# Modern Wiener–Hopf Design of Optimal Controllers Part I: The Single-Input-Output Case

DANTE C. YOULA, FELLOW, IEEE, JOSEPH J. BONGIORNO, JR., MEMBER, IEEE, AND HAMID A. JABR, STUDENT MEMBER, IEEE

***Abstract*—An analytical feedback design technique is presented here for single-input-output processes which are characterized by their rational transfer functions. The design procedure accounts for the topological structure of the feedback system ensuring asymptotic stability for the closed-loop configuration. The plant or process being controlled can be unstable and/or nonminimum phase. The treatment of feedback sensor noise, disturbance inputs, and process saturation is another major contribution of this work.**

**The cornerstone in the development is the selection of a performance index based on sound engineering considerations. It is these considerations, in fact, which ensure the existence of an optimal compensator for the system and make the performance index a natural one for the problem at hand.**

## I. Introduction

An ANALYTICAL feedback design technique is presented for single-input-output processes which are characterized by their rational transfer functions. The design procedure accounts for the topological structure of the feedback system *and ensures the* asymptotic stability of the closed-loop configuration. The plant or process being controlled can be unstable and/or nonminimum phase. The treatment of feedback sensor noise, disturbance inputs, and process saturation is another major contribution of this work. The cornerstone of the development is the selection of a performance index based on sound engineering considerations. It is these considerations in fact which ensure the existence of an optimal compensator for the system and make the performance index a natural one for the problem at hand.

The classical treatment of the analytical feedback design problem by Newton is described in [1]. With his approach, which is inherently open loop, it is first necessary to find the transfer function $W_c(s)$ analytic in $\mathrm{Re}\, s \geqslant 0$ of the optimal equivalent cascade compensator. The transfer function $C(s)$ of the corresponding controller for the feedback loop is then calculated by means of the formula

$$C(s) = W_c(s)/[1 - F(s)P(s)W_c(s)].$$

$F(s)$ and $P(s)$ denote the transfer functions of the feedback sensor and plant, respectively. Unfortunately, this procedure is flawed because it can, and often does, yield a computed $C(s)$ which possesses a zero in $\mathrm{Re}\, s \geqslant 0$ coinciding with either a pole of the plant or feedback sensor. Clearly, if $C(s)$ possesses such a zero, the closed-loop system is unstable and the design is worthless. To exclude such a possibility Newton restricts the plant and feedback sensor to be asymptotically stable from the outset. In fact, several extensions of this idea to the multivariable case have already been made by Bongiorno and Weston [2], [3].

The earliest researchers to recognize the difficulty with right-half plane pole-zero cancellations within a feedback loop worked with sampled data systems [4]. The analogous treatment for continuous-time systems was presented by Bigelow [5].[1] His argument for ruling out pole-zero cancellations in $\mathrm{Re}\, s \geqslant 0$ is based on the fallacious reasoning that exact cancellation cannot be achieved in practice. Although the observation concerning what can be achieved in practice is of course true, it is also true that even if perfect cancellation were possible the system would nevertheless still possess unstable "hidden" modes. Despite the error in physical reasoning these two papers succeeded in focusing attention on several meaningful engineering problems.

The frequency-domain optimization procedure described herein is the first one to correctly account for the asymptotic stability of the closed-loop system and to correctly treat plants which are not asymptotically stable. It also supplies significant insight into the essential role played by the classical sensitivity function in feedback system design. Although confined to single-input-output systems, these ideas can be extended to the multivariable situation. This extension is nontrivial and is the subject of Part II. Just as in [6], the scalar solution provided the necessary insight and impetus required to effect the breakthrough in the multivariable case. It is, therefore appropriate that both cases be presented in the literature. Moreover, it is only in the single-input-output case that the unique role of the sensitivity function manifests itself so clearly.

The limitations imposed by feedback sensor noise have been known for some time. Horowitz [7] has proposed a design philosophy for single-input-output minimum-phase

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D. C. Youla and J. J. Bongiorno, Jr. are with the Department of Electrical Engineering and Electrophysics, Polytechnic Institute of New York, Long Island Center, Farmingdale, NY 11735.

H. A. Jabr was with the Department of Electrical Engineering and Electrophysics, Polytechnic Institute of New York, Long Island Center, Farmingdale, NY 11735. He is now with the University of Petroleum and Minerals, Dhaharan, Saudi Arabia.

[1]The paper by Bigelow was kindly brought to the attention of the authors by P. Sarachik.

stable plants which is quite imaginative but appears limited since it is modeled around a Bode two-terminal interstage equalization scheme. On the other hand, our approach takes the lumped character of the controller as an explicit constraint from the outset and nonminimum-phase and/or unstable plants offer no special obstacles.

A discussion of the relationship of our frequency-domain design procedure and some of the more popular state-variable techniques [14] is certainly in order and will be given in Part II. For now, we merely observe that the methods of this paper obviate the need to find state-variable representations and can handle stochastic inputs which are non-Gaussian and colored, as well as step and ramp-type disturbances. In addition, it permits the modeling and incorporation of feedback transducers such as tachometers, rate gyros, and accelerometers with nondynamical transfer functions.

## II. Problem Statement and Preliminary Results

In this paper attention is restricted exclusively to the design of controllers for single-input-output finite-dimensional linear time-invariant plants embedded in an equivalent single-loop configuration shown in Fig. 1.[2] Suppose $y_d(s)$, the *desired* closed-loop output is related to $u_i(s)$, the *actual* input set-point signal in the linear fashion

$$y_d(s)=T_d(s)u_i(s)$$

via the *ideal* transfer function $T_d(s)$. The *prefilter* $H(s)$ can be selected in advance once and for all, but irrespective of the particular choice of criterion that is employed.[3]

$$u=H(u_i+n)$$

is the best available linear version of $y_d(s)$. Any reasonable performance measure must be based on the difference

$$e(s)=u(s)-y(s) \tag{1}$$

between the actual plant output $y(s)$ and the actual smoothed input $u(s)$ driving the loop. For a given plant and overall sensor $F(s)$ the design of the controller $C(s)$ should evolve from an appropriate minimization procedure subject to a power-like constraint on $r(s)$ to avoid plant saturation.

Plant disturbance $d(s)$ and measurement noise $m(s)$ are modeled in a perfectly general way by assuming that

$$y(s)=P(s)r(s)+P_0(s)d(s) \tag{2}$$

[2]To avoid proliferating symbols, all quantities are Laplace transforms, deterministic or otherwise, all stochastic processes are zero-mean second-order stationary with rational spectral densities and $\langle,\rangle$ denotes ensemble average.

[3]Function arguments are omitted wherever convenient.

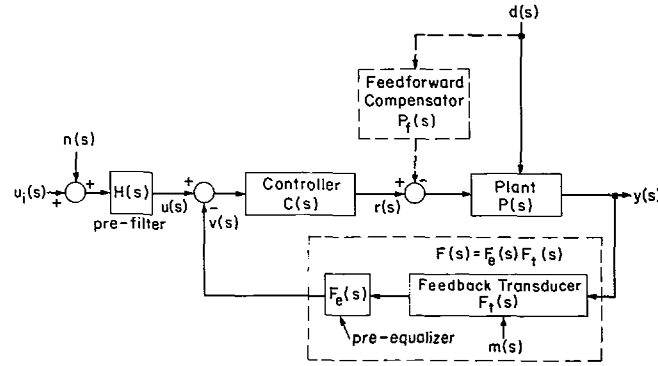


Fig. 1. Basic single-loop feedback configuration.

and

$$v(s)=F(s)y(s)+F_0(s)m(s) \tag{3}$$

where $P(s)$, $P_0(s)$, $F(s)$, and $F_0(s)$ are four real rational functions in the complex variable $s=\sigma+j\omega$. Moreover, by changing $P_0(s)$ into

$$P_0(s)-P(s)P_f(s) \tag{4}$$

it is also possible to envisage any desirable feedforward compensation $P_f(s)$.

Straightforward analysis yields

$$y=\frac{1-S}{F}(u-F_0m)+SP_0d, \tag{5}$$

$$r=\frac{1-S}{PF}(u-F_0m-FP_0d) \tag{6}$$

and

$$e=\left(\frac{F-1+S}{F}\right)u-SP_0d+\left(\frac{1-S}{F}\right)F_0m \tag{7}$$

where

$$S(s)=\frac{1}{1+F(s)P(s)C(s)} \tag{8}$$

is the closed-loop sensitivity function. In process control, the actual choice of a reliable feedback transducer $F_t(s)$ is more or less dictated by the problem at hand. However, as explained in greater detail later, some low-power-level preequalization $F_e(s)$ is almost always necessary to model delay in the feedback path, to improve stability margin and to assure zero steady-state error. In other words

$$F(s)=F_e(s)F_t(s). \tag{9}$$

We therefore assume that $P_0(s)$, $F_0(s)$, $F(s)$, and $P(s)$ are prescribed in advance. Equations (5)–(7) reveal the possibilities for tradeoff in the various frequency bands. Observe, that with unity feedback ($F=1$), $e(s)$ is the sum of the two errors

$$e_1(s)=S(u-P_0d) \tag{10}$$

and

$$e_2(s)=(1-S)F_0m \tag{11}$$

whose different origins are betrayed by the prefactors $S(s)$ and $1\text{-}S(s)$. The impossibility of making both $S(s)$ and $1-S(s)$ arbitrarily "small" over any frequency band is partly intrinsic and partly conditioned by the plant restrictions [8], [9]. This fundamental conflict is inevitable and largely responsible for a great deal of the difficulty surrounding practical feedback design.

Let

$$P(s)=\frac{n_p(s)}{d_p(s)} \tag{12}$$

$$F(s)=\frac{n_f(s)}{d_f(s)} \tag{13}$$

and

$$C(s)=\frac{n_c(s)}{d_c(s)} \tag{14}$$

where each numerator polynomial is relatively prime to its respective denominator mate. It is well known [10]–[12] that if the plant, controller and feedback sensor are free of unstable hidden modes, the closed loop of Fig. 1 is asymptotically stable iff the "reduced" characteristic polynomial

$$\varphi(s)=d_f(s)d_p(s)d_c(s)+n_f(s)n_p(s)n_c(s) \tag{15}$$

is strict Hurwitz; i.e., iff $\varphi(s)$ has no zeros in $\mathrm{Re}\,s\geqslant 0$. Hence, the pair $d_f(s)$, $n_p(s)$ as well as the pair $d_p(s)$, $n_f(s)$ must be devoid of common zeros in $\mathrm{Re}\,s\geqslant 0$ in which case $P(s)$ and $F(s)$ are said to be *admissible*.[4] Observe that once $H(s)$, $P(s)$, $F(s)$, $P_0(s)$, $F_0(s)$ and the statistics of $u_i(s)$, $n(s)$, $d(s)$, and $m(s)$ are specified, $y(s)$, $r(s)$, and $e(s)$ are uniquely determined by the choice of sensitivity function $S(s)$. Consequently, the following definition and its accompanying lemma have an obvious importance and are fundamental to our entire approach.

*Definition 1:* $S(s)$ is said to be *realizable* for an admissible pair $P(s)$, $F(s)$ if the closed-loop structure of Fig. 1 is asymptotically stable for some choice of controller $C(s)$ and possesses the sensitivity function $S(s)$.

*Lemma 1 (Appendix):* The function $S(s)\not\equiv 0$ is realizable for the admissible pair $P(s)$, $F(s)$ iff

1) $S(s)$ is analytic in $\mathrm{Re}\,s\geqslant 0$;
2) Every zero of the polynomial $d_f(s)d_p(s)$ in $\mathrm{Re}\,s\geqslant 0$ is a zero of $S(s)$ of *at least* the same multiplicity;
3) Every zero of the polynomial $n_f(s)n_p(s)$ in $\mathrm{Re}\,s\geqslant 0$ is a zero of $1-S(s)$ of *at least* the same multiplicity.

Let $G_i(-s^2)$, $G_n(-s^2)$, $G_u(-s^2)$, $G_d(-s^2)$, and $G_m(-s^2)$ denote the rational spectral densities of $u_i(s)$, $n(s)$, $u(s)$, $d(s)$, and $m(s)$, respectively. Setting aside for the moment all questions of convergence,[5]

$$2\pi jE_t=\int_{-j\infty}^{j\infty}\langle e_*(s)e(s)\rangle ds \tag{16}$$

is the usual quadratic measure of steady-state response. Similarly,[6] if $\boldsymbol{P}_s(s)$ represents the column-vector transfer matrix coupling the plant input $r(s)$ to those "sensitive" plant modes which must be especially protected against excessive dynamic excursions,

$$2\pi jE_s=\int_{-j\infty}^{j\infty}\langle r_*(s)\boldsymbol{P}_{s*}(s)\boldsymbol{P}_s(s)r(s)\rangle ds \tag{17}$$

is a proven useful penalty functional for saturation [1]. More explicitly,

$$2\pi jE_s=\int_{-j\infty}^{j\infty}Q(-s^2)\langle r_*(s)r(s)\rangle ds \tag{18}$$

where

$$Q(-s^2)=\boldsymbol{P}_{s*}(s)\boldsymbol{P}_s(s). \tag{19}$$

Thus,

$$E=E_t+kE_s. \tag{20}$$

$k$, a positive constant, serves as a weighted cost combining both factors. Using (6) and (7) and assuming all processes to be independent, a simple calculation yields the compact four-term expression

$$2\pi jE=\alpha+2\int_{-j\infty}^{j\infty}\frac{(F-1)S_*}{FF_*}G_u ds + \int_{-j\infty}^{j\infty}SS_*G_a ds+\int_{-j\infty}^{j\infty}(1-S)(1-S)_*G_b ds \tag{21}$$

where

$$\alpha=\int_{-j\infty}^{j\infty}\frac{(F-1)(F-1)_*}{FF_*}G_u ds, \tag{22}$$

$$G_a=\frac{G_u}{FF_*}+P_0P_{0*}G_d,\quad G_u=HH_*(G_i+G_n) \tag{23}$$

$$G_b=\frac{F_0F_{0*}}{FF_*}G_m+\frac{kQ}{PP_*}\left(P_0P_{0*}G_d+\frac{G_u+F_0F_{0*}G_m}{FF_*}\right). \tag{24}$$

Our entire physical discussion revolves around the implications of (21), and our assumptions are as follows.

*Assumption 1:* Rate gyros and tachometers are examples of practical sensing devices which are not modeled as dynamical systems.[7] Yet almost invariably, sensors are stable and their associated transfer functions $F_t(s)$ are analytic in $\mathrm{Re}\,s\geqslant 0$. For our purposes it suffices to restrict

[4]When $F_0(s)$, $H(s)$, $P_f(s)$, and $P_0(s)$ represent distinct physical blocks, these blocks must be stable: their transfer functions must be analytic in $\mathrm{Re}\,s\geqslant 0$. On the other hand if $F_0(s)$, $H(s)$, $P_f(s)$, and $P_0(s)$ are merely part of the paper modeling it is often possible to relax the analyticity requirements.

[5]If $A(s)$ is a real rational (or meromorphic) matrix in $s$, $A_*(s)\equiv A'(-s)$, the transpose of $A(-s)$.

[6]Column-vectors are written $\boldsymbol{a}$, $\boldsymbol{b}$, etc., and det $A$, $A'$, $\bar{A}$, $A^*$ $(\equiv\bar{A}')$ denote the determinant, transpose, complex conjugate, and adjoint of the matrix $A$, respectively. Note that for $A(s)$ real and meromorphic, $A_*(j\omega)=A^*(j\omega)$, $\omega$ real.

[7]A system with transfer function $A(s)$ is dynamical if $A(s)$ is proper; i.e., if $A(\infty)$ is finite.

$F(s)$ to be analytic on the finite $j\omega$-axis and to insist that the component of the cost $\alpha$ be finite. In particular, the integrand in (22) must be analytic on the $j\omega$-axis and $0(1/\omega^2)$ for $\omega^2 \to \infty$.

Suppose that parameter variations induce a change $\Delta\varphi(s)$ in the characteristic polynomial $\varphi(s)$. Clearly, if the nominal design is stable and structural changes are precluded

$$\eta(s) = \frac{\Delta\varphi(s)}{\varphi(s)}$$

is proper and analytic in $\mathrm{Re}\, s \geq 0$. Invoking the standard Nyquist argument it is immediately concluded that $\varphi(s) + \Delta\varphi(s)$, the reduced characteristic polynomial of the perturbed closed loop, is strict Hurwitz iff the normal plot of $\eta(j\omega)$ does not encircle the point $-1+j0$ in a clockwise direction. It is imperative therefore that at the nominal setting $|\varphi(j\omega)|$ be comparably large over those frequency ranges where $|\Delta\varphi(j\omega)|$ is expected to be large. Unfortunately, it does not appear possible to translate any nontrivial stability-margin criteria directly into manageable integral restrictions reconcilable with $E$. However, once the formula for the optimal $S(s)$ is available, the role played by $F(s)$ in securing adequate stability margin will be clarified and further discussion along these lines is postponed until the next section.

*Assumption 2:* A pole of $P(s)$ in $\mathrm{Re}\, s > 0$ reveals true plant instability but a pole on the $j\omega$-axis is usually present because of intentional preconditioning and is not accidental. For example, with unity feedback ($F=1$) and $d(s)=m(s)=0$, a stable loop enclosing a plant whose transfer function possesses a pole of order $\nu$ at the origin will track any causal linear combination of the inputs $1, t, \cdots, t^{\nu-1}$ with zero steady-state error. Similarly, if $s = j\omega_0$, $\omega_0$ real, is a pole of order $\nu$ of $P(s)$, a unity-feedback stable loop will track any linear combination of $e^{j\omega_0 t}, te^{j\omega_0 t}, \cdots, t^{\nu-1}e^{j\omega_0 t}$ with zero steady-state error. These generalized ramp-modulated sinusoids constitute an important class of shape-deterministic information-bearing signals and play a key role in industrial applications. In a nonunity-feedback loop this perfect accuracy capability is lost unless $F_t(s)$ is also preconditioned compatibly. From (7) with $d(s)=m(s)=0$,

$$e = \left(\frac{F-1+S}{F}\right)u. \tag{25}$$

Now according to Lemma 1, a finite pole $s=j\omega_0$ of $P(s)$ of multiplicity $\nu$ must be a *zero* of $S(s)$ of order at least $\nu$. Thus, if $s=j\omega_0$ is *also* a zero of $F(s)-1$ of order $\nu$ or greater, (25) shows that the loop is again capable of acquiring any linear combination of the inputs $e^{j\omega_0 t}$, $te^{j\omega_0 t}, \cdots, t^{\nu-1}e^{j\omega_0 t}$ with zero steady-state error. By setting $u(s)=m(s)=0$ in (7) we obtain

$$e = -SP_0 d \tag{26}$$

the loop error under load disturbance $d(s)$. In many areas, such as process control, the recovery of steady state under load changes is a requirement of paramount importance. As is seen from (26), if the shape deterministic component of $P_0(s)d(s)$ is envisaged to be the transform of a sum of ramp-modulated sinusoids, bounded zero steady-state error is possible iff $e(s)$ vanishes at infinity and is analytic in $\mathrm{Re}\, s \geq 0$. Assuming $SP_0$ proper and $SP_0 d$ analytic in $\mathrm{Re}\, s \geq 0$ is evidently sufficient. In particular, reasoning as above, the $j\omega$-axis poles of $P_0(s)d(s)$, multiplicities included, must be contained in those of $P(s)$. Summing up, $(F-1)P$,

$$d_p d_{p*}\langle uu_*\rangle = d_p G_u d_{p*} \tag{27}$$

and

$$d_p d_{p*}\langle P_0 dd_* P_{0*}\rangle = d_p P_0 G_d P_{0*} d_{p*} \tag{28}$$

must be $j\omega$-analytic. Equivalently, in view of Assumption 1 and (23),

$$d_p G_a d_{p*} \tag{29}$$

is analytic on the finite $j\omega$-axis.

*Assumption 3:* In general, the effects of parameter uncertainty on $P(s)$ and $F(s)$ are more pronounced as $\omega$ increases and closed-loop sensitivity is an important consideration. This sensitivity is usually expressed in terms of the percentage change in the loop transfer function

$$T(s) = \frac{P(s)C(s)}{1+F(s)P(s)C(s)}. \tag{30}$$

A straightforward calculation yields

$$\frac{\delta T}{T} = \frac{\delta(PC)}{PC}\cdot S - \frac{\delta F}{F}\cdot(1-S) \tag{31}$$

and once again $S(j\omega)$ and $1-S(j\omega)$ emerge as the pertinent gain functions for the forward and return links, respectively. Clearly then, to combat the adverse effects of high-frequency uncertainty in the modeling of $F(j\omega)$ and $P(j\omega)$ it is sound engineering practice to design $S(j\omega)$ proper and equal to 1 at $\omega=\infty$. This requirement is easily introduced into the analytic framework by imposing the restrictions $G_b(-s^2) \not\equiv 0$ and

$$G_b(-s^2) = 0(\omega^{2l}), \qquad l \geq 0 \tag{32}$$

for large $\omega^2$.

Our final assumptions are fashioned for the express purpose of excluding from consideration certain mathematically possible but physically meaningless degeneracies. They are also motivated by Lemma 1, the structure of (21), and the requirement of finite cost.

*Assumption 4:* For large $\omega^2$,

$$G_a(\omega^2) = 0(1/\omega^{2\nu}), \qquad \nu \geq 1. \tag{33}$$

*Assumption 5:* $Q$ is analytic on the finite $j\omega$-axis, has no purely imaginary zeros in common with $n_p$, and the constant $k$ is positive.

*Assumption 6:* Let

$$G = P_0 G_d P_{0*} + \frac{G_u + F_0 G_m F_{0*}}{FF_*}; \tag{34}$$

then

$$(d_p n_f) G (d_p n_f)_*$$

is analytic and nonzero on the finite $s = j\omega$-axis.

For later reference we record the useful formula

$$G_a + G_b = \left(1 + \frac{kQ}{PP_*}\right) G \tag{35}$$

which drops out of (23), (24), and (34).

## III. The Wiener–Hopf Solution

Recall that any rational function $A(s)$ possesses a Laurent expansion constructed from all its poles, finite or infinite and as is customary, $\{A(s)\}_+$ denotes that part of the expansion associated with all the *finite* poles of $A(s)$ in $\mathrm{Re}\, s < 0$. Thus, $\{A(s)\}_+$ is analytic in $\mathrm{Re}\, s \geq 0$ and vanishes for $s = \infty$. The remainder of the expansion is written $\{A(s)\}_-$ and of course,

$$A(s) = \{A(s)\}_+ + \{A(s)\}_-.$$

*Theorem 1 (Appendix):* Let

$$d(s) = d_f(s) d_p(s) \tag{36}$$

$$n(s) = n_f(s) n_p(s) \tag{37}$$

$$\chi(s) = d(s) n(s) \tag{38}$$

and write

$$\chi(s) = \chi_l(s) \chi_r(s). \tag{39}$$

The polynomial $\chi_l(s)$ absorbs all the zeros of $\chi(s)$ in $\mathrm{Re}\, s < 0$ and $\chi_r(s)$ all those in $\mathrm{Re}\, s \geq 0$. Perform the spectral factorization

$$\chi_r \chi_{r*} (G_a + G_b) = \Omega \Omega_* \tag{40}$$

where $\Omega(s)$ is free of zeros and poles in $\mathrm{Re}\, s > 0$.

1) Under Assumptions 1–6, the optimal closed-loop sensitivity function $S_0(s)$ associated with any admissible pair $P(s)$, $F(s)$ is given by

$$S_0 = \frac{\left\{ \dfrac{\chi_r \chi_{r*}}{\Omega_*} \left( G_b - \dfrac{F-1}{FF_*} G_u \right) \right\}_+ + f}{\Omega}. \tag{41}$$

$f(s)$ a real polynomial. The requirements $E < \infty$ (finite cost) and $S_0(s)$ realizable for $P(s)$, $F(s)$ determine $f(s)$ *uniquely*.

2) The optimal controller $C_0(s)$ which realizes $S_0(s)$ for the pair $P(s), F(s)$ is obtained from the formula

$$C_0(s) = \frac{1 - S_0(s)}{P(s) F(s) S_0(s)} \tag{42}$$

and can be improper, unstable or both.[8] Nevertheless, the closed-loop structure is always asymptotically stable and $S_0(s)$ is proper and analytic in $\mathrm{Re}\, s \geq 0$. (Assumptions 1–6 actually force $\Omega(s)$ to be free of zeros in $\mathrm{Re}\, s \geq 0$.)

With exact arithmetic the finite zeros and poles of $P(s)F(s)$ in $\mathrm{Re}\, s \geq 0$ are cancelled exactly by the zeros of $S_0(s)$ and $1 - S_0(s)$, respectively. Thus, in any computer implementation of (41) and (42) it is necessary that all these exact arithmetic cancellations in $\mathrm{Re}\, s \geq 0$ be effected automatically by suitable preparation. Failure to do so will result in a nonstrict-Hurwitz stability polynomial $\varphi(s)$ and a corresponding unstable closed-loop design.

An examination of (41) reveals that the zeros of $\Omega(s)$ and the poles of

$$\left\{ \frac{\chi_r \chi_{r*}}{\Omega_*} \left( G_b - \frac{F-1}{FF_*} G_u \right) \right\}_+ \tag{43}$$

constitute the poles of $S_0(s)$. Since the poles of $S_0(s)$ are all zeros of $\varphi(s)$, the stability margin of the optimal design is ascertainable *in advance*. This important feature cannot be overemphasized. From the formula

$$\Omega \Omega_* = \chi_r \chi_{r*} \left(1 + \frac{kQ}{PP_*}\right) G \tag{44}$$

it is seen that the zeros of $\chi_r \chi_{r*} G$ and $1 + kQ/PP_*$ in $\mathrm{Re}\, s < 0$ emerge as poles of $S_0(s)$. The locations of these zeros depend on the choice of $F(s)$, the spectral density $G(-s^2)$ and the value of $k$. Changing $k$ means compromising saturation (and accuracy). A more detailed analysis shows generally that the *negative images* of the right-half plane poles of $P(s)$ and $F(s)$ are zeros of $\chi_r \chi_{r*}$ and therefore poles of $S_0(s)$ unless $G(-s^2)$ is properly preconditioned. If some of these poles lie close to the $s = j\omega$-axis, it may be impossible to attain adequate stability margin. This difficulty can be circumvented by simply incorporating the offending poles into $G(-s^2)$. Hence, the rule, any pole of $P(s)F(s)$ in $\mathrm{Re}\, s > 0$ which lies "too close" to the imaginary axis must be made a pole of $G(-s^2)$ of exactly twice the multiplicity. Last, we mention that delay $\tau$ in the feedback path can be simulated by introducing right-half plane zeros into $F_e(s)$ through one of the many available rational function approximations to $e^{-s\tau}$.

## IV. Example

The theory developed in the preceding sections is now used to design the controller $C(s)$ for the system shown in Fig. 2. Since the theory is based on rational transfer

[8] $C(s)$ is proper if the integer $l$ in Assumption 3 equals the order of the zero of $F(s)P(s)$ at infinity. This is often the case.

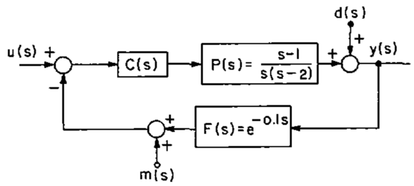


Fig. 2. Example.

functions, the first step is the selection of a suitable approximation for the ideal delay represented by $F(s) = e^{-0.1s}$. Highly satisfactory results are obtained with the second-order Padé approximation

$$F(s)=\frac{\frac{12}{(0.1)^2}-\frac{6}{(0.1)}s+s^2}{\frac{12}{(0.1)^2}+\frac{6}{(0.1)}s+s^2}\equiv\frac{p_f(-s)}{p_f(s)}. \tag{45}$$

Note that $F(s)$ and $P(s)$ are an admissible pair and (45) satisfies the condition $F-1=0$ at $s=0$, the only pole of $P(s)$ on the imaginary axis.

Because of the plant pole at the origin and the choice of rational approximation for $F(s)$, the closed loop is capable of following step inputs $u$ with zero steady-state error when $m=d=0$. The simplest example calls for

$$G_u=-\frac{1}{s^2}. \tag{46}$$

For the remaining spectral densities we choose

$$G_d=\frac{1}{100-s^2} \tag{47}$$

and

$$G_m=1. \tag{48}$$

We also assume that the plant input is the signal most likely to cause saturation and put $Q=1$. The only remaining quantities needed for the calculation of the optimal controller are $k$, $F_0$, and $P_0$. Comparing Figs. 1 and 2, it is seen that $F_0=P_0=1$. With regard to $k$, we note that the performance index $E$ is actually an auxilary cost function. The design objective is to minimize $E_t$ subject to the constraint $E_s \leqslant N_s$, $N_s$ a specified bound. Thus, $k$ is a Lagrangian multiplier chosen to meet the design objectives.

The first step in the design of the optimal controller $C(s)$ is the determination of $S_0$ given by (41). For the determination of the optimal sensitivity function we need the quantities $\chi_r$, $\Omega$, $G_a$, and $G_b$. From (45) and Fig. 2 it follows that

$$\chi=n_p d_p n_f d_f=s(1-s)(2-s)p_f(s)p_f(-s) \tag{49}$$

and

$$\chi_r=s(1-s)(2-s)p_f(-s). \tag{50}$$

Substitution of the given data into (23), (24), (34), and (35) yields

$$G_a=G_u+G_d=\frac{-2(50-s^2)}{s^2(100-s^2)} \tag{51}$$

$$G_b=G_m+\frac{k}{PP_*}(G_m+G_u+G_d)=1+\frac{k(4-s^2)p_1(s)}{(1-s^2)(100-s^2)} \tag{52}$$

and

$$G_a+G_b=\frac{-kp_1(s)p_2(s)}{s^2(100-s^2)(1-s^2)} \tag{53}$$

where

$$p_1(s)=100-102s^2+s^4 \tag{54}$$

and

$$p_2(s)=\frac{1}{k}-\left(\frac{1}{k}+4\right)s^2+s^4. \tag{55}$$

It now follows from (40) that

$$\Omega=\frac{\sqrt{k}\;p_1^+p_2^+(2+s)p_f(s)}{10+s} \tag{56}$$

where

$$p_1^+=10+\sqrt{122}\;s+s^2 \tag{57}$$

and

$$p_2^+=\frac{1}{\sqrt{k}}+\sqrt{\frac{1}{k}+\frac{2}{\sqrt{k}}+4}\;s+s^2 \tag{58}$$

are the factors containing all the zeros of $p_1(s)$ and $p_2(s)$ in $\mathrm{Re}\,s<0$, respectively.

In addition,

$$\frac{(F-1)G_u}{FF_*}=\frac{120}{sp_f(s)} \tag{59}$$

where

$$\left\{\frac{\chi_r\chi_{r*}}{\Omega_*}\left(G_b-\frac{F-1}{FF_*}G_u\right)\right\}_+=\frac{k_0}{10+s} \tag{60}$$

$$k_0=-\left.\frac{\sqrt{k}\;s^2(2+s)(4-s^2)p_f(s)p_1^+(s)}{p_2^+(-s)}\right|_{s=-10} \tag{61}$$

We now find that

$$S_0=\frac{k_0+(10+s)f(s)}{\sqrt{k}\;(2+s)p_f(s)p_1^+(s)p_2^+(s)}\equiv\frac{h(s)}{g(s)} \tag{62}$$

in which the sixth-degree polynomial $f(s)$ is uniquely determined by the interpolatory conditions

$$S_0(0)=S_0(2)=0 \tag{63}$$

$$S_0(1)=S_0(z_0)=S_0(\bar{z}_0)=1 \tag{64}$$

$$1-S_0(s)=0\left(\frac{1}{s^2}\right), \quad |s| \text{ large}. \tag{65}$$

In (64), $z_0$ and $\bar{z}_0$ are the zeros of $p_f(-s)$.

Due to roundoff error, the computed polynomial $f(s)$ will, in general, not satisfy conditions (63)–(65) exactly. However, we know from the theory that the poles of $F(s)$ and $P(s)$ in Re $s \geqslant 0$ must be zeros of $S_0(s)$ and the zeros of $F(s)$ and $P(s)$ in Re$s \geqslant 0$ must be zeros of $1-S_0(s)$. The computations are therefore conditioned so that

$$h(s)=s(s-2)h_1(s) \tag{66}$$

and

$$g(s)-h(s)=(s-1)(s-z_0)(s-\bar{z}_0)h_2(s). \tag{67}$$

This is accomplished by dividing the computed $h(s)$ by $s(s-2)$, setting $h_1(s)$ equal to the quotient and ignoring the remainder.[9] The same procedure is followed in obtaining (67), and (42) then yields

$$C_0(s)=\frac{p_f(s)h_2(s)}{h_1(s)} \tag{68}$$

for the optimal controller.

The design described above depends parametrically on $k$. The values of $E_t$ and $E_s$ have been computed for various values of $k$ and the results are shown in Fig. 3. It is clear from the curves that the choice $k=4$ leads to a suitable compromise between the desire to minimize $E_t$ while limiting $E_s$. (Note that all transfer functions in the frequency domain and all signals in the time domain are taken to be dimensionless quantities but time is measured in seconds. It follows that $E_t$ and $E_s$ have the dimensions of seconds.) The transient response of the optimally designed system to a unit step input has also been investigated. The error and plant input responses for several values of $k$ are shown in Fig. 4 and Fig. 5. Since these responses are obtained with $d=m=0$ they do not, and should not, reflect the optimality of the design. They do show, however, that reasonable transient performance is obtained with the choice $k=4$.

It has been pointed out in Assumption 1 that $|\phi(j\omega)|$ should be large or, equivalently, its reciprocal should be small for good stability margin. We have in fact computed and plotted $\phi^{-1}(j\omega)$ for several values of $\omega$ in the range zero to infinity. The results are shown in Fig. 6 and are highly satisfactory. With $k=4$, $|\phi^{-1}(j\omega)| \leqslant 10^{-4}$ and the system remains stable no matter what the phase of $\Delta\phi(j\omega)$ provided only that $|\Delta\phi(j\omega)| < 10\ 000$. For completeness, plots are also shown in Fig. 7 of $|S_0(j\omega)|$ versus $\omega$ for several choices of $k$.

Evidently, in the light of these observations, the choice $k=4$ makes engineering sense and the final step in the design is to compute the optimal controller transfer function with $k=4$. Using (68) we get

$$C_0(s)=\frac{K(s-\sigma_1)(s-\sigma_2)(s-s_0)(s-\bar{s}_0)}{(s-\rho_1)(s-\rho_2)(s-\rho_3)(s-s_p)(s-\bar{s}_p)}, \tag{69}$$

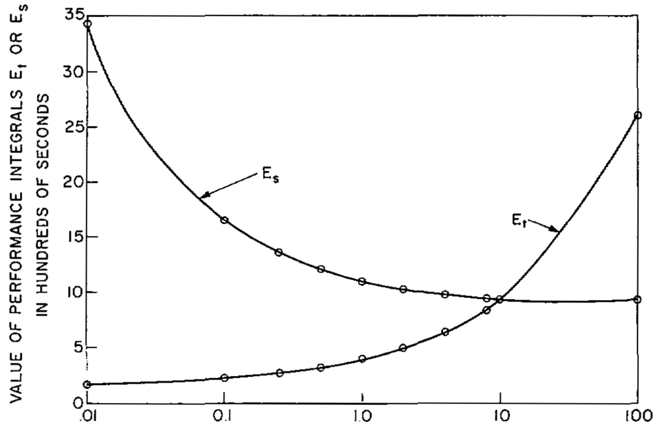

Fig. 3. Variation of performance integrals.

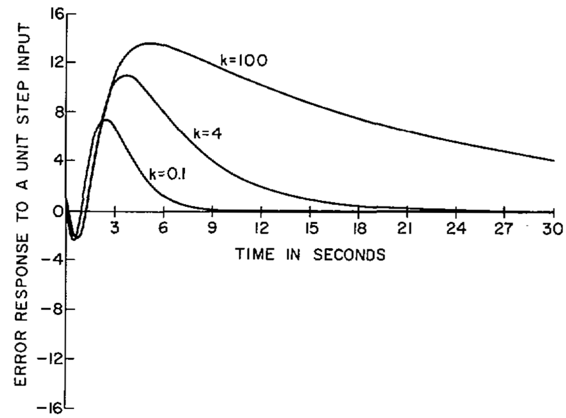

Fig. 4. Error responses.

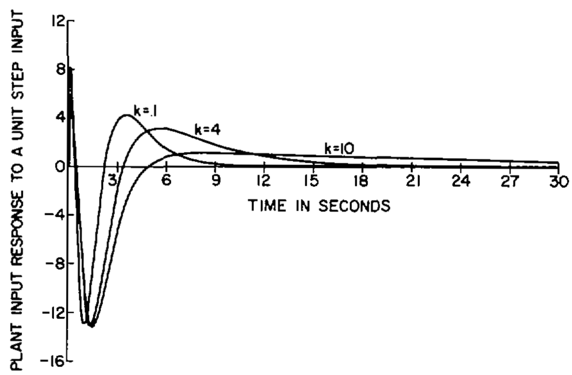

Fig. 5. Plant input responses.

[9] The accuracy of the computations on a digital computer is such that this remainder is quite small (coefficients less than $10^{-5}$ in this example).

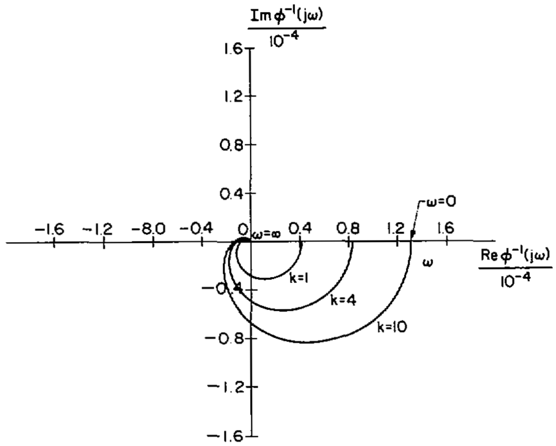


Fig. 6. Nyquist plot of $\phi^{-1}(j\omega)$.

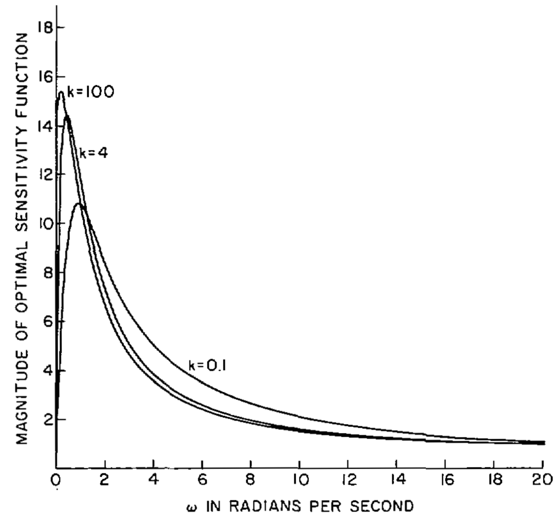


Fig. 7. Sensitivity function amplitude response.

where

$$\left.\begin{aligned} K &= 67.228808647 \\ \sigma_1 &= 0.014874634 \\ \sigma_2 &= -9.9999638 \\ \rho_1 &= 2.413271030575 \\ \rho_2 &= -9.9806403944 \\ \rho_3 &= -33.65463165144 \\ s_0 &= -30+j17.320508076 \\ s_p &= -18.05732390209+j14.991623794. \end{aligned}\right\} \tag{70}$$

The reader has probably noticed that $\sigma_2$ and $\rho_2$ are very nearly equal. Although we have in fact verified theoretically that these two quantities cannot be equal, it is nevertheless natural to inquire whether any significant deterioration in performance results by putting $\sigma_2=\rho_2$ and using the suboptimal controller

$$C(s)=\frac{67.2(s-0.015)(s^2+60s+1200)}{(s-2.4)(s+33.7)(s^2+36s+549)}. \tag{71}$$

Another aspect of the design which should be clarified is the use of the Padé approximation (45) for the delay $e^{-0.1s}$ in the feedback loop. These points have been taken up and the results are presented in Table I. A comparison of the first two columns in the table reveals that the use of the Padé approximation is certainly satisfactory while a comparison of the last two shows that the use of the suboptimal controller is justified. This is gratifying since no analog controller can be designed to the accuracy demanded by the values in (70).

We have also studied the stability margin with respect to variations in the delay $\tau=0.1$ s. With the suboptimal controller the feedback loop remains stable for $0\leqslant\tau\leqslant 0.155$ s. This stability margin is clearly satisfactory and the example indicates that the design procedure is a practical one.

One final point. It is quite obvious that the design equations are substantially simpler if $F(s)=1$ is used instead of the Padé approximation and the delay is ignored. However, when the corresponding controller is employed it is found that the system remains stable only for $0\leqslant\tau\leqslant 0.08$ s. Thus, with an actual delay $\tau=0.1$ s the system designed optimally with $F=1$ would be unstable. This facility to incorporate delay is of significant practical value.

TABLE I
COMPARISON OF RESULTS

| | Optimal $C(s)$ | | Suboptimal $C(s)$ |
|---|---|---|---|
| | Padé Approximation for $F(s)$ | $F(s)=e^{-0.1s}$ | $F(s)=e^{-0.1s}$ |
| $E_t$ | 646.9 | 646.1 | 676.8 |
| $E_s$ | 986.7 | 957.6 | 952.2 |

## V. Discussion

It appears from Fig. 4 that the transient performance of the optimally compensated loop in our example is poor. In fact, for $k=4$ a peak error response of 10.7 is obtained. Is this poor transient performance a consequence of the design procedure, inherent limitations imposed by the plant, or both? To answer this question the example described in the previous section is considered once again but with $G_m=G_d=k=0$. (Note that although the conditions $k>0$ and $G_b(-s^2)\not\equiv 0$ are violated, it is still possible to obtain an optimal solution. Optimal solutions can exist for cases which do not satisfy our assumptions on the data. Assumptions 1–6 are *sufficient* to guarantee the existence of an optimal controller and they hold in most cases of interest.)

The optimal solution obtained for $G_m=G_d=k=0$ is the one which minimizes the integral square error with a unit step input. The optimal controller in this case is

$$C_0=\frac{-\hat{K}(s-\sigma_3)(s-s_0)(s-\bar{s}_0)}{(s-\rho_4)(s-\rho_5)} \tag{72}$$

where $s_0$ is given in (70) and

$$\left.\begin{aligned}\hat{K}&=1.136452161649\\ \sigma_3&=0.240139\\ \rho_4&=56.64885134963\\ \rho_5&=9.55199217963.\end{aligned}\right\} \quad (73)$$

The error response to a unit step input with this controller in the system is shown in Fig. 8. The initial value of the error is $e(0)=-7.33$ and differs from unity because with $G_m=G_d=k=0$ the performance index is finite and $S(\infty)\neq 1$. In fact, the optimal sensitivity function is

$$S_0=-\tilde{K}\frac{s(s-2)(s-\rho_4)(s-\rho_5)}{(s+1)(s+2)(s-s_0)(s-\bar{s}_0)} \quad (74)$$

where

$$\tilde{K}=-7.328575728765. \quad (75)$$

It is clear from Fig. 8 that even in the best of circumstances, no disturbance inputs, no measurement noise, and no plant saturation constraints, the best possible transient performance is poor. The reason is that this particular nonminimum-phase unstable plant is one of the most difficult to control irrespective of whether the policy is optimal or suboptimal. Indeed, since it is impossible to stabilize this plant $P(s)=(s-1)/s(s-2)$ by means of any dynamical stable compensation whatsoever [6], lead-lag methods are futile. In our opinion, a design methodology which can accomodate disturbance inputs, feedback sensor noise, rms restrictions on plant inputs, and also yield results as encouraging as those shown in Figs. 4 and 5, is a valuable engineering tool.

## Appendix

*Proof of Lemma 1:* The controller $C(s)=n_c(s)/d_c(s)$ is determined from the formula

$$S(s)=\frac{1}{1+P(s)F(s)C(s)}=\frac{d_f(s)d_p(s)d_c(s)}{\varphi(s)} \quad (76)$$

$$\varphi(s)=d_f(s)d_p(s)d_c(s)+n_f(s)n_p(s)n_c(s). \quad (77)$$

What must be shown is that $\varphi(s)$ is strict Hurwitz. According to Assumption 1, $S(s)$ is analytic in Re $s\geqslant 0$ and any zero $s_0$ of $\varphi(s)$ in Re $s\geqslant 0$ must be a zero of $d_f(s)d_p(s)d_c(s)$ and therefore of $n_f(s)n_p(s)n_c(s)$. If it is a zero of $d_f(s)d_p(s)$, it cannot be a zero of $n_f(s)n_p(s)$ because $P(s)$ and $F(s)$ are assumed to be admissible. Thus, $s_0$ must be a zero of $n_c(s)$ and consequently not one of $d_c(s)$ which is relatively prime to $n_c(s)$. But this means that the multiplicity of $s=s_0$ as a zero of $S(s)$ is less than its multiplicity as a zero of $d_f(s)d_p(s)$ which contradicts Assumption 2. If instead $s_0$ is assumed to be a zero of $d_c(s)$, it then follows that it is a zero of $n_f(s)n_p(s)$, but not a zero of $n_c(s)$. However, the expression

$$1-S(s)=\frac{n_f(s)n_p(s)n_c(s)}{\varphi(s)} \quad (78)$$

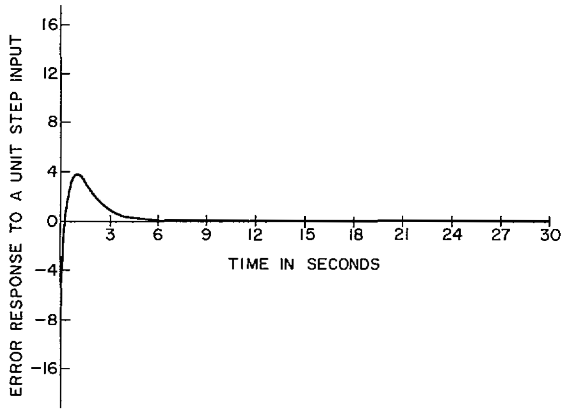


Fig. 8. Optimal error response.

then reveals that the multiplicity of $s=s_0$ as a zero of $1-S(s)$ is less than its multiplicity as a zero of $n_f(s)n_p(s)$, a contradiction with Assumption 3. Q.E.D.[10]

*Proof of Theorem 1:* In (21), the candidate functions $S(s)$ must all be realizable for the prescribed admissible pair $P(s)$, $F(s)$, Hence, if $S_0(s)$ is minimizing and $\epsilon$ is any real number,

$$S(s)=S_0(s)+\epsilon\chi_r(s)\delta(s)$$

is a legitimate competitor provided $\delta(s)$ is analytic in Re$s\geqslant 0$ and the resulting cost is finite. The proof of this assertion is simple. Both $S_0(s)$ and $S(s)$ must include all the zeros of $d_f(s)d_p(s)$ in Re $s\geqslant 0$ and so must the difference $S-S_0$. Again, all the zeros of $n_f(s)n_p(s)$ in Re$s\geqslant 0$ must be zeros of $1-S_0(s)$ and $1-S(s)$ and therefore of $(1-S_0)-(1-S)=S-S_0$. But $d_f(s)d_p(s)$ and $n_f(s)n_p(s)$ are relatively prime in Re$s\geqslant 0$ and it follows that $S(s)-S_0(s)$ is divisible by

$$(d_fd_p)_r\cdot(n_fn_p)_r=\chi_r.$$

This quotient $\epsilon\delta(s)$ is analytic in Re$s\geqslant 0$ and the order of its zero at $s=\infty$ must be sufficiently high to guarantee the finiteness of $E$. To exploit the optimality of $S_0(s)$, we set

$$\left.\frac{dE}{d\epsilon}\right|_{\epsilon=0}=0 \quad (79)$$

and use the standard Wiener–Hopf variational argument [3] to obtain

$$\chi_{r*}S_0(G_a+G_b)-\chi_{r*}\left(G_b-\frac{F-1}{FF_*}G_u\right)=X_*(s) \quad (80)$$

$X(s)$ analytic in Re$s\geqslant 0$. Performing the spectral factorization

$$\chi_r\chi_{r*}(G_a+G_b)=\Omega\Omega_* \quad (81)$$

where $\Omega(s)$ is free of zeros and poles in Re$s>0$ and dividing both sides of (80) by $\Omega_*/\chi_r$ gives, after re-

[10]The proof of necessity follows immediately from (76)+(78) and is trivial.

arrangement,

$$\Omega S_0 - \left\{ \frac{\chi_r \chi_{r*}}{\Omega_*}\left(G_b - \frac{F-1}{FF_*}G_u\right)\right\}_+ = \frac{\chi_r X_*}{\Omega_*} + \left\{ \frac{\chi_r \chi_{r*}}{\Omega_*}\left(G_b - \frac{F-1}{FF_*}G_u\right)\right\}_- . \quad (82)$$

Using (35) and (38) it is seen that

$$\chi\chi_*(G_a+G_b) = (n_p n_{p*} + kQ d_p d_{p*})\cdot(d_p n_f)G(d_p n_f)_*\cdot(d_f d_{f*}) \quad (82a)$$

and invoking Assumptions 1, 5, and 6 we conclude that $\Omega(s)$ is actually free of zeros and poles in $\mathrm{Re}\, s \geqslant 0$. Thus, $S_0(s)\Omega(s)$ must be analytic in $\mathrm{Re}\, s \geqslant 0$. But then the left-hand side of (82) is also analytic in $\mathrm{Re}\, s \geqslant 0$ and equals the right-hand side which is analytic in $\mathrm{Re}\, s<0$. Being analytic in the entire finite $s$-plane,

$$\Omega S_0 - \left\{ \frac{\chi_r \chi_{r*}}{\Omega_*}\left(G_b - \frac{F-1}{FF_*}G_u\right)\right\}_+ = f(s) \quad (83)$$

$f(s)$ a real polynomial; or

$$S_0 = \frac{\left\{ \frac{\chi_r \chi_{r*}}{\Omega_*}\left(G_b - \frac{F-1}{FF_*}G_u\right)\right\}_+ + f}{\Omega}, \quad (84)$$

which is (41).

Clearly, $S_0(s)$ is analytic in $\mathrm{Re}\, s \geqslant 0$. Since $G_b(\omega^2) = 0(\omega^{2l})$, $l \geqslant 0$, the convergence of (21) forces

$$1 - S_0(j\omega) = 0(1/\omega^{l+1}). \quad (85)$$

Write

$$\left\{ \frac{\chi_r \chi_{r*}}{\Omega_*}\left(G_b - \frac{F-1}{FF_*}G_u\right)\right\}_+ = \frac{h(s)}{g(s)} \quad (86)$$

and

$$\Omega(s) = \frac{a(s)}{b(s)}. \quad (87)$$

$h(s)$, $g(s)$, $a(s)$, and $b(s)$ are four real polynomials. Then, as is easily checked,

$$\text{degree } g \geqslant \text{degree } h \quad (88)$$

$$\text{degree } a = \text{degree } b + \text{degree } \chi_r + l \quad (89)$$

and

$$S_0 = \frac{(h+gf)b}{ag}. \quad (90)$$

To insure a proper $S_0(s)$, we must impose the degree restriction[11]

$$\delta(f) \leqslant \delta(a) - \delta(b) \quad (91)$$

or, using (89),

$$\delta(f) \leqslant \delta(\chi_r) + l. \quad (92)$$

To guarantee that $S_0(s)$ be realizable for $P(s)$, $F(s)$ every zero of $(d_f d_p)_r$ must be a zero of $S_0(s)$ and every zero of $(n_f n_p)_r$ must be a zero of $1-S_0(s)$. Coupling this with (85) we get a total of $\delta(\chi_r)+l+1$ interpolatory constraints on $f(s)$ which is one more than its permitted maximum degree, $\delta(\chi_r)+l$. Thus, $f(s)$ is the unique *Lagrange interpolation* polynomial [13] satisfying these conditions and if a minimizing $S_0(s)$ exists, it must be the one given by (84) because the cost functional is quadratic in $S(s)$.

Now part 2 is obviously correct and to complete the proof of Theorem 1 it suffices to show that $X_*(s)$, as given by (84), is actually analytic in $\mathrm{Re}\, s \leqslant 0$. Rearranging (80) with the aid of (81) and (84) leads to

$$\frac{\Omega\Omega_*}{\chi_r}\left(\frac{\{\Psi\}_+ + f}{\Omega}\right) - \frac{\Omega_*}{\chi_r}\Psi = \frac{\Omega_*}{\chi_r}(f - \{\Psi\}_-) \quad (93)$$

where

$$\Psi = \frac{\chi_r \chi_{r*}}{\Omega_*}\left(G_b - \frac{F-1}{FF_*}Gu\right). \quad (94)$$

It is apparent that (93) is analytic in $\mathrm{Re}\, s<0$ and it only remains to show that the same is true on the $s=j\omega$-axis. We observe first that the analyticity of $\{\Psi\}_-$ for $s=j\omega$ is implied by Assumption 2, (82a), and Assumption 6. Thus, (93) is analytic on the $j\omega$-axis if the purely imaginary zeros of $\chi_r$ are also zeros of $f-\{\Psi\}_-$ of at least the same multiplicities. But this is automatic whenever $f$ is chosen to make $S_0(s)$ realizable for the pair $F(s)$, $P(s)$. For suppose that $s=j\omega_0$ is a zero of $\chi_r$ of order $\nu$. Then, invoking Assumption 1, it is either a pole of $P$ or a zero of $FP$ of order $\nu$.[12] Suppose it is a pole of $P$. Since the only $j\omega$-poles of $G_b$ are either zeros of $F$ or zeros of $P$, (94) shows that $s=j\omega_0$ is a zero of $\Psi$ of order at least $\nu$. It now follows from the identity

$$\{\Psi\}_+ + f = \Psi + (f - \{\Psi\}_-) \quad (95)$$

that $s=j\omega_0$ is a zero of $f-\{\Psi\}_-$ of multiplicity $\nu$ or higher [see (84)].

Suppose instead that $s=j\omega_0$ is a zero of $\chi_r$ which is a zero of $FP$ of order $\nu$. A direct calculation yields

$$1 - S_0 = \left(1 - \frac{\Psi}{\Omega}\right) - \frac{1}{\Omega}(f - \{\Psi\}_-) \quad (96)$$

and from (81) and (94) we obtain

$$1 - \frac{\Psi}{\Omega} = \frac{G_a + \frac{F-1}{FF_*}G_u}{G_a + G_b}. \quad (97)$$

With the aid of (82a) and Assumption 6 it is seen that

[11] $\delta(\cdot) \equiv$ degree $(\cdot)$.

[12] According to Assumption 1, $F$ is $j\omega$-analytic.

$s=j\omega_0$ is a zero of the right-hand side of (97) of order $\nu$ or more. Since $f$ has been constructed to make $S_0$ realizable for $F$ and $P$, $s=j\omega_0$ is a zero of $1-S_0$ of order at least $\nu$ and therefore, in view of (96), a zero of $f-\{\Psi\}_-$ of multiplicity $\nu$ or greater. Q.E.D.

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**Dante C. Youla** (SM'59–F'75) was born in Brooklyn, NY, on October 17, 1925. He received the B.E.E. degree from the City College of New York, New York, NY, in 1947 and the M.S. degree from New York University, New York, NY, in 1960.

From 1947 to 1949 he was employed as an Instructor in the Department of Electrical Engineering at C.C.N.Y. He attended the N.Y.U. Graduate School of Mathematics as a full-time student from 1948 to 1950, and for the next two years, worked at Fort Monmouth, NJ and the Brooklyn Naval Shipyard on problems of UHF and microphonics. In 1952 he joined the Communication Group at the Jet Propulsion Laboratory, Pasadena, CA, and participated in the design of antijam radio links for guided missiles. In 1955 he began his association with the Microwave Research Institute, Polytechnic Institute of Brooklyn, where he engaged in the practical and theoretical study of codes for combating noise and improving efficiency. He is now Professor of Electrical Engineering at the Polytechnic Institute of New York, Farmingdale, NY, working actively on network synthesis and control problems, stability of time-variable systems, solid-state devices, and $n$-port filtering.

Prof. Youla is a member of Tau Beta Pi and Sigma Xi.

**Joseph J. Bongiorno, Jr.** (S'56–M'60) was born in Brooklyn, NY, on August 3, 1936. He received the B.E.E., M.E.E., and D.E.E. degrees from the Polytechnic Institute of Brooklyn, Brooklyn, NY, in 1956, 1958, and 1960, respectively.

While working toward his graduate degrees, he was first a Junior Research Fellow at the Microwave Research Institute and later an Instructor in the Department of Electrical Engineering at Polytechnic. He was supported by the Grumman Aircraft Engineering Corporation during the period in which his doctoral dissertation work was completed. He is currently Professor of Electrical Engineering at the Polytechnic Institute of New York, Farmingdale, NY. He is presently also a consultant at Sperry Systems Management Division. He has been a consultant to Bell Laboratories and the Sperry Gyroscope Company.

Dr. Bongiorno is a member of Sigma Xi and Eta Kappa Nu.

**Hamid A. Jabr** (S'73) was born in Arab Palestine in 1945. He received the B.S. degree in electrical engineering (power systems) from Ain Shams University, Cairo, Egypt, in 1966, and the M.S. and the Ph.D. degrees, both in electrical engineering–system science from the Polytechnic Institute of New York, Brooklyn, NY, in 1973 and 1975, respectively.

From 1967 to 1970 he worked as an Electrical Field Engineer for Nitrogen Fertilizer Plant, Homs, Syria. From 1970 to 1971 he was employed as an Electrical Engineer with the Public Works Department, Riyadh, Saudi Arabia. From 1973 to 1975 he was a Senior Research Fellow at the Polytechnic Institute of New York. In September 1975 he joined the faculty of the University of Petroleum and Minerals, Dhahran, Saudi Arabia. His current research interests are in multivariable control theory with applications to power systems, process control, and nuclear power plants.

Dr. Jabr is a member of Sigma Xi.