Second degree price discrimination

Yossi Spiegel

1. Introduction

Second degree price discrimination refers to cases where a firm does not have precise information about the preferences of individual customers but it can use non-linear tariffs in order to extract the relevant information from its customers. At the optimum, the firm typically offers a menu of offers and buyers self-select from this menu. Second degree price discrimination is undoubtedly much more complex to analyze than first degree or third degree price discrimination. This is because the combination of non linear tariffs and asymmetric information about buyers requires certain techniques to solve the firm's problem. These techniques were not available until the early 70's. The major breakthrough was by Mirrlees in his 1971 paper about non linear taxation under asymmetric information. In 1996 Mirrlees got the Nobel prize for this contribution. The methodology that Mirrlees developed is the basis for the analysis of second degree price discrimination. Since this methodology is so widely used we shall study it in detail.

2. The model

A monopoly sells a single product to a continuum of buyers. Buyers differ from one another with respect to their utilities: the utility of a representative buyer if he buys is given by

$$U(\theta) = V(q,\theta) - p, \tag{1}$$

where p is the price and θ is the buyer's type. If a buyer does not buy his utility is 0. As for q, we can give this parameter at least two interpretations. First we can think of q as a quantity (i.e., the number of units the buyer buys). In that case, p is the price of a bundle that contains q units and $V(q,\theta)$ is the gross consumer surplus (i.e., the area under the inverse demand function). This interpretation corresponds to Maskin and Riley's (1984) paper. Second we can think of q as a measure of the quality of the product. Under this interpretation, each buyer is

interested in buying only one unit, p is the price of that unit, and $V(q,\theta)$ is the direct utility from consumption, ignoring the loss of income due to the payment to the monopoly. The interpretation of q as quality corresponds to Mussa and Rosen's (1978) paper. As usual, we will assume that $V_q > 0 > V_{qq}$ so the direct utility from consumption increases with q (the number of units the buyer buys or the quality of the good he buys) at a decreasing rate. Since V_q is the marginal willingness to pay, then if we interpret q is quantity then V_q is simply the inverse demand function so the assumptions that $V_q > 0 > V_{qq}$ amount to assuming that the inverse demand function is positive and decreasing.[1] Similarly, if we interpret q is quality, the assumption that $V_q > 0 > V_{qq}$ implies that the inverse demand function for quality is positive and decreasing.

The variable θ (the buyer's type) is distributed on the interval $[\theta_0, \theta_1]$ according to a distribution function $f(\theta)$ and a cumulative distribution function $F(\theta)$. Assume that $V_\theta > 0$ and $V_{q\theta} > 0$ so higher types have higher utility levels and higher marginal utilities.[2] Since V_q can be thought of as the inverse demand function, the assumption that $V_{q\theta} > 0$ means that higher types have higher demand functions. This assumption is important because it guarantees that the inverse demand functions for two different types never cross one another. In addition we will assume that $V_{qq\theta} \leq 0$. Keeping in mind the interpretation of V_q as the inverse demand function, the assumption that $V_{qq\theta} \leq 0$ means that the slope of the inverse demand functions is steeper for lower types. This situation is shown in Figure 2.

The firm has a the same cost for serving each buyer (so there is no interaction between different buyers through the monopoly's cost like in the case of third degree price discrimination). The cost of serving a buyer depends on q and it is given by $c(q)$, where $c' > 0$, $c'' \geq 0$. Hence, if we adopt the interpretation of q as quality, this assumption means that it is

---

[1] Note that given a uniform price, a buyer's utility is given by $V(q,\theta)-pq$. The quantity that the buyer would like to buy at that price is given implicitly by $V_q(q,\theta) = p$. The equation can be also interpreted as describing the highest price that the buyer will agree to pay when he already buys q units. This is by definition the inverse demand function of the buyer.

[2] If the utility of the buyer when he does not buy is not 0 as I have assumed above (i.e., $V(0,\theta) \neq 0$), then the assumption that $V_\theta > 0$ should be replaced with the assumption that $d/d\theta(V(q,\theta)-V(0,\theta)) = V_\theta(q,\theta)-V_\theta(0,\theta) > 0$ for all $q > 0$.

gross consumer surplus
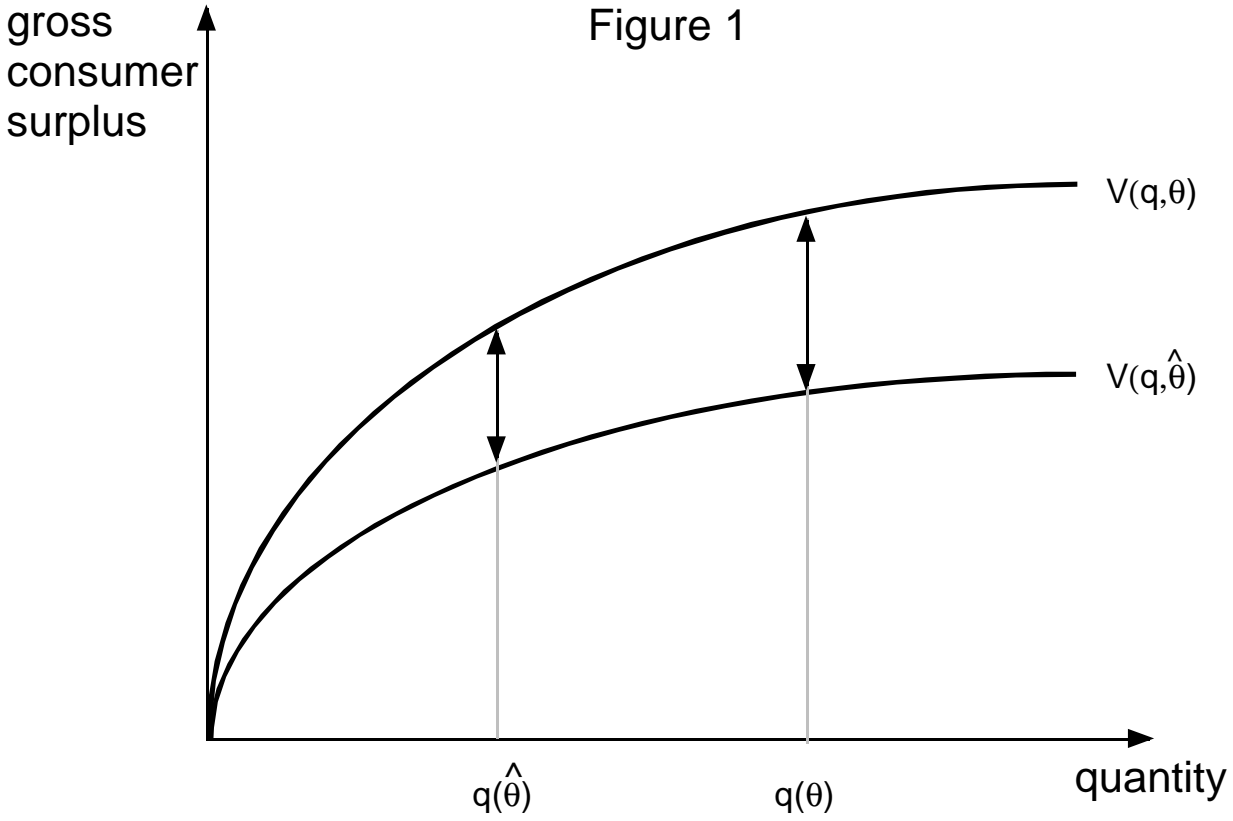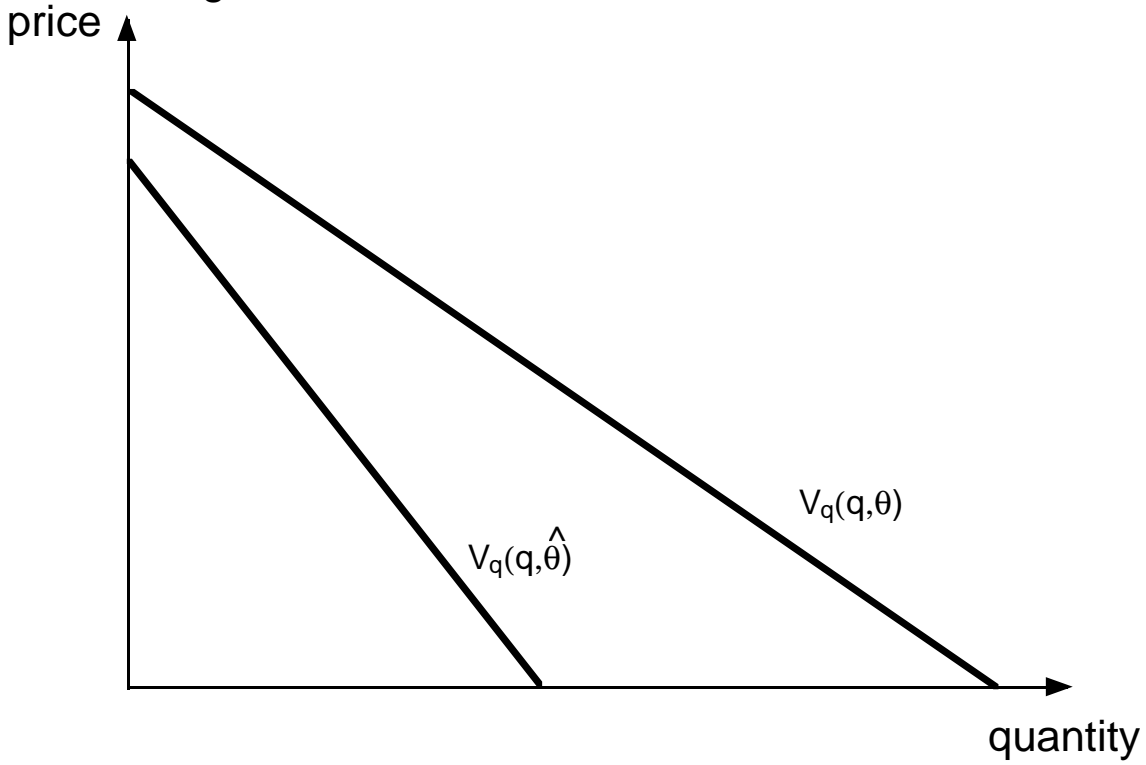
Figure 1

$V(q,\theta)$

$V(q,\hat{\theta})$

$q(\hat{\theta})$ $q(\theta)$ quantity

Figure 2: The inverse demand functions

price

$V_q(q,\theta)$

$V_q(q,\hat{\theta})$

quantity

The aggregate demand function

more costly to produce higher quality products and the marginal cost increases at an increasing rate. When we'll treat q as quantity in a bundle, it will be more natural to assume that c" = 0. (It is hard to believe that a 2 liter bottle costs to produce more than twice than a one liter bottle - in fact quite the opposite might be true. But to keep up with the assumption that c" ≥ 0 which is natural in the case of quality, let's assume that when q is quantity, c" = 0.)

3. The full information benchmark

When the monopoly knows the buyers' type, we are in a situation of first degree price discrimination. The monopoly's problem then is as follows:

$$\underset{p,q}{Max} \ p - c(q) \tag{2}$$

$$s.t. \ V(q,\theta) - p \ \geq \ 0.$$

That is the monopoly maximizes profits subject to the constraint that the buyer will buy. Clearly at the optimum it must be the case that the constraint holds with equality otherwise the monopoly can raise p and make more money. Substituting from the constraint into the objective function and maximizing with respect to q, the optimal q is given implicitly by the following first-order condition:

$$V_q(q,\theta) \ = \ c'(q). \tag{3}$$

Let $q^*(\theta)$ be the solution to this equation. The optimal price is then given by $p^*(\theta) = V(q^*(\theta),\theta)$. Equation (3) shows that at the optimum the monopoly equates the marginal benefit of the consumer from q with the marginal cost of q. The choice of q is socially optimal because the monopoly captures the entire social surplus by making sure that at the optimum the buyer makes zero utility and is just indifferent between buying and not buying.

4. Asymmetric information

Now suppose that the buyer's type is private information (i.e., only the buyer knows his own θ). The monopoly only knows the distribution of types but cannot tell the type of an individual

buyer. In order to find out the prices and quantities/qualities that the monopoly will offer we will model the situation as a Bayesian game. In this game, the monopoly chooses a menu of price and quantity/quality pairs, and each buyer selects at most one pair from the menu. If a buyer does not select any pair from the menu, he does not buy anything and does not pay anything. But if he selects the pair (p, q), then he pays the monopoly p and gets in return a quantity/quality q. Since the monopoly can put himself in the buyers shoes, he anticipates their selections for every menu he offers them. Therefore, we can characterize the optimal menu offered by the monopoly by using the revelation approach which is the standard way to solve these kind of problems. According to this approach, instead of offering the buyers a menu with pairs of p and q, the monopoly can ask each buyer for his type and select for the buyer the pair that the buyer whose type coincides with the report would have selected in the original game. To illustrate, suppose that in the original game, a buyer whose type is θ' chose the pair (p', q'). Now in the revelation game, if a buyer will report that his type is θ' the monopoly will select for him the pair (p', q'). Of course if some type, say θ", chose not to select anything from the original menu, the monopoly will select nothing for buyers who will report that their type is θ".

In short, the monopoly will design a menu of price and quantity/quality pairs, (p(θ), q(θ)), such that each pair will be geared towards one type of a buyer. Then each buyer will report his type to the monopoly, and the monopoly will then charge him the price and will give him the quantity/quality that was designed that reported type. The monopoly's strategy along with the buyer's report are referred to as a mechanism and will be denoted by M = {p(θ), q(θ), θ ∈ [θ_0, θ_1]}.

In order to find the optimal mechanism from the monopoly's point of view, we can invoke the revelation principle. This principle says that we can restrict attention to mechanisms in which the buyers report their types truthfully.[3] Such mechanisms are called **Incentive**

---

[3] This of course does not mean that the buyers will always report their types truthfully. It only means that at the optimum there is no loss of generality in looking at mechanisms in which they do report their types truthfully. The reason for this is straightforward: suppose that we had a mechanism in which the monopoly offers buyers a menu that contains pairs of p and q and each buyer selects one pair from this menu. Suppose that among all possible such mechanism (all the possible menus of pairs of p and q) there is one that is best for the monopoly and suppose moreover that in this optimal mechanism, the buyer whose type is θ' selects the pair (p',

Compatible (IC) and they satisfy the following set of (IC) constraints:

$$U(\theta) \equiv U(\theta \,|\, \theta) \geq U(\hat{\theta} \,|\, \theta), \qquad \forall \hat{\theta} \in [\theta_0, \theta_1], \quad \forall \theta \in [\theta_0, \theta_1]. \qquad \textbf{(4)}$$

The IC constraints state the each buyer is better-off reporting his type truthfully than pretending to be some other type. Or using an alternative interpretation, each type prefers to pick the pair that geared towards him than picking any other pair. Since buyers can always get a zero level of utility it must be the case that

$$U(\theta) \geq 0, \qquad \forall \theta \in [\theta_0, \theta_1]. \qquad \textbf{(5)}$$

These type of constraints are called **Individual Rationality** (IR) or **Voluntary Participation** (VP) constraints. A mechanism is said to be **feasible** is it satisfies the IC and IR constraints.

4.1 The monopoly's problem

Given an incentive compatible mechanism $M = \{p(\theta),\, q(\theta),\, \theta \in [\theta_0,\, \theta_1]\}$, the monopoly's problem is to solve the following problem:

---

q'). Since the monopoly can always puts himself in the buyers' shoes, he realizes that this is what the buyer will do, so he can design an equivalent revelation mechanism. In this mechanism, each buyer announces his type to the monopoly and the monopoly in turn selects for the buyer a pair of p and q from the menu, based on the buyers report. The monopoly can always design this revelation mechanism in such a way that once a buyer announces that his type is θ' he gets the pair (p', q'). That is, the monopoly designs the new mechanism such that p(θ') = p' and q(θ') = q'. Since type θ' selects the pair (p', q') in the original game he will have an incentive to report his true type in the revelation mechanism in order to receive again the pair (p' q') which is the most preferred pair from his point of view. Hence, the monopoly can design the revelation mechanism in a way that leads to exactly the same outcome as in the original mechanism. Since the original mechanism was the best from the monopoly's point of view, so will be the new mechanism. Note that in the new mechanism, each buyer reports his type truthfully, since otherwise he gets a pair that is not optimal from his point of view.

$$\underset{p(\theta),q(\theta)}{Max} \int_{\theta_0}^{\theta_1} [p(\theta) - c(q(\theta))] f(\theta) d\theta \qquad (6)$$

$$s.t. \quad IC$$

$$IR.$$

In general the problem is very messy. Hence we shall now simplify it in order to characterize the optimal menu that the monopoly will offer. We will do so by simplifying the IC and IR constraints in a series of claims. This will enable us in the end to solve a much simpler problem. This approach follows the methodology developed by Baron and Myerson's (1982) classic paper on optimal regulation under asymmetric information (see also Baron 1989).

4.2 Simplifying the constraints

Claim 1: *IC implies that $U(\theta)$ is monotonically increasing, so $U(\theta_0) \geq 0$ is sufficient to ensure that all the IR constraint are satisfied.*

Proof of Claim 1: Using IC and recalling that $V_\theta > 0$, it follows that for all $\theta > \theta_0$:

$$U(\theta) \equiv V(q(\theta),\theta) - p(\theta) \geq V(q(\theta_0),\theta) - p(\theta_0) > V(q(\theta_0),\theta_0) - p(\theta_0) \equiv U(\theta_0). \quad (7)$$

Hence if IR holds for type θ_0, it surely holds for all types greater than θ_0. ∎

Remark: Claim 1 implies that we can replace all IR constraints with the single constraint $U(\theta_0) \geq 0$. Moreover, we can already tell that at the optimum it must be the case that $U(\theta)_0 = 0$, otherwise the monopoly can raise $p(\theta_0)$ further.

Claim 2: *Incentive compatibility implies*

$$U(\theta) = \int_{\theta_0}^{\theta} V_\theta(q(\tilde{\theta}),\tilde{\theta}) d\tilde{\theta} + U(\theta_0), \qquad (8)$$

where $U(\theta_0)$ is the utility of the lowest type of the buyer.

Remark: The first term on the right side of equation (8) is referred to in the literature as the "information rent" of the type θ buyer because we know that under full information each buyer gets a zero utility. Thus the buyers can have a positive surplus only because the monopoly needs to "bribe" them in order to induce them to reveal their private information regarding their types.

Proof of Claim 2: Suppose that type $\hat{\theta}$ reports truthfully. Then, his utility is

$$U(\hat{\theta}) = V(q(\hat{\theta}),\hat{\theta}) - p(\hat{\theta}), \tag{9}$$

Using this expression, the IC constraint for type θ can be written as

$$U(\theta) \geq U(\hat{\theta}|\theta) = U(\hat{\theta}) + V(q(\hat{\theta}),\theta) - V(q(\hat{\theta}),\hat{\theta}). \tag{10}$$

Similarly, the IC constraint of type $\hat{\theta}$ can be written as

$$U(\hat{\theta}) \geq U(\theta|\hat{\theta}) = U(\theta) + V(q(\theta),\hat{\theta}) - V(q(\theta),\theta). \tag{11}$$

Combining the inequalities (10) and (11) we get

$$V(q(\theta),\theta) - V(q(\theta),\hat{\theta}) \geq U(\theta) - U(\hat{\theta}) \geq V(q(\hat{\theta}),\theta) - V(q(\hat{\theta}),\hat{\theta}). \tag{12}$$

Assuming without a loss of generality that $\theta > \hat{\theta}$, dividing the inequalities by $\theta-\hat{\theta}$ and taking the limit as $\theta-\hat{\theta}$ goes to zero yields

$$U'(\theta) = V_\theta(q(\theta),\theta), \tag{13}$$

almost everywhere. Since by assumption, $V_\theta > 0$, equation (13) shows that, as we found in Claim 1, higher types indeed get a higher level of utility at the optimum. Thus, we can integrate (13) from θ_0 to θ to obtain equation (8). ∎

Remark: If we assume that $p(\theta)$ and $q(\theta)$ are differentiable, we can derive equation (13) more easily from the IC constraints. To see how, note that incentive compatibility means that

$$\theta = \underset{\hat{\theta}}{Argmax}\ U(\hat{\theta}|\theta). \tag{14}$$

That is, the best thing that a buyer can do is to report his type truthfully. Therefore it must be the case that

$$\frac{dU(\hat{\theta}|\theta)}{d\hat{\theta}}\bigg|_{\hat{\theta}=\theta} = V_q(q(\hat{\theta}),\theta)\frac{dq(\hat{\theta})}{d\hat{\theta}}\bigg|_{\hat{\theta}=\theta} - \frac{dp(\hat{\theta})}{d\hat{\theta}}\bigg|_{\hat{\theta}=\theta}$$

$$= V_q(q(\theta),\theta)\frac{dq(\theta)}{d\theta} - \frac{dp(\theta)}{d\theta} = 0. \tag{15}$$

Differentiating $U(\theta)$ with respect to θ and using this equation, yields equation (13). The proof of the claim is more general however because it does not rely on the assumption that $q(\theta)$ and $p(\theta)$ are differentiable. Indeed one of the purposes of the proof is to establish that $U(\theta)$ is differentiable almost everywhere.

Claim 3: *IC implies that*

$$p(\theta) = V(q(\theta),\theta) - \int_{\theta_0}^{\theta} V_\theta(q(\tilde{\theta}),\tilde{\theta})d\tilde{\theta} - U(\theta_0), \tag{16}$$

where $U(\theta_0) = 0$ is the utility of the lowest type.

Proof of Claim 3: Equation (16) can be derived from equations (1) and (13). Now let's verify that $p(\theta)$ indeed induces truthful revelation. To this end, suppose that a buyer whose true type is θ reports that his type is $\hat{\theta}$ and buys according to the pair $(p(\hat{\theta}),\ q(\hat{\theta}))$. The buyer's utility then is given by

$$U(\hat{\theta}|\theta) = V(q(\hat{\theta}),\theta) - p(\hat{\theta}) = V(q(\hat{\theta}),\theta) - V(q(\hat{\theta}),\hat{\theta}) + \int_{\theta_0}^{\hat{\theta}} V_\theta(q(\tilde{\theta}),\tilde{\theta})d\tilde{\theta} + U(\theta_0). \tag{17}$$

The optimal report for the buyer is given by the following first-order condition:

$$\frac{dU(\hat{\theta}\,|\,\theta)}{d\hat{\theta}} = \left[V_q(q(\hat{\theta}),\theta) - V_q(q(\hat{\theta}),\hat{\theta})\right]\frac{dq(\hat{\theta})}{d\hat{\theta}} + V_\theta(q(\hat{\theta}),\hat{\theta}) - V_\theta(q(\hat{\theta}),\hat{\theta}) = 0. \quad \textbf{(18)}$$

This equation implies that at a local optimum, $\hat{\theta} = \theta$, so the buyer will indeed have an incentive to report his type truthfully at least locally. Truthful revelation is optimal at this point only locally because we did not show up to now that $\hat{\theta} = \theta$ is a global optimum (indeed we are yet to show that the second order condition is satisfied).

Claim 4: *A monotonically increasing q(θ) is a necessary and sufficient condition for truthful revelation to be globally optimal.*

Remark: Claim 4 establishes that any mechanism such that the q(θ) is monotonically increasing in θ is incentive compatible. This claim simplifies the monopoly's problem immensely since it will allow us to replace the IC constraints with the constraint that q(θ) must be an increasing function (in fact we will not even do that: we will simply solve an unconstrained maximization problem and then verify that the optimal q(θ) is monotonically increasing in θ). It implies that higher types buy bigger bundles or if q represents quality, they buy better products. Of course, it can't be that higher types get higher q and pay lower prices as this would give lower types an incentive to mimic them, thus violating IC. This can be verified by differentiating p(θ) with respect to θ:

$$p'(\theta) = V_q(q(\theta),\theta)\,q'(\theta) + V_\theta(q(\theta),\theta) - V_\theta(q(\theta),\theta) = V_q(q(\theta),\theta)\,q'(\theta), \quad \textbf{(19)}$$

which is positive when q'(θ) > 0.

Proof of Claim 4: First we will show that a necessary condition for global incentive compatibility is that q(θ) is a monotonically increasing function. From (12) we know that for all $\hat{\theta}$, $\theta \in [\theta_0, \theta_1]$, the following inequality must hold if the mechanism is incentive compatible:

$$V(q(\theta),\theta) - V(q(\theta),\hat{\theta}) \geq V(q(\hat{\theta}),\theta) - V(q(\hat{\theta}),\hat{\theta}). \quad \textbf{(20)}$$

Now assume without a loss of generality that $\theta > \hat{\theta}$. Because $V_\theta > 0$, both sides of the inequality

are positive. Moreover because $V_{q\theta} > 0$, the left side can exceed the right side only if $q(\theta) > q(\hat{\theta})$. Hence, we showed that global incentive compatibility is satisfied only if $q(\theta)$ is a monotonically increasing function.

Next we will show that the necessary condition is also sufficient. In other words, we need to show that if $q(\theta)$ is a monotonically increasing function then the IC constraints given by (4) hold. To this end, assuming again (without a loss of generality) that $\theta > \hat{\theta}$, and subtracting equation (17) from (8) yields,

$$U(\theta) - U(\hat{\theta}|\theta) = -\left[V(q(\hat{\theta}),\theta) - V(q(\hat{\theta}),\hat{\theta})\right] + \int_{\hat{\theta}}^{\theta} V_{\theta}(q(\tilde{\theta}),\tilde{\theta})d\tilde{\theta}$$

$$\text{(21)}$$

$$= \int_{\hat{\theta}}^{\theta} \left[V_{\theta}(q(\tilde{\theta}),\tilde{\theta}) - V_{\theta}(q(\hat{\theta}),\tilde{\theta})\right]d\tilde{\theta}.$$

Since $V_{q\theta} > 0$ it is clear that $q(\theta) > q(\hat{\theta})$ ensures that $U(\theta) - U(\hat{\theta}|\theta) > 0$ so the monotonicity of $q(\theta)$ is also a sufficient condition for global incentive compatibility. ∎

The first part of Claim 4 is demonstrated in Figure 1. The two sides of inequality (20) describe the vertical distance between the gross consumer surpluses of two buyers and are represented by the two arrows. Given that $\theta > \hat{\theta}$, we know that from the assumption that $V_{\theta} > 0$, that the gross consumer surplus of the θ type buyer lies above the gross consumer surplus of the $\hat{\theta}$ type buyer. Moreover, from the assumption that $V_{q\theta} > 0$, we also know that the vertical distance between the two curves must increase as we move to the right since $V(.,\theta)$ is steeper than $V(.,\hat{\theta})$. Consequently, inequality (20) can hold only if $q(\theta) > q(\hat{\theta})$. Graphically, the properties of the gross consumer surplus imply that the left arrow in Figure 1 is shorter than the right arrow so it must be that $q(\theta) > q(\hat{\theta})$.

4.3 Rewriting the monopoly's maximization problem

Having simplified the set of constraints, we are now in a position to rewrite the monopoly's problem. Since $U(\theta_0) = 0$ is sufficient to ensure that all IR constraints hold and since requiring that $q(\theta)$ is a monotonically increasing function is necessary and sufficient for global incentive

compatibility, the monopoly's problem can be rewritten as follows:

$$\underset{p(\theta),q(\theta)}{Max} \int_{\theta_0}^{\theta_1} [p(\theta) - c(q(\theta))] f(\theta) d\theta$$

(22)

$$s.t. \quad q(\theta) \text{ is increasing in } \theta$$

$$U(\theta_0) = 0.$$

Substituting from equation (16) for p(θ) into the objective function and imposing the constraint that $U(\theta_0) = 0$, the monopoly's problem becomes:

$$\underset{q(\theta)}{Max} \int_{\theta_0}^{\theta_1} \left[V(q(\theta),\theta) - \int_{\theta_0}^{\theta} V_\theta(q(\theta),\theta) d\theta - c(q(\theta)) \right] f(\theta) d\theta$$

(23)

$$s.t. \quad q(\theta) \text{ is increasing in } \theta.$$

We therefore turned the problem into a problem of finding an optimal function, q(θ), subject to a monotonicity constraint. The common "trick" now (due to Mirrlees, 1971) is to ignore the constraint, solve the problem and verify later that the solution satisfies the constraint. But before we solve the problem we can simplify it a bit further. To this end note that after integration by parts, the expression

$$\int_{\theta_0}^{\theta_1} \left[\int_{\theta_0}^{\theta} V_\theta(q(\theta),\theta) d\theta \right] f(\theta) d\theta,$$

(24)

becomes

$$\left[\int_{\theta_0}^{\theta} V_\theta(q(\theta),\theta) \, d\theta \right] F(\theta) \Bigg|_{\theta_0}^{\theta_1} - \int_{\theta_0}^{\theta_1} V_\theta(q(\theta),\theta) F(\theta) d\theta$$

$$= \int_{\theta_0}^{\theta_1} V_\theta(q(\theta),\theta) d\theta - \int_{\theta_0}^{\theta_1} V_\theta(q(\theta),\theta) F(\theta) d\theta \qquad (25)$$

$$= \int_{\theta_0}^{\theta_1} V_\theta(q(\theta),\theta)(1 - F(\theta)) d\theta.$$

Substituting this expression in (23), the monopoly's problem becomes:

$$\underset{q(\theta)}{Max} \int_{\theta_0}^{\theta_1} I(q(\theta),\theta) f(\theta) d\theta, \qquad (26)$$

where

$$I(q(\theta),\theta) = V(q(\theta),\theta) - c(q(\theta)) - V_\theta(q(\theta),\theta) \frac{(1 - F(\theta))}{f(\theta)}. \qquad (27)$$

4.4 The optimal menu

Let q*(θ) be the solution to the monopoly's maximization problem written in (26). We can solve for q*(θ) by maximizing the integral in (26) pointwise. That is, for every θ, q*(θ) is the value of q that maximizes the expression I(q(θ),θ). Since we repeat this procedure for every value of θ we obtain the optimal q as a function of θ. Pointwise maximization is possible since there is no "interaction" between the optimal choices for different types in the sense that the integrand in (26) involve terms that depend only on one value of θ. The first order condition for an interior solution for q*(θ) is

$$V_q(q(\theta),\theta) = c'(q(\theta)) + V_{q\theta}(q(\theta),\theta) H(\theta), \qquad (28)$$

where H(θ) $\equiv$ (1-F(θ))/f(θ) is the inverse of the "hazard rate" of the distribution of θ. The left side of the equation is simply the marginal willingness of the buyer to pay for an extra unit of

q (or the inverse demand function of the buyer). The first expression is the marginal cost of an extra unit of q. These two expression are exactly the same as in the full information case. The difference between the current case and the full information case is that we now have an additional term on the right side of the equation which represents an additional cost from the monopoly's point of view. This term is the marginal "informational rent" of type θ. The optimal choice of $q(\theta)$ is illustrated Figure 3: $q^*(\theta)$ is simply the quantity at which the inverse demand function, $V_q(.)$, intersects the sum of the marginal cost of production, $c'(.)$, and the marginal information rents, $V_{q\theta}(.)H(\theta)$.

The second order condition for a maximum is given by

$$I_{qq}(q^*(\theta),\theta) = V_{qq}(q^*(\theta),\theta) - c''(q^*(\theta)) - V_{qq\theta}(q^*(\theta),\theta)\,H(\theta) < 0. \tag{29}$$

This condition is certainly satisfied if $V_{qq\theta} \leq 0$ because then along with the assumptions that $V_{qq} < 0$ and $c''(q) \geq 0$, it ensures that $I(q,\theta)$ is concave in q. In what follows we shall assume that this condition holds so the first order condition is also sufficient.

Note from equation (28) that at the top where $\theta = \theta_1$, $H(\theta_1) = 0$. Hence $q(\theta_1)$ is exactly as in the full information case. We therefore establish the important property that the highest type gets the optimal quantity/quality without distortions. This is known as the "no distortion at the top" property. This property is due to the fact that the highest type has the highest wiliness to pay. Consequently, the monopoly has an incentive to induce this type to make an optimal decision on q which in turn generates the highest surplus for this type which the monopoly can partly share through properly setting $p(\theta_1)$. All other types get lower q's than they get at the full information case. This is because the monopoly needs to distort their bundles in order to ensure that the high type does not wish to mimic them (otherwise the monopoly cannot separate the types from one another).

It might be that for some types, the quantity needs to be distorted downward to the point where the monopolist would like to exclude them from the market altogether. For such types the solution to equation (28) will be negative (i.e., the two curves in Figure 3 intersect to the left of the y-axis) which is of course impossible since $q \geq 0$. Thus whenever the solution to equation (28) yields negative values of q for low types, then these types will not be served at all: the monopolist will offer them empty bundles (and of course they will not buy at all). Whenever
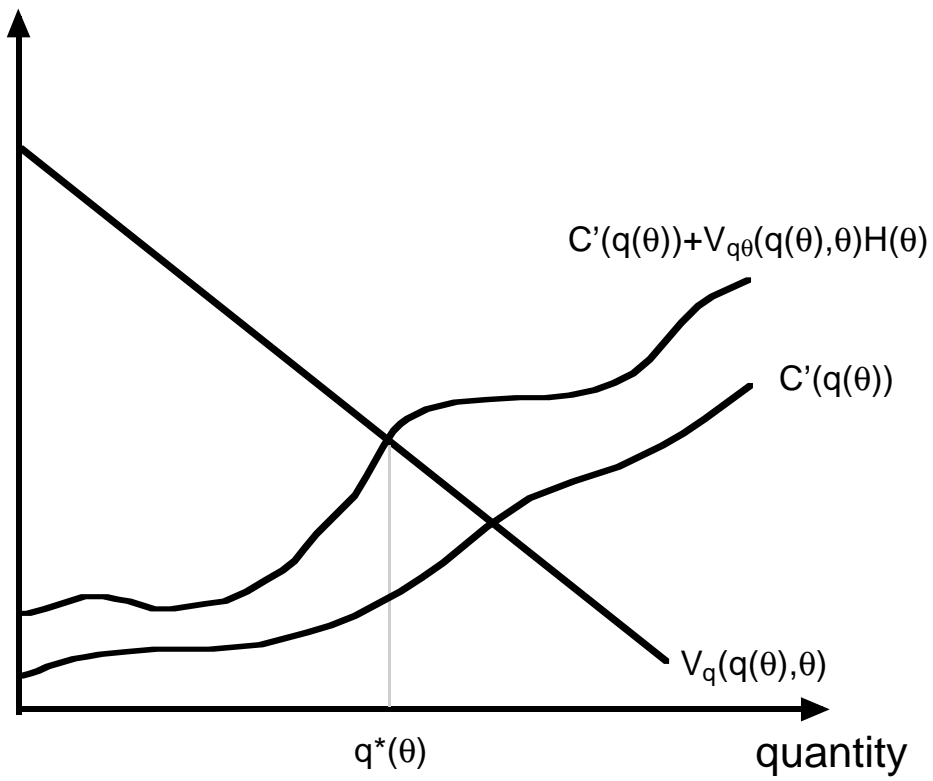
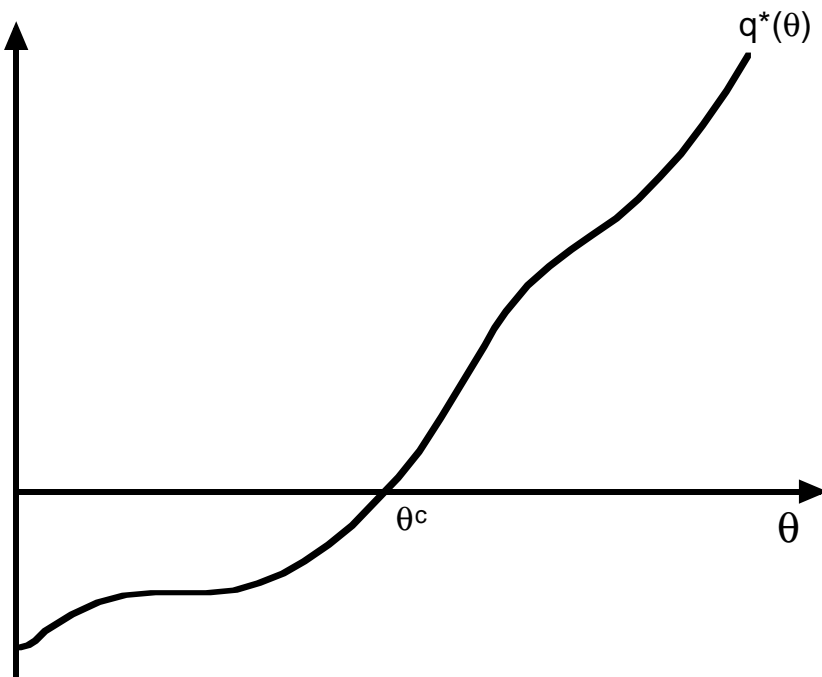Figure 3: Determining the optimal size of the bundle, $q^*(\theta)$

Figure 4: Determining the lowest type which is still served, θ^c

this is the case, we need to find the critical value of θ beyond which types are served. Assuming that $q^*(\theta)$ is indeed monotonically increasing as in Figure 4, this critical value of θ, denoted θ^c, is defined implicitly by $q^*(\theta^c) = 0$. Alternatively, using equation (28), the lowest type which is still served by the monopoly is given by

$$V_q(0,\theta^c) = c^{\,\prime}(0) + V_{q\theta}(0,\theta^c) H(\theta^c). \tag{30}$$

Thus, only types in the interval $[\theta^c, \theta_1]$ buy positive quantities.

Having solved for the optimal size of bundles we now need to solve for the prices of different bundles. First it is obvious that types who do not buy will not have to pay either (otherwise their IR constraint is violated). Thus we only need to solve for the prices that buyers that do buy positive quantities will have to pay. To this end note that so long as $q(\theta)$ is differentiable, then

$$
\begin{aligned}
V(q^*(\theta),\theta) &= \int_{\theta_0}^{\theta} \frac{dV(q^*(\tilde\theta),\tilde\theta)}{d\tilde\theta} d\tilde\theta \\
&= \int_{\theta_0}^{\theta} \left[V_q(q^*(\tilde\theta),\tilde\theta)\, q^{*\prime}(\tilde\theta) + V_\theta(q^*(\tilde\theta),\tilde\theta) \right] d\tilde\theta \\
&= \int_{\theta_0}^{\theta} V_q(q^*(\tilde\theta),\tilde\theta)\, q^{*\prime}(\tilde\theta) d\tilde\theta + \int_{\theta_0}^{\theta} V_\theta(q^*(\tilde\theta),\tilde\theta) d\tilde\theta.
\end{aligned} \tag{31}
$$

Substituting this expression in equation (16), recalling that at the optimum $U(\theta_0) = 0$, and evaluating the expression at $q^*(\theta)$, the optimal price function is given by:

$$p^*(\theta) = \int_{\theta^c}^{\theta} V_q(q^*(\tilde\theta),\tilde\theta) q^{*\prime}(\tilde\theta) d\tilde\theta. \tag{32}$$

Note that the lower bound in the integral is θ^c since $q^*(\theta) = 0$ for all $\theta < \theta^c$. The optimal menu is now fully characterized by pairs of $q^*(\theta)$ and $p^*(\theta)$, where $q^*(\theta)$ is given implicitly by equation (28) and $p^*(\theta)$ is given by (32).

4.5 Showing that q(θ) is monotonically increasing

We now need to verify that q*(θ) is monotonically increasing in θ to ensure that the mechanism is incentive compatible. Differentiation the first-order condition with respect to q and θ yields:

$$q^{*'}(\theta) \; = \; \frac{V_{q\theta}(q^*(\theta),\theta)\,\big(1-H'(\theta)\big) - V_{q\theta\theta}(q^*(\theta),\theta)\,H(\theta)}{-I_{qq}(q^*(\theta),\theta)}, \tag{33}$$

where I(q*(θ),θ) < 0 by the second order condition for the monopoly's problem. Thus, q*'(θ) provided that the numerator is positive. Sufficient conditions for this are: $V_{q\theta\theta} \le 0$ and H'(θ) ≤ 0. The latter condition says that the inverse of the hazard rate of the distribution of θ is non-increasing, or put differently, the hazard rate of the distribution of θ is non-decreasing. This condition is satisfied by many distributions including, uniform, normal, log-normal, exponential, logistic, Pareto, and all distributions with non-decreasing density functions.

4.6 What happens when q*(θ) is not monotonically increasing?

Since by Claim 4 the monotonicity of q*(θ) is necessary and sufficient for truthful revelation, it is clear that when q*(θ) is not monotonically increasing in θ, the optimal menu we computed is not incentive compatible. To deal with this problem we need to apply a procedure that is referred to in the literature as the **Ironing Procedure**. According to this procedure we modify q*(θ) to preserve monotonicity by picking an interval [θ$_2$, θ$_3$] that contains those types for which q*(θ) is decreasing (the interval however also contains types for whom q*(θ) is increasing) and setting q*(θ) constant over this interval. After "ironing" q*(θ) in this way, q*(θ) will be increasing up to θ$_2$, will be constant at q*(θ$_2$) for all θ ∈ [θ$_2$, θ$_3$], and will begin to increase again from θ$_3$ onward.

The full proof that this procedure indeed yields an optimal menu is provided in the Appendix to Maskin and Riley (1984). Here we shall only discuss the main ideas and provide the algorithm to compute the interval [θ$_2$, θ$_3$]. To this end, consider Figure 5 which replicates Figure 5 in Maskin and Riley. According to this figure, q*(θ) first increases with θ, reaches a local maximum at θ$_{max}$, then decreases with θ up to a local minimum at θ$_{min}$ and from then on it increases again. Let θ$_l$ be the lowest type whose quantity according to q*(θ) is equal to the

quantity that θ_{min} is supposed to buy, i.e., $q^*(\theta_l) = q^*(\theta_{min})$. Similarly let θ_h be the highest type whose quantity according to $q^*(\theta)$ is equal to the quantity that type θ_{max} is supposed to buy, i.e., $q^*(\theta_h) = q^*(\theta_{max})$. Note that for types outside the interval $[\theta_l, \theta_h]$, $q^*(\theta)$ is monotonically increasing as required. Since $q^*(\theta)$ was obtained by pointwise optimization of $I(q(\theta),\theta)$, it is clear that for types outside the interval $[\theta_l, \theta_h]$, $q^*(\theta)$ remains optimal. In other words, for types outside $[\theta_l, \theta_h]$, the monotonicity constraint is not binding and can be ignored. Hence we only need to deal with the "problematic" type inside the interval $[\theta_l, \theta_h]$ where $q^*(\theta)$ is not monotonic. Since in this interval the monotonicity constraint is binding, it is clear that we need to modify $q^*(\theta)$ and make it monotonic. The question then is how to do it optimally?

To address this question, note that since $q^*(\theta)$ increases up to θ_l and from θ_h onward, and since overall $q(\theta)$ has to be monotonically increasing, the "ironed" $q^*(\theta)$ must intersect the original $q^*(\theta)$ somewhere along the interval along which $q^*(\theta)$ is decreasing. Let's denote this intersection point by $\hat{\theta}$. Having defined $\hat{\theta}$, we next define θ_2 and θ_3 as the types for whom $q^*(\hat{\theta})$ $= q^*(\theta_2) = q^*(\theta_3)$. Note that by construction, the original $q^*(\theta)$ lies above $q^*(\hat{\theta})$ for types in the interval $[\theta_2, \hat{\theta}]$ but below $q^*(\hat{\theta})$ for types in the interval $[\hat{\theta}, \theta_3]$. Assuming as we do that $V_{qq} < 0$, $V_{qq\theta} \leq 0$, and $c''(q) \geq 0$, $I(q,\theta)$ is concave in q, so given the monotonicity constraint it is optimal to lower $q^*(\theta)$ just enough in the interval $[\theta_2, \hat{\theta}]$ to make it monotonic. Hence over the interval $[\theta_2, \hat{\theta}]$, it is optimal to set $q^*(\theta) = q^*(\hat{\theta})$. Similarly, because $I(q,\theta)$ is concave in q, it is optimal to increase $q^*(\theta)$ just enough for types in the interval $[\hat{\theta}, \theta_3]$ to ensure that it is monotonic. Hence over the interval $[\hat{\theta}, \theta_3]$, it is also optimal to set $q^*(\theta) = q^*(\hat{\theta})$. Consequently, $q^*(\theta) = q^*(\hat{\theta})$ for all $\theta \in [\theta_2, \theta_3]$. The "ironed" $q^*(\theta)$ is shown in Figure 5 by the thick line: it coincides with the original $q^*(\theta)$ to the left of θ_2 and to the right of θ_3 and is constant at $q^*(\theta_2)$ $= q^*(\theta_3)$ over the interval $[\theta_2, \theta_3]$.

Finally we need to find the boundaries of the interval $[\theta_2, \theta_3]$. To this end note that given the "ironed" $q^*(\theta)$, the monopoly's problem becomes
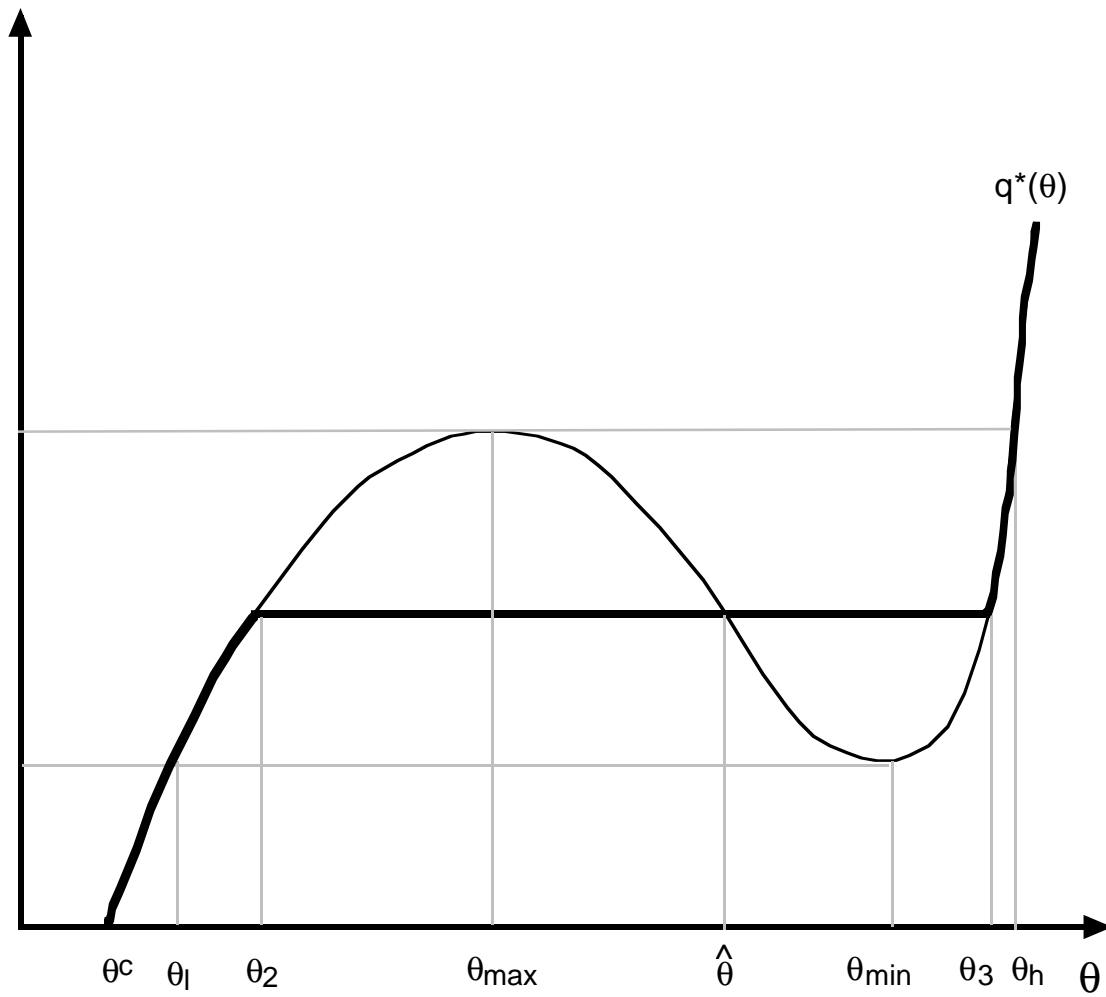
Figure 5: The ironing procedure

Figure 3: Determining the optimal size of the bundle, q*(θ)

$$Max_{\theta_2,\theta_3} \int_{\theta_0}^{\theta_2} I(q^*(\theta),\theta)f(\theta)d\theta + \int_{\theta_2}^{\theta_3} I(q^*(\theta_2),\theta)f(\theta)d\theta + \int_{\theta_3}^{\theta_1} I(q^*(\theta),\theta)f(\theta)d\theta,$$ (34)

$$s.t. \quad q^*(\theta_2) = q^*(\theta_3).$$

The first order conditions for this maximization problem are:

$$\int_{\theta_2}^{\theta_3} I_q(q^*(\theta_2),\theta)q^{*/}(\theta_2)f(\theta)d\theta + \lambda q^{*/}(\theta_2) = 0,$$ (35)

and

$$I(q^*(\theta_2),\theta_3)f(\theta_3) - I(q^*(\theta_3),\theta_3)f(\theta_3) - \lambda q^{*/}(\theta_3) = 0,$$ (36)

where λ is the Lagrange multiplier associated with the equality constraint. Noting that $q^*(\theta_2) = q^*(\theta_3)$, the first two terms in equation (36) vanish. Together with the fact that $q^{*'}(\theta_3) > 0$, it follows that $\lambda = 0$. Substituting $\lambda = 0$ in equation (35) and noting that $q^{*'}(\theta_2) > 0$, the boundaries of the interval for which $q^*(\theta)$ is "ironed" are determined by the equations

$$\int_{\theta_2}^{\theta_3} I_q(q^*(\theta_2),\theta)f(\theta)d\theta = 0, \qquad q^*(\theta_2) = q^*(\theta_3).$$ (37)

4.7 An Example

Suppose that $V(q,\theta) = \theta V(q)$ and $C(q) = cq$. Then the first order condition for $q^*(\theta)$ is given by:

$$\theta V_q(q(\theta)) = c + V_q(q(\theta)) H(\theta)$$

$$\Leftrightarrow$$ (38)

$$(\theta - H(\theta)) V_q(q(\theta)) = c.$$

Now, since c is a constant and $V_q < 0$, it follows that $q^{*'}(\theta) > 0$ provided that $\theta-H(\theta)$ is increasing with θ. For instance, if θ is distributed uniformly over the unit interval (i.e., $\theta_0 = 0$ and $\theta_1 = 1$), then $F(\theta) = \theta$ and $f(\theta) = 1$, so $H(\theta) = 1-\theta$ and $\theta-H(\theta) = 2\theta-1$ which is increasing

with θ. In the uniform distribution case, q*(θ) is given implicitly by

$$V_q(q(\theta)) = \frac{c}{2\theta - 1}.$$ **(39)**

If we assume in addition that $V(q) = q^\alpha$, where $\alpha < 1$ (to ensure that $V < 0$),

$$q^*(\theta) = \begin{cases} \left(\dfrac{c}{\alpha(2\theta - 1)} \right)^{\frac{1}{\alpha - 1}} & \theta \geq \dfrac{1}{2}, \\[4mm] 0 & \theta < \dfrac{1}{2}. \end{cases}$$ **(40)**

Thus the monopoly sells only to types above 1/2 (so in this case, $\theta^c = 1/2$) and since $\alpha < 1$, the quantity sold increases with θ and decreases with c. The price that a buyer whose θ exceeds θ^c pays can now be obtained from equation (32) (the computations are messy though).

4.8 Quantity discounts

Casual observation suggests that firms typically give quantity discounts. That is, the average price per unit when a large bundle is purchased is lower than the average price per unit when the bundle is small. To see under what conditions this is so, let's adopt the interpretation of q as representing the quantity being purchased. Using equation (32), the average price per unit paid by a buyer is therefore given by

$$AP^*(\theta) = \frac{\displaystyle\int_{\theta^c}^{\theta} V_q(q^*(\tilde{\theta}),\tilde{\theta})q^{*\prime}(\tilde{\theta})d\tilde{\theta}}{q^*(\theta)}.$$ **(41)**

Now, to see if the monopoly uses quantity discounts or not, recall that q*'(θ) > 0. Thus the monopoly uses quantity discounts if AP*'(θ) < 0 because then higher types who buy larger quantities also pay lower average prices. Differentiating AP*(θ) yields:

$$AP^{*'}(\theta) = \frac{V_q(q^*(\theta),\theta)\, q^{*'}(\theta)\, q^*(\theta) - q^{*'}(\theta) \int_{\theta^c}^{\theta} V_q(q^*(\tilde{\theta}),\tilde{\theta})\, q^{*'}(\theta)\, d\tilde{\theta}}{(q^*(\theta))^2}$$

$$= \frac{q^{*'}(\theta)}{(q^*(\theta))^2} \left[V_q(q^*(\theta),\theta)\, q^*(\theta) - \int_{\theta^c}^{\theta} V_q(q^*(\tilde{\theta}),\tilde{\theta})\, q^{*'}(\tilde{\theta})\, d\tilde{\theta} \right] \qquad (42)$$

$$= \frac{q^{*'}(\theta)}{(q^*(\theta))^2} \left[\int_{\theta^c}^{\theta} \left[V_q(q^*(\theta),\theta) - V_q(q^*(\tilde{\theta}),\tilde{\theta}) \right] q^{*'}(\tilde{\theta})\, d\tilde{\theta} \right],$$

where the last inequality follows because

$$V_q(q(\theta),\theta)q(\theta) = V_q(q(\theta),\theta) \int_{\theta^c}^{\theta} q'(\tilde{\theta})\, d\tilde{\theta} = \int_{\theta^c}^{\theta} V_q(q(\theta),\theta)\, q'(\tilde{\theta})\, d\tilde{\theta}. \qquad (43)$$

Recalling that q*'(θ) < 0, equation (42) indicates that AP*'(θ) < 0 provided that $V_q(q^*(\theta),\theta) < V_q(q^*(\tilde{\theta}),\tilde{\theta})$ for all θ > $\tilde{\theta}$. In other words, the monopoly uses quantity discounts if the marginal utility of high types is lower than the marginal utility of low types. Mathematically, this means that we need to show that

$$\frac{dV_q(q^*(\theta),\theta)}{d\theta} = V_{qq}(q^*(\theta),\theta)q^{*'}(\theta) + V_{q\theta}(q^*(\theta),\theta) < 0. \qquad (44)$$

Substituting from equation (33) for q*'(θ) and using equation (29) we get

$$\frac{dV_q(q^*(\theta),\theta)}{d\theta} = \frac{V_{qq}(q^*(\theta),\theta)\left[V_{q\theta}(q^*(\theta),\theta)(1 - H'(\theta)) - V_{q\theta\theta}(q^*(\theta),\theta)\, H(\theta) \right]}{-I_{qq}(q^*(\theta),\theta)}$$

$$+ \frac{V_{q\theta}(q^*(\theta),\theta)\left[-V_{qq}(q^*(\theta),\theta) + c''(q^*(\theta)) + V_{qq\theta}(q^*(\theta),\theta) \right]}{-I_{qq}(q^*(\theta),\theta)}. \qquad (45)$$

Assuming for simplicity that c''(.) = 0 (the monopoly has a constant marginal cost), and simplifying terms we get:

$$\frac{dV_q(q^*(\theta),\theta)}{d\theta} = \frac{-V_{qq}(q^*(\theta),\theta)\left[V_{q\theta}(q^*(\theta),\theta)\,H^/(\theta) + V_{q\theta\theta}(q^*(\theta),\theta)\,H(\theta)\right]}{-I_{qq}(q^*(\theta),\theta)}$$

$$+ \frac{V_{q\theta}(q^*(\theta),\theta)\,V_{qq\theta}(q^*(\theta),\theta)\,H(\theta)}{-I_{qq}(q^*(\theta),\theta)} \qquad (46)$$

$$= - \frac{\dfrac{d}{d\theta}\left(\dfrac{V_{q\theta}(q^*(\theta),\theta)\,H(\theta)}{V_{qq}(q^*(\theta),\theta)}\right)}{-I_{qq}(q^*(\theta),\theta)}.$$

Hence if we assume that $V_{q\theta}(q,\theta)H(\theta)/V_{qq}(q,\theta)$ is decreasing with θ (note that we fix q so in the expression, $V_{q\theta}(q,\theta)$ and $V_{qq}(q,\theta)$ depend on θ only directly but not through q) then the monopoly will give quantity discounts. This result makes sense since in many application it would be impossible to set prices such that $AP^{*'}(\theta) > 0$ because that would mean that buyers are better off buying many small bundles than buying one large bundle. In other words, if the average price increases with quantity, a buyer would be better off buying two packages that contain one unit each than buying one unit that contain two units. Of course quantity discounts may also be problematic since several buyers can buy one large bundle and share the units in the bundle among themselves thereby saving money relative to the case where each buys a small bundle. However in many real life cases, this does not happen since the hassle of sharing units is not worth the trouble (think for example on how many occasions you went to a store to buy coke and instead of buying a can you offered some other buyer who happened to buy coke at the same time to share a large bottle in order to save money).

4.9 Interpreting the optimal menu as a menu of two-part tariffs

If we interpret q as quantity, the optimal menu we computed contains different bundles (one for each type of a buyer) where each bundle contains a different number of units and is priced differently. However the firm does not necessarily have to offer buyers bundles in order to discriminate. Another way to create second degree price discrimination is by offering buyers a menu of two-part tariffs such that each item on the menu is a pair $(T(\theta), t(\theta))$, where $T(\theta)$ is a fixed charge and $t(\theta)$ is a price per unit. Given a two-part tariff $(T(\theta), t(\theta))$, a buyer needs to

pay T(θ) for the right to be served by the monopoly and then he can buy as many units as he likes to at a price of t(θ) per unit. However the monopoly anticipates that given t(θ), a type θ buyer will buy up to the point where

$$V_q(q(\theta),\theta) \;=\; t(\theta). \tag{47}$$

That is, the buyer will buy up to the point where his willingness to pay which is equal to $V_q(q(\theta),\theta)$, will be equal to the price per unit. Since $V_{qq} < 0$ while t(θ) is independent of q, there is a one-to-one relationship between t(θ) and q(θ), so allowing the buyer to buy at the price t(θ) is equivalent to allowing him to buy a bundle that contains q(θ) units. Therefore instead of offering the buyer a bundle with q*(θ) units the monopoly can offer him a price per unit t*(θ) such that

$$V_q(q^*(\theta),\theta) \;=\; t^*(\theta). \tag{48}$$

The amount that the buyer pays then is t*(θ)q*(θ), or $V_q(q^*(\theta),\theta)q^*(\theta)$. Previously we found that a θ type buyer pays p*(θ) when he buys q*(θ) units. Hence if T*(θ) = p*(θ) - $V_q(q^*(\theta),\theta)q^*(\theta)$, the two-part tariff (T*(θ), t*(θ)) is completely equivalent to the bundle (q*(θ), p*(θ)) in the sense that the buyer buys exactly the same number of units and pays exactly the same amount.

5. Using demand profiles to design optimal tariffs

An alternative method to design optimal tariffs under asymmetric information has been developed by Wilson (1993). The building block for this method is demand profiles. A demand profile, denoted N(p,q), specifies the fraction of buyers that purchase at least q units when the price per unit is p. Alternatively, N(p,q) can be interpreted as the fraction of buyers that are willing to pay a price p for the q-th unit. To define N(p,q) let θ(p,q) be the buyer's type who is willing to buy at least q units when the price per unit is p; θ(p, q) is implicitly defined by the equation $V_q(q,\theta(p,q))$ = p. Given this definition, we have

$$N(p,q) \;=\; 1 - F(\theta(p,q)). \tag{49}$$

That is, the fraction of buyers that buy at least q units is the fraction of buyers whose θ's exceed

θ(p,q). Given a demand profile and assuming for simplicity that the marginal cost of production is constant, the monopoly's profit from selling the q-th unit is given by:

$$\pi(p,q) = MN(p,q)(p-c), \tag{50}$$

where M is the total mass of buyers. The profit from selling the q-th unit is therefore the number of buyers who buy it, MN(p,q), times the per-unit price cost margin. Using this expression, the optimal price for the q-th unit, denoted p*(q), is given by the following first order condition:

$$\pi_p(p,q) = N(p,q) + N_p(p,q)(p-c) = 0. \tag{51}$$

A consumer who buys q units will therefore have to pay a total price of

$$P^*(q) = \int_0^q p^*(\tilde{q})d\tilde{q}. \tag{52}$$

Thus, we computed the optimal price for each unit by treating it as a completely separate good from the other units that are being sold and we found the optimal price for that unit alone.

To find the connection between this method and the method we saw thus far, note that from the definition of N(p,q) it follows that $N_p(p,q) = -f(\theta(p,q))\partial\theta(p,q)/\partial p$. Recalling that θ(p,q) is defined implicitly by $V_q(q,\theta(p,q)) = p$, and differentiating the equation with respect to p and θ yields $\partial\theta(p,q)/\partial p = 1/V_{q\theta}(q,\theta(p,q))$. Hence, $N_p(p,q) = -f(\theta(p,q))/V_{q\theta}(q,\theta(p,q))$. Substituting for N(p,q) and $N_p(p,q)$ in equation (51) we get

$$\pi_p(p,q) = 1 - F(\theta(p,q)) - \frac{f(\theta(p,q))(p-c)}{V_{qt}(q,\theta(p,q))} = 0. \tag{53}$$

Substituting $p = V_q(q,\theta(p,q))$ in the equation and rearranging terms yields equation (28) which was the first order condition for the firm's problem under the previous method. Hence the solutions are the same.

To illustrate Wilson's method, consider again the example that was analyzed above. In this example, $V(q,\theta) = \theta q^\alpha$, $F(\theta) = \theta$, and the total mass of consumers is 1. Hence,

$$N(p,q) = 1 - F(\theta(p,q)) = 1 - \frac{p}{\alpha q^{\alpha-1}}. \tag{54}$$

The monopoly's profit from selling the q-th unit is given by:

$$\pi(p,q) = \left(1 - \frac{p}{\alpha q^{\alpha-1}}\right)(p - c). \tag{55}$$

Maximizing with respect to p, the optimal price of the q-th unit is

$$p^*(q) = \frac{\alpha q^{\alpha-1} + c}{2}. \tag{56}$$

At this price, a consumer whose type is θ will buy up to the point where $V_p(q,\theta) = p^*(\theta)$. Assuming that there exists an interior solution and solving for $q^*(\theta)$ yields

$$\theta \alpha q^{\alpha-1} = \frac{\alpha q^{\alpha-1} + c}{2} \Leftrightarrow q^*(\theta) = \left(\frac{c}{\alpha(2\theta - 1)}\right)^{\frac{1}{\alpha-1}}, \tag{57}$$

Clearly, an interior solution exists only if $\theta \geq 1/2$. Otherwise, $q^*(\theta) = 0$. Hence we found that at the optimum, buyers buy exactly the same quantity as in the case where we computed optimal bundles (see equation (40)). This illustrates the equivalence of the two methodologies.

6. Problems

Problem 1

A monopoly produces a single good at a constant marginal cost per unit, c. The monopoly knows the distribution of buyers' types but cannot tell the specific type of each buyer. Suppose that the monopoly offers the buyers a menu of bundles, i.e., pairs of q and p. Compute the optimal menu of bundles that the monopoly will offer in the following cases:

(i) The utility of buyers and the cumulative distribution of θ are given by

$$U(\theta) = 1 - k e^{\theta - \frac{q}{k}}, \quad k > 0; \quad and \quad F(\theta) = 1 - e^{-\frac{\theta}{m}}, \quad \theta \geq 0, \quad 0 < m < 1.$$

(ii) The utility of buyers and the cumulative distribution of θ are given by

$$U(\theta) = \frac{\theta q^{1-\beta}}{1-\beta}, \quad \beta < 1; \quad and \quad F(\theta) = \theta^2, \quad \theta \in [0,1].$$

In each case:

(a) Solve for q(θ), verify that it is monotonically increasing in θ, and identify the lowest type which is still served, θ^c.

(b) Solve for the price of each bundle.

(c) Check whether the monopoly offers quantity discounts.

(d) Find the equivalent menu of two-part tariffs.

(e) Find the optimal non-linear tariff using Wilson's method.

Problem 2

<u>Remark</u>: The problem is meant to illustrate first degree price discrimination (part (a)) and monopsony behavior under uniform prices (part (b)).

A monopoly sells a homogenous good. The inverse demand function for the good is $p = 1-q$, where p denotes the price and q denotes the quantity that the monopoly sells (the monopoly cannot use price discrimination on the market - all units must be sold at the same price). The monopoly buys the good from a single manufacturer whose cost per unit is $c < 1$. The sequence of events is as follows: the monopoly first offers the manufacturer to buy from him q units and offers him in return a payment T. If the manufacturer accepts the offer, the monopoly takes the q units and sells them on the market. If the manufacturer rejects the offer there is no trade and all parties get a zero payoff.

(a)   Suppose that the monopoly knows c. How many units will the monopoly buy from the manufacturer and how much will he pay him? Compute the profits of the monopoly and the manufacturer.

(b)   Now suppose that the monopoly does not know c but knows that it is drawn from a uniform distribution on the interval [0, 1]. How many units will the monopoly buy from the manufacturer now and how much will he pay him? Compute the expected profit of the monopoly and the ex ante expected profit of the manufacturer's (by ex ante I mean before the manufacturer knows his type; hence the expectation is taken on all possible types).

References

Baron D. (1989), "Design of Regulatory Mechanisms and Institutions," in *The Handbook of Industrial Organization*, R. Schmalansee and R. Willig (eds.), Volume II, Chapter 24, 1347-1447, Elsevier Science Publishers B.V. (North Holland).

Baron D. and Myerson R. (1982), "Regulating a Monopolist with Unknown Costs," *Econometrica,* 50: 911-930.

Maskin E., and Riley J. (1984), "Monopoly with Incomplete Information," *Rand Journal of Economics,* 15: 171-196.

Mirrlees J. (1971), "An Exploration in the Theory of Optimal Taxation," *Review of Economic Studies,"* 38: 175-208.

Mussa M. and Rosen S. (1978), "Monopoly and Product Quality," *Journal of Economic Theory,* 18: 301-317.

Wilson R. (1993), *Nonlinear Pricing*, Oxford University Press.