ for all $q \in \Gamma^+$. Let $\mathfrak{r}_p =$ rad Φ_p and $\mathfrak{r}_{ip}$ the radical of the restriction of Φ_p to $\mathfrak{h}_i$. Since $Q_p(X) = \sum_i Q_p(X_i)$ where X_i is the component of X in $\mathfrak{h}_i$, it follows that $\mathfrak{r}_p = \oplus_i \mathfrak{r}_{ip}$. We now claim that $\mathfrak{r}_{ip} = (\mathfrak{h}_i)_1$, namely, the subspace of elements of $\mathfrak{h}_i$ fixed by T_1. Since $\mathfrak{r}_{ip}$ is a L_0^p-submodule it suffices, in view of the lemma above, to show that $0 \neq \mathfrak{r}_{ip} \neq \mathfrak{h}_i$. If $\mathfrak{r}_{ip}$ were 0, Φ_p would be strictly positive definite on $\mathfrak{h}_i$, and hence the action of L_0^p will have an invariant positive definite quadratic form. So the action of L_0^p on $\mathfrak{h}_i$ will be semisimple, implying that T_1 will act trivially on $\mathfrak{h}_i$. This is impossible, since, by the preceding lemma, $\mathfrak{h}_i \neq (\mathfrak{h}_i)_1$. If $\mathfrak{r}_{ip} = \mathfrak{h}_i$, then $\Phi_p = 0$ on $\mathfrak{h}_i$, and this will imply that $p = 0$. Thus $\mathfrak{r}_{ip} = (\mathfrak{h}_i)_1$, hence $\mathfrak{r}_p = (\mathfrak{g}_1)_1$. The other assertions except the table are now clear. For the table we need to observe that $\mathfrak{g}_{1p} = \oplus_i \mathfrak{h}_i/\mathfrak{r}_{ip}$ and that $\mathfrak{r}_{ip} \simeq \mathfrak{h}_i/\mathfrak{r}_{ip}$ except when $D \equiv 2, 6 \bmod 8$; in these cases, the two modules are the two real or quaternionic spin modules which are dual to each other. The table is worked out in a similar manner to Proposition 47. We omit the details. ■

Remark 21 *The result that the dimension of $\mathfrak{g}_{1p}$ has $1/2$ the dimension of $\mathfrak{g}_1$ extends the known calculations when $D = 4$ (see [FSZ81]).*

The role of the R-group in classifying the states of κ_p

In the case of N-extended supersymmetry we have two groups acting on $\mathfrak{g}_1$: L_0^p, the even part of the little super group at p, and the R-group ([Del99],[Var04]) R. Their actions commute and they both leave the quadratic form Q_p invariant. In the massive case we have a map

$$L_0^p \times R \longrightarrow \mathrm{Spin}(\mathfrak{g}_{1p})$$

so that one can speak of the restriction κ_p' of the spin representation of $\mathrm{Spin}(\mathfrak{g}_{1p})$ to $L_0^p \times R$. The same is true in the massless case except we have to replace L_0^p by a two-fold cover of it. It is thus desirable to not just determine κ_p as we have done but actually determine this representation κ_p' of $L_0^p \times R$. We have not done this but there is no difficulty in principle. However, when $D = 4$, we have a beautiful formula [FSZ81]. To describe this, assume that we are in the massive case. We first remark that $\mathfrak{g}_1 \simeq \mathbb{H}^N \otimes_{\mathbb{H}} S_0$ where S_0 is the quaternionic irreducible of $\mathrm{SU}(2)$ of dimension 4. Thus the R-group is the unitary group $\mathrm{U}(N, \mathbb{H})$. Over $\mathbb{C}$ we thus have $\mathfrak{g}_{1\mathbb{C}} \simeq \mathbb{C}^{2N} \otimes \mathbb{C}^2$ where the R-group is the symplectic group $\mathrm{Sp}(2N, \mathbb{C})$ acting on the first factor and $L_0^p \simeq \mathrm{SU}(2)$ acts as $D^{1/2}$ on the second factor. The irreducible representations of $\mathrm{Sp}(2N, \mathbb{C}) \times \mathrm{SU}(2)$ are outer tensor products of irreducibles a of the first factor and b of the second factor, written as (a, b). Let $\mathbf{k}$ denote the irreducible of dimension k of $\mathrm{SU}(2)$ and $[2N]_k$ denote the irreducible representation of the symplectic group in the space of traceless antisymmetric tensors of rank k over $\mathbb{C}^{2N}$; by convention for $k = 0$ this is the trivial representation and for $k = 1$ it is the vector representation. Then

$$\kappa_p' \simeq ([2N]_0, \mathbf{N+1}) + ([2N]_1, \mathbf{N}) + \ldots ([2N]_k, \mathbf{N+1-k}) + \ldots ([2N]_N, \mathbf{1}).$$

To see how this follows from our theory note that $\mathbb{C}^{2N} \otimes e_1$ is a subspace satisfying the conditions of Lemma 29 for the symplectic group and so $\kappa_p \simeq \Lambda(\mathbb{C}^{2N})$. It is known that

$$\Lambda(\mathbb{C}^{2N}) \simeq \sum_{0 \leq k \leq N} (N+1-k)[2N]_k.$$

On the other hand we know that the representations of $\mathrm{SU}(2)$ in κ_p are precisely the $\mathbf{N+1-k}(0 \leq k \leq N)$. The formula for κ_p' is now immediate. In the massless case the R-group becomes $\mathrm{U}(N)$ and

$$\kappa_p' \simeq \sum_{0 \leq k \leq N} ((N-k)/2, [N]_k)$$

where $r/2$ denotes the character denoted earlier by χ_r and $[N]_k$ is the irreducible representation of $\mathrm{U}(N)$ defined on the space $\Lambda^k(\mathbb{C}^N)$. We omit the proof which is similar.

Conclusions and further developments

E' l'intrecio terminato
lieto fine ha il dramma mio
e contento qual son io
forse il pubblico sarà

Il poeta
Il turco in Italia

This thesis is centered on the representations of super Lie groups. A unitary representation of a (usual) Lie group G can be defined as a continuous action of G on a Hilbert s pace V. For a super Lie group G, the definition cannot be that simple. It is necessary to take into account the Grassmanian (anticommuting) part of G, other than the fact that V has to be a super Hilbert space. This introduces essential complications that have made difficult and retarded the birth of the theory of representations of super Lie groups, contrary to the case of super Lie algebras. Among the few contributions to this problem, the ones of Kostant ([Kos77]) and Dobrev ([DP87]) have to be mentioned. The point of view of the former has not been pursued, until now. The work of Dobrev relies on the results of Berezin, and, although very interesting, it lacks of the mathematical precision that we feel it is necessary. Our point of attack to the theory of representations of super Lie groups is that the category of super Lie groups is equivalent to the category of super Harish-Chandra pairs. A super Harish-Chandra pair is made by an ordinary Lie group and a super Lie algebra; moreover, these two objects fulfill some compatibility relations among them. The equivalence among the categories means (roughly speaking) that a super Lie group can be replaced by the corresponding super Harish-Chandra pair, and vice versa. Assuming this point of view, a representation of a super Lie group, in a super Hilbert space, can be defined as a representation of the corresponding super Harish-Chandra pair. Quite obviously, a representation of a super Harish-Chandra pair is a pair consisting of a representation of the ordinary Lie group and a representation of the super Lie algebra. Moreover, these two representations are linked together by compatibility relations. This is one of the possible ways for taking into account of the anticommuting part of the super group. The problem is that the representation of the super algebra brings into play unbounded operators that have to be controlled with great care. This issue has been solved in the first part of [CCTV06] where the theory of representations of super Harish-Chandra pairs is exposed. This allows to deal with the representations of the super Poincare' group, which is a super semi direct group. The classical theory of representations of semi direct products is based on the use of the induced representations and the imprimitivity theorem. Hence, it is necessary to extend these constructions to super Lie groups. When we define an induced representation, we have a super Lie group G and its subgroup H. In general, G/H is a super differentiable manifold. On the contrary, in dealing with representations of super semi direct products, it turns out that we have only to face cases G/H is a classical manifold. We say, in this case, that H is a special subgroup. The fact that it is enough considering only representations induced by special subgroups is a simplification of the problem of classifying the representations of the super Poincare' group (in general, of a super semi direct product). In the same way, it has been enough proving the imprimitivity theorem only for special subgroups. In (12) we have been able to work out completely the theory of representations of super semi direct products, which has been applied to the super Poincare' group. In a subsequent paper (13), we have extended to the supersymmetric case the classical theory of extensions of normal abelian subgroups. In this case, we added the hypothesis that the subgroup is special. We would like to stress that the idea of using the super Harish-Chandra pairs to study the representations is quite original, and it allowed for the first time the complete classification of the unitary representations (in super Hilbert spaces) of a non trivial family of super Lie groups, namely the super semi direct products. A fundamental step toward the construction of a complete theory of representations of super Lie groups, is extending the theory of induced representations beyond the case of special subgroups.

For this, it could be useful also the approach developed in chapter 6. There we give a rigorous basis to a construction that first appeared in Berezin's work [Ber87], and we clarify the relationship between this method and another one used in modern algebraic geometry (the functor of points approach (see [DM99a])). Let us now briefly explain what we obtained in chapter 6, and the reasons why we are so interested in this approach. It is well known (see ([Var04])) that a super Lie algebra $\mathfrak{g}$ can also be viewed as an infinite family $\{g(\Lambda)\}$ of purely even Lie algebras, indexed by Grassmann algebras, and compatible with respect to Grassmann algebras transformations. The precise statement of the above sentence is that a super Lie algebra can also be viewed as a functor from the category of Grassmann algebras to the category of ordinary Lie algebras. In Physics literature this construction is known under the name of "even rules principle". Even if it is a quite difficult fact to be proved, it is natural to expect that such a result holds in the context of super Lie groups too. In other words, we expect that a super Lie group G can be replaced by an infinite "compatible" family $\{G(\Lambda)\}$ of ordinary Lie groups, indexed by Grassmann algebras and such that $\mathrm{Lie}(G(\Lambda)) = \mathfrak{g}(\Lambda)$. Such results were proved by generalizing the theory of A-near points (due to A. Weil) to the super setting. We believe that this approach presents the following advantages:

a) it allows to view G as an ordinary group with matrix entries taking values in Grassmann algebras, if G is a matrix supergroup . This approach is the one commonly used in Theoretical Physics ([DP87]) and ([CDF91]);

b) it establishes a formal link between the theory of super Lie groups and modern algebraic geometry. This can be useful in order to understand further developments of the theory;

c) it allows to reduce problems and definitions in the category of super Lie groups to equivalent (but hopefully simpler) problems and definitions in the category of ordinary Lie groups. This is exactly what is usually done in the theory of super Lie algebras.

Reamark c) is the basis of the approach we plan to follow in order to develop a general theory of induced representations. Let us briefly explain this point in more detail. Suppose r a representation of the super Lie group G in the super vector space H. It is very easy to construct a family $\{r(\Lambda)\}$ of representations such that each $r(\Lambda)$ is an ordinary representation of the Lie group $G(\Lambda)$ in the vector space $H(\Lambda)$. This approach parallels quite a lot the way in which representations are defined in modern algebraic groups theory. It is natural to expect that an induced representation can be defined in terms of a compatible family $\{r(\Lambda)\}$ of ordinary Lie groups representations. Actually, there are some subtleties that seem to arise at this point. There are clues that further regularity properties have to be imposed on the family $\{r(\Lambda)\}$ in order to ensure that it arises from a representation r of G, in the way described above. Such regularity properties should be related to a rigorous formulation of what Berezin calls "Grassmann analytic continuation". This part of the program is probably the most difficult one. Nevertheless, it should make us able to treat in a rigorous way and in full generality the representation theory of supergroups. In order to obtain a reasonably large amount of irreducible representations of the conformal supergroup $SU(2,2\,|\,N)$, it is reasonable to proceed in analogy with the classical case. Hence, the first basic step consists in obtaining a super version of the Iwasawa decomposition for the conformal supergroup. Nevertheless, such a generalization presents some pitfalls. In fact, in [Ser83] it is proved that there are no compact real forms of complex semisimple Lie superalgebras. Due to the importance of the compact factor in the ordinary Iwasawa decomposition there has been until now no satisfying attempt to develop this part of the theory. In spite of this, some recent works ([Pel05]) have shown that this drawback can be overcome by working with the family $\{G(\Lambda)\}$. Once the Iwasawa decomposition of $SU(2,2\,|\,N)$ is obtained, it is natural to expect that it is possible to construct a large amount of representations of the conformal supergroup using the theory of induced representations exactly as in the non super setting.

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Grazie, grazie; che piacer!
Che girandola ho nel cor.
Si venga a scrivere
Quel che dettiamo...

Don Magnifico
La Cenerentola

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Appendix A

The isomorphism $\tilde{\lambda}: E(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1) \longrightarrow E(\mathfrak{g})$

We define the map

$$\begin{aligned} \lambda: \Lambda(\mathfrak{g}_1) &\longrightarrow E(\mathfrak{g}) \\ Z_1 \wedge ... \wedge Z_k &\longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\sigma} Z_{\sigma(1)} \cdot ... \cdot Z_{\sigma(k)} \end{aligned}$$

and notice that $E(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1)$ is a filtered vector space, whose filtration is that induced by $\Lambda(\mathfrak{g}_1)$. On the other hand, we consider the filtration on $E(\mathfrak{g})$ defined by (let $q = \dim \mathfrak{g}_1$):

$$\begin{aligned} E(\mathfrak{g})^{(0)} &= E(\mathfrak{g}_0) \\ E(\mathfrak{g})^{(1)} &= E(\mathfrak{g}_0)(1 + \mathfrak{g}_1) \\ &... \\ E(\mathfrak{g})^{(q)} &= E(\mathfrak{g}_0)(1 + \mathfrak{g}_1 + ... + \mathfrak{g}_1^q) = E(\mathfrak{g}) \end{aligned}$$

Lemma 32 *The map*

$$\begin{aligned} \tilde{\lambda}: E(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1) &\longrightarrow E(\mathfrak{g}) \\ X \otimes \omega &\longmapsto X\lambda(\omega) \end{aligned}$$

is an isomorphism of filtered vector spaces.

Proof. The proof is by induction. Clearly

$$\tilde{\lambda}\left([E(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1)]^{(0)}\right) = E(\mathfrak{g}_0)^{(0)}$$

Suppose now $n \leq q$ and $\tilde{\lambda}\left([E(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1)]^{(n-1)}\right) = E(\mathfrak{g})^{(n-1)}$ It follows from graded PBW theorem that if $Z_n \in E(\mathfrak{g})^{(n)}$, then there exist $w_n \in E(\mathfrak{g}_0) \otimes \Lambda^n(\mathfrak{g}_1)$ and $Z_{n-1} \in E(\mathfrak{g})^{(n-1)}$ such that $Z_n = \tilde{\lambda}(w_n) + Z_{n-1}$ In fact it is easy to check that

$$\lambda(Z_1 \wedge ... \wedge Z_n) = Z_1 \cdot ... \cdot Z_n \left(\mod E(\mathfrak{g})^{(n-1)}\right)$$

and, due to PBW, this gives

$$\tilde{\lambda}\left(\left[E\left(\mathfrak{g}_0\right)\otimes\Lambda^n\left(\mathfrak{g}_1\right)\right]^{(0)}\right) \quad = \quad E\left(\mathfrak{g}\right)^{(n)}\left(\text{ mod } E\left(\mathfrak{g}\right)^{(n-1)}\right)$$

It is then enough to apply the inductive hypothesis. ■

Before coming to the proof of the formula we need, we fix some notation. In general we denote with $(i_1,...,i_k)$ an *ordered* set of indexes. We say that $(i_1,...,i_k)$ is *increasing* if $i_1 < ... < i_k$. If $J=(i_1,...,i_r)\subseteq\{1,...,k\}$, we denote with $\overline{J}$ the increasing set of indexes $\{1,...,k\}\backslash J$. If $J=(i_1,...,i_r)\subseteq\{1,...,k\}$ is increasing and $\overline{J}=(j_1,...,j_{k-r})$, we denote with $\left(J,\overline{J}\right)$ the ordered set $(i_1,...,i_r,j_1,...,j_{k-r})$. If $Z_1...Z_k$ are vectors in $\mathfrak{g}_1$ and $J=(i_1,...,i_r)$, we define $Z_J = Z_{i_1}\wedge...\wedge Z_{i_r}$. Finally

$$(1...k) \quad\longrightarrow\quad \left(J,\overline{J}\right)$$

is the permutation in S_k bringing $(1,...,k)$ in $\left(J,\overline{J}\right)$.

Proposition 49 *With the above notations, the following formula holds:*

$$\Delta\left(\lambda\left(Z_1\wedge...\wedge Z_k\right)\right) \quad = \quad \sum_{r=0}^{k}\sum_{\substack{J\subseteq\{1...k\}\\ incr.\ ord.\\ |J|=r}}(-1)^{p\left((1...k)\to\left(J,\overline{J}\right)\right)}\lambda\left(Z_J\right)\otimes\lambda\left(Z_{\overline{J}}\right)$$

where $Z_{\sigma(I)}=Z_{\sigma(i_1)}...Z_{\sigma(i_r)}$ *with* $I=(i_1,...,i_r)$.

Fixe an increasing set $I\subseteq\{1...k\}$. Given $\sigma\in S_k$ there exists a unique increasing set $J\subseteq\{1...k\}$, with $|J|=|I|$ and there exists a unique permutation $\sigma_{I,J}$ in the subgroup $(S_k)_J\subset S_k$ fixing the set J but not its ordering, such that

$$\sigma(I)=\sigma_{I,J}\left(J\right) \qquad \sigma\left(\overline{I}\right)=\sigma_{I,J}\left(\overline{J}\right)$$

(as ordered sets).
Viceversa, if $J\subseteq\{1...k\}$ is increasing and $|J|=|I|$, given $\sigma\in(S_k)_J$ we define $\sigma^{J,I}\in S_k$ as

$$\sigma^{J,I}(I):=\sigma\left(J\right) \qquad \sigma^{J,I}\left(\overline{I}\right):=\sigma\left(\overline{J}\right) \quad \text{(as ordered sets)}$$

If we denote with σ_J and $\sigma_{\overline{J}}$ the restrictions of σ to permutations of J and $\overline{J}$ respectively, then

$$p\left(\sigma^{J,I}\right)=p\left(\left(I,\overline{I}\right)\to\left(J,\overline{J}\right)\right)+p(\sigma) \quad \text{and} \quad p\left(\sigma\right)=p\left(\sigma_J\right)+p\left(\sigma_{\overline{J}}\right)$$

Fix an increasing set $I\subseteq\{1...k\}$, then

$$\begin{aligned}\sum_{\sigma\in S_k}(-1)^{\sigma}Z_{\sigma(I)}\otimes Z_{\sigma(\overline{I})} \quad &= \quad \sum_{\substack{J\subseteq\{1...k\}\\ \text{incr. ord.}\\ |J|=|I|}}\sum_{\sigma\in(S_k)_J}(-1)^{\sigma^{J,I}}Z_{\sigma^{J,I}(I)}\otimes Z_{\sigma^{J,I}(\overline{I})}\\ &= \quad \sum_{\substack{J\subseteq\{1...k\}\\ \text{incr. ord.}\\ |J|=|I|}}(-1)^{p\left(\left(I,\overline{I}\right)\to\left(J,\overline{J}\right)\right)}\sum_{\sigma\in(S_k)_J}(-1)^{\sigma}Z_{\sigma(J)}\otimes Z_{\sigma(\overline{J})}\end{aligned}$$

Denoting with S_J the group of permutations of the set J, we have

$$\begin{aligned}\sum_{\sigma\in(S_k)_J}(-1)^{\sigma}Z_{\sigma(J)}\otimes Z_{\sigma(\overline{J})} &= \sum_{\sigma'\in S_J}\sum_{\sigma''\in S_{\overline{J}}}(-1)^{\sigma'+\sigma''}Z_{\sigma'(J)}\otimes Z_{\sigma''(\overline{J})}\\ &= |J|!\left|\overline{J}\right|!\lambda\left(Z_J\right)\otimes\lambda\left(Z_{\overline{J}}\right)\end{aligned}$$

Collecting these facts

$$\begin{aligned}\Delta\left(\lambda\left(Z_1\wedge...\wedge Z_k\right)\right) &= \frac{1}{k!}\sum_{\sigma\in S_k}(-1)^{\sigma}\left(Z_{\sigma(1)}\otimes 1+1\otimes Z_{\sigma(1)}\right)...\left(Z_{\sigma(k)}\otimes 1+1\otimes Z_{\sigma(k)}\right)\\ &= \frac{1}{k!}\sum_{\substack{I\subseteq\{1...k\}\\ \text{incr. ord.}}}(-1)^{p\left((1...k)\to\left(I,\overline{I}\right)\right)}\sum_{\sigma\in S_k}(-1)^{\sigma}Z_{\sigma(I)}\otimes Z_{\sigma(\overline{I})}\\ &= \frac{1}{k!}\sum_{\substack{I\subseteq\{1...k\}\\ \text{incr. ord.}}}(-1)^{p\left((1...k)\to\left(I,\overline{I}\right)\right)}\times\\ &\quad\sum_{\substack{J\subseteq\{1...k\}\\ \text{incr. ord.}\\ |J|=|I|}}\times(-1)^{p\left(\left(I,\overline{I}\right)\to\left(J,\overline{J}\right)\right)}\sum_{\sigma\in(S_k)_J}(-1)^{\sigma}Z_{\sigma(J)}\otimes Z_{\sigma(\overline{J})}\\ &= \frac{1}{k!}\sum_{\substack{I\subseteq\{1...k\}\\ \text{incr. ord.}}}\sum_{\substack{J\subseteq\{1...k\}\\ \text{incr. ord.}\\ |J|=|I|}}(-1)^{p\left((1...k)\to\left(J,\overline{J}\right)\right)}|J|!\,|J|!\lambda\left(Z_J\right)\otimes\lambda\left(Z_{\overline{J}}\right)\\ &= \frac{1}{k!}\sum_{r=0}^{k}\begin{pmatrix}k\\ r\end{pmatrix}\sum_{\substack{J\subseteq\{1...k\}\\ \text{incr. ord.}\\ |J|=r}}(-1)^{p\left((1...k)\to\left(J,\overline{J}\right)\right)}r!(k-r)!\lambda\left(Z_J\right)\otimes\lambda\left(Z_{\overline{J}}\right)\\ &= \sum_{r=0}^{k}\sum_{\substack{J\subseteq\{1...k\}\\ \text{incr. ord.}\\ |J|=r}}(-1)^{p\left((1...k)\to\left(J,\overline{J}\right)\right)}\lambda\left(Z_J\right)\otimes\lambda\left(Z_{\overline{J}}\right)\end{aligned}$$

Appendix B

A theorem on classical manifolds

In this appendix we state a theorem that allows to endow a given set with a differentiable manifold structure.

Theorem 11 *Let M be a set such that*

i) there exists a countable cover $\{U_i\}_{i\in\mathbb{N}}$

ii) for each U_i there exists an injective map

$$h_i : U_i \longrightarrow \mathbb{R}^n$$

such that

a) $h_i(U_i \cap U_j) \subseteq \mathbb{R}^n$ is open $\forall i, j$

b) the map

$$h_i h_j^{-1} : h_j(U_i \cap U_j) \longrightarrow h_i(U_i \cap U_j)$$

is a diffeomorphism.

c) for each x and y in M $x \neq y$, there exist $V_1 \subseteq U_i$ and $V_2 \subseteq U_j$ such that $x \in V_1$, $y \in V_2$, $V_1 \cap V_2 = \emptyset$, $h_i(V_1)$ and $h_j(V_2)$ are open

then there exists a unique manifold structure on M such that $\{U_i, h_i\}$ is a differentiable atlas for M.

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