The elastica: a mathematical history

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Abstract

This report traces the history of the elastica from its first precise formulation by James Bernoulli in 1691 through the present. The complete solution is most commonly attributed to Euler in 1744 because of his compelling mathematical treatment and illustrations, but in fact James Bernoulli had arrived at the correct equation a half-century earlier. The elastica can be understood from a number of different aspects, including as a mechanical equilibrium, a problem of the calculus of variations, and the solution to elliptic integrals. In addition, it has a number of analogies with physical systems, including a sheet holding a volume of water, the surface of a capillary, and the motion of a simple pendulum. It is also the mathematical model of the mechanical spline, used for shipbuilding and similar applications, and directly insipired the modern theory of mathematical splines. More recently, the major focus has been on efficient numerical techniques for computing the elastica and fitting it to spline problems. All in all, it is a beautiful family of curves based on beautiful mathematics and a rich and fascinating history.

This report is adapted from a Ph. D. thesis done under the direction of Prof. C. H. Séquin.

1 Introduction

This report tells the story of a remarkable family of curves, known as the elastica, Latin for a thin strip of elastic material. The elastica caught the attention of many of the brightest minds in the history of mathematics, including Galileo, the Bernoullis, Euler, and others. It was present at the birth of many important fields, most notably the theory of elasticity, the calculus of variations, and the theory of elliptic integrals. The path traced by this curve illuminates a wide range of mathematical style, from the mechanics-based intuition of the early work, through a period of technical virtuosity in mathematical technique, to the present day where computational techniques dominate.

There are many approaches to the elastica. The earliest (and most mathematically tractable), is as an equilibrium of moments, drawing on a fundamental principle of statics. Another approach, ultimately yielding the same equation for the curve, is as a minimum of bending energy in the elastic curve. A force-based approach finds that normal, compression, and shear forces are also in equilibrium; this approach is useful when considering specific constraints on the endpoints, which are often intuitively expressed in terms of these forces.

Later, the fundamental differential equation for the elastica was found to be equivalent to that for the motion of the simple pendulum. This formulation is most useful for appreciating the curve's periodicity, and also helps understand special values in the parameter space.

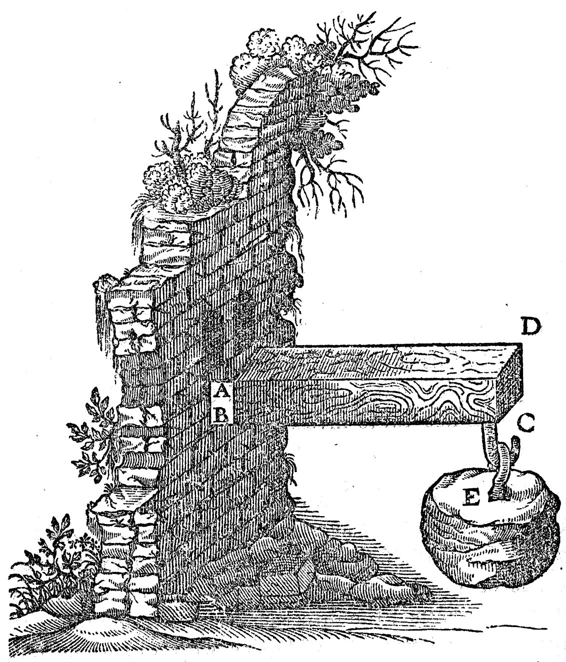
In more recent times, the focus has been on efficient numerical computation of the elastica (especially for the application of fitting smooth spline curves through a sequence of points), and also determining the range of endpoint conditions for which a stable solution exists. Many practical applications continue to use brute-force numerical techniques, but in many cases the insights of mathematicians (many working hundreds of years earlier) still have power to inspire a more refined computational approach.

2 Jordanus de Nemore—13th century

In the recorded literature, the problem of the elastica was first posed in *De Ratione Ponderis* by Jordanus de Nemore (Jordan of the Forest), a thirteenth century mathematician. Proposition 13 of book 4 states that "when the middle is held fast, the end parts are more easily curved." He then poses an incorrect solution: "And so it comes about that since the ends yield most easily, while the other parts follow more easily to the extent that they are nearer the ends, the whole body becomes curved in a circle." [8]. In fact, the circle is one possible solution to the elastica, but not to not for the specific problem posed. Even so, this is a clear statement of the problem, and the solution (though not correct) is given in the form of a specific mathematical curve. It would be several centuries until the mathematical concepts needed to answer the question came into existence.

3 Galilieo sets the stage—1638

Three basic concepts are required for the formulation of the elastica as an equilibrium of moments: moment (a fundamental principle of statics), the curvature of a curve, and the relationship between these two concepts. In the case of an idealized elastic strip, these quantities are linearly related.



(From the *Discorsi*, Leiden 1638.)

Figure 1: Galileo's 1638 problem.

Galileo, in 1638, posed a fundamental problem, founding the mathematical study of elasticity. Given a prismatic beam set into a wall at one end, and loaded by a weight at the other, how much weight is required to break the beam? The delightful Figure 1 illustrates the setup. Galileo considers the beam to be a compound lever with a fulcrum at the bottom of the beam meeting the wall at B. The weight E acts on one arm BC, and the thickness of the beam at the wall, AB, is the other arm, "in which resides the resistance." His first proposition is, "The moment of force at C to the moment of the resistance... has the same propotion as the length CB to the half of BA, and therefore the absolute resistance to breaking... is to the resistance in the same proportion as the length of BC to the half of AB..."

From these basic principles, Galileo derives a number of results, primarily a scaling relationship. He does not consider displacements of the beam; for these types of structural beams, the displacement is neglible. Even so, this represents the first mathematical treatment of a problem in elasticity, and firmly establishes the concept of moment to determine the force on an elastic material. Many researchers elaborated on Galileo's results in coming decades, as described in detail in Todhunder's history [33].

One such researcher is Ignace-Gaston Pardies, who in 1673 posed one form of the elastica problem and also attempts a solution: for a beam held fixed at one end and loaded by a weight at the other, "it is easy to prove" that it is a parabola. However, this solution isn't even approximately correct, and would later be dismissed by James Bernoulli as one of several "pure fallacies." Truesdell gives Pardies credit for introducing the elasticity of a beam into calculation of its resistance, and traces out his influence on subsequent researchers, particularly Leibniz and James Bernoulli [34, p. 50–53].

4 Hooke's law of the spring—1678

Hooke published a treatise on elasticity in 1678, containing his famous law. In a short Latin phrase posed in a cryptic anagram three years earlier to establish priority, it reads, "*ut tensio sic vis*; that is, The Power of any Spring is in the same proportion with the Tension thereof... Now as the Theory is very short, so the way of trying it is very easie." (spelling and capitalization as in original). In modern formulation, it is understood as $F \propto \Delta l$; the applied force is proportional to the change in length.

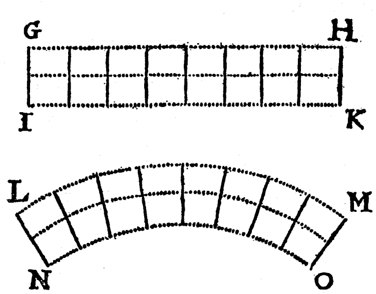


Figure 2: Hooke's figure on compound elasticity.

Hooke touches on the problem of elastic strips, providing an evocative illustration (Figure 2), but according to Truesdell, "This 'compound way of springing' is the main problem of elasticity for the century following, but Hooke gives no idea how to relate the curvature of one fibre to the bending moment, not to mention the reaction of the two fibres on one another." [34, p. 55]

Indeed, mathematical understanding of curvature was still in development at the time, and mastery over it would have to wait for the calculus. Newton published results on curvature in his "Methods of Series and Fluxions," written 1670 to 1671 [17, p. 232], but not published for several more years. Leibniz similarly used his competing version of the calculus to derive similar results. Even so, some results were possible with pre-calculus methods, and in 1673, Christiaan Huygens published the "Horologium oscillatorium sive de motu pendulorum ad horologia aptato demonstrationes geometrica," which used purely geometric constructions, particularly the involutes and evolutes of curves, to establish results involving curvature. The flavor of geometric construction pervades much of the early work on elasticity, particularly James Bernoulli's, as we shall see.

Newton also used his version of the calculus to more deeply understand curvature, and provided the formulation for radius of curvature in terms of Cartesian coordinates most familiar to us today [17]:

$$\rho = (1 + y'^2)^{\frac{3}{2}} / y'' \qquad (1)$$

An accessible introduction to the history of curvature is the aptly named "History of Curvature" by Dan Margalit [24].

Given a solid mathematical understanding of curvature, and assuming a linear relation between force and change of length, working out the relationship between moment and curvature is indeed "easie," but even Bernoulli had to struggle a bit with both concepts. For one, Bernoulli didn't simply accept the linear law of the spring, but felt the need to test it for himself. And, since he had the misfortune to use catgut, rather than a more ideally elastic material such as metal, he found significant nonlinearities. Truesdell [36] recounts a letter from James Bernoulli to Leibniz on 15 December 1687, and the reply of Leibniz almost three years later. According to Truesdell, this exchange is the birth of the theory of the "curva elastica" and its ramifications. Leibniz had proposed: "From the hypothesis elsewhere substantiated, that the extensions are proportional to the stretching forces..." which is today attributed as Hooke's law of the spring. Bernoulli questioned this relationship, and, as we shall see, his solution to the elastica generalized even to nonlinear displacement.

5 James Bernoulli poses the elastica problem—1691

James Bernoulli posed the precise problem of the elastica in 1691:

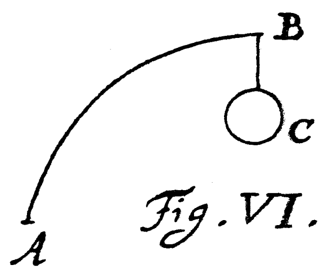


Figure 3: Bernoulli poses the elastica problem.

"Si lamina elastica gravitatis espers AB, uniformis ubique crassitiei & latitudinis, inferiore extremitate A alicubi firmetur, & superiori B pondus appendatur, quantum sufficit ad laminam eousque incurvandam, ut linea directionis ponderis BC curvatae laminae in B sit perpendicularis, erit curvatura laminae sequentis naturae:"

And then in cipher form:

"Portio axis applicatam inter et tangentem est ad ipsam tangentem sicut quadratum applicatae ad constans quoddam spatium."[1]

> Assuming a lamina AB of uniform thickness and width and negligible weight of its own, supported on its lower perimeter at A, and with a weight hung from its top at B, the force from the weight along the line BC sufficient to bend the lamina perpendicular, the curve of the lamina follows this nature:
>
> The rectangle formed by the tangent between the axis and its own tangent is a constant area.

This poses one specific instance of the general elastica problem, now generally known as the *rectangular elastica*, because the force applied to one end of the curve bends it to a right angle with the other end held fixed.

The deciphered form of the anagram is hardly less cryptic than the original, but digging through his 1694 explanation, it is possible to extract the fundamental idea: at every point along the curve, the product of the radius of curvature and the distance from the line BC is a constant, i.e. the two quantities are inversely proportional. And, indeed, that is the key to unlocking the elastica; the equation for the shape of the curve follows readily, given sufficient mathematical skill.

6 James Bernoulli partially solves it—1692

By 1692, James Bernoulli had completely solved the rectangular case of the elastica posed earlier. In his Meditatione CLXX dated that year, titled *"Quadratura Curvae, e cujus evolutione describitur inflexae laminae curvatura"* [3], or, "Quadrature of a curve, by the the evolution of which is traced out the curve

[1] The cipher reads "Qrzumu bapt dxqopddbbp..." and the key was published in the 1694 *Curvatura Laminae* with the detailed solution to the problem. Such techniques for establishing priority may seem alien to academics today, but are refreshingly straightforward by comparison to the workings of the modern patent system.

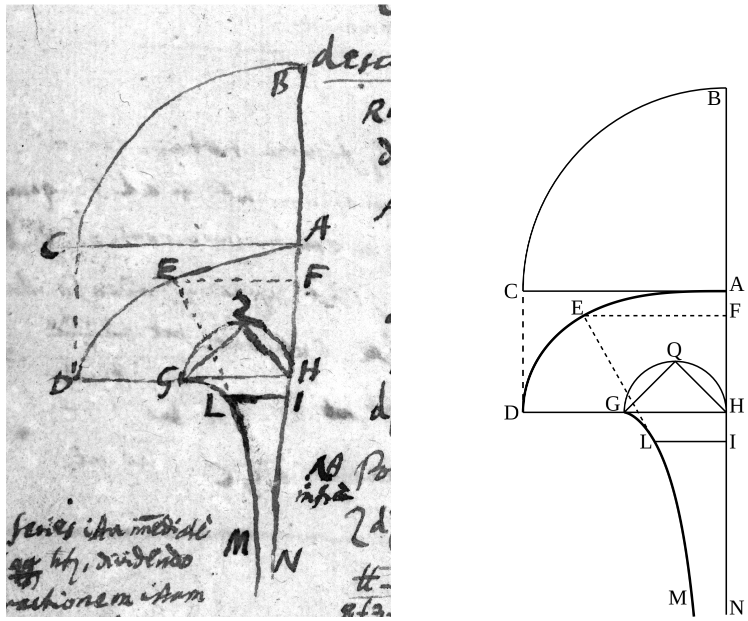


Figure 4: Drawing from James Bernoulli's 1692 Med. CLXX, with modern reconstruction.

of a bent lamina" he draws a figure of that curve (reproduced here as Figure 4), and gives an equation for its quadrature[2].

> Radius circuli $AB = a$, AED lamina elastica ab appenso pondere in A curvata, GLM illa curva, ex cujus evolutione AED describitur: $AF = y$, $FE = x$, $AI = p$, $IL = z$. Aequatio differentialis naturam curvae AED exprimens,
>
> $$dy = \frac{xx\ dx}{\sqrt{a^4 - x^4}},$$
>
> ut suo tempore ostendam:

A rough translation:

> Let the radius of the circle $AB = a$, AED be an elastic lamina curved by a suspended weight at A, and GLM that curve, the involute of which describes AED. $AF = y$, $FE = x$, $AI = p$, $IL = z$. The differential equation for the curve AED is expressed,
>
> $$dy = \frac{x^2\ dx}{\sqrt{a^4 - x^4}}, \tag{2}$$
>
> as I will show in due time.

Equation 2 is a clear, simple statement of the differential equation for the rectangular instance of the elastica family, presented in readily computable form; y can be obtained as the integral of a straightforward function of x.

Based on the figure, AED is the elastica, and GLM is its evolute. Thus, the phrase "evolutione describitur" appears to denote taking the *involute* of the curve GLM to achieve the elastica AED; L is the point on the evolute corresponding to the point E on the elastica itself. L has Cartesian coordinates (p, z), likewise E has coordinates (x, y). It is clear that Bernoulli was working with the evolute due as the geometric construction for curvature, which is of course central to the theory of the elastica. In particular, by the definition of the evolute, the length EL is the radius of curvature at point E on the curve AED.

[2] Today, we would say simply "integral" rather than quadrature.

7 James Bernoulli publishes the first solution—1694

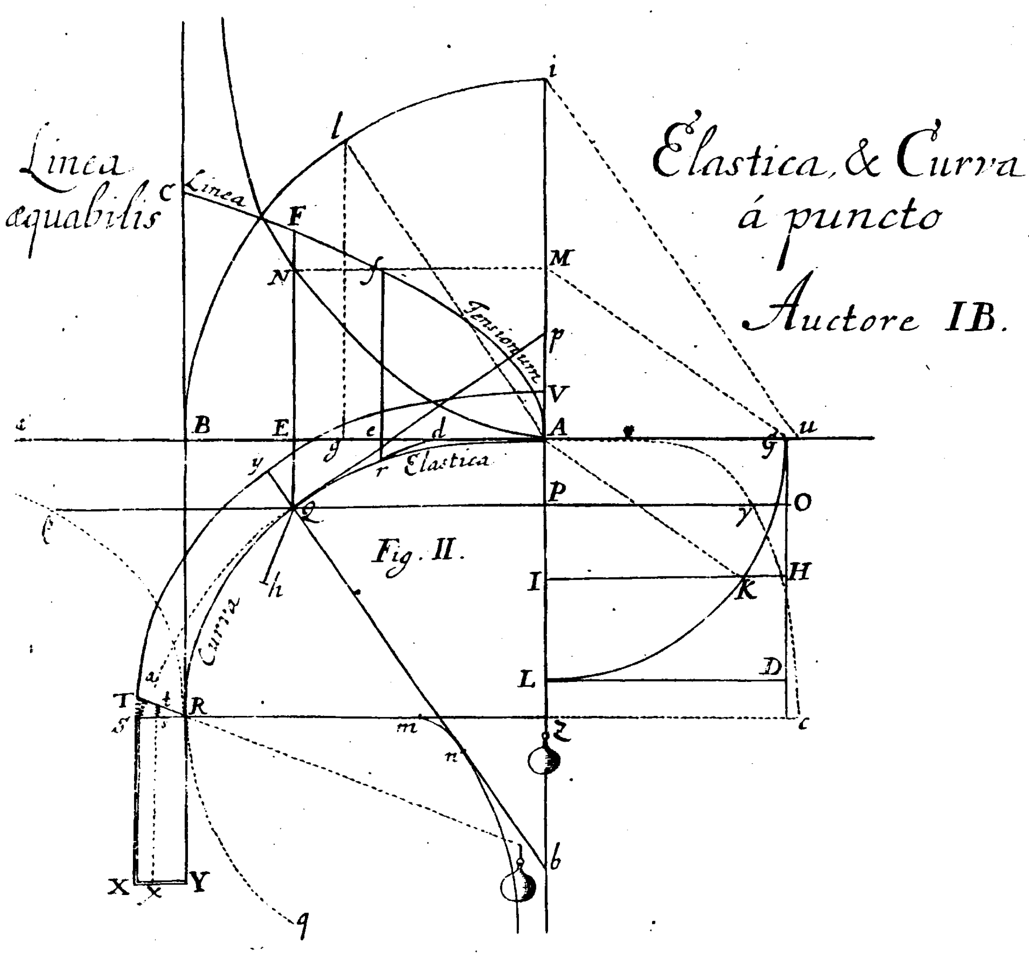


Figure 5: Bernoulli's 1694 publication of the elastica.

Bernoulli held on to this solution for a couple of years, and finally published in his landmark 1694 *Curvatura Laminae Elasticae*[3]. See Truesdell [34, pp. 88–96] for a detailed description of this work, which we will only outline here.

Bernoulli begins the paper by giving a general equation for curvature (illustrated in Figure 6), which he introduces thus, "in simplest and purely differential terms the relation of the evovent of radius of the osculating circle of the curve... Meanwhile, since the immense usefulness of this discovery in solving the velaria, the problem of the curvature of springs we here consider, and other recondite matters makes itself daily more and more manifest to me, the matter stands that I cannot longer deny to the public

[3] *Curvatura Laminae Elasticae. Ejus Identitas cum Curvatura Lintei a pondere inclusi fluidi expansi. Radii Circulorum Osculantium in terminis simplicissimis exhibiti, una cum novis quibusdam Theorematis huc pertinentibus, &c.",* or "The curvature of an elastic band. Its identity with the curvature of a cloth filled out by the weight of the included fluid. The radii of osculating circles exhibited in the most simple terms; along with certain new theorems thereto pertaining, etc.". Originally published in the June 1694 *Acta Eruditorum* (pp. 262–276), it is collected in the 1744 edition of his *Opera* [4, p. 576–600], now readily accessible online.

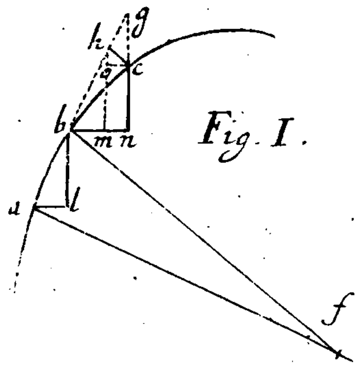


Figure 6: Bernoulli's justification for his formula for curvature.

the golden theorem." Bernoulli's formulation is not entirely familiar to the modern reader, as it mixes infinitesimals somewhat promiscuously. He sets out this formula for the radius of curvature z:

$$z = \frac{dx\,ds}{ddy} = \frac{dy\,ds}{ddx} \tag{3}$$

From Bernoulli's tone, it is clear he thought this was an original result, but Huygens and Leibniz were both aware of similar formulas; Huygens had already published a statement and proof in terms of pure Cartesian coordinates (which would be the form most familiar today). However, in spite of this knowledge, both considered the problem of the elastica impossibly difficult. Huygens, in a letter to Leibniz dated 16 November 1691, wrote, "I cannot wait to see what Mr. Bernoulli the elder will produce regarding the curvature of the spring. I have not dared to hope that one would come out with anything clear or elegant here, and therefore I have never tried." [34, p. 88, footnote 4]

Bernoulli's treatment of the elastica is fairly difficult going (as evidenced by the skeptical reaction and mistaken conclusions from Leibniz and others), but again, it is possible to tease out the central ideas. First, the idea that the moment at any point along the curve is proportional to the distance from the line of force. Bernoulli writes in a 1695 paper [4] explaining the 1694 publication (and referring to Figure 5 for the legend): "I consider a lever with fulcrum Q, in which the thickness Qy of the band forms the shorter arm, the part of the curve AQ the longer. Since Qy and the attached weight Z remain the same, it is clear that the force stretching the filament y is proportional to the segment QP." Next, Bernoulli carefully separates out the force and the elongation, and, in fact, allows for a completely arbitrary functional relationship, not necessarily linear (this function is represented by the curve AFC in Figure 5, labelled "Linea Tensionum"). "And since the elongation is reciprocally proportional to Qn, which is plainly the radius of curvature, it follows that Qn is also reciprocally proportional to x." A central argument here is that the moment (and hence curvature, assuming a linear relationship) is proportional solely to the amount of force and the distance from the line of that force; the shape of the curve doesn't matter.

The moments can be seen in the simplified diagram in Figure 7, which shows the curve of the elastica itself (corresponding to AQR in Figure 5), the force F on the elastica (corresponding to AZ in Figure 5), and the distance x from the line of force (corresponding to PQ in Figure 5). According to the simple lever principle of statics, the moment is equal to the force F applied on the elastica times the distance x from the line of force.

Here we present a slightly simplified version of Bernoulli's argument (see Truesdell [34, p. 92] for a more complete version). Write the curvature as a function of x, understanding that in the idealized case it is linear, i.e. $f(x) = cx$. Then, using Equation 3, the "golden theorem" for curvature:

[4] *Explicationes, Annotationes et Additiones ad eas quae in actis superiorum annorum de curva elastica, isochrona paracentrua, et velaria, hinc inde memorata, et partim controversa leguntur; ubi de linea mediarum directionum, aliisque novis.* Originally published in Acta. Eruditorum, Dec. 1695, pp. 537-553, and reprinted in the 1744 Complete Works [4, pp. 639–663].

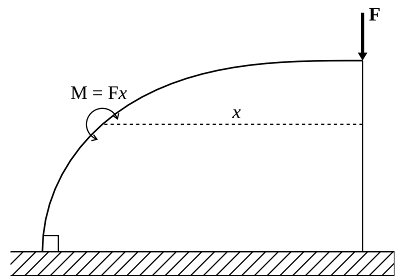


Figure 7: A simplified diagram of moments.

$$\frac{d^2y}{dx\,ds} = f(x) \tag{4}$$

Integrating with respect to dx and assuming $dy/ds = 0$ at $x = s = 0$,

$$\frac{dy}{ds} = \int_0^x f(\xi)d\xi = S(x) \tag{5}$$

Substituting the standard identity (which follows algebraically from $ds^2 = dx^2 + dy^2$),

$$dy/dx = \frac{dy/ds}{\sqrt{1-(dy/ds)^2}} \tag{6}$$

we get:

$$\frac{dy}{dx} = \frac{S(x)}{\sqrt{1-S(x)^2}} \tag{7}$$

And in the ideal case where $f(x) = 2cx$, we have $S(x) = cx^2$, and thus:

$$\frac{dy}{dx} = \frac{cx^2}{\sqrt{1-c^2x^4}} \tag{8}$$

Which is the same as the Equation 2, with straightforward change of constants. As mentioned before, the use of both ds and dx as the infinitesimal is strange by modern standards, even though in this case it leads to a solution quite directly. A parallel derivation using similar principles but using only Cartesian coordinates can be found in Whewell's 1833 Analytical Statics [37, p. 128]; there, Bernoulli's use of ds rather than dx can be seen as the substitution of variable that makes the integral tractable.

It is worth noting that, while James Bernoulli's investigation of the mathematical curve describing the elastica is complete and sound, as a more general work on the theory of elasticity there are some problems. In particular, Bernoulli assumes (incorrectly) that the inner curve of the elastica (AQR in Figure 5 is the "neutral fiber", preserving its length as the elastica bends. In fact, determining the neutral fiber is a rather tricky problem whose general solution can still be determined only with numerical techniques (see Truesdell [34, pp. 96–109] for more detail). Fortunately, in the ideal case where the thickness approaches zero, the exact location of the neutral fiber is immaterial, and all that matters is the relation between the moment and curvature.

The limitations of Bernoulli's approach were noted at the time [13]. Huygens published a short note in the *Acta eruditorum* in 1694, very shortly after Bernoulli's publication in the same forum, illustrating several of the possible shapes the elastica might take on, and pointing out that Bernoulli's quadrature only expressed the rectangular elastica. His accompanying figure is reproduced here as Figure 8. The shapes are shown from left to right in order of increasing force at the endpoints, and shape A is clearly the rectangular elastica.

Bernoulli acknowledged this criticism (while pointing out that he described these other cases explicitly in his paper, something that Huygens apparently overlooked) and indicated that his technique could be extended to handle these other cases (by using a non-zero constant for the integration of Equation 5), and went on to give an equation with the general solution, reproduced by Truesdell [34, p. 101] as:

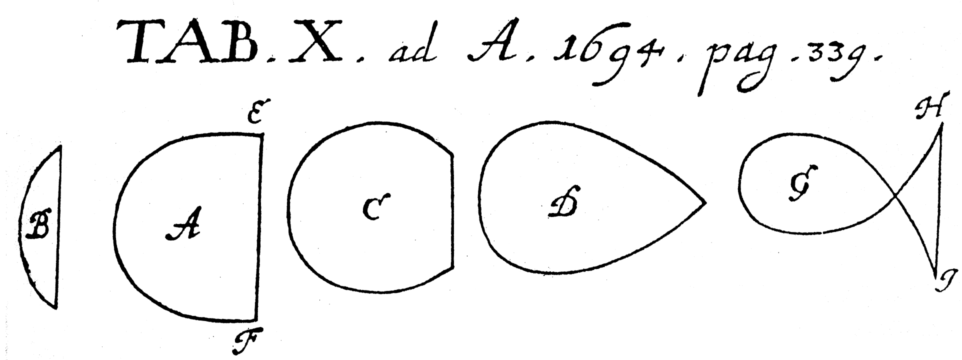


Figure 8: Huygens's 1694 objection to Bernoulli's solution.

$$\pm dy = \frac{(x^2 \pm ab)dx}{\sqrt{a^4 - (x^2 \pm ab)^2}} \tag{9}$$

It is not clear that the publication of this more general solution had much impact. Bernoulli did not graph the other cases, nor did he seem to be aware of other important properties of the solution, notably its periodicity, nor the fact that it includes solutions with and without inflections. (Another indication that these results were not generally known is that Daniel Bernoulli, in a November 8, 1738 letter to Leonhard Euler during a period of intense correspondence regarding the elastica, wrote, "Apart from this I have since noticed that the idea of my uncle Mr. James Bernoulli includes all elasticas." [34, p.109])

Without question, the Bernoulli family set the stage for Euler's definitive analysis of the elastica. The next breakthrough came in 1742, when Daniel Bernoulli (the nephew of James) proposed to Euler solving the general elastica problem with the the technique that would finally crack it: variational analysis. This general problem concerns the family of curves arising from an elastic band of arbitrary given length, and arbitrary given tangent constraints at the endpoints.

8 Daniel Bernoulli proposes variational techniques—1742

Daniel Bernoulli, in an October 1742 letter to Euler [2], discussed the general problem of the elastica, but had not yet managed to solve it himself:

> Ich möchte wissen ob Ew. die curvaturam laminae elasticae nicht könnten sub hac facie solviren, dass eine lamina datae longitudinis in duobus punctis positione datis fixirt sey, also dass die tangentes in istis punctis such positione datae seyen. ... Dieses ist die idea generalissima elasticarum; hab aber sub hac facie noch keine Solution gefunden, als per methodum isoperimetricorum, da ich annehme, dass die vis viva potentialis laminae elasticae insita müsse minima seyn, wie ich Ew. schon einmal gemeldet. Auf diese Weise bekomme ich eine aequationem differentialem 4ti ordinis, welche ich nicht hab genugsam reduciren können, um zu zeigen, dass die aequatio ordinaria elastica general sey.

A rough translation into English reads:

> I'd like to know whether you might not solve the curvature of the elastic lamina under this condition, that on the length of the lamina on two points the position is fixed, and that the tangents at these points are given. ... This is the idea of the general elastica; I have however not yet found a solution under this condition by the isoperimetric method, given my assumption that the potential energy of the elastic lamina must be minimal, as I've mentioned to you before. In this way I get a 4th order differential equation, which I have not been able to reduce enough to show a regular equation for the general elastica.

This previous mention is likely his 7 March 1739 letter to Euler, where he gave a somewhat less elegant formulation of the potential energy of an elastic lamina, and suggested the "isoperimetric method," an early name for the calculus of variations. Many founding problems in the calculus of variations concerned finding curves of fixed length (hence isoperimetric), minimizing or maximizing some quantity such as area enclosed. Usually additional constraints are imposed to make the problem more challenging, but, even in the unconstrained case, though the answer (a circle) was known as early as Pappus of Alexandria around 300 A.D, rigorous proof was a long time coming.

In any case, Bernoulli ends the letter with what is likely the first clear mathematical statement of the elastica as a variational problem in terms of the stored energy[5]:

> Ew. reflectiren ein wenig darauf, ob man nicht könne, sine interventu vectis, die curvaturam ABC immediate ex principiis mechanicis deduciren. Sonsten exprimire ich die vim vivam potentialem laminae elasticae naturaliter rectae et incurvatae durch $\int \frac{ds}{RR}$, sumendo elementum ds pro constante et indicando radium osculi per R. Da Niemand die methodum isoperimetricorum so weit perfectionnieret, als Sie, werden Sie dieses problema, quo requiritur ut $\int \frac{ds}{RR}$ faciat minimum, gar leicht solviren.
>
> You reflect a bit on whether one cannot, without the intervention of some lever, immediately deduce the curvature of ABC from the principles of mechanics. Otherwise, I'd express the potential energy of a curved elastic lamina (which is straight when in its natural position) through $\int \frac{ds}{RR}$, assuming the element ds is constant and indicating the radius of curvature by R. There is nobody as perfect as you for easily solving the problem of minimizing $\int \frac{ds}{RR}$ using the isoperimetric method.

Daniel was right. Armed with this insight, Euler was indeed able to definitively solve the general problem within the year (in a letter dated 4 September 1743, Daniel Bernoulli thanks Euler for mentioning his energy-minimizing principle in the "supplemento"), and this solution was published in book form shortly thereafter.[6]

9 Euler's analysis—1744

Euler, building on (and crediting) the work of the Bernoullis, was the first to completely characterize the family of curves known as the elastica, and published this work as an appendix [10] in his landmark book on variational techniques. His treatment was quite definitive, and holds up well even by modern standards.

Closely following Daniel Bernoulli's suggestion, he expressed the problem of the elastica very clearly in variational form. He wrote [10, p. 247]:

> ut, inter omnes curvas ejusdem longitudinis, quæ non solum per puncta A & B transeant, sed etiam in his punctis a rectis positione datis tangantur, definiatur ea in qua sit valor hujus expressionis $\int \frac{ds}{RR}$ minimus.

In English:

> That among all curves of the same length that not only pass through points A and B but are also tangent to given straight lines at these points, it is defined as the one minimizing the value of the expression $\int \frac{ds}{RR}$.

Here, s refers to arclength, exactly as is common usage today, and R is the radius of curvature ("radius osculi curvæ" in Euler's words), or κ^{-1} in modern notation. Thus, today we are more likely to write that the elastica is the curve minimizing the energy $E[\kappa]$ over the length of the curve $0 \leq s \leq l$:

[5] That said, James Bernoulli in 1694 expressed the equivalence of the elastica with the lintearia, which has a variational formulation in terms of minimizing the center of gravity of a volume of water contained in a cloth sheet, as pointed out by Truesdell [34, p. 201]

[6] Truesdell also points out that this formulation wasn't entirely novel; Daniel Bernoulli and Euler had corresponded in 1738 about the more general problem of minimizing $\int r^m\, ds$, and they seemed to be aware that the special case $m = -2$ corresponded to the elastica [34, p. 202]. The progress of knowledge, seen as a grand sweep from far away, often moves in starts and fits when seen up close.

$$E[\kappa(s)] = \int_0^l \kappa(s)^2 \; ds \tag{10}$$

While today we would find it more convenient to work in terms of curvature (intrinsic equations), Euler quickly moved to Cartesian coordinates, using the standard definitions $ds = \sqrt{1+y'^2}\,dx$ and $\frac{1}{R} = \frac{y''}{(1+y'^2)^{\frac{3}{2}}}$, where y' and y'' represent dy/dx and d^2y/dx^2, respectively. Thus, the variational problem becomes finding a minimum for:

$$\int \frac{y''^2}{(1+y'^2)^{5/2}}\,dx \tag{11}$$

This equation is written in terms of the first and second derivatives of $y(x)$, and so the simple Euler-Lagrange equation does not suffice. Daniel Bernoulli had run into this difficulty, as he expressed in his 1742 letter. In hindsight, we now know that expressing the problem in terms of tangent angle as a function of arclength yields to a first-derivative varational approach, but this apparently was not clear to either Bernoulli or Euler at the time.

In any case, by 1744, Euler had discovered what is now known as the Euler-Poisson equation, capable of solving variational problems in terms of second derivatives, and he could apply it straightforwardly to Equation 11. He thus derived the following general equation, where a and c are parameters (see Truesdell for a more detailed explanation of Euler's derivation [34, pp. 203–204]):

$$\frac{dy}{dx} = \frac{a^2 - c^2 + x^2}{\sqrt{(c^2-x^2)(2a^2-c^2+x^2)}} \tag{12}$$

This equation is essentially the same as James Bernoulli's general solution, Equation 9.

Euler then goes on to classify the solutions to this equation based on the parameters a and c. To simplify constants, we propose the use of a single λ to replace both a and c; this formulation has a particularly simple interpretation as the Lagrange multiplier for the straightforward variational solution of Equation 10.

$$\lambda = \frac{a^2}{2c^2} \tag{13}$$

Euler observes that there is an infinite variety of elastic curves, but that "it will be worth while to enumerate all the different kinds included in this class of curves. For this way not only will the character of these curves be more profoundly perceived, but also, in any case whatsoever offered, it will be possible to decide from the mere figure into what class the curve formed ought to be put. We shall also list here the different kinds of curves in the same way in which the kinds of algebraic curves included in a given order are commonly enumerated." [10, §14]. According to the note in Oldfather's translation, Euler is referring to Newton's famous classification of cubic curves [28, p. 152, Note 5]. In any case, Euler finds nine such classes, enumerated in the table below:

Euler's species #	Euler's Figure	Euler's parameters	λ	comments
1		$c = 0$	$\lambda = \infty$	straight line
2	6	$0 < c < a$	$0.5 < \lambda$	
3		$c = a$	$\lambda = 0.5$	rectangular elastica
4	7	$a < c < a\sqrt{1.651868}$	$.302688 < \lambda < .5$	
5	8	$c = a\sqrt{1.651868}$	$\lambda = .302688$	lemnoid
6	9	$a\sqrt{1.651868} < c < a\sqrt{2}$	$.25 < \lambda < .302688$	
7	10	$c = a\sqrt{2}$	$\lambda = .25$	syntractrix
8	11	$a\sqrt{2} < c$	$0 < \lambda < .25$	
9		$a = 0$	$\lambda = 0$	circle

Euler includes figures for six of the nine cases, reproduced here in Figures 9 and 10. Of the three remaining, species #1 is a degenerate straight line, and species #9 is a circle. Species #3, the *rectangular elastica,* is of special interest, so it is rather disappointing that Euler did not include a figure for it.

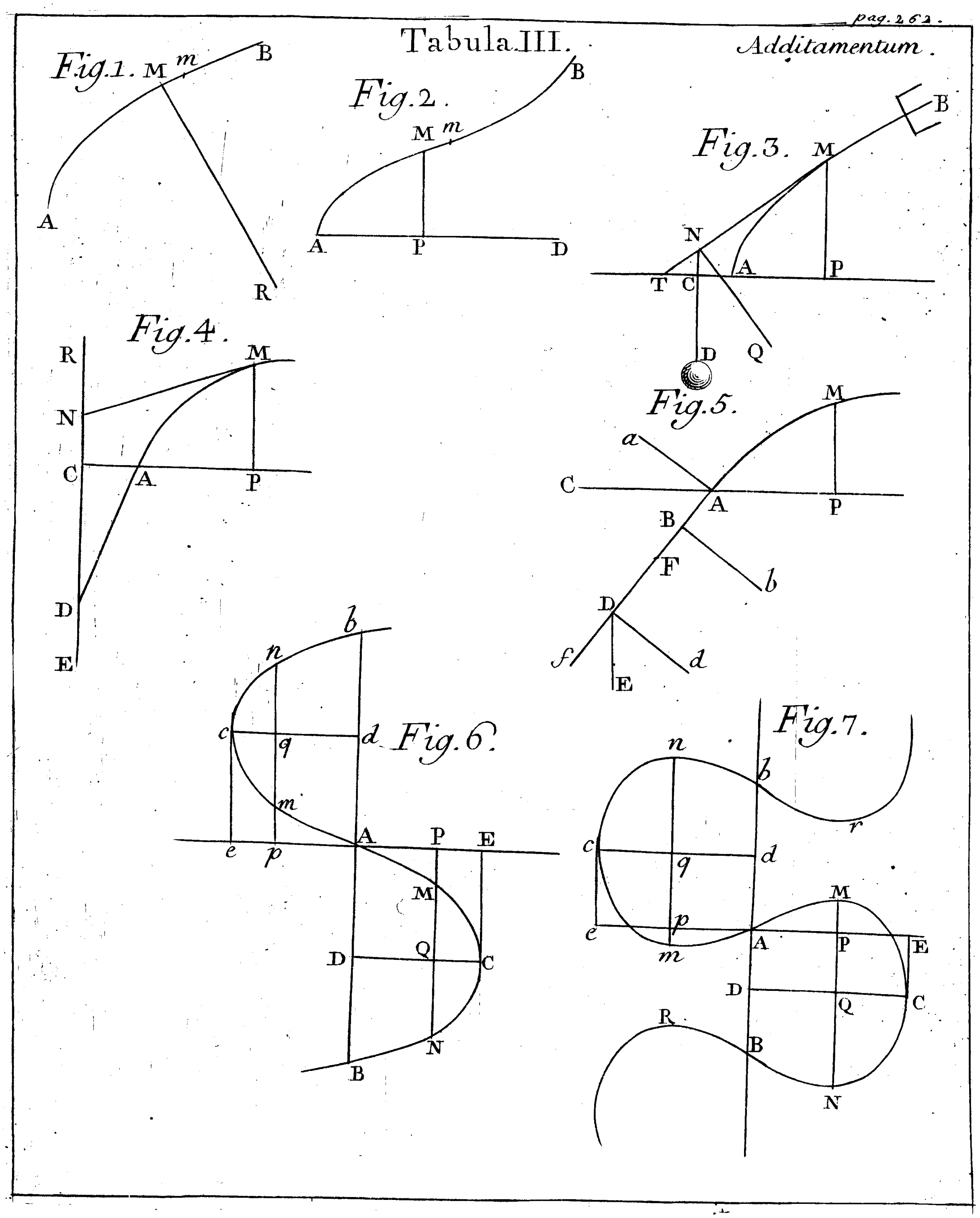


Figure 9: Euler's elastica figures, Tabula III.

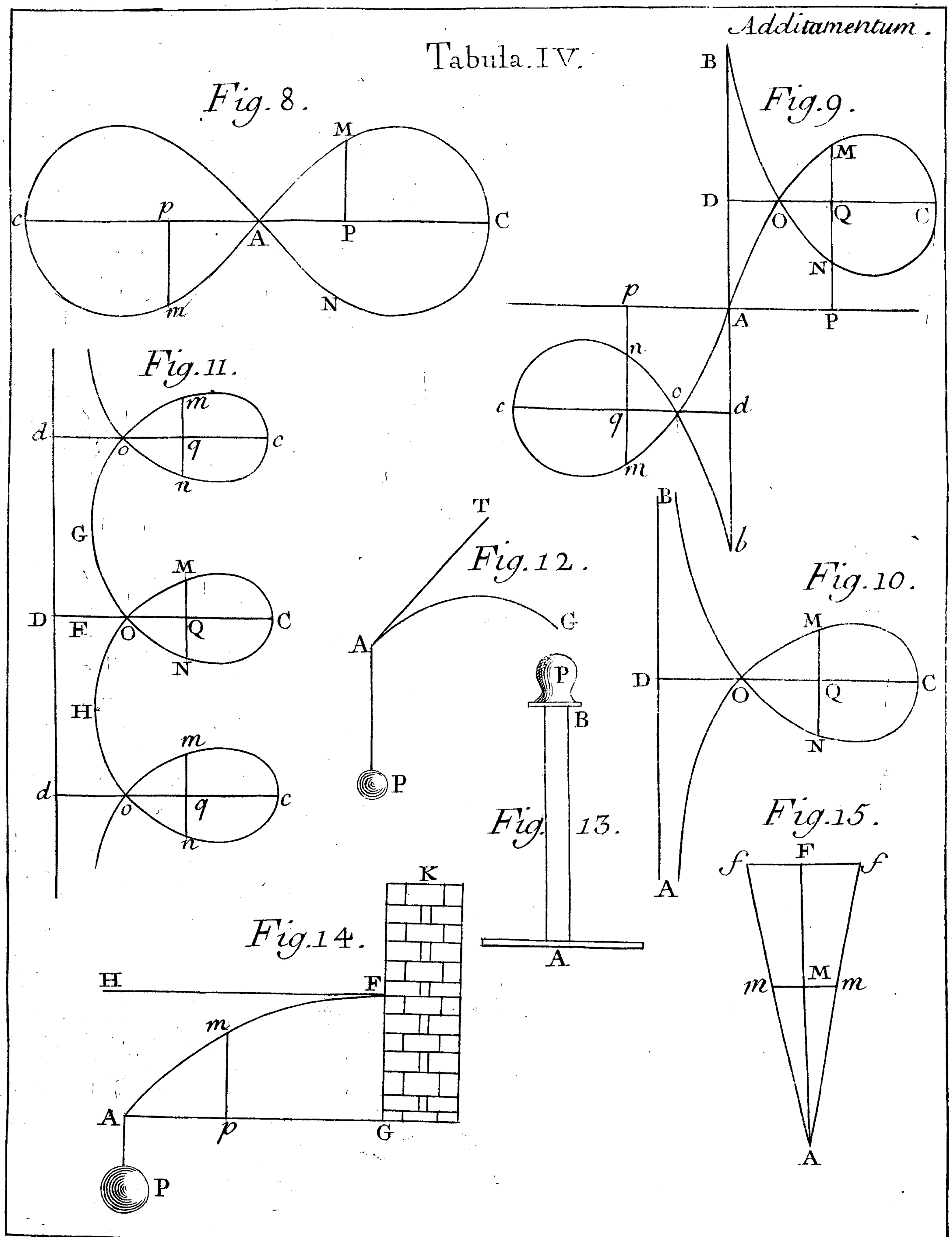


Figure 10: Euler's elastica figures, Tabula IV.

Possibly, it was considered already known due to Bernoulli's publication. Note also that in this special case, $c = a$, the equation becomes equivalent to Bernoulli's Equation 2.

Compare these figures with the computer-drawn Figure 11. It is remarkable how clearly Euler was able to visualize these curves, even 250 years ago. It is likely that the distortions and inaccuracies are due primarily to the draftsman engraving the figures for the book rather than to Euler himself; they display lack of symmetry that Euler clearly would have known. In fact, Euler computed a number of values to seven or more decimal places, including the values of a and c for the lemnoid shape (species #5).

Note that for $c^2 < 2a^2$, or $\lambda > .25$, the inflectional cases, the equation is well-defined for $-c < x < c$. In these cases, the parameter c is half the width of the figure. In the non-inflectional cases ($\lambda < .25$), the equation has no solutions at $x = 0$, and in Euler's figure the curve is drawn to the right of the y axis.

The curve described by species #7, the only nonperiodic solution, was known to Poleni in 1729, and is also known as the "syntractrix of Poleni," or, in French, "la courbe des forçats" (the curve of convicts, or galley slaves). Euler gives its equation (in section 31) in closed form (without citing Poleni):

$$y = \sqrt{c^2 - x^2} - \frac{c}{2} \log \frac{c + \sqrt{c^2 - x^2}}{x} \tag{14}$$

9.1 Moments

Euler was also clearly aware of the simple moment approach to the elastica, and in section 43 of the Additamentum, demonstrated its equivalence to the variational approach. To do so, he manipulated the quadrature formulation of the elastica, Equation 12, grouping all the terms involving first and second derivatives of the curve into a single term $Sq(1-p^2)^{-\frac{3}{2}}$. Recall that $p = \frac{dy}{dx}$ and $q = \frac{dp}{dx} = \frac{d^2y}{dx^2}$. Thus, this term is S times the curvature, yielding an equation relating curvature to Cartesian coordinates. Euler explains:

> "But $\frac{-(1+pp)^{3:2}}{q}$ is the radius of curvature R; whence, by doubling the constants β and γ, the following equation will arise:
>
> $$\frac{S}{R} = \alpha + \beta x - \gamma y\,. \tag{15}$$
>
> This equation agrees admirably with that which the second or direct method supplies. For let $\alpha + \beta x - \gamma y$ express the moment of the bending power, taking any line you please as an axis, to which moment the absolute elasticity S, divided by the radius of curvature R must be absolutely equal. Thus, therefore, not only has the character of the elastic curve observed by the celebrated BERNOULLI been most abundantly demonstrated, but also the very great utility of my somewhat difficult formulas has been established in this example."

This result certainly is confirmation of the variational technique, but in fairness it must be pointed out that James Bernoulli was able to derive essentially the identical equation (again, note the similarity between Equations 9 and 12) by using a combination of mechanical insight and clever integration.

10 Elliptic integrals

The elastica, having been present at the birth of the variational calculus, also played a major role in the development of another branch of mathematics: the theory of elliptic functions.

Even as the quadratures of these simple curves came to be revealed, analytic formulae for their lengths remained elusive. The functions known by the first half of the 18th century were insufficient to determine the length even of a curve as well-understood as an ellipse.

James Bernoulli set himself to this problem and was able to pose it succinctly (and even compute approximate numerical values), if not fully solve it himself [32]. If the quadrature of a unit-excursion rectangular elastica is given by this equation:

$$y = \int_0^x \frac{x^2 dx}{\sqrt{1 - x^4}}, \tag{16}$$

then the arc length is given by:

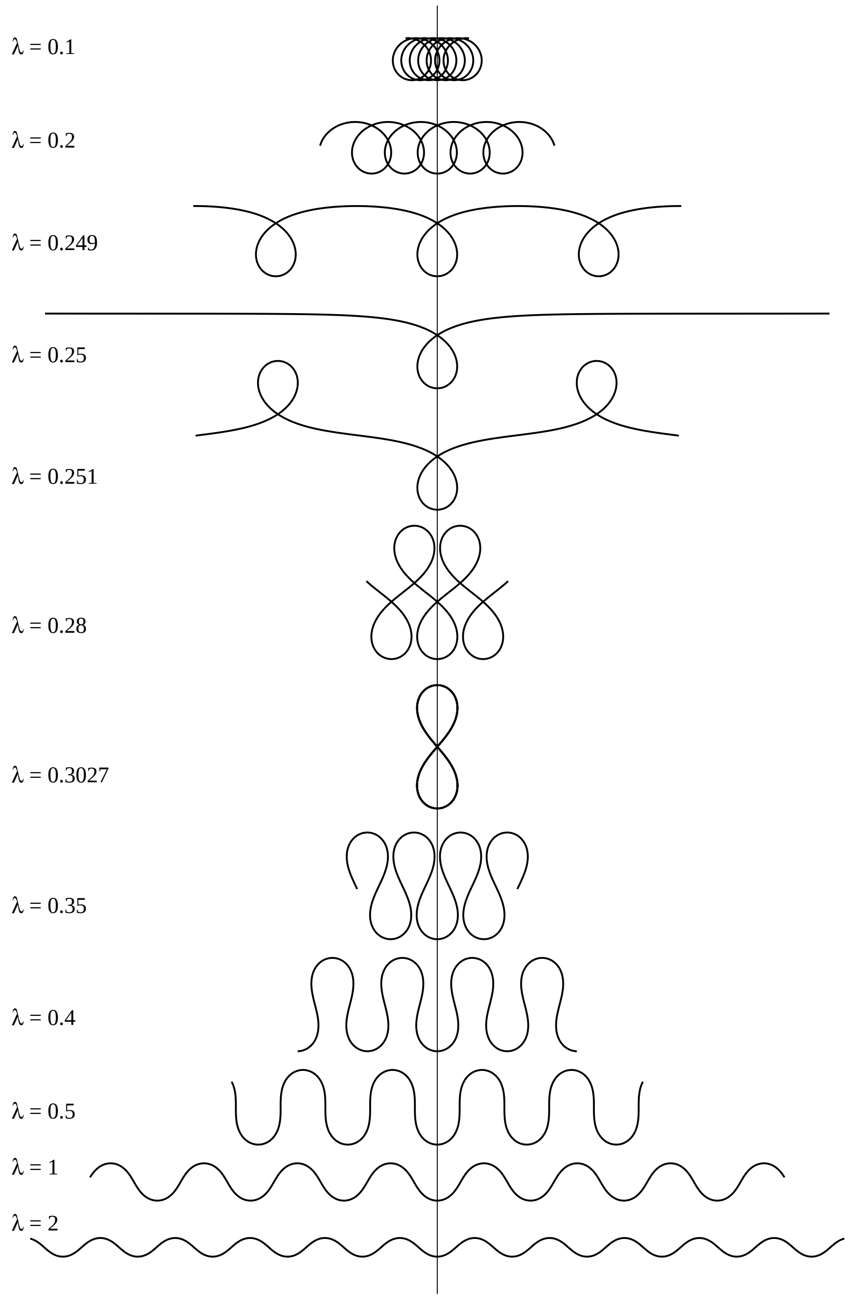


Figure 11: The family of elastica solutions.

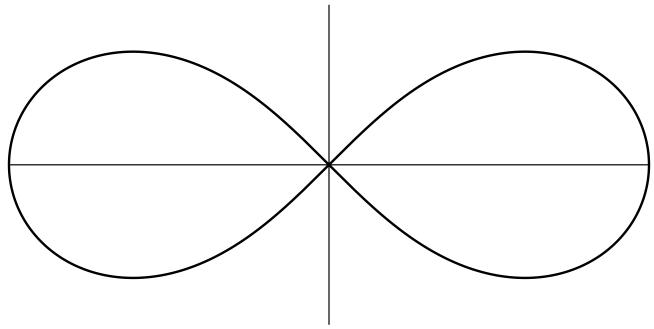


Figure 12: Bernoulli's lemniscate.

$$s = \int_0^x \frac{dx}{\sqrt{1 - x^4}}. \tag{17}$$

This integral is now called the "Lemniscate integral", because of its connection with the lemniscate, another beautiful curve studied by James Bernoulli (Figure 12). The length of the lemniscate is equal to that of the rectangular elastica; while Equation 17 gives the arclength of the elastica as a function of the x coordinate, this nearly identical equation relates arclength to the radial coordinate r in the lemniscate:

$$s = \int_0^r \frac{dr}{\sqrt{1 - r^4}}. \tag{18}$$

Among the lemniscate's other representations, its implicit equation in Cartesian coordinates is a simple polynomial (no such corresponding formulation exists for the elastica):

$$(x^2 + y^2)^2 = x^2 - y^2. \tag{19}$$

Bernoulli approximated the integral for the arclength of the lemniscate using a series expansion and determined upper and lower numerical bounds, but felt the calculation of them would not fall to standard analytical techniques. He wrote, "I have heavy grounds to believe that the construction of our curve depends neither on the quadrature nor on the rectification of any conic section." [35]. Here, "quadrature" means the area under the curve, or the indefinite integral, and "rectification" means the computation of the length of the curve. Bernoulli's prediction would not turn out to be entirely accurate; as we shall see, the curve would later be expressed in terms of Jacobi elliptic functions, which in turn are deeply related to the question of determining the arclength (rectification) of the ellipse.

Fagnano took up the problem of finding the length of the lemniscate, and achieved some impressive results. Indeed, Jacobi fixes the date for the birth of elliptic functions as 23 December 1751, when Euler was asked to review Fagnano's collected works. However, Euler had begun study of elliptic integrals as early as 1738, when he wrote to the Bernoullis that he had "noticed a singular property of the rectangular elastica" having unit excursion [35]:

$$\text{length} \cdot \text{height} = \int_0^1 \frac{dx}{\sqrt{1 - x^4}} \cdot \int_0^1 \frac{x^2 dx}{\sqrt{1 - x^4}} = \frac{1}{4}\pi. \tag{20}$$

After receiving Fagnano's work, Euler caught fire and started his remarkable research, first on lemniscate integrals, then on the more general problem of elliptic integrals, especially the discovery of the addition theorems for elliptic functions in the 1770s. The reader interested in more details, including mathematical derivations, is directed to Sridharan's delightful historical sketch [32].

These integrals would ultimately be the basis for closed-form solutions of the elastica equation, with both curvature and Cartesian coordinates given as "special functions" of the arclength parameter, as will be described in Section 13.

11 Laplace on the capillary—1807

Remarkably, the elastica appears as yet another shape of the solution of a fundamental physics problem—the capillary. Pierre Simon Laplace investigated the equation for the shape of the capillary in his 1807 *Supplément au dixième livre du Traité de mécanique céleste. Sur l'action capillaire*[7]. See I. Grattan Guinness [15, p. 442] for a thorough description of these results. In particular, Laplace considers the surface of a fluid trapped between two vertical plates, and obtains this equation (722.16 in [15]):

$$z'' = 2(\alpha z + b^{-1})(1 + z'^2)^{3/2} \tag{21}$$

Here, z represents height, and z' and z'' represent first and second derivatives with respect to the horizontal coordinate. With suitable renaming of coordinates and substution of Equation 1 for curvature in Cartesian coordinates, this equation can readily seen to be equivalent to Euler's Equation 15.

Laplace also recognized that at least one instance of his equation was equivalent to the elastica. He derives Equation 7 (save that Laplace writes Z for S) and notes that it is identical to the elastic curve [19, p. 379].

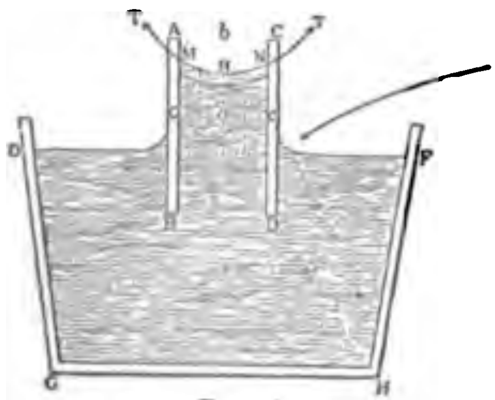


FIG. 6.

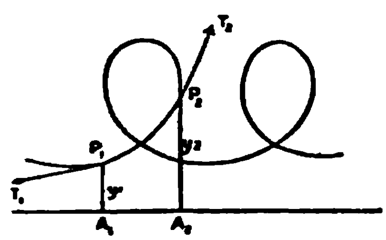


FIG. 8.

Figure 13: Maxwell's figures for capillary action.

It is not clear when the general equivalence between the capillary surface and the elastica was first appreciated. The special case of a single plate is published in *The Elements of Hydrostatics and Hydrodynamics* [27, p. 32] in 1833. In any case, by James Clerk Maxwell was fully aware of it, including it his article "Capillary Action" in the 9th edition of the Encyclopædia Britannica [25], as well as the result that the cross-section of the capillary surface in a cylindrical tube is also an elastica. His figures (reproduced here as Figure 13) illustrate the problem and clearly show a non-inflectional, periodic instance of the elastica.

12 Kirchhoff's kinetic analogy—1859

Surprisingly, the differential equation for the elastica, expressing curvature as a function of arclength, are equivalent to those of the motion of the pendulum, as worked out by Kirchhoff in 1859 (see [8] for historical detail). Using simple variational techniques, taking arclength as the independent variable and angle θ from the horizontal coordinate as the dependent, Equation 10 yields (the sine and cosine arise from the specification of endpoint location constraints, which are described as an integral of the sine and cosine of θ over the length of the curve):

$$\theta'' + \lambda_1 \sin\theta + \lambda_2 \cos\theta = 0 \tag{22}$$

[7] reprinted in Volume 4 of Laplace's Complete Works [19, pp. 349–401]

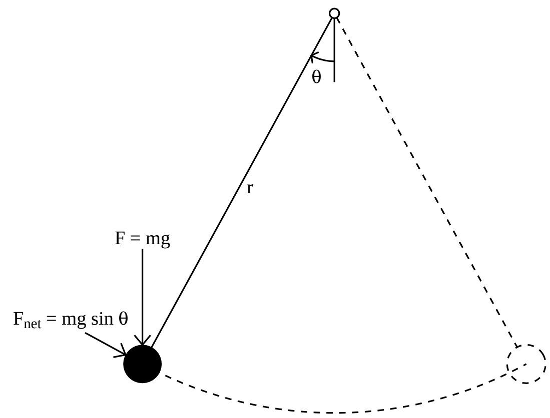


Figure 14: An oscillating pendulum.

This differential equation for the shape of the elastica is mathematically equivalent to that of the dynamics of a simple swinging pendulum. This kinetic analog is useful for developing intuition about the solutions of the elastica equation. In particular, it's easy to see that all solutions are periodic, and that the family of solutions is characterized by a single parameter (modulo scaling, translation, and rotation of the system).

Consider, as shown in Figure 14, a weight of mass m at the end of a pendulum of length r, with angle from vertical specified as a function of time $\theta(t)$.

Then the velocity of the mass is $r\theta'$ (where, in this section, the $'$ notation represents derivative with respect to time), and its acceleration is $r\theta''$. The net force of gravity, acting with the constraint of the pendulum's angle is $mg \sin\theta$, and thus we have:

$$F_{net} = ma = mr\theta'' = mg \sin\theta \tag{23}$$

With the substitution $\lambda_1 = -g/r$, and the replacement of arclength s for time t, this equation becomes equivalent to Equation 22, the equation of the elastica. Note that angle (as a function of arclength) in the elastica corresponds to angle (as a function of time) in the pendulum, and the elastica's curvature corresponds to the pendulum's angular momentum.

The swinging of a pendulum is perhaps the best-known example of a periodic system. Further, it is intuitively easy to grasp the parameter space of the system. Transformations of scaling (varying the pendulum length) and translation (assigning different phases of the pendulum swing to $t = 0$) leave the solutions essentially unchanged. Only one parameter, how high the pendulum swings, changes the fundamental nature of the solution.

The solutions of the motion of the pendulum, as do the solutions of the elastica Equation 22, form a family characterized by a single scalar parameter, once the trivial transforms of rotation and scaling are factored out. Without loss of generality, let $t = 0$ at the bottom of the swing (i.e. maximum velocity) and let the pendulum have unit length. The remaining parameter is then the ratio of the total energy of the system (which remains unchanged over time) to the potential energy of the pendulum at the top of the swing (the maximum possible), in both cases counting the potential energy at the bottom of the swing as zero. Thus, the total system energy is also equal to the kinetic energy at the bottom of the swing.

For mathematical convenience, define the parameter λ as one fourth the top-of-swing potential energy divided by the total energy (the constant is chosen so that λ matches the Lagrange multiplier λ_1 in Equation 22). This convention is equivalent to varying the force of gravity while keeping the maximum kinetic energy fixed, which may be justified by noting that the zero-gravity case (which would require infinite kinetic energy if we were assuming unit gravity) is far more relevant in the context of splines (it corresponds, after all, to a circular arc, so any elastica-based spline meeting the criterion of roundness must certainly have this solution) than the converse of zero kinetic energy.

The potential energy at the top of the pendulum is $2mgr$. The kinetic energy at the bottom of the swing is $\frac{1}{2}mv^2$, where, as above, $v = r\theta'$. Thus, according to the definition above,

$$\lambda = \frac{1}{4}\frac{2mgr}{\frac{1}{2m(r\theta')^2}} = \frac{g}{r(\theta')^2}$$

In other words, assuming (without loss of generality) unit pendulum length and unit velocity the bottom of the swing, λ simply represents the force of gravity. And this λ is the same as defined in Equation 13.

13 Closed form solution: Jacobi elliptic functions—1880

The closed-form solutions of the elastica, worked out by Saalschütz in 1880 [30], rely heavily on Jacobi elliptic functions (see also [8] for more historical development, and Greenhill [16] for a contemporary presentation of these results in English, suggesting the kinetic analog was a major motivator to deriving the closed form solutions).

The *Jacobi amplitude* $\mathrm{am}(u, k)$ is defined as the inverse of the Jacobi elliptic integral of the first kind:

$$\begin{gathered} \mathrm{am}(u, m) = \text{the value of } \phi \text{ such that} \\ u = \int_0^{\phi} \frac{1}{\sqrt{1 - m\sin^2 t}} dt \end{gathered} \tag{24}$$

Here, m is known as the *parameter*. Jacobi elliptic functions are also commonly written in terms of the *elliptic modulus* $k = \sqrt{m}$.

From this amplitude, the doubly periodic generalizations of the trigonometric functions follow:

$$\mathrm{sn}(u, m) = \sin(\mathrm{am}(u, m)) \tag{25}$$

$$\mathrm{cn}(u, m) = \cos(\mathrm{am}(u, m)) \tag{26}$$

$$\mathrm{dn}(u, m) = \sqrt{1 - m\sin^2(\mathrm{am}(u, m))} \tag{27}$$

Note that, when m is zero, sn and cn are equivalent to sin and cos, respectively, and when m is unity, sn and cn are equivalent to tanh and sech (1/cosh)([1, p. 571], §16.6).

13.1 Inflectional solutions

The closed form solutions are given separately for inflectional and non-inflectional cases. The intrinsic form, curvature as a function of arclength, is fairly simple:

$$\frac{d\theta}{ds} = \kappa = 2\sqrt{m}\ \mathrm{cn}(s, m), \tag{28}$$

Expressing the curve in the form of angle as a function of arclength is also straightforward:

$$\sin\frac{1}{2}\theta = \sqrt{m}\ \mathrm{sn}(s, m) \tag{29}$$

Through an impressive feat of analysis, the curve can be expressed in cartesian coordinates as a function of arclength. First, we'll need the elliptic integral of the second kind, $E(\phi, k)$:

$$E(\phi, k) = \int_0^{\phi} \sqrt{1 - m\sin^2\theta}\ d\theta \tag{30}$$

$$\begin{aligned} x(s) &= s - 2E(\mathrm{am}(s, m), m) \\ y(s) &= -2\sqrt{m}\ cn(s, m) \end{aligned} \tag{31}$$

In Love's presentation of these results, $E(\mathrm{am}(s, m), m)$ is defined in terms of an integral of $\mathrm{dn}(u, m)$. The equivalence with Equation 30 follows through a standard identity of the Jacobi elliptical functions ([1, p. 576], 16.26.3).

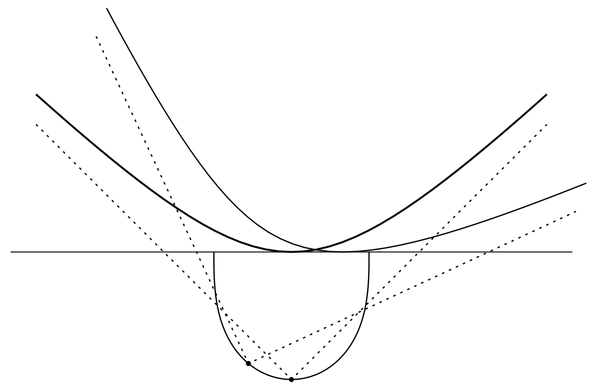


Figure 15: Rectangular elastica as roulette of hyperbola.

$$E(\mathrm{am}(s,m),m) = \int_0^s (\mathrm{dn}(u,m))^2 \, du \tag{32}$$

The solution in terms of elliptic integrals opened the way to a few more curious results. For one, the *roulette* of the center of a rectangular hyperbola is a rectangular elastica [16]. More precisely, let the hyperbola roll along a line without slipping. Then, the curve traced by its center is a rectangular elastica, as shown in Figure 15. The general term for a roulette formed from a conic section is a Sturm's roulette.[8]

13.2 Non-inflectional solutions

The equations in the non-inflectional case are similar.

$$\frac{d\theta}{ds} = \kappa = \frac{2}{\sqrt{m}}\,\mathrm{dn}\Big(\frac{s}{\sqrt{m}}, m\Big) \tag{33}$$

$$\sin\frac{1}{2}\theta = \mathrm{sn}\Big(\frac{s}{\sqrt{m}}, m\Big) \tag{34}$$

For the Cartesian version, and for a more detailed derivation, refer to Love [21].

Truesdell considers the popularity of the elliptic function approach to be a mixed blessing for mechanics and for mathematics. Where Euler's method used direct nonlinear thinking based on physical principles, much of the elliptic literature explored properties of special functions, which "came to be ends of research rather than means to solve a natural problem." [35] Indeed, elliptic functions are rarely used today for computation of elastica, in favor of numerical methods. Indeed, I've used a simple 4th-order Runge Kutte differential equation solver to draw the figures for this work, due to its good convergence and efficiency and simple expression in code. One particularly unappealing aspect of the elliptic approach is the sharp split between inflectional and non-inflectional cases, while one differential equation smoothly covers both cases.

Even so, Jacobi elliptic functions are now part of the mainstream of special functions, and fast algorithms for computing them are well-known [29]. Elliptic functions are still the method of choice for the fastest computation of the shape of an elastica.

14 Max Born—1906

In spite of the equation for the general elastica being published as early as 1695, the curves had not been accurately plotted until Max Born's 1906 PhD thesis, "Investigations of the stability of the elastic line in the plane and in space" [6].

[8] Visit http://www.mathcurve.com/courbes2d/sturm/sturm.shtml for an animated demonstration of this property.

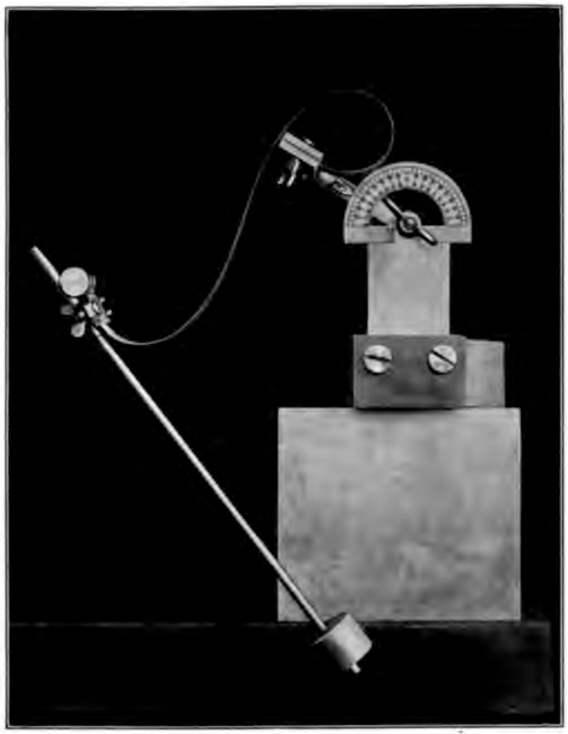


Figure 16: Born's experimental apparatus for measuring the elastica.

Born also constructed experimental apparatus using weights and dials to place the elastic band in different endpoint conditions, and used photographs to compare the equations to actual shapes. An example setup is shown in Figure 16, which shows an inflectional elastica under fairly high tension, $\lambda \approx 0.26$.

Years later, Born wrote, "...I felt for the first time the delight of finding a theory in agreement with measurement—one of the most enjoyable experiences I know." [7, p. 21]. Born used the best modern mathematical techniques to address the problem of the elastica, and, among other things, was able to generalize it to the three dimensional case of a wire in space, not confined to the plane.

15 Influence on modern spline theory—1946 to 1965

Mechanical splines made of wood or metal have long been an inspiration for the mathematical concept of spline (and for its name). Schoenberg's justification for cubic splines in 1946 was a direct appeal to the notion of an elastic strip. Schoenberg's main contribution was to define his spline in terms of a variational problem closely approximating Equation 10, but making the small-deflection approximation. [9]

> **3.1 Polynomial spline curves of order k.** A spline is a simple mechanical device for drawing smooth curves. It is a slender flexible bar made of wood or some other elastic material. The spline is placed on the sheet of graph paper and held in place at various points by means of certain heavy objects (called "dogs" or "rats") such as to take the shape of the curve we wish to draw. Let us assume that the spline is so placed and supported as to take the shape of a curve which is nearly parallel to the x-axis. If we denote by $y = y(x)$ the equation of this curve then we may neglect its small slope y', whereby its curvature becomes

[9] Note: what follows is the corrected version as appears in his selected papers [31].

$$1/R = y''/(1+y'^2)^{3/2} \approx y''$$

> The elementary theory of the beam will then show that the curve $y = y(x)$ is a polygonal line composed of cubic arcs which join continuously, with a continuous first and second derivative[10]. These junction points are precisely the points where the heavy supporting objects are placed.

While Schoenberg's splines are excellent for fitting the values of functions (the problem of appoximation), it was clear that they were not ideal for representing shapes. Birkhoff and de Boor wrote in 1965 [5] that "linearized interpolation schemes have a basic shortcoming: they are *not intrinsic* geometrically because they are not invariant under rigid rotation. Physically it seems more natural to replace linearized spline curves by *non-linear* splines (or "elastica"), well known among elasticians," citing Love [21] as an authority. They also stated the result that in a mechanical spline constrained to pass through the control points only by pure *normal forces*, only the rectangular elastica is needed.

Birkhoff and de Boor also noted some shortcomings in the elastica as a replacement for the mechanical spline, particularly the lack of an existence and uniqueness theory for non-linear spline curves with given endpoints, endslopes, and sequence of internal points. Indeed, they state that "nor does it seem particularly desirable" to have techniques for intrinsic splines approximate the elastica, and they proposed other techniques, such as Hermite interpolation by segments of Euler spirals joined together with continuous curvature.

According to notes taken by Stanford professor George Forsythe [11], in the conference presentation of this work, they also referenced Max Born's PhD thesis [6], and also described a goal of the work as providing an "automatic French curve." The presentation must have made an impression on Forsythe, judging from the laconic sentence, "Deep." Forsythe, working with his colleague Erastus Lee from the Mechanical Engineering college at Stanford, would go on to analyze the nonlinear spline considerably more deeply, deriving it from both the variational formulation of Equation 10 and an exploration of moments, normal forces, and longitudinal forces [20]. That publication also included a result for energy minimization in the case where the elastica is a closed loop.

16 Numerical techniques—1958 through today

The arrival of the high-speed digital computer created a strong demand for efficient algorithms to *compute* the elastica, particularly to compute the shape of an idealized spline constrained to pass through a sequence of *control points.*

Birkhoff and de Boor [5] cited several approximations to non-linear splines, including those of Fowler and Wilson [12] and MacLaren [22] at Boeing in 1959, and described their own work along similar lines. None of these was a particularly precise or efficient approximation.

Around the same time, and very possibly working independently, Mehlum and others designed the Autokon system [26] using an approximation to the elastica (called the KURGLA 1 algorithm). However, this algorithm was not an accurate simulation of the mechanical spline, as did not minimize the total bending energy of the curve, and, in fact, could generate (approximations to) the entire family of elastica curves in service of its splines. An objection to this technique is that the spline was not *extensional;* adding a new control point conciding with the generated spline would slightly perturb the curve. Thus, this approximate elastica was soon replaced by the KURGLA 2 algorithm implementing precisely the Euler spiral Hermite interpolation suggested by Birkhoff and de Boor.

A sequence of papers refining the numerical techniques for computing the elastica-based nonlinear spline followed: Glass in 1966 [14], Woodford in 1969 [38], Malcolm in 1977 [23]. Most of these involve discretized formulations of the problem. Interestingly, Malcolm reports that Larkin developed a technique based on direct evaluation of the Jacobi elliptic integrals which is nonetheless "probably quite slow" compared with the discretized approaches, "due to the large number of transcedental functions which must be evaluated." Malcolm's paper contains a good survey of known numerical techniques, and is recommended to the reader interested in following this development in more detail.

In spite of the fairly rich literature available on the elastica, at least one researcher, B. K. P. Horn in 1981, seems to have independently derived the rectangular elastica from the principle of minimizing the

[10] Schoenberg is indebted for this suggestion to Professor L. H. Thomas of Ohio State University.

strain energy [18], going through an impressive series of derivations, and using the full power of elliptic integral theory, to arrive at exactly the same integral as Bernoulli had derived almost three hundred years previously.

Edwards [9] proposes a spline solution in which the elastica in the individual segments are computed using Jacobi elliptic functions. He also gives a global solver based on a Newton technique, each iteration of which solves a band-diagonal Jacobian matrix. A particular concern was exploring the cases when no stable solution exists, a weakness of the elastica-based spline compared to other techniques. Edwards claims that his numerical techniques would always converge on a solution if one exists.

17 Sources and acknowledgements

This historical sketch draws heavily on Truesdell's history in an introduction to Ser 2, Volume X of Euler's *Opera Omnia* [34]. Indeed, a careful reading of that work reveals virtually all that needs to be known about the problem of the elastica and its historical development through the time of Euler.

Euler's 1744 *Additamentum* is a truly remarkable work, deserving of deep study. The original Latin is available in facsimile thanks to the Euler archive, but the 1933 English translation by Oldfather [28] makes it much more understandable to the English-speaking reader.

I am deeply grateful to Alex Stepanov, Seth Schoen, Martin Meijering and Ben Fortson for help with the translations from the Latin. Special mention is also due the excellent "Latin words" program by William Whitaker, which lets anyone with a rudimentary knowledge of Latin and a good deal of patience and determination puzzle out the meaning of a text. Of course, all errors in translation are my own responsibility.

Patricia Radelet has been most helpful in supplying high-quality images for James Bernoulli's original figures, reproduced here as Figures 4 and 5, and for providing scans of the original text of his 1694 publication.

Thanks also to the Stanford Special Collections Library for giving me access to Prof. Forsythe's notes and correspondence, which were invaluable in tracing the influence of the elastica through early work on non-linear splines.

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