On the Notion of Perfect Bayesian Equilibrium*

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Abstract

Often, perfect Bayesian equilibrium is loosely defined by stating that players should be sequentially rational given some beliefs in which Bayes rule is applied "whenever possible". We argue that there are situations in which it is not clear what "whenever possible" means. Then, we provide a simple definition of perfect Bayesian equilibrium for general extensive games that refines both weak perfect Bayesian equilibrium and subgame perfect equilibrium.

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1 Introduction

Perfect Bayesian equilibrium is profusely used to analyze the game theoretical models that are derived from a wide variety of economic situations. The common understanding is that a perfect Bayesian equilibrium must be sequentially rational given the beliefs of the players, which have to be computed using Bayes rule "whenever possible". However, the literature lacks a formal and tractable definition of this equilibrium concept that applies to general extensive games and, hence, it is typically the case that perfect Bayesian equilibrium is used without providing a definition beyond the above common understanding.

When game theory started to analyze models with imperfect information, there was a need to refine the classic concepts of Nash equilibrium and subgame perfect equilibrium. Weak perfect Bayesian equilibrium can be regarded as the first step in this direction (though it does not even refine subgame perfection). This equilibrium concept was introduced by Myerson (1991) when preparing the ground for the definition of sequential equilibrium.[1]

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[1] Originally, Myerson called this equilibrium concept weak sequential equilibrium.

Also, Mas-Colell et al. (1995) follow this approach and introduce weak perfect Bayesian equilibrium as a bridge between the classic equilibrium concepts and the belief-based ones. Informally, a strategy profile is a weak perfect Bayesian equilibrium when it is sequentially rational given a system of beliefs that is consistent with Bayes rule on the path of the strategy (no restriction is imposed on the beliefs at information sets that are off-path). Though pedagogically useful, weak perfect Bayesian equilibrium has many shortcomings as an equilibrium concept; remarkably, it does not even imply subgame perfection. Thus, this equilibrium concept also has to be refined.

Sequential equilibrium is probably the most widely used equilibrium concept for games with imperfect information and, yet, there are classes of games where less demanding equilibrium concepts that are easier to handle select reasonable strategy profiles. Even setting aside the fact that it is often hard to deal with sequential equilibrium, there are natural economic settings in which it cannot even be defined. The notion of consistent beliefs cannot be (trivially) extended to games in which the players have a continuum of strategies; the models of auctions being the most outstanding example of the latter type of games.

Despite the common practice of using perfect Bayesian equilibrium without providing a formal definition, there have also been some exceptions. To the best of our knowledge, the first paper introducing a formal definition is Harris and Townsend (1981), in the context of mechanism design.[2] A more elaborate approach is taken in Fudenberg and Tirole (1991) for multistage games with observed actions. They impose some restrictions on how off-path beliefs can be formed and present a definition of perfect Bayesian equilibrium that is natural within the class of games to which their analysis is confined and, moreover, it is easier to study than sequential equilibrium. Yet, these definitions of perfect Bayesian equilibrium cannot be easily extended to general extensive games. Finally, Battigalli (1996) analyzes some natural restrictions on how off-path beliefs should be computed in general extensive games and derives several refinements of subgame perfection which, as he says, may be generically called perfect Bayesian equilibria. The main idea underlying these refinements is what Battigalli called strategic independence: when forming beliefs, the strategic choices of different players should be regarded as independent events. Although this approach has been conceptually insightful, it is relatively hard to use in practice, since it requires the use of conditional probability systems on the set of strategy profiles.

Possibly because of the technical complications associated with a formal definition of perfect Bayesian equilibrium, there are many papers in the literature that carry out their analysis for equilibrium concepts that lie in between weak perfect Bayesian equilibrium and sequential equilibrium. They generically referred to as perfect Bayesian equilibrium and, as mentioned above, they are typically defined as a strategy profile that is sequentially rational given a system of beliefs that is obtained using Bayes rule "whenever possible" (as opposed to do it only on the path as in weak perfect Bayesian equilibrium).

In this note we argue that there are games (indeed very simple ones) in which it is not clear what "whenever possible" is supposed to mean. Then, we introduce a simple definition of perfect Bayesian equilibrium that works for all extensive games and that refines both subgame perfect equilibrium and weak perfect Bayesian equilibrium. The main idea is to refine weak perfect Bayesian equilibrium in the same spirit in which subgame perfection refines Nash equilibrium, but to do so in such a way that it has bite also for imperfect information games. From our point of view, this new equilibrium concept provides a minimal

[2] In their setting, all the uncertainty a player faces during the game is about the types of the other players. Once a player knows his type, he forms a prior over the types of the other players and this prior is updated using Bayes rule as the game unfolds. If Bayes rule cannot be applied, *i.e.*, after a history that is inconsistent with the type of the player and the equilibrium strategies, the beliefs of the player are set to coincide with his prior.

requirement that should be imposed on equilibrium concepts that are based on Bayesian rationality. In addition, further requirements might be imposed depending on the specific characteristics of the games being analyzed.

2 The meaning of "whenever possible"

Since our contribution is primarily conceptual, we introduce as few notations as possible. We think of extensive games as modeled in Selten (1975) and Kreps and Wilson (1982) and refer the reader to those papers for the definitions of the concepts we discuss here.[3] An *assessment* is defined as a pair (b, μ) where b is a (behavior) strategy profile and μ is a system of beliefs.

As discussed in the introduction, it is quite common to see papers in which perfect Bayesian equilibrium is defined as a sequentially rational assessment (b, μ) in which the beliefs are computed using Bayes rule "whenever possible". Although this definition is clear for some classes of games, it is not precise enough for general extensive games. The reason, as illustrated in Example 1 below, is that it is not clear what "whenever possible" is supposed to mean in rigorous mathematical terms.

Example 1. Consider a game whose initial part corresponds with the one depicted in Figure 1. The strategy profile b stands for anyone in which player 1 plays D and player 2 plays d at his two decision nodes. The only nontrivial issue about beliefs has to do with the

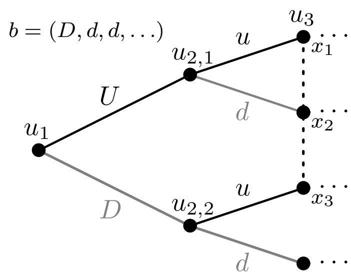

Figure 1: "Whenever possible" is imprecise.

information set u_3.[4] As far as weak perfect Bayesian is concerned, since u_3 is not on the path of strategy b, there is no restriction for the beliefs at the nodes in u_3, namely x_1, x_2, and x_3. Yet, should we impose any restriction on the beliefs in u_3 if Bayes rule is to be applied "whenever possible"? On the one hand, it can be argued that Bayes rule cannot be applied at u_3 since, given b, u_3 is reached with probability 0 and we cannot condition on probability 0 events. On the other hand, given b and conditional on either x_1 or x_2 being reached, *i.e.*, conditional on player 1 having played U, any belief consistent with Bayes rule should put probability 0 at x_1, since player 2 is playing d at $u_{2,1}$. Note that, even in the latter case, Bayes rule imposes no restriction on the relative probabilities of x_2 and x_3. Hence, what approach should we take? We consider that the natural interpretation of "whenever possible" goes in the lines of the second reasoning. In any case, this illustrates that providing a formal definition of what is meant by "whenever possible" that can be applied to any

[3] Nonetheless, unlike the definition of sequential equilibrium, our approach can be naturally extended to more general settings, such as games in which players have a continuum of actions.

[4] We denote by $u_{i,k}$ the k-th information set of player i. For simplicity, if a player i has only one information set, we denote it by u_i.

extensive game is by no means a trivial exercise. Moreover, any such definition would probably be difficult to work with and, since any such equilibrium concept would always be refined by sequential equilibrium, its applicability would probably be very limited. $\diamond$

Remark. Example 1 above is similar to the example used in Battigalli (1996) to illustrate the notion of *strategic independence*: information about player k's strategic behavior is irrelevant for probability assessments exclusively concerning player j's strategic behavior ($j \neq k$). In our example this would mean that, even if player 3 knows that player 1 has played U instead of D, player 3 still thinks that player 2 has played D at $u_{2,1}$. Then, Battigalli continues his analysis by exploring the implications of the notion of strategic independence and its relationship with the notion of consistency of assessments introduced by Kreps and Wilson (1982) when defining sequential equilibrium.

3 Simple Perfect Bayesian Equilibrium

Even after giving up on the objective of finding a compelling and useful definition of "whenever possible", there is still some room for a definition that pushes Bayesian requirements further than weak perfect Bayesian equilibrium and that does it in a clean and practical way. On the other hand, one cannot expect the simple definition of perfect Bayesian equilibrium that we present below to imply (or coincide with) the perfect Bayesian equilibrium introduced in Fudenberg and Tirole (1991). The reason is that, in order to decide what restrictions on beliefs are reasonable, they use the specific structure of the games in the class to which they have restricted. Somehow, by formally disentangling the meaning of "whenever possible" in multistage games with observed actions and independent types, they get to something which is very close to sequential equilibrium. Since we want a definition that is valid for every extensive game and that is easy to deal with, we abstract from the implicit implications of Bayes rule such as the one presented in Example 1.

Let G be an extensive game and Γ its game tree. Given a node x, we denote by $u(x)$ the information set that contains x. We say that a node y *comes after* a node x if x is on the path from the root to y; equivalently, we say that y is a *successor* of x. In particular, a node comes after itself. Similarly, a node comes after an information set if it comes after one of the nodes in the information set. The definitions below generalize the definitions of subtree and regular subtree in Selten (1975). Given an information set u, the *quasi-subtree* that begins at u, Γ_u, consists of all the nodes that come after u in Γ and all edges connecting them. The quasi-subtree Γ_u is *regular* if every information set of Γ that contains one node of Γ_u does not contain nodes outside Γ_u, *i.e.*, if an information set v has a node that comes after u, then all the nodes in v come after u. Note that, in the special case in which u is a singleton, the quasi-subtree Γ_u is indeed a subtree and, if Γ_u is regular, there is a well defined subgame of G that begins at u. If Γ_u is a regular quasi-subtree we say that u is itself a *regular* information set. Given an assessment (b, μ) and a regular information set u, we can associate a game with Γ_u in a very natural way. More specifically, let $G_u(\mu)$ be the game defined as follows. First, nature moves and selects each node x in the information set u with probability $\mu(x)$. Second, the game unfolds in Γ_u and all remaining elements of the game are the restrictions of the corresponding ones in G.[5]

For the sake of completeness we present now the definition of weak perfect Bayesian equilibrium.

[5] Note that if Γ_u is not a regular quasi-subtree, then it is not so clear how the game $G_u(\mu)$ should be defined. For instance, in the game in Figure 1, $\Gamma_{u_{2,1}}$ is not a regular quasi-subtree, since it does not contain x_3. Nonetheless, what happens after x_3 might be important to define the game $G_{u_{2,1}}(\mu)$ since, conditional on u_3 being reached, player 3 might put positive probability at x_3 ($\mu(x_3) > 0$).

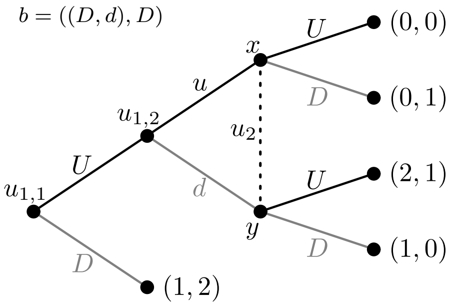

Figure 2: A weak perfect Bayesian equilibrium that is not a simple perfect Bayesian equilibrium.

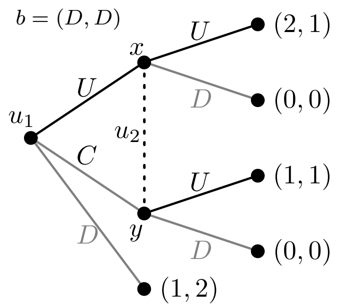

Figure 3: A subgame perfect equilibrium that is not a simple perfect Bayesian equilibrium.

Definition 1. *Let G be an extensive game. An assessment (b,μ) is a* weak perfect Bayesian equilibrium *if it is sequentially rational and, on the path of b, μ is derived from b by Bayes rule.*

As we have already argued above, the main drawback of weak perfect Bayesian equilibrium is that it does not impose any restriction on the beliefs at off-path information sets and, hence, although it is an equilibrium concept stronger than Nash equilibrium, a weak perfect Bayesian equilibrium does not even need to be subgame perfect. Below we define a version of perfect Bayesian equilibrium that imposes some simple restrictions on the off-path beliefs in a natural way.

Definition 2. *Let G be an extensive game. An assessment (b,μ) is a* simple perfect Bayesian equilibrium *if, for each regular information set u, the restriction of (b,μ) to $G_u(\mu)$ is a weak perfect Bayesian equilibrium.*

To some extent, simple perfect Bayesian equilibrium is to weak perfect Bayesian equilibrium in imperfect information games what subgame perfection is to Nash equilibrium in perfect information games.

Clearly, simple perfect Bayesian equilibrium implies both subgame perfect equilibrium and weak perfect Bayesian equilibrium. Consider the game in Figure 2. The strategy $b = ((D,d),D)$ is part of a weak perfect Bayesian equilibrium; it suffices to take the beliefs such that $\mu(x) = 1$; but it is not part of any simple perfect Bayesian equilibrium since it is not a weak perfect Bayesian equilibrium in the subgame that begins at $u_{1,2}$. Moreover, the unique simple perfect Bayesian equilibrium is $((U,d),U)$, which leads to the outcome $(2,1)$. Take now the game in Figure 3. The strategy $b = (D,D)$ is a subgame perfect equilibrium but, since choice D of player 2 is strictly dominated in the information set Γ_{u_2}, it is not simple perfect Bayesian. Now, the unique simple perfect Bayesian equilibrium is (U,U), which leads again to the outcome $(2,1)$.

Remark. When discussing the properties of sequential equilibrium, Kreps and Wilson (1982) define what they call *extended subgame perfect equilibrium* and show that sequential equilibrium is a refinement of it. Interestingly, extended subgame perfect equilibrium and simple perfect Bayesian equilibrium build upon the same idea but the former goes one step further and is a refinement of the latter. This is because extended subgame perfection imposes restrictions also at information sets that are not regular. The role played by regular information sets in our definition is played by subforms in the definition of extended subgame perfect equilibrium. Roughly speaking, a subform is a collection of information sets that is closed under *succession*; in particular, given a regular information set u, Γ_u is a subform. Then, similarly to what we did for regular information sets, one can associate a subgame to each subform. Yet, since a subform can have multiple "initial" information sets, one has to specify the relative probabilities with which they are chosen. Hence, to check that a given strategy profile is an extended subgame perfect equilibrium, it does not suffice to provide a system of beliefs, but probability measures over the "initial" information sets of the different subforms have to be specified as well.

4 Conclusions

It is easy to find extensive games in which simple perfect Bayesian equilibria selects unreasonable strategy profiles. One such example might be constructed from Figure 1. For an assessment (b, μ) to be a simple perfect Bayesian equilibrium of a game starting with the game tree in Figure 1, conditional on u_3 being reached, there is complete freedom in the beliefs, *i.e.*, it is not necessary that $\mu(x_1) = 0$. Nonetheless, we consider that this equilibrium concept can be good enough in different situations and, because of its simplicity, it is much easier to deal with than sequential equilibrium. Indeed, also weak perfect Bayesian equilibrium is sometimes good enough and, because of this, it is often used in literature.

Concerning the way in which equilibrium concepts are presented in most specialized books, there is a positive feature of simple perfect Bayesian equilibrium. Namely, it allows to restore the inclusion relation for the equilibrium concepts. Most books start by defining Nash equilibrium and subgame perfection comes immediately afterwards. Then, since subgame perfection does not perform well in imperfect information games, they introduce the systems of beliefs and, with them, weak perfect Bayesian equilibrium as a first step towards sequential equilibrium. Yet, we consider that it would be more pedagogic to use the concept of simple perfect Bayesian equilibrium instead of weak perfect Bayesian equilibrium.

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