

The pressure curve for a rubber balloon

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An equation is derived relating the internal pressure of a rubber balloon to its radius. The theoretical pressure curve is shown to be experimentally verifiable in the case of low to moderate extensions, as long as the effects of hysteresis are ignored. The problem of the equilibrium configuration of two interconnected balloons is also discussed.

I. INTRODUCTION

It has long been known that rubberlike materials do not deform according to Hooke's Law. (A rubberlike material is defined as any which, after being stretched to many times its original dimensions, will resume those dimensions immediately on release of tension.) After the discovery of vulcanization in 1839, a great deal of effort was spent to experimentally determine the properties of natural rubber, but no adequate analytical treatment of the subject appeared until a hundred years later, in a paper by James and Guth.¹ Their model treated rubber as a network of long, flexible molecular chains, which are randomly linked to form a coherent structure. It is the links, introduced by the curing process, which determine the overall form of the material under zero stress. Using statistical methods, a set of equations was derived relating the force on a cube of rubber to its temperature and relative elongation. The equations have since been confirmed experimentally,² at least in the case of small to moderate linear extensions, and have served as a basis for numerous experimental and theoretical investigations.

The purpose of this paper is to describe the application of James and Guth's theory to a simple problem, namely, finding the air pressure inside of a spherical rubber balloon as a function of its radius. In Sec. II, we obtain the pressure equation analytically. In Sec. III, we describe a simple apparatus for measuring a balloon's pressure, and the results obtained with it. In Sec. IV, the pressure curves are compared, and the discrepancies are explained in a qualitative fashion. The results are then applied to a simple example, that of the equilibrium configuration of two interconnected balloons.

II. THEORETICAL PRESSURE CURVE

James and Guth's stress-strain relations for a rectangular parallelepiped of rubber may be written³

$$f_i = \frac{1}{L_i} \left[kKT \left(\frac{L_i}{L_i^0} \right)^2 - pV \right], \quad (1)$$

where f_i is the externally applied force in the i direction, L_i is a linear dimension, k is Boltzmann's constant, K is a constant related to the number of possible network configurations of the sample, T is the absolute temperature, L_i^0 is an unstretched dimension, p is the internal (hydrostatic) pressure, and V is the volume of the sample (assumed here to be a constant⁴). Thus, the force consists of two parts: the first one (caused by the network) gives a tendency to contract, while the second gives a tendency to expand.

Assuming an isothermal extension, Eq. (1) becomes

$$f_i = (C_1/L_i)(\lambda_i^2 - C_2p), \quad (2)$$

with $\lambda_i = L_i/L_i^0$ the relative extension. In the case of a thin-walled spherical shell, all the force which acts to stretch the rubber is directed tangentially to the surface. The radial force (i.e., the force acting to compress the shell wall) may therefore be set equal to zero, so that

$$\lambda_r^2 = (t/t_0)^2 = C_2p, \quad (3)$$

where t_0 and t refer to the initial and final thicknesses, respectively. Assuming once again that the volume of the shell does not change, then r^2t is constant, and Eq. (3) becomes

$$p = \frac{1}{C_2} \left(\frac{r_0}{r} \right)^4, \quad (4)$$

where r is the balloon's radius. We may therefore write for the tangential force

$$f_t \propto (r/r_0^2) [1 - (r_0/r)^6]. \quad (5)$$

Integrating the internal air pressure over one hemisphere gives⁵

$$P_{\text{in}} - P_{\text{out}} \equiv P = \frac{f_t}{\pi r^2} = \frac{C}{r_0^2 r} \left[1 - \left(\frac{r_0}{r} \right)^6 \right], \quad (6)$$

where r_0 is the balloon's uninflated radius.

This equation is graphed in Fig. 1.⁶

Equation 6 predicts that the internal pressure P will reach a maximum for

$$r = r_p = 7^{1/6} r_0 \approx 1.38r_0 \quad (7)$$

and will drop to zero as r approaches infinity.

III. EXPERIMENTAL PRESSURE CURVE

In order to test the theory, several more-or-less spherical rubber balloons of the dime-store variety were attached to a device consisting of a U -tube manometer, an air hose, and a two-way valve. The internal pressure was then measured at various degrees of inflation. Each balloon was worked thoroughly before testing to insure a moderate degree of reproducibility during the test runs; as is well known, a balloon which has never been stretched is much harder to inflate than one which has been blown up a few times. The external dimensions were measured at each step along three mutually perpendicular diameters, and the average radius was taken to be

$$r = (D_1 D_2 D_3)^{1/3}/2. \quad (8)$$

It was quickly discovered that the pressure is not a single-valued function of the radius; there are considerable losses from hysteresis⁷ which is such that the internal

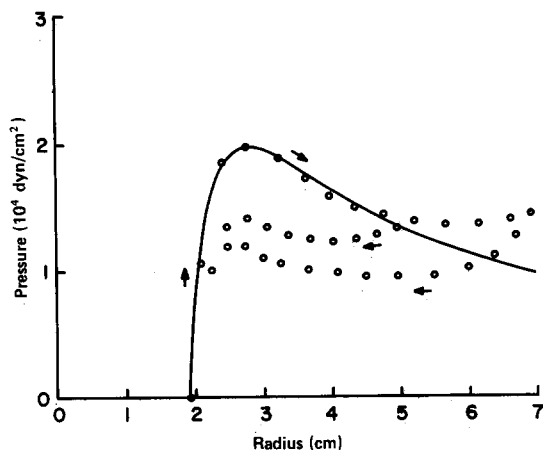


Fig. 1. Pressure curves for a rubber balloon. Circles are experimental points, obtained for two different turnaround radii. Solid curve is Eq. (6), adjusted to pass through the observed pressure maximum.

pressure during inflation is always larger than during deflation for a given radius. As long as the balloon was inflated and deflated smoothly, however, a continuous hysteresis loop was always obtained. Figure 1 shows some typical results. The size and shape of the loop depend on the radius at which turnaround occurs. Figure 2 shows the effect of the opposite sort of turnaround, i.e., from decreasing to increasing load.

In all cases, the curves measured under inflation can be fitted quite well by an equation of the form of Eq. (6), for radii up to about $2.5r_0$. For the curves of Fig. 2, an equation of the form

$$P = P_0 + \frac{C}{(r_0')^2 r} \left[1 - \left(\frac{r_0'}{r} \right)^6 \right], \quad (9)$$

with P_0 and r_0' the pressure and radius at the second turnaround, gives a good fit. In this case, C must be adjusted so that each secondary curve passes through the primary turnaround point, since this was observed to occur in all cases.

IV. DISCUSSION

The derived curve in Fig. 1 was fitted to the measured ones by matching the points of maximum pressure. The lack of fit at the low-radius end is no doubt due to the nonsphericity of the test samples in this region. At large radii, an upswing in the measured pressure always occurred, which cannot be explained in this way. There are, however, several molecular effects which become important at large extensions and which James and Guth ignored: crystallization, imperfect flexibility of the molecular chains, steric hindrances, and the like.⁸ The net result of these effects is to increase the stress at large extensions.

An interesting application of these results may be found in the problem, posed, for example, by J. S. Miller,⁹ of what occurs when two balloons of unequal radii are connected via an open tube. Gas transference will naturally occur from the balloon at higher pressure to the one at lower pressure, with the immediate result that the lower-pressure balloon will become larger; but to an observer this can appear to occur in one of two ways, depending on whether it is the smaller or the larger balloon which is initially at the higher pressure. In other words, a casual inspection of the relative sizes of the two balloons is not enough to determine which

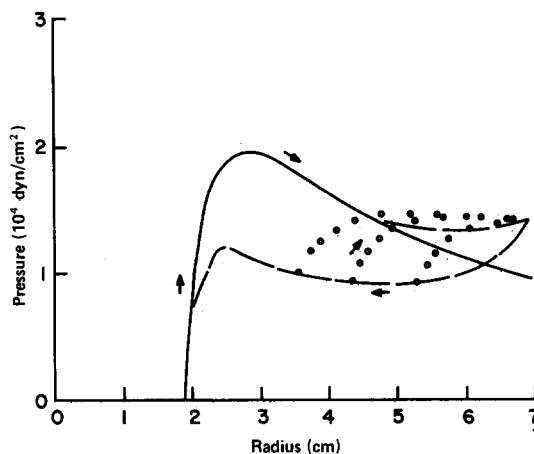


Fig. 2. Hysteresis in a rubber balloon. Solid curve is Eq. (6); broken curve reproduces one of the measured pressure curves in Fig. 1. Circles are the measured pressure points for reinflation from three different turnaround points.

will shrink and which will expand. Furthermore, the configuration at equal pressures can take one of two forms, depending on the initial conditions. Consider two balloons which are described by the ideal curve in Fig. 1. If the lower-pressure balloon is initially of *greater* radius, then it is clear from the pressure curve that at equilibrium the radii will be even more unequal; the only attainable configuration is that in which the two balloons are on opposite sides of the pressure peak. If the lower-pressure balloon is initially of *lesser* radius, then at equilibrium the radii may either be equal or unequal, depending on whether the total quantity of air in the balloons constrains them to come to rest on the left side or on opposite sides of the peak.

In the case of nonideal balloons the approach to equilibrium is more complicated, as may be seen from an inspection of Figs. 1 and 2. Once inflated, the two balloons will move along their separate hysteresis curves, the exact forms of which are dependent upon the manner in which inflation took place and the initial direction of change. It is clear that equilibrium will generally be obtained with a much lesser change in radius than would have occurred in the ideal case. Indeed, so greatly does hysteresis act to dampen changes in size of a real balloon that there is usually only a very small variation in the radii, unless one balloon is initially very close to r_0 .

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¹H. M. James and E. Guth, *J. Polym. Sci.* **4**, 153 (1949).

²See, for example, S. M. Gumbrell *et al.*, *Trans. Faraday Soc.* **49**, 1495 (1953); or F. P. Baldwin *et al.*, *J. Appl. Phys.* **26**, 750 (1954).

³D. ter Haar, *Elements of Statistical Mechanics* (Rinehart, New York, 1954), p. 328.

⁴Reference 3, p. 320.

⁵See, for example, F. Sears and M. Zemansky, *University Physics*, 4th ed. (Addison-Wesley, Reading, MA, 1973), p. 190.

⁶The interested reader may wish to compare this equation to one obtained by R. Kubo [*J. Phys. Soc. Jpn.* **8**, 312 (1949)], whose derivation was based, in a less straightforward way, on the results of earlier statistical theories.

⁷Hysteresis is discussed in R. Houwink and H. K. de Decker, *Elasticity, Plasticity and Structure of Matter*, 2nd ed. (Cambridge U.P., London, 1971), p. 226.

⁸See Ref. 1 for a discussion of James and Guth's assumptions. Crystallization due to stress is discussed in Ref. 7, p. 225.

⁹J. S. Miller, *Am. J. Phys.* **29**, 115 (1952).