Descent into Cache-Oblivion

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Outline

What is an External Memory Algorithm?

- The External Memory Model
- Two (simple) examples

Searching Cache-Obliviously

- Why aren't classic Binary Search Trees good enough?
- Static Cache-Oblivious Search Trees
- Making things Dynamic - allowing insertions and deletions

Sorting Cache-Obliviously

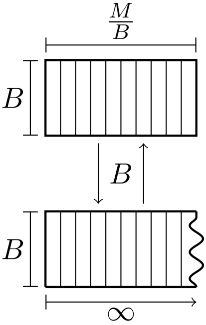
- Why isn't (binary) MergeSort good enough?
- FunnelSort - a Cache-Oblivious MergeSort

Summary and some Empirical Results

External Memory (EM) algorithms

The Problem

- Analysis in the RAM model relies on constant time memory access.



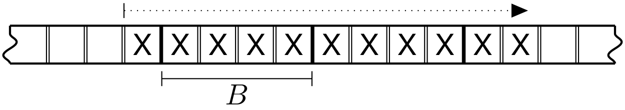
The Solution: The External Memory Model

- Two levels of memory, internal (of size M) and external (unbounded).
- Data transfer occurs in blocks of size B
- We analyse asymptotic I/O complexity.

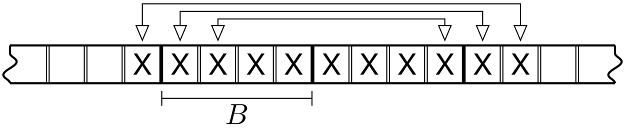
- A *Cache-Aware* algorithm knows M and B
- A *Cache-Oblivious* algorithm doesn't

Two (simple) examples of Cache-Oblvious algorithms

- Example 1: *Scanning* requires $\lceil \frac{N}{B} \rceil + 1 \in \Theta(\frac{N}{B})$ I/Os



- Example 2: *Array Reversal* requires $\lceil \frac{N}{B} \rceil + 1 \in \Theta(\frac{N}{B})$ I/Os



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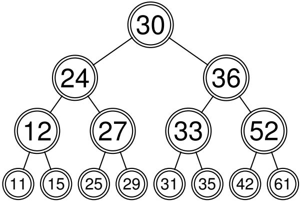

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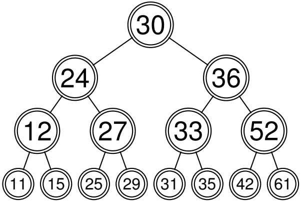

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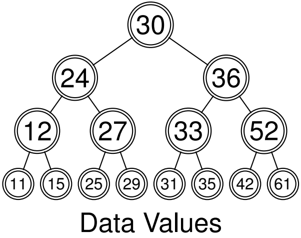

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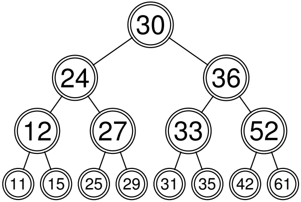

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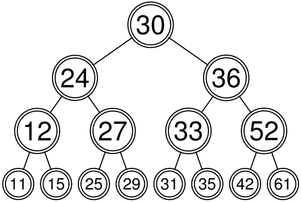

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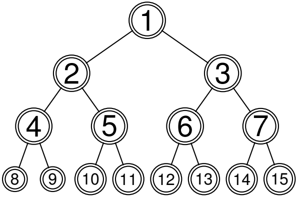

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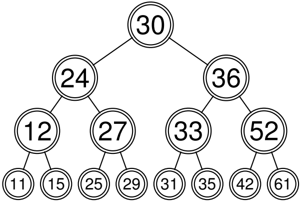

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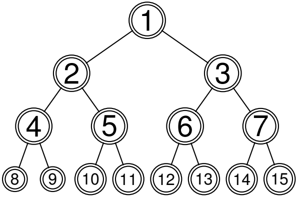

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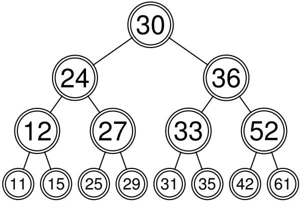

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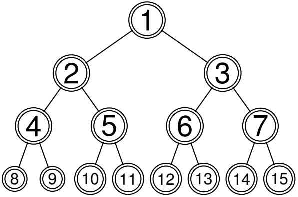

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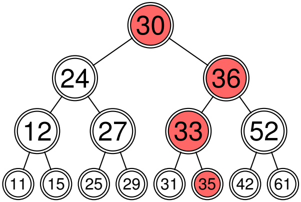

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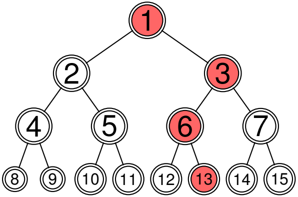

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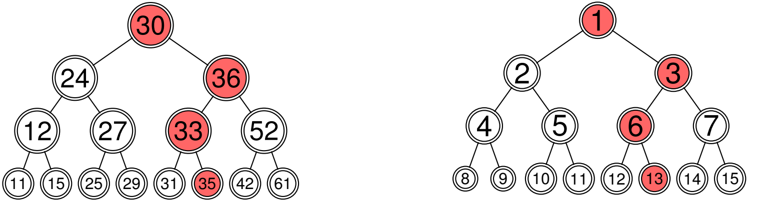

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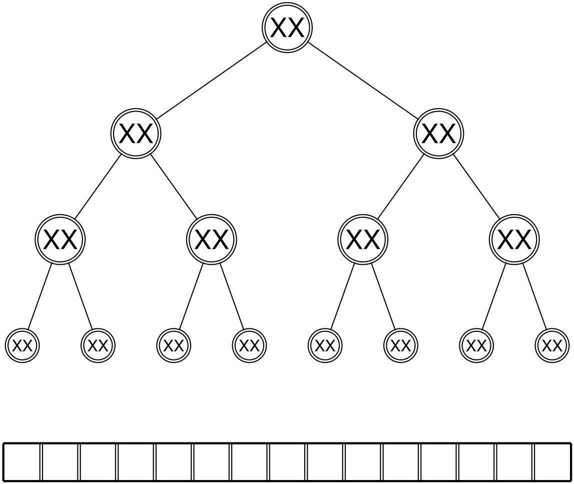
$$O(\log N - \log B) = O(\log(N/B)) \text{ I/Os}$$

An example of the Van Emde Boas layout



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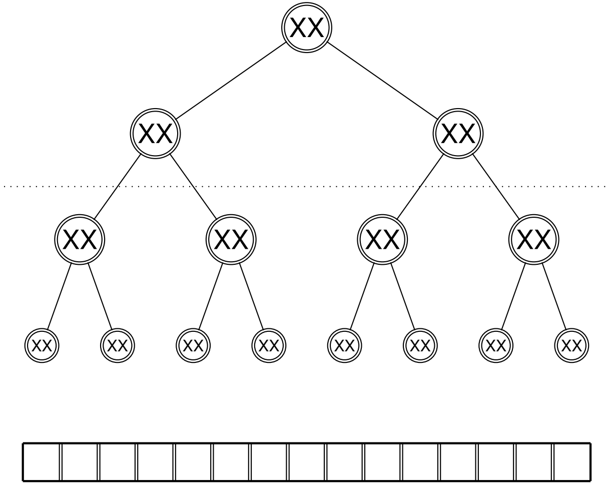

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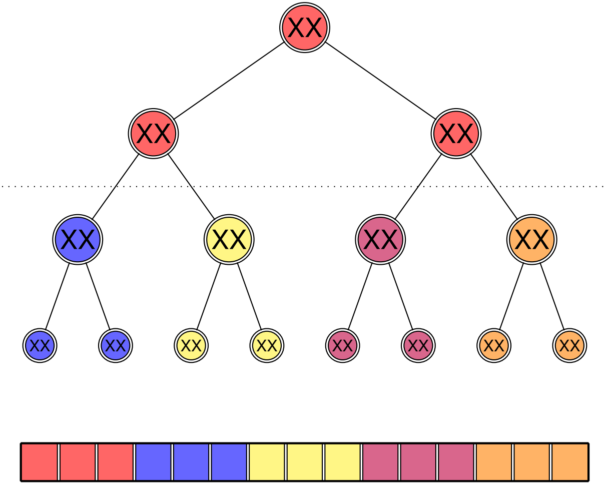

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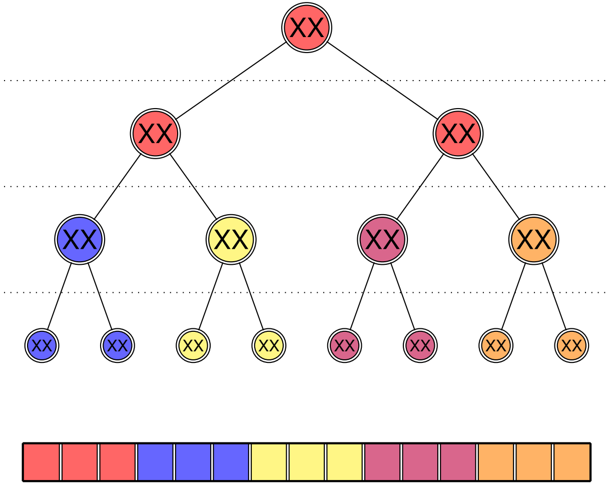

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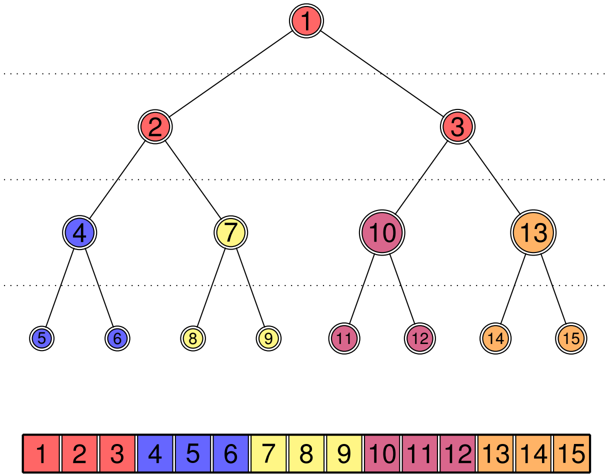

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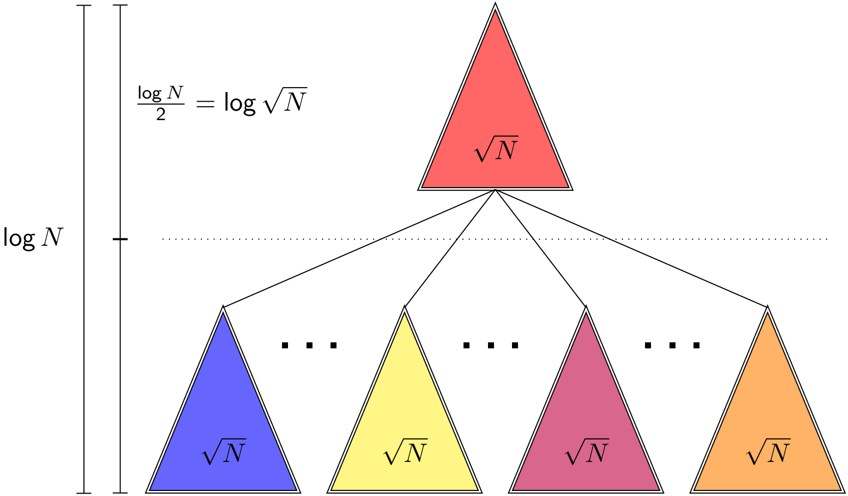
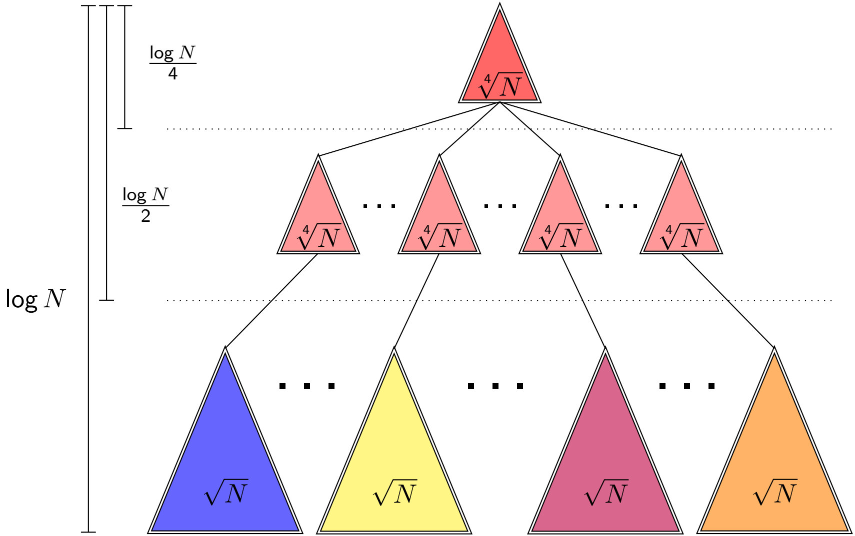
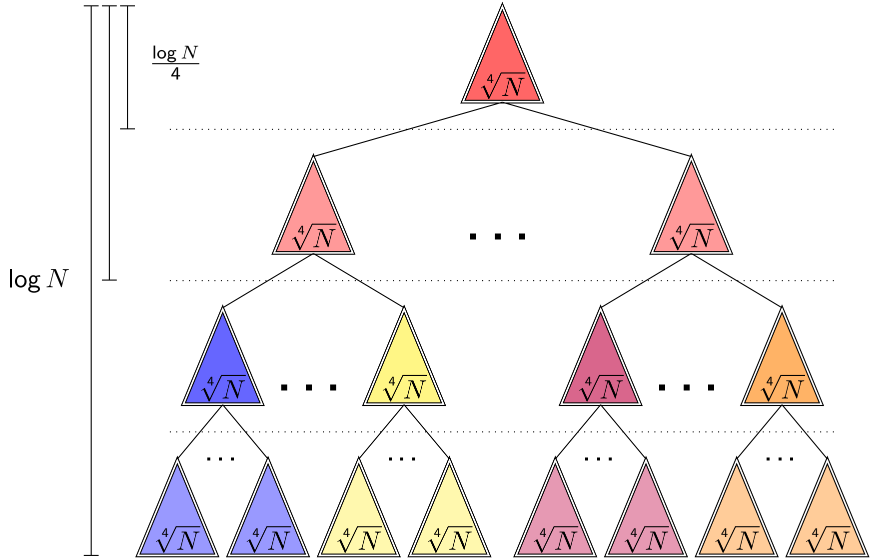


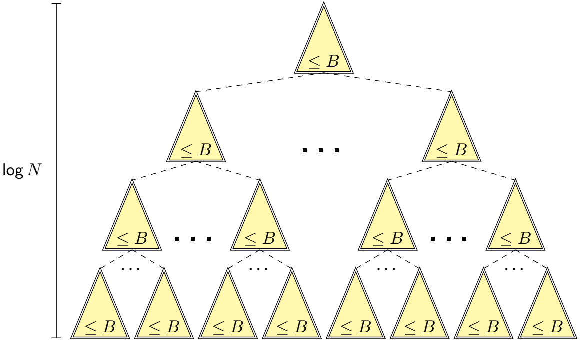
The Van Emde Boas layout -top expanded



The Van Emde Boas layout - level of detail $\sqrt[4]{N}$

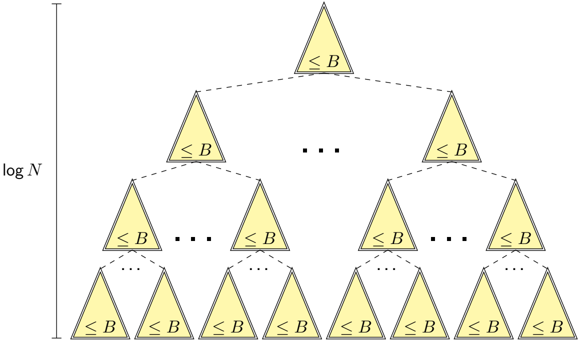


The Van Emde Boas layout - level of detail B



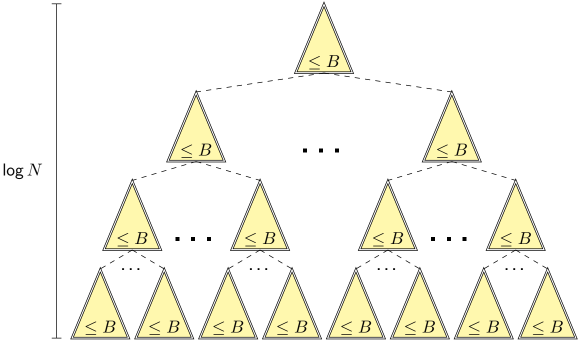
The Van Emde Boas layout - level of detail B

- conceptually recurse until all subtrees are smaller than B



The Van Emde Boas layout - level of detail B

- conceptually recurse until all subtrees are smaller than B
- the subtrees are at least as large as $\sqrt{B}$



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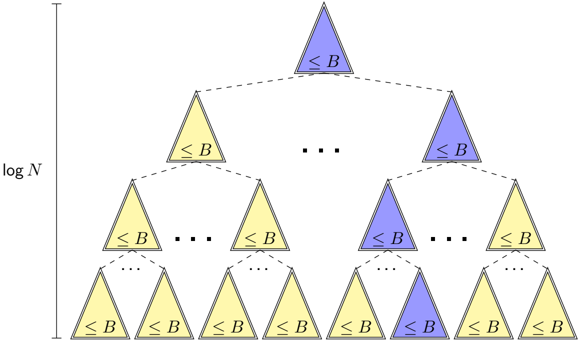
- how many such subtrees are on a path from root to leaf?



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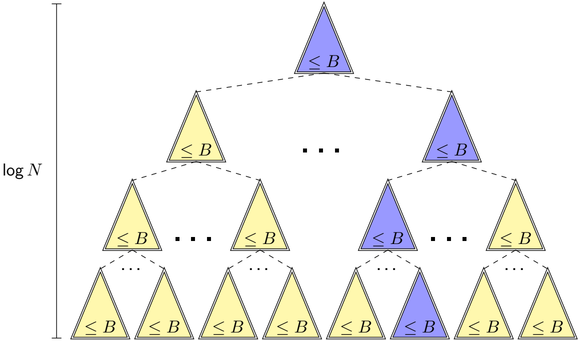
$$\sim \log N / \log \sqrt{B} \in O(\log N / \log B) = O(\log_B N)$$



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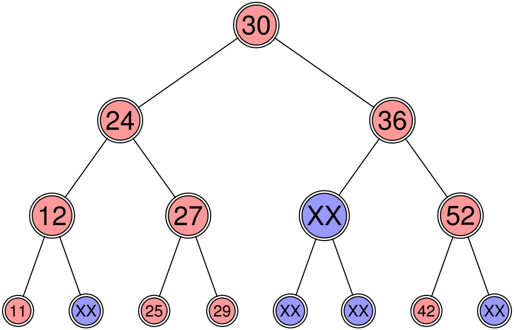
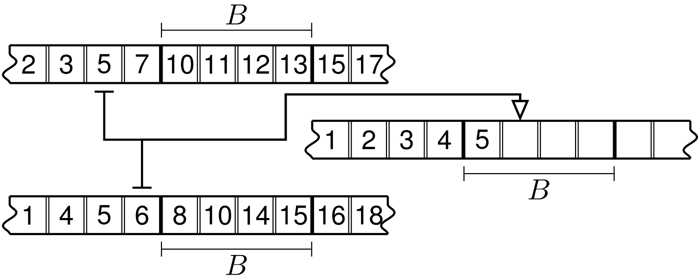


- Idea : embed the dynamic tree in a larger static tree
- If the dynamic tree becomes too unbalanced,re-distribute nodes
- If the dynamic tree becomes too large, re-construct
- Insertions and Deletions can be performed in $O(\log^2 N/B)$ I/Os

Why isn't (binary) MergeSort good enough?



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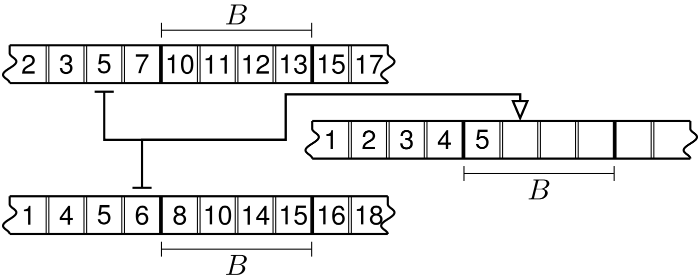

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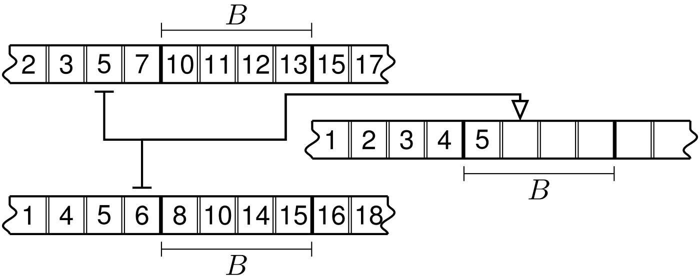

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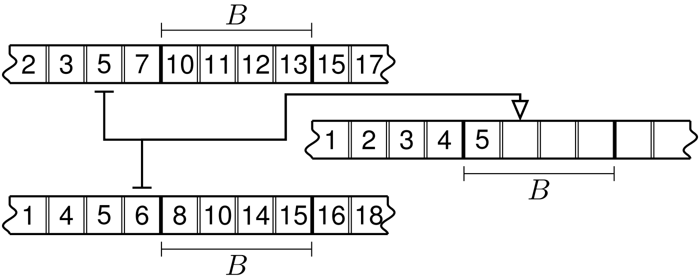

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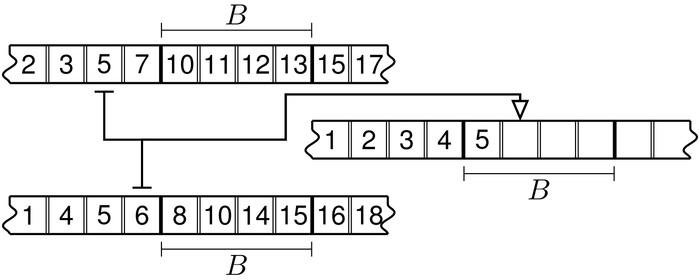

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(M/B)-way MergeSort gives $\Theta(\frac{N}{B} \log_{\frac{M}{B}} \frac{N}{B})$ I/Os (Cache-Aware)

A Cache-Oblivious MergeSort?

What if we had a black box which merged K sorted lists of total size K^3 in $O(\frac{K^3}{B} log_{\frac{M}{b}} \frac{K^3}{B} + K)$ I/Os?

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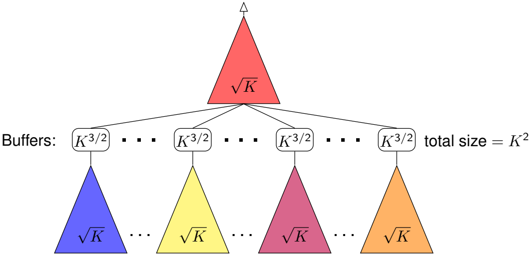
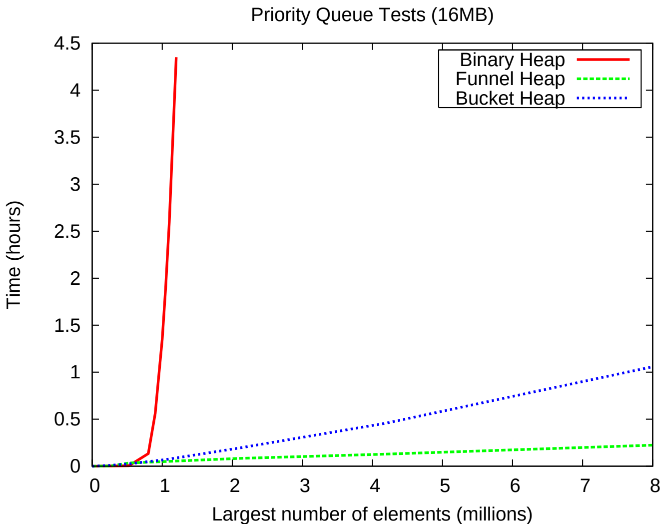
(assuming the tall cache assumption that $M > B^2$)

A K-funnel - the idea



- Look at the largest level of detail J such that a J-funnel occupies less than $M/4$ space
- We can hold a J-funnel and one block of each of its input buffers in memory.
- Empty input buffers are completely refilled. This may push the original Funnel out of memory, which the 'new' J^3 elements pay for.

Some empirical results (1)



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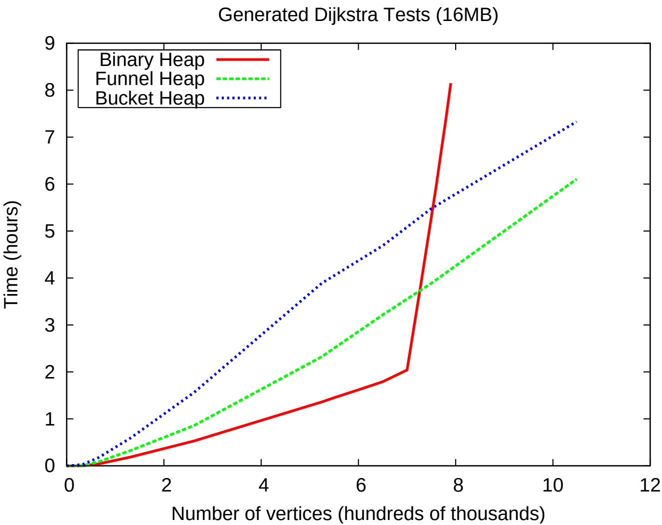

Summary and Questions

- Scanning/Array Reversal: $\Theta(N/B)$ I/Os
- Searching: $\Theta(\log_B N)$ I/Os
- Sorting: $\Theta(\frac{N}{B} \log_{\frac{M}{B}} \frac{N}{B})$ I/Os

Questions?