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Editorial

Gert Schubring

This is the first issue of the Journal, which intends to serve as an international forum for scholarly studies on the history of mathematics teaching and education, a field which hitherto was marginally represented by the existing journals. It was the rousing success of the Topic Study Group 29, *The History of Learning and Teaching Mathematics*, at the 10th International Congress on Mathematics Education in Copenhagen in 2004, which demonstrated the need for a permanent and stable international forum for such research. We feel therefore confident that an international journal devoted to the history of mathematics education, complementary to journals in mathematics education, mathematics, the history of mathematics, and the history of education, will be of substantial interest to educators, policy makers, historians, and mathematicians.

The Programme

The major aim of the *International Journal for the History of Mathematics Education* is to provide mathematics teaching and mathematics education with its *memory,* in order to reveal insights attained in earlier periods (ranging from Antiquity to the late 20th century), and to unravel the fallacies of past events (e.g., reform euphoria). This journal will inform mathematics educators and other interested parties about political, social, and cultural constraints (as evidenced by historical events, processes, and periods) with the goal of improving mathematics instruction. In doing so, the Journal aims to overcome disconnected national, cultural, and social histories, and to contribute to establishing common themes and characteristics of the development of mathematics instruction in many cultures, differentiating between what constitutes national specificities or particularities, and what may be indicative of global trends. Moreover, given the intimate relationship between production and dissemination of new and/or enhanced mathematical knowledge, theoretical reflections on the function of teaching will contribute greatly to understanding concrete and practical forms of the relationships.

The Relevance

Practically all the questions of research in mathematics education have a historical dimension that too often remains implicit, or is treated too superficially. Research can be improved by explicit consciousness of the history of teaching and learning mathematics. And, probably even more important, the history of mathematics instruction should constitute one of the dimensions of the professional knowledge of mathematics teachers. In order to be able to handle the problems they encounter in their professional life, mathematics teachers should be aware of how their profession evolved. They should know how the profession emerged historically, how it developed, what problems were encountered along this development, and what obstacles had to be overcome for the effective establishment of mathematics teaching.

The Focus

The primary focus of the Journal will be the learning and teaching of mathematics in schools (primary and secondary grades as well as their functional equivalents), and hence also the training of teachers for this instruction. Moreover, the institutional history of mathematics in higher education may be considered. All historical time periods and all cultures and nations are considered; evidently also those where a formal school system had not yet been established.

The field of research into the history of mathematics teaching and learning is still developing and hence in need of methodological reflection and refinement. The following paragraphs should present a rough outline of challenges with which research into this area is confronted and which might instigate new and in-depth studies.

National focus of studies

It is quite natural that most research, past and present, concentrates on the history within a given nation or a given culture, as the history of mathematics teaching and learning constitutes a part of the educational history of that country or culture. But lest this amount to a collection of separate, isolated histories without interconnections, it is necessary to establish relations between the different histories, and to reveal what is "general" in them and what constitutes, say, a cultural, social, or political peculiarity. We call for studies, which present national histories in an international, comparative perspective.

Approaches

A traditional focus of historical studies has been to analyze the evolution of syllabi for mathematics instruction used in a certain type of institution and in a given geographical region. Due to the type of sources relatively easily accessible, this focus of historiography amounted basically to studying decisions taken by central authorities of the respective state.

Even if the broad spectrum of historical issues is reduced to the syllabi, the real challenge for research is whether, and how, centralized decisions were implemented in school practice, and this opens again the immense range of contextual dimensions relevant to historical development.

Even the entity corresponding to the structured set of mathematical concepts, namely "school mathematics," is far from being just a derivation or a projection of the "scholarly knowledge"—well to the contrary, school mathematics develops as a product of numerous interactions, and even pressures, from and between various sectors of society.

But what further complicates the research in our field is the fact that mathematics never appears in educational systems in an independent way. Rather, it always functions within structures characterized by a compound of several school disciplines. It is thus the social and cultural esteem of mathematics within that compound which is so highly variable and which we encourage to study in different political settings.

Interdisciplinary

These contextual fields, which influence and determine to a high degree the historical evolution, show, however, that the history of the teaching and learning of mathematics constitutes an interdisciplinary field of study. The principal disciplines concerned are the history of mathematics and the history of education, but the science of history contributes as well; moreover, sociology is quite essential, in particular the sociology of religion.

Functions and status of mathematics teaching

The perennial point in all the different moments of the long history of mathematics education proves to be the *rank* attributed to mathematics within the respective set of social and cultural values. And this rank is intimately related to the *function* exerted by mathematics instruction.

One can assert that mathematics (together with the second key discipline of language) enjoyed an unquestioned, firm, and central position as a key discipline in the historically first forms of systematic instruction: the scribal schools of Mesopotamian Sumer and of Egypt. On the other hand, this instruction was clearly oriented professionally towards the concrete needs of the state, for its well developed system of administration. The high rank attributed to mathematics instruction hence derived from this administrative, professional function.

In later societies and cultures where higher social classes became established, enjoying leisure time and using it for learning became common, and thus certain forms of liberal (or general) education emerged. It is deeply characteristic that mathematics occupied, in most cases, only a marginal role—be it propaedeutic, be it auxiliary—within non-professional education and that it was a profoundly complex process, which eventually transformed mathematics into one of the major teaching subjects of general education.

This rise in status and the ensuing transformed function of learning mathematics was connected with epistemological debates about the nature of mathematical knowledge, and in Europe in particular with the advent of the Enlightenment and the beginning of rationalism's dominance in some countries. The first firm establishment of mathematics learning within a public school system as a consequence of the French Revolution documents not only the intimate relationship of the history of mathematics education to the history of education in general, but also the strong influence of cultural and political factors.

The two conflicting directions of professional training versus general education continue, however, to characterize status and function of mathematics in schools, and we invite authors to unravel the mechanisms of how their interaction marks the concrete forms of mathematics teaching.

The Name of the Journal

For a time we pondered what name to choose for the Journal. The expression *History of Learning and Teaching Mathematics*, like the title of TSG 29 at ICME 10, indicated clearly the thematic field envisaged, but was too long. First we reduced it to *History of Mathematics Teaching*, then we were confronted with *History of Mathematics Education*, which encompasses the whole educational system as it relates to mathematics and better indicates the broad range of issues, such as the history of textbooks, the history of professional organizations of mathematics teachers, or the history of teacher education programs. On the other hand, *mathematics education* is used sometimes in the sense of the scientific discipline, analogous to "mathematik-didaktik" in German, "didattica della matematica" in Italian, or "didactique des mathématiques" in French. To avoid such a narrowing sense we introduced the subtitles in various languages as shown on the cover page.

Some Practical Advice

The Journal will publish three types of papers:

- research articles (in general up to 15–25 pages), as refereed publications
- notes, in particular reports on sources (up to 5 pages), and
- book reviews (2 to 3 pages)

We should like to remind prospective authors that the audience of the Journal is international, and therefore may not necessarily know particular details about the author's country. If you write for a Russian audience you need not explain that Lunacharsky was the first Soviet minister of culture and education. A German audience knows that "Humboldt" means Wilhelm von Humboldt, the reformer of the educational system, and not his scientist brother Alexander. French audiences will be aware that Jules Ferry was the minister of education who separated school from church. But for an international audience you must explain these things. Authors are also encouraged to emphasize why their study is of interest to a broad international audience.

The language of papers to be submitted is English, but we invite the authors to give not only one abstract in English but also up to two versions in other languages.

Acknowledgments

As editors of the Journal, we should like to express our gratitude to everyone who in any way supported its birth and helped establish it. In particular, our heartfelt thanks go to Bruce Vogeli, of Teachers College, Columbia University, for his valuable support at all stages of its gestation.

Competing Arguments for the Geometry Course:
Why Were American High School Students Supposed to Study Geometry in the Twentieth Century?

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Abstract

This study contributes to the historical examination of the justification question for the particular case of the high school geometry course in the United States. The 20th century saw the emergence of competing arguments to justify the geometry course. Four modal arguments are identified including that geometry provides an opportunity for students to learn logic, that it helps develop mathematical intuition, that it affords students experiences that resemble the activity of the mathematician, and that it allows connections to the real world. Those arguments help understand what is put at stake by the various kinds of mathematical work that one can observe in contemporary geometry classes. The underlying assumptions of those arguments also help locate the influences in contemporary reform movements with regard to the study of geometry.

The geometry course has been a constant of the American high school curriculum throughout the 20th century but the arguments that justify it have been diverse. This paper explores the question "What are the stakes to be claimed by the study of geometry in the American high school?" We examine that question historically, looking at how the study of geometry in high school was justified during the 20th century. The geometry course is a particularly interesting case to examine different reasons for teaching mathematics because it was a locus of conflict, when, nearing the end of the 19th century, considerations of purpose, audience, and the emergence of new disciplines began to percolate the traditional humanist curriculum.

Herbert Kliebard (1995) has described the ensuing struggle for the American curriculum as the confrontation of various interest groups, which among other things, varied in the extent to which they supported the teaching of mathematics to every child. George Stanic (1986b) has argued that the sympathies that mathematics educators may have expressed toward one or another of those interest groups shaping the general curriculum at the beginning of the 20th century may have been just superficial ways to channel the real debate on the mathematics curriculum: whether or not to fuse the study of the various mathematical disciplines (algebra, geometry, trigonometry) into an integrated course of mathematical studies. The integrated mathematics movement of the beginning of the 20th century made only a minor impact on the mathematics curriculum before being marginalized itself into a course for the less able (Stanic & Kilpatrick, 1992). The high school geometry course with its promise of training in mathematical reasoning was the beacon of non-integration. Marie Gugle, the first female president of the NCTM, said plainly "demonstrative geometry is logic and it will not fuse" (Gugle, 1926, p. 323).

The high school geometry course has survived in practice through the 20th century in spite of the arguments against its transfer value by educational psychologists and the arguments for integration with other mathematical domains by some mathematics educators. What is the value of the study of geometry that high school students can avowedly derive? In what way have the benefits expected from the study of geometry argued for the existence of a separate course of studies? To address those questions is of theoretical interest to us because the study of geometry in the US high school presents an interesting case where, in spite of a relatively well-defined mathematical domain, the influences on the curriculum as regards to that domain have been heterogeneous. An account of those justificatory arguments may help explain the diversity of practices that one can observe in the day-to-day work done in various versions of the geometry course in the United States, in spite of its apparent stability. Additionally, the current Standards movement in the US (NCTM, 1989, 2000) has advocated for stronger connections across mathematical strands of the curriculum. An answer to the question of what has been put at stake by the geometry course over the 20th century can help in understanding relevant issues in redesigning the geometry curriculum to respond to the demands for connections.

The High School Geometry Course

At the end of the 19th century, the *Report from the Mathematics Conference of the Committee of Ten* (Newcomb et al., 1893; Eliot et al., 1893/1969; Eliot, 1905) had argued the need for the geometry course on instrumental grounds: Being structured as an axiomatic-deductive body of knowledge, the study of geometry could educate the mental faculties of deductive reasoning and was therefore of value to all high school students (Halsted, 1893; Hill, 1895; Quast, 1968). The 20th century opened with the promise that geometry would achieve the goal of developing students' capacities for deductive reasoning unlike any other subject, which would then transfer into reasoning capacity in other areas. The *Report of the National Committee of Fifteen on the Geometry Syllabus*, published in 1912, proved to be influential in successive years, especially in the writing of syllabi and textbooks (Quast, 1968). Despite the strong challenge to the notion of transfer issued by Thorndike's research (Kilpatrick 1992; Stanic, 1986a; Thorndike, 1906, 1921, 1924; Thorndike & Woodworth, 1901), the geometry course has continued to exist as a main staple of the college preparatory curriculum.

At the turn of a new century, the publication of *Principles and Standards for School Mathematics (PSSM)* attempted to provide a new vision of the school mathematics curriculum. Rather than limiting the study of geometry to a particular course, *PSSM* establish new expectations for the teaching and learning of geometry across grade levels. According to *PSSM*, the study of geometry is meant to involve students in the experience of mathematical inquiry as well as make apparent to them how a mathematical domain changes over time.

By comparing the report of the Committee of Fifteen (Slaught et al., 1912) and *PSSM* (NCTM, 2000), a reader might find changes in the goals and outcomes of geometry instruction over the years. That contrast between expectations at the beginning and end of the 20th century serve as an initial illustration of our main point: Whereas the geometry course has endured across the 20th century, it has done so in spite of changing expectations.

Methodological Considerations

We produced this account through an analysis of historical documents that trace the path connecting the report of the Committee of Fifteen and *Principles and Standards*. These sources include professional articles on school geometry, curriculum documents, and geometry textbooks[2]. They constitute an archive of the conventional wisdom regarding the geometry curriculum in the United States. We sampled key articles from this corpus, and examined those adapting some of the ideas that Michel Foucault developed for the historiography of ideas (Foucault, 1972). To analyze this corpus we grouped statements portraying similar views about the domain of school geometry or about the geometry student. We analyzed those groups of statements in search of the underlying views that answered the justification question of why students need to study geometry (Kliebard, 1977, 1982, 1995; Stanic, 1983/1984, 1986b).

We derived from that analysis a group of four arguments that have been used over the 20th century to justify the geometry course. We call these arguments "modal arguments" because they are lines of argument drawn upon by various pieces of text. The "modal arguments" are in that sense like the "ideal types" described by Max Weber (1949). We identified different arguments and the circumstances surrounding the birth of each of them by locating issues that each new argument seemed to address in providing a justification for the study of geometry.

An option in writing about the historical developments in the curriculum could have been to focus on key events, such as the meetings of committees, conferences, and boards. The report of the Committee of Fifteen is one such key event in the developments of the geometry curriculum. Other events during the 20th century have affected the geometry curriculum even though their avowed purpose was rarely focused on geometry. Our focus on the written record is an approximation to the historical development that circumvents the problem of determining the significance of any one of the events that punctuate the period of interest through examining the dispersion of ideas about the geometry course written in the period. As Foucault suggests, we consider those documents as monuments—in need of description, comparison, and connection of similar kind as the one used by an archaeologist (Foucault, 1972, p. 7).

Our examination of the written record has concentrated on the *Mathematics Teacher* and we have used those records to trace others by following references. This journal, directed to an audience of mathematics teachers, and whose publication history spans the 20th century, is likely to refer us to the stakes of instruction in classrooms. Our inspection of that archival record could be complemented by inspections that use a similar approach but position themselves on different grounds[3]. Our choice has been to reduce complexity in the number of sources in favor of keeping the amplitude of the period of interest wide.

Branching Out: From the "Mental Discipline" Argument Towards a Manifold of Arguments for the Geometry Course

The *Committee of Fifteen on the Geometry Syllabus* was formed in 1909 during the meeting of the National Education Association in Denver and in response to the request of the National Education Association and the American Federation of Teachers of the Mathematical and Natural Sciences in their respective meetings the previous year. The Committee worked for three years to write and publish a report that gave detailed information about expectations for the geometry course, fleshing out the more general recommendations from previous efforts by the Committee of Ten and the Committee on Mathematical Requirements (Eliot et al.1893/1969; Nightingale et al., 1899).

Led by Herbert E. Slaught[4], a mathematics professor at the University of Chicago, the report started with a preview of the teaching of geometry in European countries and finished by providing a syllabus of the geometry course. Professional commitments among the members of the Committee of Fifteen varied; some were university professors, others were high school teachers in public or private, regular or technical high schools, and yet others were school administrators. Florian Cajori[5], a renowned historian of mathematics, wrote the historical review. Other notable members included William Betz, David Eugene Smith, and Eugene Randolph Smith.

The recommendations of the report of the Committee of Fifteen were understandably more specific than those of the Committee of Ten in providing a vision for the geometry course. For example, the report of the Committee of Ten had acknowledged students' different abilities and interests but it had nevertheless suggested the same curriculum for all (Newcomb et al., 1893, p. 115-116). The uniformity in the syllabus constrained having different geometry courses for "various classes of students in the high schools" (Slaught et al., 1912, p. 89).

The report of the Committee of Fifteen suggested that students should spend less time solving original exercises. This suggestion defied the tendency of the late 19th century to engage students in original problems, which required students to use their reasoning to develop a novel proof rather than to replicate a proof they had studied in the textbook (Herbst, 2002, p. 290). The members of the Committee of Fifteen argued that, "in accordance with the common practice of the past twenty years, this class of exercises has been magnified and extended, especially with reference to the more difficult exercises, beyond the interest and appreciation of the average pupil" (Slaught et al., 1912, p. 95). Instead, the Committee suggested, students could work on applications.

The Committee of Fifteen recognized that some geometrical notions had historically emerged from solutions to practical problems and showed examples of geometry problems in real world situations such as designing architectural elements, surveying, and sailing. The stress on applications included forging relationships between the topics included in the course and the skills for students to develop. For example, scale drawings were connected to the notion of similar triangles and labeled as "essential in surveying" (Slaught, et al., 1912, p. 120).

In various research articles published during the first quarter of the 20th century, Edward Thorndike had issued a challenge to the notion that a curriculum for every pupil should be designed on account of each discipline's capacity to train the mental faculties. Thorndike's studies attempted to show that transfer of training was not general but specific to the situations in which the training had occurred. In regard to its implications for the curriculum, he concluded,

> By any reasonable interpretation of the results, the intellectual values of studies should be determined largely by the special information, habits, interests, attitudes, and ideas which they demonstrably produce. The expectation of any large difference in general improvement of the mind from one study rather than another seems doomed to disappointment. (Thorndike, 1924, p. 98)

Thorndike and Woodworth published in 1901 the first results that put to question the curriculum theory of the humanists. But his progress on these issues certainly fed the discourse of interest groups that pushed to differentiate the curriculum according to students' current needs or to students' anticipated roles after their years of school education (Kliebard, 1995, p. 94).

In apparent acknowledgement of the challenge issued to the humanist curriculum by Thorndike's work on transfer, the Committee of Fifteen took some distance from justifying the geometry course for its mental discipline value. Their stance on incorporating applied work as complementary to the teaching of rigorous proofs is justified by considering students' abilities. This excerpt illustrates some of the underlying tensions between Eliot's philosophy and Thorndike's research.

> In the high school[,] geometry has long been taught because of its mind-training value only. This exclusive attention to the disciplinary side may be fascinating to mature minds, but in the case of young pupils it may lead to a dull formalism [,] which is unfortunate. On the other hand those who are advocating only a nominal amount of formal proof, devoting their time chiefly to industrial applications, are even more at

> fault. The committee feels that a judicious fusion of theoretical and applied work, a fusion dictated by common sense and free from radicalism in either direction, is necessary. (Slaught, et al., 1912, p. 85)

Mental disciplinarians had held a "fundamentally optimistic view of human intelligence" (Stanic, 1986a, p. 40). Thus, from the perspective of mental disciplinarians all students were able to develop their abilities by having "access to the same knowledge" (ibid, p. 40). The report of the Committee of Fifteen proposed more of a balance between expectations for students to be taught rigorous proofs and applications of geometry. The combination of applied and theoretical work seems to have been the essential element distinguishing the report of the Committee of Fifteen from the report of the Committee of Ten. The report of the Committee of Ten had suggested that a major aim of demonstrative geometry was for students to discipline their logical reasoning faculties. Geometry, unlike algebra, was seen as an introduction to students of the "art of rigorous demonstration" (Newcomb et. al., 1893, p. 115).

A main goal of learning of geometrical concepts in the report of the Committee of Fifteen was the development of reasoning skills that could transfer to other domains. Applications, while important, should not replace or decrease students' exposure to formal proofs.

> Certain writers on education have claimed that geometry has no distinctive disciplinary value, or that the formal side is so intangible that algebra and geometry should be fused into a single subject (not merely taught parallel to each other), which subject should occupy a single year and be purely utilitarian. These writers fail to recognize the fundamental significance of mathematics in either its intellectual or its material bearing. (Slaught, et al., 1912, p. 86)

Hence whereas the Committee of Fifteen encouraged attention to applications and the making of connections between algebra and geometry, it fundamentally endorsed a geometry course whose main commitment was with students' development of reasoning skills. The syllabus included a set of theorems that provided the organization of topics to be taught, as a sort of backbone of the course, separating theorems that should be rigorously proved from theorems that could be presented informally. Thus, it seems as if making knowledge accessible to the learner preceded the need to preserve logic. Then, applications were to illustrate how geometric notions showed up in real life. Shibli (1932) argues that in doing this, the authors also took distance from the views of the Committee of Ten: The Committee of Fifteen did not expect students to just "be trained to draw inferences and follow short chains of reasoning" (p. 54).

The report of the Committee of Fifteen related to the issues raised by the Committee of Ten in different ways. On the one hand, as regards to the justification for the geometry course, the Committee of Fifteen did not ascribe to geometry a role in training students in the art of proving. While the report of the Committee of Ten justified the geometry course on a "mental discipline" argument, the report of the Committee of Fifteen did so on a "fusion" (Slaught, et al., 1912, p. 85) of applied and rigorous aspects of geometry. Trying to find common ground between applications of geometrical notions to the real world and formal aspects of geometry continued to be a theme in discussions about the geometry course in the future. On the other hand, as regards to the access question, the Committee of Fifteen endorsed the same principle that had been foundational for the Committee of Ten, namely that all students should have access to the same geometry course regardless of their career orientation or their ability.

Four Modal Arguments for the Geometry Course

Four "modal" arguments surfaced in the 20th century offering justification for the geometry course. By modal arguments we mean not necessarily ideologies explicitly promulgated by individuals but central tendencies around which the opinion of various individuals could converge. First, a *formal argument* defined the study of geometry as a case of logical reasoning (Christofferson, 1938; Fawcett, 1935, 1938, 1970; Meserve, 1962, 1972; Schlauch, 1930; Upton, 1930). Second, a *utilitarian argument* stated that geometry would provide tools for the future work or non-mathematical studies (Allendoerfer, 1969; Breslich, 1938). Third, a *mathematical argument* justified the study of geometry as an opportunity to experience the work of doing mathematics (Fehr, 1972, 1973; Henderson, 1940, 1947; Moise, 1975). Finally, an *intuitive argument* aligned the geometry course with opportunities to learn a language that would allow students to model the world (Betz, 1908, 1909, 1930; Cox, 1985; Hoffer, 1981; Usiskin, 1980; Usiskin & Coxford, 1972).

A Formal Argument: Geometry Teaches to Use Logical Reasoning

Proponents of a *formal argument* inherited the mental discipline argument and tried to fashion it in ways that could accommodate what was being learned about transfer. A major step from the publication of the Committee of Ten within those who favored the *formal argument* was the articulation of ways to enact the idea of teaching for transfer. The value of studying geometry was located in becoming skilled at building arguments, applying the same reasoning used in the geometry course. Proofs were not important because of the leverage they gave to understand particular mathematical concepts but because of the opportunity they created for students to learn, practice, and apply deduction. That is, geometric ideas were not as important as the method for making a logical argument. This method was said to be transferable to other domains such as newspaper reading and democratic participation.

In spite of Thorndike's challenge to the notion of transfer of training, many of the mathematics educators who opposed the notion of an integrated mathematics curriculum and defended the need for every student to study mathematics advocated for the study of geometry on account of its formal training capabilities. Their argument was that if the transfer expected of geometry had not yet been shown, it was because the course had not been taught with transfer in mind, but that geometry could be taught for transfer. Harold Fawcett, who designed a geometry course that would teach geometry for transfer, said that

> If the real purpose of teaching demonstrative geometry is to give the pupil an understanding of the nature of proof, the emphasis should not be placed on the conclusions reached, but rather on the kind of thinking used in reaching these conclusions. (Fawcett, 1935, p. 466)

Fawcett's seminal study of a geometry course taught for transfer became the 13th NCTM yearbook, *The Nature of Proof*. This book built on the *formal argument* by connecting the goals of geometry with the need for all students to learn the reasoning habits with which they could support and exercise the values of a democratic society (Fawcett, 1938, p. 75). Fawcett argued that learning how to do proofs in geometry is a skill needed by educated citizens because it prepares them for the task of analyzing a text logically and to reach conclusions. In another report, a group of mathematics educators that included Fawcett argued that,

> Students should therefore learn geometry in order to learn to reason with equal rigor in other fields. Fundamentally the end sought is for the student to acquire both a thorough understanding of certain aspects of logical proof and such related attitudes and abilities as will encourage him to apply this understanding in a variety of life situations. (Bennett, et al., 1938, p. 188)

According to William Betz, who served as a president of NCTM (1932 -1934), the main goal of geometry was to combine experiences in the real world with abstract knowledge. In "The transfer of training, with particular reference to geometry," Betz (1930) discussed the relevance of theories on transfer at the time. He concluded that teachers had an important role in helping students to experience the values of education in a democracy. Betz said that " geometry is a unique laboratory of thinking, and as such it fosters the persistent and systematic cultivation of the mental habits which are so essential to all those who would claim mental independence and genuine initiative as their birthright" (Betz, 1930, p. 194). He stressed that geometry is a special venue for the training of minds in developing tools that could be applied to other domains.

Bruce E. Meserve agreed with this perspective of teaching geometry for transfer. Meserve, who presided NCTM from 1964 to1966, proposed as the goal of teachers "to help each student develop his or her mathematical abilities (whether these abilities be very extensive or very limited) so that the student may have a greater potential for being an effective citizen in our modern society" (1962, p.

452). That is, regardless of students' individual differences they should become productive members of society.

Ten years later, Meserve modified some of his recommendations including a program where informal geometry permeated all grade levels before a proof-based high school geometry course. Some of these changes included "to make increasing use of student explorations and conjectures, to stress logical concepts without becoming more formal, to welcome coordinate proofs and vector proofs as well as others, and to treat geometry as a part of mathematics" (1972, p. 181). While the new ideas suggest a move towards a justification centered on mathematics, Meserve continued to describe geometry as a case of logical reasoning.

Similarly, Halbert Christofferson[6] who presided over NCTM from 1938 to 1940, argued that all students would be citizens required to reason and that geometry was essential for developing their reasoning skills. He provided many examples where the logical thinking of geometry was applied to the study of other kinds of propositions. Proofs were both a resource for developing thinking and a goal of the geometry course. Within his view, geometry "shows how thinking must be done if it is to be sound, dependable, rigorous" (Christofferson, 1938, p. 155).

Proponents of the *formal argument* stressed that the geometry course was the place to learn logical reasoning, unlike other courses in high school. W. S. Schlauch, honorary president of NCTM (1948-1953), argued that training in logical thinking was one of the main reasons for teaching demonstrative geometry. "In geometry more than in any other school subject, the learner is led to a belief in reason, and is made to feel the value of demonstration" (Schlauch, 1930, p. 134). Schlauch suggested to cover less theorems and to focus on discussing "numerous original exercises" (p. 142). While this expectation seems to be similar to the proponents of the *mathematical argument*, Schlauch's emphasis on the development of formal mathematical thinking transferable to other domains exemplifies the views of those within the *formal argument.* His hope that students would engage in crafting original proofs seems to be a reminiscence of expectations in the report of the Committee of Ten and the practice of many, simple proof-exercises that became standard shortly thereafter (Herbst, 2002).

In sum, the main goal of the geometry course according to proponents of the *formal argument* was to have students learn to transfer skills and ways of thinking learned in geometry to other domains. None other high school mathematics course, this argument said, would carry on this responsibility as the geometry course.

A Utilitarian Argument: Geometry Prepares Students for the Workplace

A *utilitarian argument* was advanced to justify the geometry course on account of the need to prepare students for the needs of the workforce. This theme had its roots in the work of the Committee of Fifteen's recommendation of applications of geometry, but it also resonates with the workplace preparation pitch of the social efficiency movement, one of the interest groups that Kliebard (1995) has identified as claiming a stake on the general curriculum debate of the 20th century. Within *the utilitarian argument*, decisions regarding the content of the geometry course were to be detached from any notion of mathematical activity. Rather, decisions as to what the geometry course should include were to be made according to the relevance of the topics in applying geometrical concepts or geometrical thinking to students' future occupations or professions or, as it became apparent during war times, to the needs of the country (Osborne & Crosswhite, 1970). While there is some overlap between the aims for the geometry course within the *utilitarian argument* and other arguments, such as the expectation for students to develop particular skills, to write proofs or to use their intuition, the expectation was to match students' experiences in geometry with the demands of their future jobs.

Proponents of the *utilitarian argument* considered the geometry students as the future workers that they would become. For Ernst Rudolph Breslich[7],one of the aims of geometry was to provide tools for having educated citizens who would participate actively in the work force by applying practical notions of geometry. His suggestions for changes to the content of the geometry course were determined by the relevance of the topics in applying geometrical concepts or geometrical thinking to students' future occupations or professions. "Many adults firmly believe that in the training in reasoning and attacking problems in geometry they received something that was of definite value and help to them later in their occupations and professions" (Breslich, 1938, p. 312). Logical reasoning was one of the important elements of the geometry course within Breslich's perspective. Yet, his emphasis on logic was different than among the proponents of the *formal argument*. Breslich (1938) focused on students' use of these skills in their future jobs making constant references to future professions as opportunities to use geometric notions even when students may not show interest in knowing these applications.

The singularity of the *utilitarian argument* lies on the way these skills would be developed and the ultimate purpose for developing them. Rather than doing problems from the textbook, Breslich suggested to have students experience the work of professionals and to "train" students in acquiring skills as workers who are to do their job (Breslich, 1938, p. 311). Under the provisions of such utilitarian argument, Breslich stressed the process of using "the right kind of problems taken from life" (1938, p. 313). Thus, geometrical knowledge had the purpose of helping students to deal with everyday life.

In their chapter "Mathematics Education on the Defensive: 1920-1945," Alan Osborne and Joe Crosswhite (1970) document that some circumstances around the coming of the Second World War (e.g., that induction testing by the military showed evidence of incompetence in mathematics) came into the mathematics education rhetoric as arguments to teach "*mathematical content with military uses*" (p. 231). They also indicate, "concern for the mathematical competence of American youth extended beyond military needs to encompass the employment and training problems of increasingly technical industries" (p. 231). In particular, Euclidean geometry was then accused of being too abstract and, at the same time, recommendations were made to teach students methods of indirect measurement and basic principles of engineering and military work (Osborne & Crosswhite, 1970, pp. 232-233). The high school geometry course was under pressure to accommodate more practice in the study of formulas associated with measures of plane and solid figures and their applications, and to reduce the role of proof.

Another proponent of the utilitarian argument, Carl B. Allendoerfer, a mathematician at the University of Washington and former President of the MAA (1959-60), stated that making connections between geometry and other subject matters, especially science or technical careers, was of utmost importance. His suggestions about the content of the geometry course were influenced by applications of geometrical skills to other domains, as it is the case for including solid geometry. The means for getting students acquainted with geometric concepts were also related to applications. He stressed the notion that geometry ought to be taught in agreement with methods and goals of the workforce, by forging connections with teachers of vocational areas (Allendoerfer, 1969, p. 168). However, different from Breslich, Allendoerfer wanted the course to remain within mathematical activity when he said, "We must strive to teach our geometry courses with a truly geometric flavor, and not merely as an exercise in algebra or in logic" (Allendoerfer, 1969, p. 169). His emphasis on having a "formal deductive" (Allendoerfer, 1969, p. 169) course on plane geometry distinguished him from Breslich. Starting with informal experiences with geometry in earlier grades, students should have opportunities to have a formal geometry course. The ultimate goal would be to "apply our geometry to algebra, calculus, science, art architecture, and elsewhere" (Allendoerfer, 1969, p. 169).

The place of proof in the *utilitarian argument* is problematic. Breslich argued that the emphasis on producing proofs limits students' ability to engage in creative work and excludes many capable students. Though he accepted that deductive proofs provide reasons for why something is true, he insisted that they should follow an informal understanding of the geometrical notions. Similarly, Allendoerfer stated that good teachers should "not bury the geometry under an avalanche of rigor" (1969, p. 167). His worries about students seemed to take precedence over the norms or values from the discipline. While proponents of the *utilitarian argument* might argue in favor of including proofs within the geometry course, their concerns with preparing students for their future careers tended to be stronger than other aims.

A Mathematical Argument: Geometry for the Experience and the Ideas of Mathematicians

Within proponents of the *mathematical argument,* the geometry course had as a major goal that of having students experience the activity of mathematicians. Ways of attaining this goal varied. Some proponents argued that Euclidean geometry is an optimal context for students to engage in making and proving conjectures (Henderson, 1947; Moise, 1975). Others claimed that if students needed to follow the work of mathematicians they ought to give preeminence to non-Euclidean geometries by modeling the developments in the discipline (Fehr, 1972). One common notion among proponents of the *mathematical argument* was that the study of geometry remained within the realm of mathematical activity and focused on knowing geometry. Kenneth B. Henderson[8], who had authored a geometry textbook, expected students "to discover and test possible courses of action" (1947, p. 177) when they worked on a geometry problem, just like mathematicians do. Henderson highlighted distinctions between the work of mathematicians and that of empirical scientists. "The difference is that the scientist relies chiefly on experimental corroboration while the mathematician demonstrates the theorem as a necessary consequence of other theorems, postulates, or definitions" (1947, p. 177).

Henderson also stressed the importance of public discussion in the development of postulates and theorems in the course. Within his view, classroom discussion was so important that what textbooks prescribed had to be subordinated to the geometrical notions developed in class. Debates among students as they tried to produce convincing arguments were essential in learning geometry "as it is made rather than by imitation of the 'canned' proofs of the textbook" (Henderson, 1947, p. 177). That is, students should go beyond getting trained to write proofs, by encountering the work of proving in the context in which ideas emerged.

Some of the ideas that contributed to the *mathematical argument* preceded the publication of the report of the Committee of Fifteen. Indeed, Henderson's views have strong resonances with those of Eugene Randolph Smith—a geometry teacher who had written a textbook that developed geometry by "the syllabus method." In the preface of his book he had stated "the hope of encouraging teachers to undertake Geometry by the 'no text method'" (1909, p. 3). Smith's book gave a list of definitions, axioms and theorems along with a variety of exercises including proofs, numerical problems and constructions. Smith's suggestions were slightly different from those later proposed by the Committee of Fifteen (which he was part of) and which proposed to keep a close connection between theorems and relevant exercises.

Edwin Moise[9], a Harvard mathematics professor who co-authored a geometry textbook (Moise & Downs, 1964), expected students to engage in mathematical activity through problem solving. Within Moise's view, students needed to face mathematics as a creative activity. Writing proofs on their own was an extension of the work done in class, the real test for understanding and the rite of passage for becoming a mathematician. "When students solve such problems—and they do—the gap between theory and homework vanishes. On these occasions the student is, probably for the first time in his life, working in his capacity as a mathematician" (Moise, 1975, p. 477). Henderson and Moise agreed upon setting the aims and the contents of the course within the realm of mathematical activity. Proofs became an important resource for students to understand geometric notions and more than a mere exercise on logic.

Moise was particularly interested in order and coherence. His proposed geometry course was mostly defined by its structure. The historical development of geometry was important in supporting his view of the geometry course as a year-long high school course and not integrated with other mathematics courses. "If the facts of elementary geometry were taught piecemeal, as digressions in other courses, with no regard to the way in which they fit together, then the educational effect would be quite different" (Moise, 1975, p. 477). Moise saw geometry as a distinct body of knowledge and thus stressed the importance of a unified geometry course.

Two other proposals are also characteristic of the *mathematical argument*: the proposal to integrate geometry with other courses, eliminating the one-year high school course on geometry and the proposal to use non-Euclidean geometries as the basis for high school geometry. Howard F. Fehr, who presided NCTM from 1956 to 1958, turned to changes in the discipline.

> The survival of Euclid's geometry rests on the assumption that it is the only subject available at the secondary school level to introduce students to an axiomatic development of mathematics. This was true a century ago. But recent advances in algebra, probability theory, and analysis have made it possible to use these topics in an elementary and simple manner, to introduce axiomatic structure. In fact, geometrical thinking today is vastly different from that used in the narrow synthetic approach. (Fehr, 1972, p. 151)

According to Fehr, high school geometry should model the work of current mathematicians. He defined geometry as it is connected to other branches of mathematics. Following Dieudonné, Fehr declared, "Mathematics is no longer conceived of as a set of disjoint branches, each evolving in its own way" (1972, p.

151). Consequently, preparing students for further studies in mathematics and related areas was of utmost importance. Students of geometry could be introduced to an axiomatic system within the framework of linear algebra. Fehr also worried about the isolation of geometry within high school mathematics as an American phenomenon.

> Of all the developed countries of the world, the only country that retains a year sequence of a modified study of Euclid's synthetic geometry is the United States. We must immediately give serious consideration to presenting our high school youth with a mathematical education that will not leave them anachronistic when they enter the university or enter the life of adult society. (Fehr, 1973, p. 379)

Thus, two different stances define the place of geometries other than Euclid's synthetic geometry within the proponents of the *mathematical argument*. One stance uses non-Euclidean geometries as a place for students to understand how to deal with assumptions in a mathematical system. The other stance incorporates other geometries as a way to align the work in high school with the current work of mathematicians. While the range of possibilities in terms of the resources available for students and the definition of mathematical activity vary, these two stances share the aim of making geometry the place of experiencing the ideas and the work of mathematicians.

An Intuitive Argument: Geometric Expression Helps Students Interpret their Experiences in the World

The interplay between geometry and intuition permeates the justifications of the geometry course among different arguments. However, proponents of what we call the *intuitive argument* made a case for geometry as a unique opportunity for students to apply the intuition of the geometric objects to describing the world. This argument can be traced back to John Dewey's (1903) views on the psychological and the logical in the teaching of geometry. There are variations among this argument regarding students' engagement in mathematical activity. Some proponents responded to the need to develop students' basic skills (e.g., calculating perimeter and area of figures) and thus call for developing geometric literacy (Cox, 1985; Hoffer, 1981). Others tended to go deeper in advocating that the course present alternative mathematical ideas that would be more aligned with students' needs (Usiskin, 1980/1995; Usiskin & Coxford, 1972). The core idea sustaining proponents of the *intuitive argument* was the principle that geometry provides lenses to understand, to experience, and to model the physical world by forging stronger connections between experiences, intuitions, skills, and geometrical notions. Mathematics, as a human activity, allows bonding with the physical world through studying the spatial features of physical objects. Unlike other branches of mathematics, geometry was said to merge empirical knowledge about physical objects and abstract ways of dealing with those objects.

Proponents of the *intuitive argument* juggled tensions between the need for all students to acquire geometric literacy and differences in students' abilities. For some, informal geometry appeared as a solution that would promote students' interest in proving some conjectures formally later rather than starting with a formal treatment of the subject (Cox, 1985; Hoffer, 1981; Peterson, 1973). According to Peterson, "The use of informal geometry in what is usually considered a formal geometry course should make the study of geometry more interesting" (Peterson, 1973, p. 90). Thus, students' motivation shaped decisions about the geometry course.

Philip Cox argued that, "No longer can geometry be considered an appropriate subject for study only by those with a special aptitude for mathematics" (1985, p. 404). According to Cox, the first semester of the geometry course should be devoted to studying the concepts informally, making the study of geometry more inclusive. "More informal geometry and informal geometry courses at the high school level are needed if we wish to have most of our students achieve some degree of geometric literacy" (Cox, 1985, p. 405). Cox also suggested various versions of the geometry course, tailored to different populations such as college-bound students and those who would pursue other careers and he wrote a textbook that illustrated the approach (Cox, 1992). In contrast with the *formal argument,* which intended to prepare educated citizens, and the *mathematical argument,* which viewed all students as budding mathematicians, the *intuitive argument* intended to cater different courses according to students' intended needs.

Some proponents of the *intuitive argument* turned to the van Hiele levels of geometry learning (see Fuys, Geddes & Tischler, 1988) for deciding the range of skills that students ought to develop (Hoffer, 1981). The van Hiele levels categorize students' experiences with geometry according to the kind of reasoning invested. Based on his experience teaching high school geometry, Hoffer recommended a transition from informal to formal geometry. Students' work with proofs ought to happen after more informal experiences. Within his view, proofs were as important in the geometry course as other experiences that might not include proofs. He concluded, "geometry is more than proof" (1981, p. 18), in contrast with the *mathematical argument* for which proofs are essential in the construction of geometric ideas.

The informal geometry course has been adopted in various schools as an alternative for students' mathematics curriculum requirements, especially for students in a low track. One of the characteristics of this course is that it minimizes and even eliminates proofs, substituting them by explorations. Usually, the informal geometry course would emphasize algebraic skills, using geometric properties as a context for reviewing or learning applications of Algebra I such as solving equations (see Hoffer & Koss, 1996).

A different perspective within the proponents of the *intuitive argument* has been supported by a mathematical examination of geometry. Zalman Usiskin has argued that the teaching of geometry should not focus solely on students being exposed to a mathematical system but should allow students to make connections between geometry and the real world. These connections lay beyond the development of particular skills and stress on the power of geometry to model real-life phenomena. For example, in their geometry textbook, Zalman Usiskin and Arthur Coxford (1972) chose a transformation approach to the geometry course because of possible connections with relevant mathematical ideas in other courses and in response to current ways of working with geometry (Coxford, 1973). This justification is closely related to the *mathematical argument*. At the same time, the opportunity to relate the high school geometry course with "previous intuitive ideas" (Usiskin & Coxford, 1972, p. 21) brings to the fore a geometry course that takes into account students' intuition.

Usiskin's suggestions about the geometry course foreshadowed the first publication of the NCTM *Standards* in 1989. Usiskin stated as reasons for teaching high school geometry that: "1. Geometry *uniquely* connects mathematics with the real physical world. 2. Geometry *uniquely* enables ideas from other areas of mathematics to be pictured. 3. Geometry *non-uniquely* provides an example of a mathematical system" (1980, p. 418). While the justifications for high school geometry had usually focused on the last reason, Usiskin argued that geometry had to provide opportunities for students to make connections with the real world.

From the quotes that accompany our discussion of each of the arguments, it is apparent that individual authors rarely subscribed to a unique, well-defined modal argument. Still, their writings permit to isolate those four modal arguments as ideal types of justifications for the study of geometry. **Table 1** shows some of the essential elements characterizing each modal argument.

Table 1. Elements within the four modal arguments justifying the geometry course.				
	Formal argument	*Utilitarian argument*	*Mathematical argument*	*Intuitive argument*
Goals of the geometry course of studies	Geometry is a case of logical reasoning.	Geometry is a tool for dealing with applications in other fields.	Geometry is a conceptual domain that permits to experience the work of mathematicians.	Geometry provides a language for our experiences with the real world.
Views about mathematical activity	Transferring formal geometry reasoning to logical abilities.	Studying concepts and problems that apply to work settings.	Applying deductive reasoning through the study of geometric concepts.	Modeling problems using geometric ideas while reasoning intuitively.
Expectations about students	All students require logical reasoning to be good citizens and to participate in a democracy.	All students will be part of the workforce in the future.	All students can simulate the work of mathematicians.	All students could develop skills but their abilities vary
Characteristics of problems in the geometry curriculum	Applying logical thinking to mathematical and real-life situations.	Relating geometric concepts and formulas to model real-world objects or to solve problems emerging in job situations.	Making conjectures and proving theorems deductively.	Exploring intuitively geometric ideas towards formality. Integrating algebra and geometry.
The place of proofs	Proofs give opportunities to practice deductive reasoning detached from geometric concepts.	Proofs are not as important as problems that apply geometry to future jobs.	Proofs of original problems provide opportunities to experience the activity of mathematicians.	Proofs follow informal appreciation of geometric concepts, blurring differences between definitions, postulates and theorems.

Principles and Standards as a Space for Convergence

Similar to the movement that proposed an integrated mathematics curriculum at the beginning of the 20th century, the recommendations of the Standards movement of the end of the 20th century include a strong impetus to connect mathematical domains, connect mathematical ideas with those of other disciplines, and connect mathematical ideas with problems from the real world (NCTM, 2000, pp. 64-66). *PSSM* does not provide a syllabus for any of the mathematics courses and does not directly disown a separated geometry course. Rather it suggests that elements of geometry should permeate the school mathematics curriculum and proposes forging stronger connections between subjects in the event that separate courses be taught (NCTM, 2000, p. 289; see also pp. 354-359). The existence of a Geometry Standard among the five main content Standards[10] confirms that students' development of geometric knowledge is still valued. One of the consequences of a call for connections is that the justifications for the teaching and learning of geometric concepts permeate the mathematics curriculum without solely allocating that responsibility to any one course. In addition to that, the five process Standards (Problem Solving, Reasoning and Proof, Communications, Connections, and Representations) are all connected to geometric content.

Some of the modal arguments developed through the 20th century to justify the geometry course have percolated into the Standards. Justifications for the study of geometry retain some of the special properties that sustained its presence within the high school mathematics curriculum. At the same time, other justifications have faded from the high school mathematics curriculum. The way in which the aims of learning and teaching geometry manifest as well as the leverage of these goals in the mathematics curriculum have changed.

The 20th century opened with issues about transfer of mental discipline as justifications for the study of geometry. The entrenched notion among mathematics educators that the main role of geometry was to train in logic was arguably one culprit for the eventual failure of attempts at unification started at the beginning of the century with the general mathematics movement of E. H. Moore (1926/1902). A notable change in the rhetoric of the Standards movement is that in spite of the value put on students' learning of geometry, the *formal argument* plays no role in the justification of the study of geometry within the rhetoric of the Standards movement. The Reasoning and Proof Standard embeds the justifications for the teaching of proof at all levels. But in contrast with the approach shown in Fawcett's *The Nature of Proof* that provided examples of using deductive reasoning in non-mathematical situations, *PSSM* includes examples that lie within the realm of mathematical activity (or that relate to mathematical models of applied problems). Mathematical reasoning is not directly exported from geometry into non-mathematical situations; rather the role of proof in creating reasonable mathematics is emphasized. For example, the Geometry Standard establishes that students should "put together a number of logical deductions" (NCTM, 2000, p. 310) when solving a problem. Students' learning of logical reasoning is a tool for them to use in their mathematics classrooms, especially when crafting proofs (NCTM, 2000, p. 342). But geometry does not carry the burden of teaching reasoning skills. Rather, students' use of logical deductions (in mathematics) should lead students to have a deeper understanding of geometric notions (NCTM, 2000, p. 311), a goal that seems to be more aligned with the *mathematical argument.*

Whereas the *formal argument* emphasized the use of logical reasoning in situations outside of mathematics, *PSSM* highlights that students will be more empowered and autonomous in their pursuit of mathematical knowledge. "In order to evaluate the validity of proposed explanations, students must develop enough confidence in their reasoning abilities to question others' mathematical arguments as well as their own" (NCTM, 2000, 345-346). Thus, *PSSM* aligns the expectation for students to engage in logical reasoning with the work of doing mathematics, getting closer to proponents of the *mathematical argument* in this regard.

The 1989 Standards had reflected some of the goals of the *formal argument* within the context of integration of technology. The 1989 Standards had mentioned computer software as useful to "develop, compare, and apply algorithms"(p. 159). In *PSSM*, however, the discussion on how to use computers to experiment the making of deductive arguments seem to be more aligned with a *mathematical argument.* The evolution of dynamic geometry software to include less aspects of programming and more aspects of manipulation appears to have affected the goals of teaching geometry. Earlier kinds of educational software required some basic programming skills for which the learning of formal logic was a resource; current dynamic geometry software tend to demand uses of logic more tied to the semantics of the domain being studied.

A strong reminiscence of the humanist discourse of the Committee of Ten, and of the mental discipline philosophy that influenced them, is the notion that all students should learn geometry, implied in the NCTM motto of "mathematics for all." This is hardly news for NCTM, which was founded to counter some educational impetus to make mathematics optional (Betz, 1936; Kilpatrick et al., 1920). *PSSM* does not establish different goals for students depending upon their ability, similar to the report of the Committee of Fifteen.

Students' work in crafting logical arguments is aligned with the work of mathematicians, making the Geometry Standard a case of the *mathematical argument.* For example, *PSSM* offers a vignette in which students fail to find a generalization as they look at polygons that result from connecting the midpoints of the sides of different polygons. The teacher in the vignette emphasizes the value of the process students were engaged with and commends students for engaging in a process that "is truly mathematical" (NCTM, 2000, p. 312).

While the Geometry Standard draws no support from the *formal argument*, the influence of the *utilitarian argument* is perhaps as salient as that of the *mathematical argument*. The Geometry Standard shows applications of geometric notions in the workplace. "Applied problems can furnish both rich contexts for using geometric ideas and practice in modeling and problem solving" (NCTM, 2000, p. 313). The mathematics of the engineers designing a pipeline, of the artist using perspective drawing, of the worker finding the minimum path, and of the banker setting the best route is the mathematics of students' future careers.

The choice of different geometries in *PSSM* is sustained by the need to solve real-world problems as within proponents of the *utilitarian argument* (NCTM, 2000, p. 316-317). The authors of *PSSM* state that "in one set of circumstances it might be most useful to think about an object's properties from the perspective of Euclidean geometry whereas in other circumstances, a coordinate or transformational approach might be more useful" (NCTM, 2000, p. 309). Thus, whereas the 1989 Standards justified allusion to non-Euclidean geometries on the need to illustrate the role of assumptions in an axiomatic system, the choice of geometries other than synthetic Euclidean geometry in *PSSM* is more aligned with the *utilitarian* notion of finding the best tool to solve a problem.

Traces of the *intuitive argument* also appear in the Geometry Standard. Students are to develop visual, spatial, and drawing skills as well as a language to communicate about their experiences in an increasingly human-made, geometry based material world (Tatsuoka et al., 2004). Computers play a special role to develop their geometric intuition. For example, the Geometry Standard emphasizes the use of computer software to develop students' visualization skills needed in job settings (NCTM, 2000, p. 316). There is the sense that geometry will help in having a better understanding of the world as Usiskin (1980/1995) had argued.

The Geometry Standard follows the trend started in the report of the Committee of Fifteen, where the fusion between geometric concepts and applications became relevant in the geometry curriculum. At the same time, the Geometry Standard incorporates some of expectations for the geometry course developed during the 20th century such as having students experience the work of mathematicians, preparing them for their future careers, and developing skills unique to the study of geometry. While the Geometry Standard has acknowledged those goals for the study of geometry, it has abandoned the notion of transfer of training in logical reasoning that characterized earlier debates about the study of geometry.

The articulation of competing arguments within the Geometry Standard might reflect tensions about the goals of schooling. Different justifications might be in conflict with each other and at the same time might help in supporting diverse versions of the geometry course.

> Competing visions—that is, competing answers to the questions of what we should teach, why we should teach one thing rather than another, and who should have access to what knowledge—can be healthy, but only if they are recognized and dealt with. It is naive, moreover, to assume that wide-ranging reform in school mathematics will result from any effort that focuses only on schools and is not somehow linked to reform of the wider society. (Stanic & Kilpatrick, 1992, p. 416)

The 21st century may also bring about new justifications for the geometry course, especially with the availability of new technology and other resources and with the advent of more stringent accountability demands for teachers, schools, and districts.

Conclusion

In this article we have described four arguments that have been used during the 20th century to justify the high school geometry course in the US. At the onset of the 20th century, geometry was justified on the grounds of a *formal argument*—that geometry helped discipline the mental faculties of logical reasoning. At various times during the 20th century other arguments emerged recurrently. A *utilitarian argument* was an incipient influence in the report of the Committee of Fifteen, which recommended the teaching of applications of geometry. The argument was that geometry would provide tools for students' future work or non-mathematical studies. A *mathematical argument* was also an incipient influence in the report of the Committee of Fifteen, but came to the fore with more force at about mid century, justifying the geometry course as an opportunity for students to experience the work and ideas of mathematicians. The *mathematical argument* recommended the study of geometry because of its capacity to engage students in making and proving conjectures or to illustrate for students how dramatic conceptual developments occur in the discipline of mathematics that permit to solve a multitude of new problems. Finally, an *intuitive argument* emphasized the role of geometry providing students with an interface language and a representation system to relate to the real world.

The value of the distinction between different arguments is apparent as we look at the argument made in *PSSM* for the study of geometry at the end of the 20th century: This argument draws on a combination of the modal arguments offered during the 20th century but has a distinctive flavor, quite different than the mental disciplinarian call for the study of geometry at the end of the 19th century. The four modal arguments can be used to describe specific curriculum approaches and note what is at stake in the geometry instruction of specific institutions. Whatever an institution puts at stake in a course of studies, interaction in classrooms develops around the procurement of those stakes, even if the mathematics "constituted through teaching" (Høyrup, 1994, p. 3) does not reduce to the claiming of those stakes. The four modal arguments thus identify expectations that might shape what a teacher and her class at work are pursuing—what they hold themselves accountable for vis-à-vis the subject of studies.

The 20th century showcases the history of the rise and fall of the formal argument as the main reason for students to study geometry. Our work shows how three other arguments emerged through the century to justify the course, adding conditions and constraints to the work that teachers and students do in classrooms. One could therefore expect that the contents of the geometry course of any institution have become more heterogeneous as these various arguments have come to the fore. But also this heterogeneity has contributed to sustain the geometry course in spite of the gradual fall of the formal argument. *PSSM* exemplifies how the other three arguments can integrate to justify the study of geometry, if not the geometry course.

But those developments in the justification for the geometry course are the discussions at the level of the opinion leaders and policy makers. Actual schools, parents, teachers, and students might well continue to hold geometry instruction accountable to procure the stakes identified by the formal argument. Our research suggests that at a minimum, instructional policy that seeks to promote the vision of *PSSM* will have to contend with those expectations and find a serious way to talk to stakeholders about the kind of transfer that is reasonable to expect from school studies.

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Notes

1. The order of authorship is alphabetical.
2. For sake of brevity we do not include here the analysis of textbooks and will present them on a separate piece.
3. It is possible for example to think that the record in *School Science and Mathematics*, *School Review*, the *American Mathematical Monthly*, or *L'Enseignement Mathématique (*if we wanted to think globally) would yield complementary categories.
4. Slaught would later be named honorary president of NCTM from 1936-1938.
5. Cajori was the only member of the Committee of Fifteen who also participated as a member of the Mathematics Commission that reported to the Committee of Ten.
6. Halbert Christofferson was a professor of mathematics in the Mathematics Department and director of the Secondary Education program at the School of Education at Miami University, Ohio. He wrote a book for geometry teachers based upon his dissertation work at Columbia University. He founded with other teachers, including Harold Fawcett, the Ohio Council of Mathematics Teachers.
7. E. R. Breslich presided the School Science and Mathematics Association between 1926 and 1927 and NCTM between 1939 and 1941.

8. Later a professor of education at the University of Illinois, Henderson became a pioneer in research on mathematics teaching.
9. Moise had studied mathematics with the celebrated topologist R. L. Moore who had advocated for a similar experiential pedagogy in the training of professional mathematicians (see Burton-Jones, 1977; Zitarelli, 2006).
10. The other content strands are number and operations, algebra, measurement, and data analysis and probability.

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The Role of Projective Geometry in Italian Education and Institutions at the End of the 19th Century

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Abstract

At the end of the 19th century, projective geometry was at the basis of most geometric research in Italy, and of much other research work in Europe. Furthermore, through its close connection with descriptive geometry, it seemed particularly responsive to social and educational needs of the time. In Italy, a reform of the Technical Institutes brought projective geometry into the syllabuses; and Cremona's book Elementi di Geometria Projettiva *helped to spread the synthetic method in Italy and in Europe.*

In this paper we will examine the link between projective geometry and education, from the personal point of view of Luigi Cremona, and from the institutional point of view of technical instruction in schools and universities. Information about the reception of Cremona's book in Europe and some letters from Cremona's estate will help us to understand the scientific climate of that period.

Introduction

System of education in the period of the unification of Italy

The first law to regulate Italian schools was the Casati Law of 1859. It was originally passed only for the Kingdom of Sardinia and for Lombardy, and was later gradually extended to the other Italian regions after their annexation had been declared. With few modifications, it was to remain in force until the Gentile Reform of 1923. The Casati Law established the general characteristics of state secondary education, distinguishing between a *classical education*, whose purpose was to provide a literary and philosophical educational background which would prepare young people for higher studies and specifically for academic courses at the State Universities (Art. 188), and a *technical education,* which sought to provide an appropriate general educational background to young people intending a career in the public services, in industry, in commerce and in agricultural management (Art. 272).

Secondary education was divided into a first and a second level. To cover classical secondary education, the Casati Law introduced the Gymnasium and the Lycée (*Ginnasio-Liceo*), which were to become the point of reference for the entire Italian secondary education (Vita, 1986, p.2). The Technical School (*Scuola Tecnica*) and the Technical Institute (*Istituto Tecnico*) were set up for technical secondary education.

Pupils entered the Gymnasium and the Technical School after a primary school extended over four years. The Technical School thus covered the same age range as the present-day middle school (11–14) while the Gymnasium lasted for five years and hence included the first two years of high school. The Technical School soon lost its characteristic of being a preparatory school for the Technical Institute (to which access could also be gained from the Gymnasium) and was transformed into a school for general education.

After finishing the Gymnasium, pupils completed their classical education by attending the Lycée for three years before going on to university. Initially, a technical education was shorter than its classical counterpart because, after the three years at Technical School, only a further three years were foreseen at the Technical Institute. However, through an 1871 reform the duration of the Technical Institute was extended to four years and, in some cases, to five.

Italian mathematics and mathematicians

The political unification of Italy in 1861 led to the various areas of Italian mathematics becoming integrated into the context of European research. The most eminent Italian mathematicians were involved, both scientifically and politically, in bringing Italy back to the forefront of international developments in the fields of science and economics. Francesco Brioschi (1834–1897), Enrico Betti (1823–1892) and Luigi Cremona (1830–1903), who had also taken part in the First War of Independence, can be considered, along with Eugenio Beltrami and Felice Casorati, as the period's leading mathematicians. Brioschi, an analyst, re-established the *Annali di Matematica* in 1857; he founded the Polytechnic of Milan, was General Secretary for Public Education and held several other political posts. Betti, who had introduced Galois' Theory to Italy and who worked on applying Riemann's ideas to mathematical physics, became head of the Scuola Normale in Pisa in 1865, an institute which produced numerous researchers in mathematics. He was also a member of the Italian Parliament, and in 1884 became a senator.

This was without a doubt an unique period in the scientific history of Italy. Patriotism intertwined with a lay and positivist mentality born with the Risorgimento[1], which was especially widespread amongst the bourgeoisie, gave scientific research a central role which it was not to have again[2]. After a period of just a few decades, Italian mathematics found its place in the vanguard of research in the early twentieth century.

Of special standing in this context were the geometric studies of a synthetic[3] nature. These achieved their greatest development in the *school of geometry* headed by Luigi Cremona. Cremona anchored his work within the classical school of projective geometry, with particular attention given to the ramifications which came about in France with Poncelet and Chasles, and in Germany with Von Staudt, Plücker, Möbius, Steiner and Clebsch. Cremona's research into birational transformations would prove to be the basis for a great deal of successive studies carried out in Italy in algebraic geometry[4]. Cremona developed his research employing *pure* methods, separating geometric properties from analytical ones. Although the excessive purism of this approach was later criticised, it continued, albeit with the toning down of the "*extreme*" positions, to characterise Italian algebraic geometry, which culminated in the famous (Roman) school of geometry at the beginning of the 20th century, amongst whose leading exponents were Federigo Enriques, Guido Castelnuovo and Francesco Severi.

Born in Pavia in 1830, Luigi Cremona graduated with degrees in civil engineering and architecture. In 1860 he was appointed to the Chair of Advanced Geometry at the University of Bologna, where he also taught descriptive geometry. He then moved to Milan in 1866, where he taught graphical statics at the Polytechnic and advanced geometry at the Scuola Normale (annexed to the Polytechnic itself to train Technical Institute teachers). Finally, in 1873 he accepted an invitation to transfer to Rome to head the School for Engineers, in whose reconstruction he played a fundamental role. He did not however abandon the teaching of graphical statics and advanced geometry. His political work was noteworthy. In Parliament he was recognised as an authority when it came to issues of education (Tricomi, 1962), and hence Cremona was able to exert an enormous influence over the organisation of mathematical studies in Italy. He became a senator in 1879, was made vice-president of the Senate and in 1898, for just one month, was Minister for Education.

Luigi Cremona was recognised as a 'maestro' (master) on an international level. His wide circle of scientific relationships is demonstrated by the many translations of his papers and books, as well as by the large collection of about 1000 letters sent to him by world-famous mathematicians, which was discovered several years ago in the Department of Mathematics at the University of Rome (Israel & Nurzia, 1983; Menghini, 1986 and 1993).

Geometry in secondary schools

In 1867, before the complete unification of Italy, a reform by Education Minister Coppino introduced the use of Euclid's *Elements* as the textbook for the teaching of geometry in the Gymnasium-Lycée. More specifically, Book I was intended for the fifth year of the Gymansium, Books II and III for the first year of the Lycée, and the successive books for the second year of the Lycée.

Luigi Cremona was a member of the commission which brought Euclid's text into the schools. He also contributed in an unofficial way to the Italian edition of Euclid's *Elements*, which was edited by Betti and Brioschi. At first the reform generated some criticism, but it has provided the model for the teaching of geometry right up to the present day (Menghini, 1996). Geometry was seen as "mental gymnastics" and was intended to get the young used to the rigour of reasoning .It was thought that only the pure geometry of the *Elements* could carry out such a task. In truth, it is not difficult to interpret the 1867 reform as a desire to create a new Italian model freed from the foreign textbooks which were then common in Italy. This point of view is also expressed in some of Cremona's letters (see, for example, his letter to Betti, later in this paper).

The successive reform in 1871 concerned the teaching of geometry in the Technical Institute. The reform presumed the explicit introduction of the fundamental principles of projective geometry, which was deemed to be a necessary theoretical preamble to the study of descriptive geometry. Cremona's *Elementi di Geometria Projettiva*, published in 1873, was later written to fulfil this syllabus.

The successful translations of Cremona's book and their reception in France (1875), Germany (1882) and England (1885) demonstrated the fact that a synthetic treatment of projective geometry was appreciated outside of Italy. Although the book was written for Italian secondary school, it was to be adopted mostly at university level outside Italy.

Foreign books in Italy

Before the unification and until 1867, Italian textbooks were few in number and moreover were hardly rigorous (Pepe, in press). Most schools in the various Italian states adopted translations of foreign books. Only a few of these books were appreciated by the leading mathematicians (Brigaglia, in press). With reference to projective geometry, it is worth mentioning the *Éléments de géométrie* by Amiot, which had been translated, with notes and addenda, by G. Novi (Amiot, 1858). Cremona, in the preface to the Italian edition, praised Amiot for having sought to transform the old books in such a way as to help young people take part in the progress achieved by geometry in the previous one hundred years. However, above all he praised Novi for having completed topics "mentioned too briefly" by Amiot, like the method of the projections, the anharmonic ratio, poles and polars, etc.

The introduction of Euclid's *Elements* in schools brought about the disappearance of most of these books, including the widely adopted *Éléments* of A. M. Legendre (Maraschini & Menghini, 1992; Schubring, 2004)[5]. In his letter to Betti of 8. 9. 1869 Cremona wrote:

> People can say what they want but Euclid's is still the most logical and most rigorous system we have: all the successive systems are impure hybrids; in seeking to remove one defect, they fall into other worse ones and more than anything else they stop being true geometric systems. Legendre suffices as an example and this even though he is the most respectable of the elementary geometry reformists. However, if we cast our minds back to the books used in our schools before 1867, which would now be re-introduced if the syllabuses are modified, who would dare to deny that the introduction of the Euclidean method has been of immense benefit for our schools?

In fact, Legendre's was a book of elementary geometry whose structure was very similar to the *Elements* of Euclid. But some proofs were "simplified" and the author sometimes made use of arithmetic and algebraic notations which hid the pureness of the geometric treatment.

Two years before the syllabuses for the Gymnasium-Lycée came out, in 1865, Cremona himself translated the *Elemente der Mathematik* by Richard Baltzer (1818–1887) into Italian. Baltzer's work covered many areas (arithmetic, algebra, trigonometry,...). With regard to geometry, the book introduced the primary notions and definitions of elementary geometry rather rapidly and then quickly progressed to more demanding current theorems in Euclidean geometry. Projective problems were not covered. Cremona tried in vain to convince his friend Betti, who was a member of the Public Education Council, to declare it useful for the schools. The expression "for secondary schools" did appear on the title page, however (Balzer, 1865). In this case, apparently unconcerned by the intrusion of foreign books, Cremona held that Baltzer's book was appropriate for the Technical Institutes. He continued to put forward this book even after 1871, holding that the geometric part could be covered in the first two years of the Technical Institute.

In a lecture given at the opening session of the *Association for the Improvement of Geometrical Teaching* and reported in the *Giornale di Matematiche* (1871), Thomas A. Hirst, objecting to the adoption of Euclid's *Elements* in England, referred to the fact that even Cremona recognized the necessity of a different geometry for the scientific curriculum, and confirmed that Balzer's book was used in most Italian Technical Institutes.

Balzer's book was a compendium for teachers, not a book for use by students in school. But this is in line with the complete absence of didactical tools in Italian schoolbooks. Although they were written for use in school, the intended audience was the teacher, rather than the students. The didactical transposition was left completely in the hands of the teacher.

Later, in 1874, another translation of a German book appeared. It was a book of descriptive geometry by Wilhelm Fiedler (1832–1912). Although it was written for the *Technische Hochschulen*, which are university level schools in Germany, the Italian edition was explicitly translated and adapted for use at the secondary school level, in the Technical Institutes of the Italian Kingdom.

It was certainly appropriate to have Fiedler's book alongside Cremona's Projective Geometry in the parallel course of descriptive geometry at the Technical Institutes. According to Fiedler, the main scope of the teaching of descriptive geometry is the scientific construction and development of "Raumanschauung"[6]. Fiedler reinforced this point of view in a paper translated and published in the *Giornale di Matematiche*. Fiedler sees a complete symbiosis between descriptive and projective geometry and holds that starting from central projection, which corresponds to the process of viewing, we can develop the fundamental part of projective geometry in a natural and complete way (Fiedler, 1878, p.248). He feels supported by Pestalozzi, who argues that teaching must start with intuition. Fiedler sees these strategies as the best method for the reform of geometry teaching at all levels.

The position of Fiedler was very close to Cremona's[7]. Fiedler never mentions Cremona in his paper, but in a letter to Cremona, at the beginning of 1873, he praises his book and the simple way in which Cremona introduces the topics. Furthermore, asking for information about Italian technical education, he adds:

> ...my interest is in this scientific and educational organization of Italy, in the foundation of unity through the school for a new generation. (Fiedler to Cremona, 1873, in Menghini, in press).

Institutions: Technical Institutes in Italy

In 1871, Minister Castagnola issued new syllabuses for Technical Institutes. The Technical Institutes had developed considerably after the unification of Italy, having undergone various reforms since 1860, all of which recognised the necessity of the separate developments of humanist and technical education, with an eye to the model of the German *Realschulen* (Morpugo, 1875, XXVI and on; Ulivi, 1978).

Coming after the syllabuses introduced by Minister Coppino in 1867, which incorporated the study of Euclid's *Elements* in the classical education, the 1871 syllabuses represented a further significant development for the teaching of mathematics.[8] The reform embodied in the syllabuses recognised the need for a general literary and scientific education in technical education too and instituted a Physics-Mathematics Section (*Sezione fisico – matematica*). This section consisted of classes in which the scientific topics, particularly mathematics and physics, would receive greater attention. Since this section did not have the aim of qualifying students to go into the professions, and permitted university entrance, it could be seen as the scientific alternative to the Lycée (the future *liceo classico*). The existence of this section could be considered a victory for the new working middle-classes (Vita, 1986, p. 38), brought about by the pressing consequences of progress of science and technology and on the wave of a mainly positivist philosophy (Carrara, 1966). The *liceo scientifico* was to be introduced much later in 1923, in a climate which was decidedly less favourable to scientific culture. At that time, the stimulus given to science by the Unification was completely lost in favour of an academic life which became increasingly self-confined.

The preface to the syllabuses of 1871 refers to the foundations of mathematics and demonstrates the deep mathematical knowledge[9] of the authors. It established that the mathematics taught in the Technical Institutes should promote "useful and not remote applications" (Ministero etc., 1871), and at the same time "enhance the faculty of reasoning"; and hence the methods used to present it had to be "rigorously precise." Nevertheless, the model was not Euclid's Elements, but was instead provided by the "new doctrine of projectivity", which "supplies graphic constructions to solve first and second order problems" in a straightforward way.

The main points are summarized here:

> The first two years of the Technical Institute are common to all the sections and have a general and preparatory character. The study of geometry starts from the

> first elementary notions (angles, circles, inscribed figures, equality, equivalence and similarity of plane figures) and includes the graphical multiplication of segments, the transformation of area given a base, preliminary notions of solid figures and their measures.
>
> In the second "advanced" two-year phase, the syllabus of the third year is common to the physics-mathematics section and to the industrial section. Geometry includes: the theory of projections of geometric forms (projective ranges and pencils, cross ratio, complete quadrilateral) with its applications to the graphical solution of the problems of first and second degree[10] and to the construction of the curves of the second order, seen as projections of the circle (this requires: projective ranges in a circle; self-corresponding elements of superposed forms); the theory of involution (conjugate points with respect to a circle); the duality principle in the plane; elements of stereometry and the graphic construction of the barycentres of plane figures.
>
> In the fourth year the program is specific for the physics - mathematics section. It provides for the focal properties of conic sections and the projective properties of conics and spheres; the principles of analytic geometry will be founded on the metric relations (by means of the cross ratio) of projective forms.
>
> In the parallel teaching of *descriptive* geometry, the teacher will start from central projection and from the projective properties of figures and will handle the theory of collinearities, of affinities, of similarities, with attention to homology, up to the construction of intersections of surfaces of the second degree. The teachers of mathematics and descriptive geometry shall cooperate, as both are concerned with the projection of geometric figures (Ministero etc., 1871, p. 52-63)

Mathematics occupied six hours per week in the school, while descriptive geometry occupied three hours. The only didactical suggestion was that pupils should be directed to do a lot of practical work and that the teacher should question them individually and help them in solving exercises.

According to Morpurgo (1875, p. 46) the syllabuses were highly commended. They undoubtedly covered a great deal of ground. Indeed, the original aim was to prepare students in the physics-mathematics classes for direct entry to the School for Engineers without having to attend the two-year preparatory course (see Section 3). This was in the end not permitted, and in 1876 a new reform took place, based on proposals coming from teacher councils. The physics-mathematics section "preserves its character of school of a general culture, to which the extensive study of Italian letters, that of modern languages, and a strong teaching of the sciences furnish the strength that the humanist education takes from the Greek and Latin literature" (Ministero etc., 1876). The aim of mathematics teaching is again that of enhancing the faculties of the mind while acquiring notions which are fundamental for further studies at the University. The syllabuses were reduced. The teaching of projective and descriptive geometry was unified and appeared only in the fourth year. After two years of plane geometry, and one year of solid geometry and trigonometry, the study of projective geometry was whittled down to the study of the projective ranges and pencils, and of the harmonic properties and projective relationships in a circle. Descriptive geometry was restricted to orthogonal and central projections, which were taught together with equalities, similarities, affinities and perspective collinearities.

Institutions: the Citadel of Science

From his arrival in Rome in 1873 onwards, Luigi Cremona was an advocate of a transformation which involved many aspects of the development, even in a 'physical' sense—of scientific studies and, in particular, of mathematical studies in the capital. Cremona accomplished this through the creation of the School for Engineers and with the aim of overcoming the separation of pure and applied science.

Based on experiences in the Polytechnical Schools in northern Italy, this project foresaw transferring a part of the structures and professors of the Faculty of Science from the old 'Sapienza' to a new site at San Pietro in Vincoli, thus setting up a sort of 'citadel of science' in which, along with other disciplines, all teaching of a mathematical nature was brought together in a newly-founded autonomous Institute of Mathematics. This decision was important not only on an institutional level, but also from a strictly conceptual point of view. Indeed, it reflected the close ties between aspects of a theoretical nature and aspects of a 'concrete' nature–linked to the practice of drawing— in the conceptions of that time and in those of Luigi Cremona in particular.

In the new setting, one could find the *School for Engineers,* the *School of Mathematics*, the *Library* and the *School of Drawing and Architecture*. All these schools were part of the University[11]. The position that mathematics should occupy in science is clearly reflected in this "physical" arrangement.

The courses of study in the School for Engineers lasted for three years, after a preliminary two-year period of studies concerning physics and mathematics at the Faculty of Science.

In the last two decades of the nineteenth century, there was a significant increase in the number of courses conducted within the Faculty of Science at the University of Rome. Cremona was also responsible for the establishment, probably for the first time in history, of a course called Projective Geometry (the former Advanced Geometry) designed for the first year of studies at the University. Later, it was his idea to merge the teachings of analytical and projective geometry, this coming about for the first time in Rome in 1888-89. Cremona was to oversee the School for Engineers until his death in 1903.

Projective Geometry and Education

As is well-known, modern projective geometry–which is generally held to begin with the publication in 1822 of the famous *Traité des propriétés projectives des figures* by Jean-Victor Poncelet—came about essentially due to the considerable developments in descriptive geometry, which had taken place previously, in revolutionary France[12]. Gaspard Monge—with his *Géométrie descriptive*, published in 1799—played a considerable role in these developments and contributed to the design of the rising *Ecole Polytechnique* (Grattan-Guinness, 2005). The military and civilian needs met by the *Ecole Polytechnique* through the training of engineers and administrators were of course not only a French phenomenon. The *Ecole* influenced the development of technical schools in many countries in 19th century Europe, like the *Technische Hochschulen* in Germany (Schubring 1989, p. 180) and the *Politecnici* in Italy.

In those technical schools, mathematics was seen as a fundamental preparatory topic. However, in the engineering schools, the triad 'drawing, geometry (descriptive and projective), engineering schools', promoted geometry as a basic theoretical subject, but also as a practical and applied activity.

So, at the end of the 19th century, the study of projective geometry was validated not only on the basis of the flourishing of theoretical research, but also because of its responsiveness to the requirements set by technical education due to the corresponding needs of society.

Moreover, the synthetic approach developed in Italy was able to satisfy those didactic and cultural needs expressed at the time of the introduction of Euclid's *Elements* into schools. The treatment of geometry was pure and did not require recourse to algebraic or analytical instruments. The proofs were rigorous and made reasoning obligatory. Furthermore, the subject was not dominated by foreign authors and was in line with Italian research into geometry.

In 1965 Campedelli (1965, p.227) listed those factors that he considered important in innovating school curricula: the linkage with the university and thus with the developments of research, the linkage with applications and the didactic and formative aims. These factors were explicit in the Bourbakist reform of the 1960s (OECE, 1959, p. 11–14), and more or less present in Klein's curricular reform around 1900, particularly in his *Erlanger Antrittsrede* (Rowe, 1985). It would appear that they were already considered by those such as Cremona, who endeavoured to bring projective geometry into schools.

Because of the considerable innovation in mathematical content, the reform which began with the Technical Institutes in Italy can be compared with the Bourbakist reform of the 1960s, which was centered on linear algebra. In the reform of the 60s, the didactic and formative aims were advocated by the pedagogues (In particular, Piaget's proposed cognitive structures were considered to correspond to the mathematical structures of Bourbakism, Piaget *et al.*, 1955). In our case, the didactic and formative aims were identified with the enhancement of mathematical reasoning through the synthetic approach. There were no specific methodological or didactical tools proposed in connection with the introduction of projective geometry in schools.

Cremona's Project

In a letter dated 8th September 1869 (Gatto, 1996) Cremona wrote to his friend Betti:

> I am convinced that modern methods, especially Steiner's and Staudt's, are destined to revolutionise the whole of geometric knowledge right from the elements themselves; with these methods even the most elementary things can be dealt with more simply, more originally and more fruitfully. However, it will not be possible to introduce these methods in schools until an elementary book has been written specifically for that purpose: such a book does not exist and at the moment there is nobody who wishes or is able to write one. Until that still far-off day when that radical reform can be carried out, I believe that Euclid is still the best guide for teaching geometry in the Lycée.

In 1873, Cremona published his *Elements of Projective Geometry*[13]. This was an elementary book introducing modern projective methods into schools (this does not mean that the book was written in such a way that a student could read it), but it did not represent that radical reform to which Cremona looked forward in his letter to Betti. The book was specifically aimed at the Technical Institutes. It contained the topics for the third-year course, in accordance with the 1871 syllabuses, and also a part of the topics and graphic constructions intended for the descriptive geometry syllabus. The focal properties of conic sections were to have been covered in the second volume, but this volume was never published because the syllabuses were reduced in 1876.

In the preface, Cremona states that he had deeply desired the reform and that he felt that it was his duty to write a proper book. The stated purpose of the book was to propagate the useful theories of projective geometry in Italian schools. Such theories, as the preface pointed out, could be found both in the works of Euclid and Apollonius, and in those of Chasles and von Staudt. Cremona stated that projective geometry, apart from scientific results and from the usefulness of its applications, also had the advantage of being very easy to learn, since it required very few preliminary notions. This is especially the case if pure methods are adopted, such as those of von Staudt. We also find in this discussion all those factors which, later in the 20th century, were to be put forward as reasons for innovating school syllabuses.

In his work, Cremona set out to go beyond the mere training of future engineers. Even at an institutional level, the intention seemed to be the creation of a scientific secondary school of a high cultural level, which would be able to compete with the Lycées in the education of Italy's future leading classes.

> Perhaps in the not too distant future, this will be the springboard for the solution to the problem of the teaching of elementary geometry: then (if I am not mistaken) and only then, will we have something with which it will be worth substituting the Euclidean method. (Cremona, 1873, p. V)

Therefore, it is possible that Cremona's book was not only aimed at enhancing the scientific culture in the Technical Institutes, but also was intended to constitute a first experiment in promoting the use of projective geometry methods in the Gymnasium–Lycée as well.

Actually, this never came about and, indeed, the contrary occurred. In the early 1900s, texts based on the *Elements* of Euclid became the norm in all types of school. Veronese's 1897 textbook, written for the first two years of the Technical Institutes and for the 4th and 5th year of the Gymnasium, also remained faithful to the Euclidean treatment of geometry and did not contain topics or problems of a projective nature.

There is not much written evidence regarding the reception of the syllabuses for the Technical Institutes and of Cremona's book, apart from the general appreciation stated by Morpurgo (see Section 2). In the *Giornale di Matematiche*, the only Italian journal addressed to school teachers (Furinghetti & Somaglia, 1992), we find no explicit reference at all, although the editor, G. Battaglini, had been involved in the syllabus' reform of 1876. On the other hand, most problems and papers published in this journal addressed pure projective geometry. Only later, in the year 1875, in volume XIII, p.341, do we find the questions assigned for the final examination of the Technical Institutes[14].

Other books appeared for the 3rd and 4th year of the Technical Institutes (Reggio, 1891 and De Franchis, 1908) in which the syllabus of 1876 was fulfilled (the syllabus was in force until 1923, with little variations). Books of descriptive geometry continued to exist separately from those of geometry. In a 1903 book, (Farisano, 1903) the author wrote that the teaching of descriptive geometry is entrusted to the teacher of "Constructions" because it must be "only technical."

Theory and Applications, Research and Methods

The scientific and didactic connection between projective geometry and descriptive geometry, or between the theoretical and graphical aspects, started to influence Italian thinking as early as the pre-unification period, as can be seen in the 1838 translation of Monge's *Géométrie descriptive*. The conceptual aspect is linked to the relationship between "pure mathematics" and "applied mathematics." From this point of view, it is possible to make a clear distinction between 19th century mathematical thought, which is characterised by a general symbiosis between theory and application, and that of the 20th century, which shows a clear division between the two aspects, especially on formalist bases. In geometry, this position was to lead to the complete separation of the teaching of projective geometry from that of descriptive geometry and, in fact, to a complete exclusion of descriptive geometry from faculties other than for engineering and architecture.

In any case, this division came somewhat later in the universities than in the Technical Institutes. In 1935, the foundation of the Faculty of Architecture in Rome saw the substantial participation of mathematicians in its establishment. Right from its opening, courses of descriptive geometry, mathematical analysis and applications of descriptive geometry were taught to architecture students by some of the leading exponents of the Roman mathematics community of the period (dell'Aglio, Emmer & Menghini, 2001).

What clearly emerges from *Elementi di Geometria Projettiva* is the belief of the importance in the inclusion of topics of a theoretical nature involving projective geometry into the curriculum of future engineers—from the Technical Institutes to the Polytechnic Schools—in which the central role nevertheless remains occupied by graphical activities. Making didactic recommendations, for example, Cremona states very clearly in the 'Preface':

> Finally, it should be noted that the graphical execution of problems always accompanies the theoretical reasoning for the proof of theorems and the deductions of corollaries. (Cremona 1873, p. XIII)

This point of view is clearly closely related to *research and methods*: a science which looks at applications cannot be too "pure." In his research, Cremona operated in the domain of algebraic geometry using pure projective methods. However, his geometry was not wholly independent of analytical algebraic support. Even though Cremona did not use coordinates, and used pure reasoning (and intuition) to prove all his propositions, he nevertheless employed several algebraic methods. Thus, he was willing to use the results of algebraic systems, and concepts taken from algebra (order and genus of algebraic forms), if they could enhance the continuity of the geometric treatment .

The same mathematical conception is at the core of his book for the Technical Institutes:

> I therefore gave greater emphasis to graphic rather than metric properties; I used the procedures of Staudt's *Geometrie der Lage* more often than Chasles' Géometrie Supérieure, although I never wholly excluded metric relations, because this would have had an adverse practical effect on teaching. […] I could have copied Staudt by doing without any preparatory notions whatsoever, but in this case my work would have been too long, and I would not have been able to adapt it to the students in Technical Institutes, who are supposed to have studied the usual fundamentals of mathematics in their first two years. (Cremona 1873, p. XIII)

For instance, Cremona maintained reference to affine formulation of the theorems as much as possible, considering projections of figures from a plane onto a parallel plane, thus using parallelism and points at infinity. For example, he introduced homothetic (similar) triangles as a significant case of homological (perspective) triangles, or parallelograms as a particular case of quadrilaterals; he used length and sign of a segment, together with similitude, to prove the invariance of cross ratio, in accordance with Moebius' barycentric calculus, rather than basing it on the complete quadrilateral, as von Staudt did.

Elementi di Geometria Projettiva in Europe

In the Italian title page of Cremona's book, the subtitle is "for the Technical Institutes of the Italian Kingdom." In the various translations there are no indications of the level to which the book is addressed. In fact, at that time, particularly for the technical education, the age of the students was not always clear, and the character of the schools could change from place to place. We have seen, for example, that in Italy there had been the idea to permit the students of

Technical Institutes to enter the third year of University directly; and in Germany the *Polytechnische Schulen* changed in the course of the 19th century, becoming *Polytechnische Hochschulen* (Schubring, 1989, p.179).

In any case it can be stated that the translations of Cremona's book were principally adopted at a university level.

The first translation of Luigi Cremona's text was into French, carried out by Eugène Dewulf and published by Gauthier-Villars (Cremona, 1875). Dewulf had also translated *Preliminari ad una teoria geometrica delle superficie* (Bologna 1866–67). There are many letters from Dewulf amongst Cremona's papers concerning the translation work, together with a few from Gauthier-Villars. These testify to the importance attributed to the diffusion of Cremona's work in teaching:

> [...] en ce moment, je prépare mes annonces, prospectives et catalogues à l'occasion de la rentrée. Et si je ne puis y comprendre votre important ouvrage, ce sera une année perdue pour la diffusion dans le monde de l'enseignement en France. (Gauthier-Villars to Cremona, 24.8.1875, in Nastasi 1992)
>
> At the moment, I am preparing my advertisements, summaries and reading lists for the new academic year. If I am unable to include your important work in these then it will be a year lost for its dissemination throughout the world of teaching in France

When the second French edition of the book was decided upon, Gauthier-Villars wrote to Dewulf:

> En dehors des ouvrages répondant à des programmes d'enseignement ou d'examen, la France offre peu de débouchés; c'est triste à dire. Par suite, les traductions françaises d'ouvrages étrangers d'un ordre elevé, n'ont d'ordinaire qu'un médiocre écoulement. Ainsi, pendant les premières années, l'édition française de la *Géométrie projective* s'est peu vendue, comme vous le savez, malgré tout son mérite; et on peut dire qu'elle serait loin d'être épuisée, si les pays étrangers, l'Italie elle-même, n'étaient venus nous prendre des exemplaires, après la fin de l'édition italienne.
>
> La difficulté sera encore plus grande, lorsque notre nouvelle édition se trouvera en face de l'ouvrage original italien et de la traduction allemande. Je n'ai pas hésité cependant à vous demander de préparer le travail; et cela pour le motif suivant que je n'exposerai pas à d'autres qu'à un ancien camarade voulant bien me comprendre.
>
> La Géométrie est absolument délaissée en France; elle n'est plus représentée à l'Académie et n'a pas une seule chaire où on la professe. J'ai pensé qu'un des meilleurs moyens de raviver, dans la limite du possible, le goût de cette Science, était de réimprimer l'ouvrage d'un maître, comme celui de M. Cremona.
>
> [...] Je cherche toujours, dans la limite de mes moyens, publier des traductions pouvant développer certains courants d'études dans notre pauvre pays, qui ne lit rien de ce qui se fait à l'Etranger et qui a si grand besoin d'être tenu au courant

> des productions nouvelles." (Gauthier-Villars to Dewulf, 27. 12. 1882, in Nastasi 1992)
>
> Sadly, France does not offer much opportunity other than for works tailored for the teaching and examination programs. Consequently, the French translations of high-level foreign works usually sell poorly. Hence, as you are well aware, in its first few years the French edition of *Projective Geometry* sold little despite all its merits. It can be said that it would be far from being sold out if not for the fact that foreign countries, Italy included, came to us to obtain some copies after the Italian edition had been sold out. The difficulty will be even greater when our new edition will find itself up against both the original Italian work and the German translation. Nevertheless it was without hesitation that I requested that you prepare the work. This is because of the following reason which I will not make known to others, except to a long-standing companion who is especially willing to understand my view. Geometry has been totally neglected in France; it is no longer represented at the Academy and there is not a single chair where it is taught. I considered that, as far as possible, one of the best ways to revive a taste for this science was to reprint the work of a master such as Mr. Cremona. [...] I am still trying, to the extent that I can, to publish translations so as to be able to develop such areas of study in our poor nation, which reads nothing of what is taking place abroad and which desperately needs to be informed of current advancements.

The second translation of *Geometria Proiettiva*, into German, was made by R. Treutvetter and published in Stuttgart by J.C. Cotta (Cremona, 1882). As Cremona himself stated in his preface, attempts had already been made to introduce the first elements of projective geometry into the German schools. These texts were specifically aimed at the *Gymnasia*. Cremona mentioned the book by E. Müller in particular, who enunciated the duality principle for projective forms, alongside topics of a metric nature, and presented the complete quadrilateral, polarity (in a circle) and the conic sections (Struve, 1994).

We do not find letters from Treutvetter amongst Cremona's papers. Hence, little is known about the use made of Cremona's book. However, it is unlikely that it was employed in schools. On the first page, the translator wrote:

> The Italian original is destined above all to the *technischen Hochschulen* of the Italian Kingdom; but the graphical method used by the author in his well-known masterly way will surely also find friends in German institutions.

The term "Hochschulen" does not seem correct, because "Hochschulen" are universities, while the Technical Institutes were secondary schools. But in 1882, Cremona's book, being no longer in accordance with the 1876 syllabus for the Technical Institutes, had surely shifted to universities.

The English translation was made by Leudesdorf on the basis of the German and French translations (Cremona, 1885). This translation, complete with a chapter on the focal properties of conics, was definitely aimed at university students (in particular those at Oxford), as we can read from the letter of B. Price, from the Clarendon Press in Oxford, to Cremona, dated May 26 1884:

"My dear Professor Cremona,

I refer to the proposal made to you by many mathematicians of Oxford, when you were on your visit here to Prof. Sylvester, I have now to say that the unanimous wish is that an English translation of your book on Projective Geometry should be published as soon as possible as the demand for such work is great, and there is no English treatise fit to supply at. You will remember that when the question was under discussion, two plans were proposed, viz. (1) to wait for the expected edition of your *whole* work in its Italian language and to translate that. (2) to translate the German edition which was revised and corrected by yourself on the understanding that you would assist the Translator and Editor with the additions and corrections which you have prepared for your new Italian Edition. Now as less time is required for the second plan than for the first the second is preferred: and Mr Leudesdorf, whom you saw and convened with, and who is a much competent person both as a mathematician and as a good linguist is ready to undertake the work." (Nurzia, 1996, p. 199)

Like Italy, England experienced the reintroduction of the Euclidean text in the schools and the polemics that went with it (see Hirst's address to the *Association for the Improvement of Geometrical Teaching* mentioned in the Introduction). However, it would appear that there were no attempts to modernise teaching by shifting it towards projective geometry. O. Henrici, an assistant and then successor of Hirst at London's University college, wrote:

"In England, pure geometry is almost unknown, excepting in the elements as contained in Euclid and in the old-fashioned geometrical conics. The modern methods of synthetic projective geometry as developed on the Continent have never become generally known here. [...] The one English mathematician whose mathematical thought is purely geometrical is Dr Hirst, a pupil of Steiner, who in the position he has just relinquished has been able to introduce, as the first, modern geometrical methods into a regular system of professional education[15], whilst showing at the same time by his original work what can be done with these methods" (Presidential Address to section A, Report of the Fifty-Third Meeting of the British Association for the Advancement of Science, Southport, 1884, reported in Nurzia, 1997/98, p. 2)

J. J. Sylvester had adopted Cremona's book in 1884, probably in French, and had promoted, like Hirst, its translation into English. In a letter to Cayley (20. 5. 1884, in Parshall, 1998, p. 251), he wrote:

"Cremona stayed with me in College for a couple of days. Leudesdorf is to translate his Géométrie Projective (now out of print) into English. I am in the thick of my lectures 3 times a week on this subject and manage to draw on the board sufficiently well for the purposes of instruction all the geometrical constructions required."

In the preface to the English edition Cremona wrote:

> "[...] My intention was not to produce a book of high theories which should to be of interest to the advanced mathematician, but to construct an elementary text-book of modest dimensions, intelligible to a student whose knowledge needs not extend further than the first books of Euclid. I aimed therefore at simplicity and clearness of exposition; and I was careful to supply an abundance of examples of a kind suitable to encourage the beginner, to make him seize the spirit of the methods, and to render him capable of employing them." (Cremona 1885 p. xiv)

It is exactly this "spirit of the methods", this masterly way, in the words of Treutvetter, of using the graphical methods, which seems to have been most appreciated in the didactics of the elementary courses of mathematics. The English translation was reprinted several times, until the early 1900s.

Conclusion

The Italian aspiration of forming, around the nucleus of projective geometry, a learned scientific class able to compete with its counterpart in the humanities cannot be considered to have been fulfilled.

On the one hand, as we have seen in the previous sections, there were factors which promoted a reform, even a radical one, of mathematics teaching:

• The moment was favourable: important mathematicians played leading roles in political life and intervened personally not only regarding the school syllabuses but even as far as the textbooks themselves were concerned. The social image of mathematics and of all the scientific disciplines was good since their contribution to the technological development of the nation was recognised.

• Projective geometry possessed some characteristics that appeared appropriate for its introduction in schools, such as the link with academic research, the link with applications through descriptive geometry and the "simplicity" of illustrating it because it required few prerequisites and because of the purity of its reasoning.

On the other hand, there were some significant negative factors:

• The didactical conceptions of the time could not provide for a didactic transposition really suitable for a pupil of high school level. Although it is possible to hold that the teachers were competent and although it is possible to consider that the high schools, the Technical Institutes included, were the schools of the few, there can be no justification for that failure to consider how this subject should be introduced in the schools.

• There was not sufficient time to fruitfully adapt the new topic to school: the fields of application of projective geometry, in particular in its synthetic treatment, did not last long enough.

• The 1871 reform's syllabuses were too vast and ambitious. The fact that the reform was promoted and planned by mathematicians involved in the most advanced research, and the reform being without an involvement on the part of the schools themselves, led to some negative effects; the emphasis was placed on contents and not on questions of didactics and methodology.

• Further, with regard to the idea of transferring topics in projective geometry to the Lycée, this was probably a personal 'dream' of Cremona's. There is no evidence that it was shared by other mathematicians of the time.

However, the idea that a mathematical topic (projective geometry) could represent an element of cultural unification between research, education and society, in an era in which scientific studies were not considered poor cousins, appears interesting.

It was an analogous idea (maybe analogously pretentious, but better supported) that led, around 1960, to a new topic (linear algebra) permeating reforms in teaching all over Europe.

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Notes

1. The Resurgence. This is the period of political and cultural activism leading to the unification of Italy in the 19th century.
2. In Brigaglia & Masotto, 1982 there is a good illustration of the interaction between politics and science at the time.
3. The term *synthetic geometry*, beyond its historical and etymological meaning, is used as a synonym of *pure geometry.* The pure methods in projective geometry are also referred to as *graphic.*
4. Italian algebraic geometry made use of pure projective methods. For instance, the intersections of curves and algebraic surfaces were mostly described and "counted", instead of being calculated by means of systems of equations.
5. In truth, as Schubring's work shows, the translations into Italian of Legendre's text, in the edition edited by Blanchet, continued well beyond 1867.
6. Space perception.
7. Cremona was inspired by Fiedler in his first University course of Descriptive Geometry in 1860.
8. Technical Schools depended at that time on the Ministry of Agriculture, Industry and Commerce.
9. The mathematician F. Brioschi was a member of the Council (Consiglio Superiore per l'Istruzione Tecnica) which edited the syllabus. But surely the syllabuses were inspired by Cremona.
10. Of course the resolution of a problem of the second degree always requires a compass. But every problem can lead to the construction of the self-corresponding elements in two

projective pencils, by means of a fixed circle. In particular, the method suggested by the syllabus is that of "false position."

11. In Rome the School for Engineers was annexed to the Faculty of Science. In other Italian cities, as Milan or Turin, these schools were independent from the University.
12. Some of the considerations contained in this paragraph, as well as in paragraph 6, were developed with L. Dell'Aglio in (Dell'Aglio, Emmer & Menghini, 2001).
13. The preface is dated 5. 11. 1872.
14. The one concerning projective geometry is the following: "Given three tangents and the direction of the diameters of a parabola, find: 1. the points of intersection of the three tangents: 2. other tangents to the curve: 3. the points of intersections of these latter tangents."
15. The reference is to a course held at the University College School of London, which had the main objective of preparing poorer students for entrance to the University of Cambridge.

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The Theorem of Thales: A Study of the Naming of Theorems in School Geometry Textbooks

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Abstract

An interesting topic for research and reconstruction in the history of mathematics in school textbooks concerns how geometrical theorems were named, and how the name became established within the educational system. In this paper, we examine the name Theorem of Thales, as it emerges at the end of the 19th century, within different cultural, mathematical and educational contexts and how it was attributed to different theorems in European geometry textbooks. In an attempt to explain this phenomenon, we draw upon the concept of didactical reconstruction, a concept which may prove to have wider application to the use of the history of mathematics in school education.

Introduction

Notes concerning history of mathematics have appeared for a long time in school textbooks, at least since 5th century A.D. (textbook of Proclus: *A Commentary on the first book of Euclid's Elements*). The study of these notes may help to clarify our understanding of the evolution of pedagogical views and didactical strategies of textbook authors. As far as we know, there has been as yet no systematic study into how these views and strategies developed[1] . As a contribution to this study, we shall present an analysis of one of the most interesting illustrations of such historical notes in geometry textbooks: how the term *Theorem of Thales* emerged and developed at the end of the 19th century.

Geometrical Education and Textbooks in the 19th Century

National public systems of education emerged during the 19th century, due at least in part to the the impact of the French Revolution. One important feature of these changes was the establishment of *secondary education*, a level of education between elementary education and higher education at universities. One of the results of the pedagogical and didactical demands of this century was that large changes took place in mathematical education, and particularly in the teaching of geometry (Cajori, 1910; Schubring, 1996, 1985). The subject of geometry became established in lower levels of secondary education, and in primary education. Also, the method of teaching geometry underwent radical change in senior grades of secondary education, frequently becoming (as in England, France, and somewhat less in Germany) an exercise in logic (Smith 1900, 303) within the context of introducing students "to the art of syllogism."

Substantial changes also took place in the system of how textbooks were authored and brought into circulation. Many new authors appeared, particularly in secondary education, and the level of circulation of textbooks increased greatly as the number of copies printed rose (Schubring, 1985). Geometry textbooks, in particular, saw growth in the number of translations of important textbooks into several languages. For example, perhaps the most important of these being the geometry textbooks by Legendre and Lacroix (Schubring, 1996, 367), which were translated from the French to several other languages.

Except in England, the content of geometry textbooks (especially of those for the senior grades of secondary education) began to deviate more and more from Euclid's *Elements*, a work that had been the a paradigm of exposition in geometry during previous centuries. As an example for this development, we can consider the significant influence Legendre's *Elements of Geometry* (first edition in 1794) (Schubring, 1996, 366) had both on European textbooks and on those of the United States during the 19th century. In Legendre's book, the study of the circle (3rd Book in *Elements*) preceded that of parallelograms (2nd Book of *Elements*). Thus, the adherents of Legendre modified the Euclidean order of presenting geometry, as they considered the concept of circle simpler and more elementary than that of the parallelogram (Smith, 1900, 230).

Historical References in Textbooks

Towards the end of 19th century, we see *historical references* begin to appear more or less systematically. We use the term *historical references* to designate the elements of a geometry textbook which refer to the history of geometry, and which are written so that omitting them from the textbook would not be detrimental to its conceptual geometrical content. During the first stage of their appearance in textbooks, historical references were never an organic part of the main part of text. Instead, they are generally contained either in the book's general introduction, or in the introductions of chapters, or at a chapter's close. If they do appear within the text, it is as commentary in brackets and/or small point, or even in footnotes (which are usually in smaller point)). As a rule, these historical references refer to the work of mathematicians of antiquity, by *naming* of theorems the latter are held to have proven, or not to have proven. Sometimes, such references contain a brief summary of the evolution of geometry, or a note relating to a specific geometrical subject, and to related dates. Such references were not altogether novel in geometry textbooks, they were present in older textbooks as well. However, it was not a recognized technique, but one used by only a few authors. For example, there are summaries of geometry's evolution in textbooks dating from the 17th century, as in Beaulieu ,1676; Leyboyrn, 1690; Le Clerc, 1690 (Kokomoor, 1928, 101). The name "Lunes of Hippocrates" appears in a textbook of the 17th century[2] as well.

We believe that the increased use of historical references in geometry textbooks at the end of 19th century is related to some extent to the substantial advances made in the historiography of mathematics after 1870, mainly in England, France, Germany and Italy, in connection with the rise of interest in historical studies in general (Allman 1877, 160- 161). This led some textbook authors and teachers of mathematics to try to "use" history in the teaching of mathematics, in a more "systematic" way than before (Dauben 1999, 110, quoting G. Eneström). The introduction, in particular, of references to *Ancient Greece* can be explained by the rising general interest, in Ancient Greece and Greek mathematical works[3] in the course of the 19th century. Some of these references simply consist of attaching a *name to theorems*. One of these peculiar historical references introduces the name *Theorem of Euclid* (only in German textbooks), and another, which we shall study below, introduces the name *Theorem of Thales*.

Geometrical Achievements Attributed to Thales by Ancient Sources

In Ancient Greek sources, we find: i) five *major references* to geometric achievements of Thales and ii) some other references concerning his calculation of the height of Egyptian pyramids in Plutarch, Hieronymus the Rhodian, and Pliny[4]. Four of the five major references are found in Proclus. They attribute to Thales the following specific theorems: the circle is bisected by its diameter, the angles at the base of an isosceles triangle are equal, the opposite angles are equal and two triangles are equal when they have one side and two adjacent angles equal (Thomas 2002, 164-167). The other major reference is found in Diogenes Laertius's biography of Thales, quoting Pamphila's testimony that Thales "was the first to inscribe in a circle a right-angled triangle" (Thomas 2002, 166-169). Among historians of mathematics, we find diverse views about the correctness of attributing the above theorems to Thales[5] .

In the case of the work (generally ascribed to Thales) which involved measuring the height of pyramids, historians of the 19th century usually concluded that Thales knew some basic principles of similarity (similar triangles), but they rarely attributed a specific theorem to him. We know of only one work in the history of Mathematics which explicitly mentions two theorems relevant to this type of measurement (Ball 1908, 14-16), and these are theorems of Euclid's *Elements:* "If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally" (VI, 2) and: "In equiangular triangles, the sides about the equal angles are proportional" (VI, 4) (Heath 1926, 194-5, 200-1).

Three historians mention the name *Theorem of Thales*, but two of these only mention it to reject it (Loria 1914, 22) (Tannery 1930, 67), and the third, G. Eneström, voices serious objections to the appropriateness of the name (Enriques 1911, 57). The case of D.E. Smith is different. He was active both as a recognized scholar of the history of mathematics and as a textbook author. It is notable, however, that he never used the name *Theorem of Thales* in his texts, despite the fact that his textbooks contain historical references to Thales (Wentworth-Smith, c1913, 32).

How the Name *Theorem of Thales* Emerged and Became Established

Early attributions of various theorems to Thales

Many years before the name *Theorem of Thales* emerged, historical references appear in geometry textbooks attributing various geometrical achievements to Thales. For example, both the Modern Greek translation of Tacquet's textbook (Voulgaris, 1805, 25) and its original in Latin (Tacquet, 1722, 20) attribute to Thales the calculation of the distance to the inaccessible points by applying the theorem which states that two triangles are equal when one of their sides and two adjacent angles are equal. Also, Benjamin of Lesbos credited Thales with the theorem about the angle inscribed in a semicircle, as well as the theorem that opposite angles are equal (Benjamin of Lesbos, 1820, 90, 21). The same author also mentions that Thales calculated the height of Egyptian pyramids by using the proportionality of the sides of similar triangles (Benjamin of Lesbos, 1820, 6).

French textbooks

The name *Theorem of Thales* first appears in a few French textbooks before the end of 19th century, as early as 1882, (Rouché and Comberousse, 1883, cited in Plane, 1995, 79). The name is attributed to the (general) *theorem of proportional line segments or theorem of proportional lines*: "Des droites parallèles déterminent sur des sécantes quelconques des segments proportionnels" (Parallel lines determine proportional segments on any lines which they cut). However, the very same name is attributed to at least two special cases of the general theorem, as e.g.: "Toute parallèle à l'un des côtes d'un triangle partage les deux autres côtes en parties proportionelles" (All lines parallel to one of the the sides of a triangle cut the other two sides proportionally) (Combette, 1882, cited in Plane 1995, 79) and "Dans le triangle, l'égalité des angles entraîne la proportionnalité des côtes" (two triangles with equal angles will have proportional sides) (Rouché and Comberousse, 1883, cited in Plane 1995, 79). By the 1920's, the name Theorem of Thales was well-established in French geometry textbooks, and mentioned in the

1925 French curriculum as well (Bkouche, 1995, 9). It also appears in textbooks of Descriptive Geometry (Cholet- Mineur, 1907- 1908, 315).

Italian textbooks

The theorem of proportional line segments also bears the name of *Theorem of Thales* in Italian geometry textbooks (Faifofer, 1890, 262), at least since 1885 (Enrico 1885, 34). It also appears in Italian textbooks on analytic geometry (Enrico, 1885, 34) and analytic-projective geometry (Burali- Forti, 1912, 92).

German textbooks

The name *Theorem of Thales* is also used in some German textbooks written at the end of 19th century, at least since 1894, but here, it is attributed to a completely different theorem: "Der Peripheriewinkel im Halbkreise ist 90° "(The angle inscribed in a semicircle is a right angle) (Schwering and Krimphoff, 1894, 53). This name is used for the same theorem (possibly with some variations in the theorem formulation), in German textbooks during the first decades of the 20th century. It also shows up in a German encyclopedia of mathematics (Weber-Wellstein- Jacobsthal, 1905, 232).

English and US textbooks

The name *Theorem of Thales* neither appears in American textbooks, nor in those published in England. In the US, however, we do have references to Thales concerning both his geometrical achievements and the measurement methods attributed to him[6]. Several of these *historical references* are due to D.E. Smith (Wentworth/Smith, c1913, 454, 466). In English textbooks, historical references were generally rare during the 19th century.

Other European textbooks

As a consequence of the cultural impact of France and Germany on several other European countries, the name *Theorem of Thales* also shows up (with different meanings) in those countries' textbooks. Thus, the name *Theorem of Thales* appears in Spanish (Deruaz-Kogej, 1995, 2390, Belgian (Cambier 1916, 142), and Russian textbooks (Kastanis, 1986, 3) in the same sense as it is used in French and Italian textbooks[7]. The same name, Theorem of Thales, is employed in Austrian, Hungarian (Howson 1991, 21) and Czech textbooks (Pomylaková 1993, 62) however, but with the meaning attributed to it in German textbooks, rather than the French and Italian meaning. Modern Greek textbooks present an exceptional case: at first they used the name of the Thales theorem in a sense adhering to that of German textbooks (Hadjidakis, 1904, 60), and later they switched to the sense given to it by French textbooks (Nikolaou, 1927, 128); (Barbastathis 1940, 136).

The Naming of Theorems in the Context of Mathematical Education of the 19th Century

The study of how theorems were *named* in different countries, some of the variation having been shown above, reflects the different (cultural) conditions prevalent in the mathematical education of the countries concerned. An important example is provided by France with its time-honored anti-Euclidean outlook (Schubring, 1996, 377; Cajori, 1910, 182). This viewpoint resulted in a tradition which caused the French order of geometry subject-matter to deviate from that of Euclid. For example, in Euclid, the theorem of the square of the hypotenuse (I, 47) precedes the theory of proportions (5th book of *Elements*) just as well as the theorem corresponding to that of proportional lines (VI, 2). In French textbooks, this order had been reversed since P. Ramus's era. This reversal was not followed by German textbooks until the beginning of 20th century[8]. Italian textbooks (after 1866) retained the Euclidean order presenting geometry subjects because, at that time, Euclid's *Elements* had been adopted as the official textbook in Italian schools (Schubring, 1996, 377-378; Cajori, 1910, 191).

In the meantime, the theorem of proportional lines had risen to a prominent position in French textbooks, because of the new developments in (academic) mathematical research in geometry. One of these developments which following the the works of G. Desargues, B. Pascal, La Hire (1685), Carnot (Coolidge, 1934, 219-220), was due to the work of J.V. Poncelet (of 1813, published in 1822) marked the beginning of *Projective Geometry and Affine Geometry*. A notion maintaining a key position in projective geometry is *harmonic separation* (Coolidge, 1934, 220), which refers to an invariant ratio of particular line segments and thus relates to the theorem of proportional lines. This theorem also relates to the new subject of affine geometry because the theorem of proportional lines implies preservation of ratios between collinear line segments which is a key notion in affine geometry. These new "structural" mathematical developments had in some sense been *institutionalized* in the teaching of geometry by the end of 19th century[9]. Simultaneously, historical interest in Thales as a mathematician grew in the 19th century, this interest possibly being one of the reasons why the name *Theorem of Thales* became the generally accepted designation for the above theorem[10].

The same situation is all the more true for Italian textbooks. Although Italian textbooks maintained the Euclidean order of presenting theorems, Italian authors, particularly L. Cremona, adopted certain elements from projective geometry (Cajori, 1910, 191). Some Italian authors, like R. de Paolis, 1884 (Candido, 1899, 204) also tried to blend elements from plane geometry and the geometry of solids, again by drawing on ideas taken from projective geometry. This helped create a "preference" in Italian authors for the theorem concerning proportional line segments, which they attributed to Thales.

The case of German textbooks is different, since the Germans did not change the Euclidean presentation. Note for example that they showed strong preference for the theorem concerning the square of the hypotenuse, the so-called *Pythagorean Theorem*[11]. This preference exemplifies a school geometry which was "more aligned" to Euclid, and thus lacked any conceptual association with projective or affine geometry. (However, this is not to suggest that academic research in

Germany did not contribute to the development of projective geometry after 1830).

It would appear that *names* are assigned to theorems considered essential and significant for school geometry[12]. Thus, French textbooks chose to ascribe the name Theorem of Thales to a theorem which was essential for the modern view of geometry (projective, affine), while German textbooks chose to ascribe the name to a theorem important for "classic" Euclidean geometry, but containing a large number of modern elements (algebraic calculations). Modern European textbooks in other countries were far removed from Euclidean geometry, and saw the necessity of stressing the importance of theorems essential to new developments in geometry.

Towards a Didactical Explanation of the Name

Our study pertains to those contents of school textbooks which do not refer directly to mathematical concepts, but to the "historical origin" of those concepts. Textbook authors were always looking for ways to emphasize the importance of certain theorems in the curricula. *Naming* a theorem is a symbolic act which goes far beyond presenting a simple historical landmark. After having introduced the history of a concept or theorem which the textbook authors had emphasized in their text, they tried "to profit" from history. Naming the theorem after an undisputed, time-honored authority to which it could be historically traced further authenticates it and establishes its importance.

In the first phase of this process, we find some simple historical references to Thales, without naming of theorems after him. A French textbook of 1866 (Rouché and Comberousse, 1866, v) for example, attributes a theory of similar triangles to Thales, while a German textbook of 1875 (Kruse, 1875, 18, 64) attributes two theorems (opposite angles are equal; in a semicircle, the angle is a right angle) to the same Thales. In a later stage, the historical references to Thales evolve into *naming* of the corresponding theorems after him, and into using of his name not in footnotes or asides, but within the main body of text. Establishing the name *Theorem of Thales* in this later stage served a didactical need by using the history of geometry to underpin the particular didactical-ideological interpretation the textbook authors favored. We are thus led to speak, in analogy to *didactical transposition* (Chevallard and Joshua, 1982) about a *didactical reconstruction* of History of geometry, i.e. a reconstruction of history of geometry for didactical requirements.

The authors of school textbooks are as a rule not historians of mathematics;. Rather, they are teachers who evidently desire to "profit" didactically from the history of geometry. They seem to know about Thales from a historical source, and try to establish a direct link between Thales and a theorem already present in school geometry. There is no mention of sources from antiquity at all upon introducing attributions of theorems in textbooks of school geometry, particularly not in connection with the name *Theorem of Thales*. What we find are only different *choices* from historical sources among these theorems ascribed to Thales, with no discussion or historical argumentation at all. Historians of mathematics proceed in a totally different way. For example, they argue at length whether the ancient sources about Thales are valid, or not, trying to see if Thales indeed gave proofs, and if he employed some general method.

We consider that the concept of *didactical reconstruction* offers a key to understanding how textbook authors used the history of mathematics. They focused on its didactical advantages, treating history as one educational tool among others. Here, we have shown that to emphasize a theorem's value, they attributed it to a famous mathematician of antiquity (Thales). In other cases, they selected topics or subjects from the history of mathematics according to their own national and cultural tradition. We believe that there is more to investigate and learn in mathematics education by examining historical references in school textbooks. The name *Theorem of Thales* is only a particular, though telling, example in this direction.

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Notes

1. M. Gebhardt's work, *Die Geschichte der Mathematik im mathematischen Unterrichte*, 1912, remains an exception, for a description of the book see (Furinghetti, 2001, 1).
2. The name *Lunes de Hippocrate de Scio* appears in the textbook *Elemens de Geometrie* of the Jesuit Pardies (1676), as well as in Tacquet's textbook (1745) (Lietzmann, 1912, 35). The name appears also in the Modern Greek translation of the Tacquet textbook as *conjugate lunes of Hippocrates of Chios* (Boulgaris, 1805, 296, 302), This translation draws on the 1710 edition of Tacquet's textbook (Karas, 1993, 70).
3. There were many editions of ancient Greek mathematical works at the end of 19th century, for examples see (Allman, 1877, 161).
4. There is an exhaustive presentation of the literature about Thales and geometry in (Tezas, 1990, 61-103).
5. We refer to three of the most important of these historians, (Cantor 1907, 134-147; Heath 1921, 128-137; Tannery 1887, 81-94). At this point, we should like to make a distinction between the research works in the history of mathematics, and the works addressed to a wider audience (popular works). In any case, we have not found a book on history of mathematics written only for the wider audience in the second half of the 19th century.
6. As examples, we may mention the calculation the height of Pyramids, and measuring inaccessible points (Betz and Webb, c1912, 281, 68). In another textbook, among other

achievements, the theorem that every diameter bisects a circle is attributed to Thales (Fletcher, 1911, 496).

7. In Russian textbooks, we read the following theorem as *Theorem of Thales*: if two transversals are given and three or more parallel lines intercept congruent segments on one of them, then they also intercept congruent segments on the other one. In addition, the theorem of French and Italian textbooks was sometimes named *Generalized Thales' Theorem* (information by professor G.Schubring). In later Greek education, the theorem about transversals had the name *Lemma of Thales* (information by professor T. Patronis).

8. This change of order had an impact on the proof of the corresponding theorems (I, 47 and VI, 2). We observe that the complete proof of the theorem VI, 2 and the analogous theorems of section 5.2 requires the theory of proportion (book V), or, in general, some theory of irrational numbers (limits or Dedekind cuts). The historical stages in the development of the proof of the theorem of proportional lines from Euclid to Dedekind may be found in Bkouche, 1995, but textbooks contain a great variety of incomplete proofs, which have their own interest and their own history. We note that textbook authors do not relate the name of a theorem to its *proof*, otherwise they would not attribute to Thales (a beginner of geometry) a theorem which requires so difficult a proof.

9. According to Patronis (2002, 68), institutionalization is, first and foremost, a symbolic act of showing what is important and respectable within human society, or within a context of social activity. In this sense, institutionalization in the context of (mathematical) education uses (names of) historical figures as Thales, Pythagoras, and sometimes also Euclid to assign a status to the subject taught.

10. More generally, there was a growing historical interest in early Greek mathematics fostered by C. A. Bretschneider's *Die Geometrie und die Geometer vor Euklides*, 1870, see (Allman, 1877, 161).

11. There are at least four books in German which study this theorem: J. Hoffmann (1819), J. Wipper (1880), H.A. Naber (1908), of course, Lietzmann, 1912. Lietzmann refers to Hoffmann, Wipper and Naber at the page 70).

12. This is also the case for the name *Pythagorean Theorem*, as well as for the name of *Theorem of Hippocrates of Chios* in Modern Greek textbooks of Geometry (Patsopoulos, 2003, 577).

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A Turning Point in Secondary School Mathematics in Brazil: Euclides Roxo and the Mathematics Curricular Reforms of 1931 and 1942

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Abstract

The purpose of this paper is to show how Felix Klein's ideas concerning the teaching of mathematics influenced a major attempt to reform the teaching of secondary school mathematics in Brazil. We follow the story of this attempt from the late 1920s to the early 1940s, centering on the role Euclides Roxo had in these reforms. Roxo had a crucial influence on the establishment of the content of the mathematics programs for two educational reforms in 1931 and 1942. One can follow how Roxo retreated from his initial positions of 1931, which had adhered closely to Klein's ideas, to a more conservative position in 1942.

Résumé

Nous montrons comme les idées de Felix Klein sur l'enseignement des mathématiques ont exercé une grande influence sur des réformes de l'enseignement des mathématiques aux écoles sécondaires au Brésil. Nous suivons l'histoire de ces réformes de la fin des années 1920 jusqu'au début des années 1940, et nous étudions le rôle joué par Euclides Roxo. Roxo, qui a exercé une grande influence sur les programmes de mathématique des réformes de 1931 et de 1942, et il est possible de voir comme il récule de ses positions initiales, qu'il avait adoptées de Klein, à une position plus conservative en 1942.

Introduction

Felix Klein's program for the modernization of the teaching of mathematics is well-known. It can be viewed as part of the first international mathematics curriculum reform movement, which was most strongly promoted by IMUK (Schubring, 1989, p. 6; 2003).

IMUK (Internationale Mathematik Unterricht Kommission, Commission Internationale pour l'Enseignement des Mathématiques) was created at the 4th International Congress of Mathematicians, which convened in Rome in April

1908. The congress appointed a committee composed of Felix Klein as President, George Greenhill as Vice-President, and Henri Fehr as Secretary-General. In 1912, Greenhill was replaced by David E. Smith. When the Commission was reestablished in 1952, its name changed to ICMI (International Commission on Mathematics Instruction) (Schubring, 1989, p. 17; Coray and Hodgson, 2003).

It is little known that Klein's ideas influenced a major attempt at reform in a country at the periphery, Brazil, which participated in IMUK only in an extremely marginal way. Although the reforms in Brazil were initially strongly based on Klein's ideas, there was a marked return, after a few years, to more conservative positions.

The reform movement in Brazil was championed by a major figure in the teaching of mathematics in Brazil at the time, Euclides Roxo. Roxo's attempts at reform provoked strong reactions, and he eventually reverted to more conservative views. It is nevertheless true that some of the changes he introduced in secondary school mathematics teaching were of lasting effect. How his ideas concerning mathematics teaching evolved can be followed through several sources: in a series of newspaper articles, in a book he wrote about mathematics in secondary school (Roxo, 1937), in his several textbooks, and in talks he gave. He participated in two major educational reforms, one in 1931, and the other in 1942. In the first, he was able to impose his reform views, both on the programs and on the corresponding pedagogical guidelines (Rocha, 2001). In the second, we see him withdrawing more and more from his initial viewpoints (Dassie, 2001).

The Setting

Secondary schooling in Brazil underwent a major change in 1837, when the Imperial Government created the Colégio Pedro II in Rio de Janeiro. This city had been the seat of the Portuguese Empire since 1808, the year the Portuguese court had had to emigrate to Brazil after Napoleon's invasion of Spain and Portugal. The Colégio Pedro II was intended to act as a model secondary school for the entire country. In fact, it established a "de facto" official curriculum, because only schools adhering to its programs could award secondary school diplomas valid for entering higher education (in professional schools, since Brazil did not have universities at the time) (Haidar, 1972). From 1837 to the middle of the 20th century, a long list of distinguished Brazilian intellectuals and scholars taught at the Colégio Pedro II, usually after having been admitted by public examination. During this period, the faculty of the Colégio Pedro II enjoyed a very high social status (Doria, 1997).

From 1837 to 1940, secondary education in Brazil underwent several changes, some of them minor, and others more sweeping. Among these are the programs instituted in 1890, right after the Empire had been overthrown (1889), and the Republic established. This reform was strongly influenced by the ideas of its promoter, Benjamin Constant, a staunch positivist, and one of the founders of the Republic.

Brazil underwent deep structural changes after the First World War. Industrialization made big advances, partly because of the war. The country needed skilled workers and a middle class consisting of doctors, engineers, accountants, etc, and thus its educational system could no longer exclusively focus on the goal of forming an intellectual elite (Nagle, 1947; Nunes, 1962). At this time, Brazilian educators discussed intensely and searchingly the course of Brazilian education and its role in molding a new society. These concerns and the hopes of a whole decade found expression in the well-known Manifesto dos Pioneiros da Educação (1932) the Pioneers of Education Manifesto, which summed up most of the ideas of several Brazilian educators, many of whom were influenced by Dewey and Montessori, among others (Xavier, 2004; Romanelli, 1991, pp. 128, 129).

The tensions in this changing society may be perceived from the political unrest of the period. There was a failed revolution in 1922, as well as widespread worker strikes. The growing crisis came to a peak in 1930, ending with a revolution which overthrew the "old republic" and founded a new one. This "new republic" created the Ministry of Education, which was initially headed by Francisco Campos, who immediately started a sweeping reorganization of secondary school education, known as "Reforma Francisco Campos". In 1937, President Getúlio Vargas, one of the leaders of the 1930 revolution, lead a *coup d'état.* Vargas remained in power until the middle 1940s. In 1942, Gustavo Capanema, who was then the Minister of Education, initiated a new educational reform effort.

We summarize the major education reforms in Brazil from 1837 to 1942 and some related events in the following table:

1837	Establishment of Colégio Pedro II in Rio de Janeiro
1889	Fall of the Empire and proclamation of the Republic
1890	Creation of the Ministry of Instruction, mail and telegraph
	Curricular Reform made by Benjamin Constant, Minister for Instruction, Mail and Telegraph
1892	The Ministry of Interior Affairs and Justice becomes responsible for Education
1925	Euclides Roxo becomes head of Pedro II
1929	Euclides Roxo reforms the mathematics curriculum at Pedro II
1930	Vargas overthrows the established government and becomes President
	Creation of the Ministry of Education and Health

1931	Francisco Campos organizes secondary education—the Francisco Campos Reform
1937	Vargas establishes a dictatorship
1942	Gustavo Capanema reorganizes secondary education—the Capanema Reform

Euclides Roxo and the *Colégio Pedro II*

Euclides de Medeiros Guimarães Roxo (1890–1950) was an engineer who graduated with honors at every stage of his education. Very soon after he received his engineering degree, he started teaching mathematics at the Colégio Pedro II, where he had completed his secondary schooling. In 1925, he became the school's director, a post which he retained until 1935 (Carvalho, 2003; Valente, 2003).

Euclides Roxo was a contemporary of Eugênio Raja Gabaglia, Joaquim de Almeida Lisboa, and Cecil Thiré, mathematics teachers at Pedro II. Raja Gabaglia represented Brazil in IMUK (Schubring, 1989, pp 6-8; Valente, 2003). He had a great influence on mathematics teaching in Brazil, partly due to his translations of French mathematical textbooks for secondary education, which were used until the late 1950s and early 1960s. He was very conservative, and his position as the representative of Brazil at IMUK did not summon winds of change for the teaching of mathematics in Brazil (Valente, 2003, p. 53).

Almeida Lisboa was younger than Raja Gabaglia. He had a mercurial temperament and eventually clashed with Roxo on issues related to the reform of mathematics teaching. Another colleague of Roxo was Cecil Thiré (1853–1924). Unlike Gabaglia and Almeida Lisboa, he was more open-minded and up to date with the reform movements of mathematics teaching in several other countries (Valente, 2003, pp. 53, 54, 55). Notwithstanding his interest in modernizing the teaching of mathematics at the Colégio Pedro II, almost nothing seems to have happened to further Thiré's ideas, even though he was responsible for developing the school's mathematics programs from 1912 to 1928 (Valente, 2003, p. 54).

We surmise that Euclides Roxo learned about the international reform movements of mathematics teaching from Raja Gabaglia or, more probably, from Thiré. As shown by the many quotations in his newspaper articles about the reform of mathematics and in his book *A Matemática na Educação Secundária* (Roxo, 1937), Roxo read extensively about mathematics, psychology, education and mathematics teaching reform, particularly focusing on Felix Klein's ideas. He quotes extensively from Klein's *Elementarmathematik vom höheren Standpunkt aus* (*Elementary Mathematics from an Advanced Standpoint*) (Klein, 1925–1928). As a matter of fact, in the first of a series of 13 newspaper articles he wrote about the reform of secondary school mathematics teaching, the first one is almost a reproduction, without any indication of quotations, of the following chapters of Klein's books: *Vol I: Zwischenstück—Über die moderne Entwicklung und den Aufbau der Mathematik überhaupt* (*About the Modern Developments in the Structure of Mathematics*), pp. 82–92; *Zusatz I—Über die Bestrebungen zur Reform des mathematischen Unterrichts* (*About the Goals of Mathematics Education Reform*), pp. 290–297; *Vol II—Zusatz II: Ergänzungen über den geometrischen Unterricht in den einzelnen Ländern* (*Addenda:Concerning Geometry Education In Each Country*), pp. 277–293. Roxo also quotes extensively from Poincaré, Boutroux, Jules Tannery, Borel, L'Enseignement Mathématique and L'Enseignement Scientifique.

Because of his position as director of the Colégio Pedro II, Roxo was aware of the educational discussions underway in Brazil. He participated in the debates of the Brazilian Academy of Education, in which he headed the secondary school division. He was called upon, as director and representative of the faculty of the Colégio Pedro II, to formulate educational policy proposals for the government. He also had family ties that guaranteed him political backing and support and allowed him to remain at his post as head of Pedro II after the 1930 revolution, even though he had publicly voiced his opposition to it.

It should also be mentioned that Roxo, as head of the Colégio Pedro II, which was the official secondary school of the central government, had direct access to the Minister of Education. Even after 1935 Roxo continued to hold important positions: in 1937 he became the director for secondary education at the Ministry of Education, and in 1944 he was made president of the national commission for textbooks, which examined all secondary school textbooks to determine if they adhered to the official curricula.

Mathematics education in Brazil up to the twenties had, in general, the following characteristics (Haidar, 1972; Beltrame, 2000):

- The subjects of secondary school mathematics—geometry, trigonometry, arithmetic and algebra—were taught in a strictly compartmentalized way, separated by school years.
- Mathematics was not taught in all secondary school years. According to the several secondary school official programs from 1837 to 1929, the distribution of mathematics in the curriculum varied widely: Sometimes it was taught during the first years, sometimes in the middle years, and sometimes in the senior years.
- Mathematics was viewed as a tool to promote the discipline of the mind, an exercise in intellectual training without regard to applications to the everyday needs of society. It was taught as a preparation for university studies, most frequently in the field of engineering.

Euclides Roxo and the Mathematics Curriculum Reform

Euclides Roxo, aware of the international debate on educational reform, saw the need for change in Brazil. But he was not the only one. Many other educators clamored for reform, campaigned for a "new school" which was to focus on the students and their psychology, and on the relevance of education for the aspirations and requirements of society (Xavier, 2004). In short, times were ripe for change, and Roxo was a catalyst for change in Mathematics Education.

In his quest for change, Roxo turned to Klein's ideas about mathematics teaching. There he found a clearly formulated proposal for mathematics education reform, backed by a distinguished mathematician, and set out in a number of writings.

In 1929, as director of Pedro II, Roxo instituted a profound reform of the mathematics curriculum. He stated his ideas very clearly in the extensive preface of the first volume of the textbook he wrote for use with the new program (Roxo, 1929, 30, 31). After quoting Poincaré on the importance of intuition as opposed to rigor and formalism, he presented Klein's ideas about mathematics teaching. According to Roxo, who practically quotes Klein, the general objectives of mathematics education should include:[1]

1. Predominance of the psychological viewpoint in the teaching of mathematics (…) one should always start with the living and concrete intuition, and only very slowly bring forward the logical side of mathematics. One should adopt the genetic method, which allows for a slow introduction of the new notions.
2. Keeping in mind the applications of mathematics to the other subjects.
3. Subordination of mathematics teaching to the goals of modern schooling which were to morally and intellectually enable individuals to cooperate within a modern civilization essentially oriented towards practical success.

He continues, stating that from these three main directives followed the recommendations:

- a) To fuse arithmetic, algebra and geometry (including trigonometry).
- b) To introduce the function concept early, since it is at the core of the modern reform movement for Klein.
- c) To abandon the rigid Euclidean method of teaching geometry .
- d) To introduce coordinates and analytic geometry very early.
- e) To introduce the notions of differential and of integral calculus.
- f) To place more emphasis on perspective drawing in the teaching of elementary geometry.
- g) To introduce laboratory resources.

h) Finally, to give precedence to one principle above all the preceding ones: the use of the historical method in the teaching of mathematics.

In this preface, he supports his suggestions with frequent references to Klein, quoting him directly. Roxo wrote nine extensive articles, of more than 7000 words each, developing these ideas in one of the most influential newspapers of Rio de Janeiro, the Jornal do Commercio, to justify his reform movement. This series of articles was written between November 1930 and March 1931, and their ideas are essentially reproduced in Roxo (1937). There were another four articles in the same newspaper during the same time-span, which constitute Roxo's part in his bitter quarrel with Almeida Lisboa, who attacked Roxo's reform very heatedly, also in the Jornal do Commercio.

The curriculum proposed by Roxo and approved by the faculty of Pedro II drastically changed several characteristics of secondary school mathematics education in Brazil:

- Mathematics was taught in all school years.
- The rigid separation between arithmetic, algebra and geometry was abolished. These subjects were all taught in each school year and were to be integrated or, in the terminology of the time, "correlated." As a consequence, there was a single textbook for each school year.
- The function concept was introduced very early in the curriculum.
- The adoption of Klein's idea that deductive geometry should be preceded by an introductory geometry course, which was to enhance intuition and the knowledge of geometrical facts instead of deductive arguments.

Klein's ideas provided Roxo with the guidelines for his program of reform. A new curriculum, however, needs new textbooks, a problem that always accompanies major curricular reforms. And textbooks need authors with actual classroom experience. Even though Roxo was an experienced teacher, he could not tackle alone the task of writing the textbooks needed for the radically new curriculum he proposed. For this task, he relied heavily on Ernst Breslich.

Breslich had been born in Germany in 1874, and become an American citizen in 1896. He was associated with the University of Chicago for almost all his professional life. He was an instructor, and for a while was the chairman, of the Mathematics Department of the University Laboratory Schools. He died in 1966.

Breslich believed in Klein's ideas and adhered to them in his several textbooks, which were used, among other places, at the laboratory schools of the University of Chicago, and which saw many editions. He was particularly enthusiastic about what was then known as "correlation," that is, the integrated teaching of arithmetic, algebra and geometry, and he authored many papers on this subject.

Besides his textbooks, Breslich wrote extensively on secondary school mathematics teaching. In May 1933, he gave a speech at the New York Society for the Experimental Study of Education (Breslich, 1933, pp. 327–349). In this speech, he stated that "the curriculum must be adapted to the changes in the social order," and he repeats the description of the reform movement presented by Klein in his *Elementarmathematik von höheren Standpunkt aus*. Since Roxo read Breslich extensively and followed his ideas, it is relevant to quote them. Breslich maintains, that "(…) algebra and geometry in the plane should not be taught as separate subjects, but in connection with arithmetic" and that mathematics should be "closely [connected] with other school subjects." He repeats Klein's ideas that algebra and geometry be joined by making the function concept the unifying idea in mathematics; and that a psychological arrangement of subject matter should be insisted on. In his speech, Breslich also repeats Klein's idea that the formal teaching of geometry theorems should be preceded by an informal and practical course.[2]

Roxo, as he himself stated several times, made extensive use of Breslich's textbooks as a source of examples, exercises, and as a model for the structure of his own textbooks. In his textbooks he adheres to these ideas and tries to implement them. In fact, in the bitter discussions which followed the new curriculum, Roxo was often accused of plagiarism by Almeida Lisboa and other secondary school mathematics teachers (Fontes, 1930; Novo, 1929, 1934).

It is not clear how Roxo learned about Breslich. Rio de Janeiro had, at the time, some very good bookstores, which imported books both from Europe (mostly from France), and from the United States. Even today, it is possible to find in used bookstores copies of mathematics textbooks and books on the teaching of mathematics well known in the 1920s and 1930s. Some of these copies were once owned by teachers mentioned in this paper. In one of his articles against Roxo, Almeida Lisboa boasts that he too had read Breslich's textbooks, and proceeds to list exercises he claims Roxo had copied, sometimes with minor adaptations.

Why did Roxo rely so heavily on Breslich? Was it lack of time to write a completely new textbook, since he was very busy at the time, as head of Pedro II? Or, was it because he was not or did not feel capable of transposing Klein's general ideas into a workable textbook? We do not know.

Euclides Roxo and his Textbooks

Roxo, as head of Pedro II, planned to establish his reform progressively, so that only students who entered the Colégio after 1929 would follow the new programs. He wrote the first three textbooks of the series he planned for the new programs (Roxo, 1929, 1930, 1931) alone. After the 1931 reform, which made the new curricula mandatory, he interrupted his series, and joined two colleagues, Cecil Thiré and Mello e Souza (another very influential mathematics teacher), in the series of textbooks they were writing, and which were then competing with Roxo's books. The first two volumes of this series, called *Matemática*, were written by Cecil Thiré and Mello e Souza. Roxo joined them starting with

Volume III, and the series was then called *Curso de Matemática*. It covered the five years of secondary school mathematics.

A perusal of the first two volumes of Roxo's *Curso de Matemática Elementar*, Roxo's first secondary school textbook series, shows how far he relied on Breslich, including even the exercises, sometimes with minor adaptations, as pointed out mainly by Almeida Lisboa in his attacks against Roxo's reforms and textbooks. Actually, Lisboa's criticism extended far beyond the textbooks—he aimed directly at Roxo's ideas concerning the teaching of mathematics (Lisboa, 1930):

> Mr. Roxo forgot the real aim of mathematics in secondary school. Its purpose is not a more or less plentiful harvest of isolated and practical knowledge. Mathematics is a discipline of the mind, the unsurpassed and irreplaceable trainer of reason to which youth should be submitted.

He also claimed that Roxo had reduced mathematics to a triviality (Lisboa, 1930):

> (…) It is as though the most eminent architects had defined the rules of elegance and proportion, defined the nobility of style and, in the heart of Africa, a savage tribe presented their poor hut as a result of these modern precepts. I beg my illustrious colleague to forgive me the unpleasant comparison, but I cannot see a more fitting one for his reform of the teaching of mathematics.

(…)

> Mr. Euclides Roxo wants to convince us that he based [his reforms] on the international congress and on Klein's uncontroversial authority, to abolish the teaching of mathematics in Brazil. Mr. Roxo does not even seek really useful applications of mathematics, which would require a wide theoretical knowledge.

Roxo opens the first volume with a lengthy introduction, which is a passionate defense of the modern reform movement. The table of contents shows at once that the book was completely different from the textbooks so far adopted in the Colégio Pedro II. These texts were all jettisoned, and a new style of textbook was introduced in Brazilian mathematics teaching.

As an example for the style of this book, let us mention that Roxo proves the familiar rule $(a + b)^2 = a^2 + 2bc + b^2$ geometrically, putting into practice what he preached, the "correlation" between algebra and geometry. This was a genuine innovation in Brazil. The function concept figures prominently in this volume intended for students around the age of eleven. We find functions given by their graphs, by tables of values and by their analytical representations. This contrasts strongly with the now dominant practice, in Brazil, of introducing the function concept only in the last year, or in the last three years of a seven-year secondary school curriculum.

The accusation, made by Almeida Lisboa and others (see Fontes, 1931), that Roxo was reducing mathematics to a set of rules, trivial applications and an indigestible mixture of different pieces, is not sustainable. For example, in the third volume of his series, Roxo presented a formal Euclidean geometry course, including full proofs. Nevertheless, he broke with the strict Euclidean tradition by placing a chapter on displacements of plane figures right at the start and, later, a chapter on symmetry.

Roxo's texts broke with a long tradition of textbooks (Beltrame, 2000; Dassie et al., 2001), and it is not surprising that they were strongly attacked. But he also had support from some influential teachers of mathematics. For example, Nivaldo Reis, a teacher of the Ginásio Mineiro, in Belo Horizonte, state of Minas Gerais, wrote a series of articles in the Revista Brasileira de Matemática Elementar, supporting Roxo, but showing concern about the implementation and success of the new programs (Reis, 1931). He stresses that Roxo only followed ideas already known in Europe:

> But this transformation of the teaching of mathematics was not the work of Euclides Roxo, as many believe. He only adopted what was known and applied in Germany since the beginning of this century [20th Century].

In his two first books (Roxo, 1929, 1930) Roxo takes seriously Klein's advice, already put in practice by Breslich in his several textbooks, that a course in deductive geometry should be preceded by an intuitive course, which was to consist of "lessons about things." He also adheres very closely to Breslich's attempts at "correlating" subjects, that is, at integrating arithmetic, algebra, geometry and trigonometry. And he tries, also following Klein's ideas, to make the concept of functional relationship "impregnate the whole curriculum."

These two books were met with widespread opposition, most of which Roxo tried to answer in several newspaper articles.[3] Some of these articles were devoted to his exposition of the aims of the reform movement, some to answer Almeida Lisboa's vitriolic critique, other to answer other critics, mainly secondary mathematics teachers.

In the third book of this series (Roxo, 1931(a)), devoted to geometry, Roxo presents elementary Euclidean geometry in a more traditional way. Was he tired of the opposition to his ideas and to his books? We do not think so, because in the preparation of the 1942 reform, promoted by Gustavo Capanema, he initially maintained his reform ideas, though he was forced to back down steadily. By then, he was no longer director of the Colégio Pedro II, and had been seriously shaken by the death of his daughter, and had to compete with the Catholic church and with the army for Capanema's attention (Horta, 2001).

Euclides Roxo, the Catholic Church and the Military

Roxo was also attacked from another side: the Catholic priest Arlindo Vieira, director of Rio de Janeiro's most prestigious Catholic school for boys, the Colégio Santo Inácio, owned by the Jesuits. Vieira attacked Roxo's mathematics program as being encyclopedic, and said that the real formation of the minds of the students would be better achieved by a return to the classics, and that the new mathematics programs were much wider in scope than the equivalent programs in France and Italy. Vieira indicated that the mathematics programs should be largely curtailed to allow more time for teaching Latin and the classics (Viera, 1934(a), (b); O Journal, 1936). Vieira also strongly opposed the introduction of the function concept at an early stage of secondary education. These ideas would find sympathetic ears in Gustavo Capanema, who promoted the next reform, in 1942. In a letter to Capanema of 1941, trying to convince him of the wrongness of Roxo's ideas, Vieira states that (FGV (b)):

> The best and most decisive argument against the ideas of the illustrious educator are the disastrous results of the mathematics programs he instituted in 1931 and which he tries to defend. A survey poll would show that more than 90% of all teachers condemn this unpalatable mixture.

One should not view this as an isolated attack, which reflected only Vieira's personal ideas. On the contrary, it should be viewed in the context of the Catholic Church's opposition to the educational ideas that had started to prevail in Brazil in the 1920s and 1930s (Cury, 1988; Horta, 2001). Vieira was part of a "counter-reformation" that fought against the spreading of ideas found dangerous by the Church: public school education, with co-education of boys and girls, and the separation of Church and State in educational matters. The Church walked a tightrope, supporting the strongly conservative government that fought communism, but at the same time opposed the new educational policies promulgated by this government (Azevedo, 1971, pp. 664-670).

It is interesting to note that Almeida Lisboa (1936(a), (b)), whose position was that mathematics was the subject appropriate to develop students' minds, also attacked Vieira. This is consistent with the ideas Lisboa had already used in some of his criticisms of Roxo.

The military also had strong reservations about Roxo's reform. Their schools for prospective officers were very conservative, and opposed any change to their well-established methods. For them, the new educational ideas that were discussed in the 1920s and 1930s threatened school discipline, and looked suspiciously liberal (Horta, 2001).

Euclides Roxo and the Teachers of Mathematics

We have already seen that the Church and the military opposed the mathematics curriculum reform made by Euclides Roxo. It was also strongly attacked by many teachers of mathematics.

Up to the 1930s, there were no university courses for secondary school teachers in Brazil. Mathematics teachers in the secondary schools were mostly engineers, including military engineers. The military schools had a strong mathematics curriculum, following the French tradition. Some of their graduates were mathematics teachers at private schools, or even had preparatory schools for young men who planned to take the entrance examinations for the engineering schools, like Sebastião Fontes (Fontes, 1930; Dassie, 2001). These examinations were highly competitive, and required a good mathematical background. So, these teachers had two reasons to oppose Roxo. In the first place, the very conservative mathematical curriculum in the military schools was long-established. Secondly, there was a palpable fear that their students, either in private secondary schools or in the preparatory schools, would not meet the requirements of the entrance examinations, since these teachers felt that Roxo was downgrading the mathematics curriculum.

The civilian engineers in Brazil usually received their education from the Escola Politécnica in Rio de Janeiro, where they had, in general, received a fair to good foundation in mathematics. Their reaction to Roxo's reform was mixed. Some of them adhered to ideas consistent with Almeida Lisboa's conception of mathematics as a lofty science of the mind, while some others saw the urgent need for reform (see Novo, 1929, 1931 for an example of such teachers).

Euclides Roxo's Return to More Conservative Views

The military and the Church were not heard in the organization of the curricula of the 1931 secondary education reform,[4] which was actually put forward by a very conservative Minister of Education, Francisco Campos. The new mathematics curriculum was proposed by Roxo and accepted completely by Campos, without discussion. By contrast, with respect to the preparation of the 1942 educational reform, made under Vargas's dictatorship, the Minister of Education, Gustavo Capanema, paid close attention both to the Church and the military in the preparation of the curricula and acknowledged their pedagogical guidelines. With respect to the coverage of mathematical topics, he did not entirely follow Roxo's proposals and accepted several of the objections voiced by the Church and the military, one of which was the opposition to the early introduction of the function concept (Dassie, 2001).

The available documentation shows how Roxo tried to fight the attempts to modify his proposals for the 1942 reform. For example, he strongly opposed Arlindo Vieira's suggestion that the function concept should be relegated to the last year of secondary school. In a very incisive letter to Capanema (FGV (a)), he states:

> As to the general ideas, there was one which I find very harmful to the good teaching of mathematics (…) it is the suppression of the function concept as an axial idea in the teaching of mathematics. Since it was proposed by Felix Klein, this pedagogical principle has been accepted in almost all civilized countries. (…) It is a pity that Brazil should give up the pedagogical conquest made by the acceptance of this principle in the reform of 1931.

Another point Roxo strongly opposed was the idea of having several textbooks for the same school year, for example one for algebra and one for trigonometry, a universal practice before the 1931 reform. In the letter to Capanema already quoted, he writes:

> I received the message from Your Excellency recommending that I include in the methodological guidelines for the mathematics programs an instruction that the subjects should be distributed among several textbooks, using any criterion but that of using only one textbook per school year.
>
> Unfortunately, Mr. Minister, I cannot do this, because I am deeply convinced that the only acceptable criterion, particularly for mathematics, is precisely the use of only one textbook per school year.

What did Roxo lose in the 1942 reform, compared to that of 1931? First, the function concept did not "impregnate the whole curriculum," since it was postponed to the last school year. In the second place, if we compare his first textbook series with the one he wrote together with three other well-known mathematics teachers immediately after the 1942 reform, we see that the "correlation" Roxo fought so hard to see in the mathematics classroom disappears almost completely. Even though each book covers at least two of the main blocks of secondary school mathematics (arithmetic, algebra, geometry and trigonometry), they are presented in separated units in the books, with no real attempt at integrating them.

Concluding Remarks

Euclides Roxo took part in two reforms that were imposed "top down," instead of evolving "bottom up" (Schubring, 1989, pp. 15-16; Krüger, 2004). This caused widespread resistance, which had to bow under the very authoritarian and centralized political regime instituted by the 1930 revolution, and made even more so under Vargas dictatorship from 1937 to 1945. This resistance gathered its forces for the upcoming 1942 reform, which was heavily debated, and in which Roxo was not the only person heeded by the Minister of Education (Gustavo Capanema), as shown in the files of Capanema's papers at the Fundação Getúlio Vargas in Rio de Janeiro.

Did Roxo really believe in Klein's ideas? Did he have a clear program for educational reform, or was he simply parroting European ideas about mathematics education? We do not think the latter is the case. Roxo explained his ideas consistently for several years. He obviously felt deeply committed to the modernization process.

When Roxo adopted Klein's ideas, he made adaptations, usually drawing on Breslich, his role model. For example, when Roxo stresses applications, he means strictly elementary applications, like the example much criticized by Almeida Lisboa of how to measure heights using a pencil held at arm's length, or how to draw a circle using a string and a peg. He never mentions the integration of secondary school education with higher education, although he was aware that, at the time, secondary school education in Brazil was—and largely remains so even today—mainly a preparation for higher courses. In addition, since he used Breslich as model, his emphasis on the integration of geometry and algebra is not surprising, since this was much stressed in Brelich's adopted home country, the United States (Schubring, 1989, 1989, p. 19).

Schubring (1989) has remarked that Klein's "(…) ultimate objective [was the] clarification and redefinition of the transition from the secondary schools to higher education, particularly to the technical colleges." This objective is completely absent from Roxo's writings. Even though he was an engineer, with a degree from the prestigious Escola Politécnica—the university-level school of engineering—Roxo was completely cut off from higher education. As a matter of fact, the two faculties, that of the Colégio Pedro II and that of the Escola Politécnica, competed for prestige and influence.

In his 1937 book, Roxo summarized the ideas he had proposed in his newspaper series, and which had been originally published in the introduction to the first volume of his *Curso de Matemática Elementar*. Even though this book clearly reveals some educational conceptions that are no longer accepted, it has a very modern flavor. In our experience, many young people who read it for the first time are surprised to find in it ideas that are considered very "modern" as, for example, the importance of graphs and data tables, the emphasis on the graphical representation of functions, and the idea that the problems and applications should be meaningful to the students.

Roxo's ideas were further summarized and refined in his lecture "*Mathematics and secondary school*" (*A Matemática e o curso secundário*), of 1937 (Peixoto, 1937; Valente, 2004 (b)).

Roxo consistently fought for a slowly progressive teaching of mathematics, from simple ideas in the first years, to a rigorous presentation of geometry and of notions of the infinitesimal calculus in the last years of secondary school. He also proposed the early inclusion of trigonometry in the curriculum.

In any case, the reforms he instituted had a lasting influence on Brazilian secondary school mathematics, which has retained some of the characteristics of Roxo's modernizing attempts to this very day:

- Mathematics is taught in all secondary school years
- In each school year, the students use only one mathematics textbook instead of several treatises, one for each branch of school mathematics.
- Compared to the old textbooks, the new series of Roxo, and those of other mathematics teachers tried to take in account the needs and level of maturity of the student, a practice that has been consistently considered and improved.

Acknowledgements

I wish to thank the referees, whose suggestions improved this work significantly. I particularly thank Prof. Gert Schubring for his comments, suggestions and encouragement, spread over the last few years. Thanks also to Bruno Dassie, Flávia Soares and José Lourenço da Rocha for the many lively discussions and interchange of ideas and material.

Notes

1. All translations from Portuguese into English by the author of this paper
2. Breslich seems to deserve more attention than he received. He is rarely mentioned even in the comprehensive recent studies on the history of school mathematics in the USA. Meantime, he was an influential textbook writer, connected to a young and dynamic institution (the University of Chicago) and played an important role in the transmission of Klein's ideas to the United States. More information on him is given in the paper of Gonzalez and Herbst in the current issue (editor).
3. It should be mentioned that educational matters were widely discussed in newspapers at the time. Many had a specific section on education and they ran articles on educational reforms. This fits with the cultural atmosphere of the time, when educators sought changes to make school more relevant to the needs of Brazilian society, as already mentioned. This subject deserves detailed study.
4. Of course they influenced the aims of the educational reform, its politics and ideology, but not the specific curricula of the secondary school subjects, particularly mathematics.

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Notes

Sources for the History of Mathematics Education in Brazil

Gert Schubring

Sources are of utmost importance for research into the history of mathematics education. In Brazil, there is a community which is particularly active in detecting and exploring relevant sources.

Let me mention Maria Ângela Miorim's book *Introdução à história da educação matemática* (São Paulo 1998) which presents the rare case of combining an overall history of mathematics education with that history within her own country. And João Bosco Pitombeira, the pioneer of research into the history of mathematics education in Brazil, succeeded in finding the son of Euclides Roxo, whom one might call the Brazilian Felix Klein, and to study material in his father's *Nachlass*. A few years ago, Wagner R. Valente even succeeded in convincing Roxo's son Stélio Roxo to hand this precious *Nachlass* over to research. Nowadays, it is easily accessible as the APER: Arquivo Pessoal de Euclides Roxo, within the PUC of São Paulo: Pontifícia Universidade Católica de São Paulo, Centro de Ciências Matemáticas Físicas e Tecnológicas, Departamento de Matemática. A description of this source is published in:

W. Valente, APER—Arquivo Pessoal Euclides Roxo—Inventário Sumário. Special issue of the journal: *Educação Matemática Pesquisa*, São Paulo, 2002.

In fact, W. Valente devotes enormous energy and initiative to exploring sources and to facilitating access. He established the research group GHEMAT: Grupo de Pesquisa de História da Educação Matemática no Brasil, affiliated to the Programa de Estudos Pós-Graduados em Matemática at the PUC of São Paulo. Meanwhile, this research group published two CDs giving access to two kinds of key sources:

Valente, W. R.; Braga, C.; Barros, H. N.; Pires, I. M. P.; Pires, L. H. C.; Meneses, R. S.; Assuncao, R. L.; Alvarez, T., *A matemática do ginásio: livros didáticos e as reformas Campos e Capanema*. 2005.

This CD-ROM provides access to the mathematics schoolbooks used during the two decisive stages of school reform in Brazil: the reform Francisco Campos

(1931) and the reform Gustavo Capanema (1942)[1]. For the first stage, five series of textbooks are analyzed and documented in extracts:

Euclides Roxo, *Curso de Mathemática Elementar*;

Cecil Thiré, Júlio César de Mello e Souza and Euclides Roxo, *Curso de Matemática*;

Jacomo Stávale, *1º, 2º, 3º ,4º e 5º ano de Matemática*

Algacyr Munhoz Maeder, *Lições de Matemática;* and

Agrícola Bethlem, *Curso de Mathemática.*

For the second stage, the objects of analysis are:

Euclides Roxo, *Thiré, Mello e Souza, Matemática Ginasial;*

Jacomo Stávale, *Elementos de Matemática*; and

Algacyr Munhoz Maeder, *Curso de Matemática.*

The most innovative of these textbooks for Brazil is the first on the list, published from 1929 on by Roxo, implementing the reform which he himself had initiated at the *Colégio Pedro II* in Rio de Janeiro, the most prestigious secondary school in Brazil and at the same time, by law, the institution which served as model for all the others.[2] And this reform transferred to Brazil the core of the international reform movement for mathematics instruction, promoted since 1908 by Felix Klein: permeation of the entire curriculum by functional thinking, by the function concept, and unification of the various branches to form a coherent body of school mathematics from arithmetic, algebra, and geometry. The second textbook, by Thiré and his team, was more conservative and was more successful economically; and for the second stage of these reforms, Roxo was eventually forced to abandon his own innovative textbook, and to infuse its contents into his competitor's book, which now assumed the novel title of "Matemática Ginasial". On the other hand, the Thiré series itself had been considerably changed, and been consistently adapted to the program of the Capanema reform. The other three textbook series (Stávale, Maeder, Bethlem) present other models to follow the reform measures and differences in implementing them. For each textbook series, the CD gives an exhaustive introduction, and an analysis of its major features. For each issue discussed and analyzed, the CD reproduces the respective pages of that textbook concerned.

The extensive documentation and analysis of the textbooks is complemented by biographical information on their authors. Moreover, in a special section, the original texts of the ministerial decrees for the two reforms are reproduced.

Valente, W. R.; Duarte, A. R. S.; Ribeiro, D.; Tavares, J. C.; Miranda, M. M.; Machado, R. C. G.; Prado, R. C.; Alvarez, T.; Santos, V. C. M.; Sório, W. F., "*Arquivo Escolar do Colégio Pedro II - Coletânea de Documentos*". 2005.

This CD presents a selection from the voluminous archive of the *Colégio Pedro II* with documents—essentially for the teaching of mathematics, but some for

Portuguese, too—which illuminate various important aspects of the historical reality of the teaching and learning process in this prestigious institution. The documentation is organized into the following sections:

- minutes of the meetings of the teaching staff of the Colégio, basically between 1928 und 1950, and with one isolated item of 1872. The minutes are not arranged chronologically.
- decisions taken by the directorate of the school, regulating current issues of school life, from 1927 to 1953, and a few items from the years 1885 to 1895 and one from 1916, also not in chronological order.
- selection procedure for professorships in mathematics and in drawing for the years 1934 and 1952, resp. for 1926 and 1940. The documents handed in by the candidates, the exams to which they submitted, the nominations for the juries, and the decisions are presented.
- the largest section is devoted to exams of students: intermediate as well as final exams in the various subjects of mathematics teaching: arithmetic, drawing, but mainly in algebra. Typical problems like frauds appear from the documents. The documents presented cover the period from 1928 to 1939 with a few items from earlier and later times.
- selected documents of various types on aspects of school life.

Two film clips showing students in their work for cataloguing and extracting data make evident the good organization of the archive of this school, and the enormous amount of material preserved.

Both the CD-ROMs are provided with effective means for navigation. They can be ordered from: e-mail: valente@pucsp.br.

It is to be hoped that this commitment for making key documents accessible for a large research community will be followed in other countries.

Notes

1. See the paper by J. B. Pitombeira in this issue.
2. See ibid.

Information

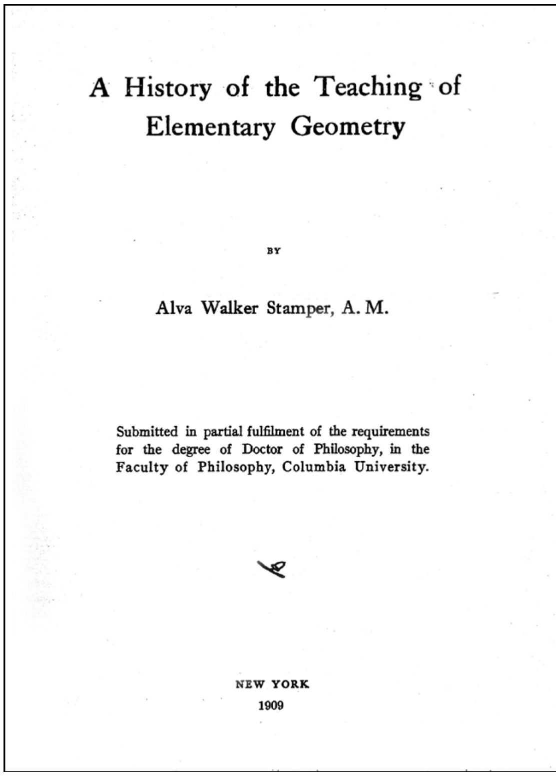
On April 25, 2006 a research symposium devoted to the study of the history of mathematics education was held at the Research Presession of the annual meeting of the National Council of Teachers of Mathematics in St. Louis. Members of the editorial board of the *International Journal for the History of Mathematics Education* and active contributors made presentations addressing the orientation and specific direction of the Journal.

Dr. Gert Schubring (Bielefeld University, Germany), who was unable to come to St. Louis in person, sent a videotaped presentation. His report examined all that has been done in the history of mathematics education up to now and discussed the principal directions for possible new research topics. One such direction is the study of textbooks. Dr. Eileen Donoghue (City University of New York) offered examples of such research, drawing on her own work on the history of American textbooks. Dr. V. Frederick Rickey (United States Military Academy, West Point) gave a presentation devoted to the study of the history of collegiate mathematics education, which also drew on his articles and suggested various research topics. These presentations formulated a number of recommendations and proposals that will undoubtedly be useful to beginning researchers in these areas. Dr. Alexander Karp (Teachers College, Columbia University) spoke about the significance of studying the history of mathematics education both for researchers in mathematics education and for practicing teachers in this field. In conclusion, Dr. Karp addressed the principal goals and distinctive features of the Journal and offered a number of recommendations for its contributors.

In upcoming issues, the Journal's editorial board hopes to publish articles containing detailed examinations of the topics addressed at the symposium.

***

This year marks one hundred years since the first doctoral degree in mathematics education was awarded in the United States. The first doctoral dissertations in this field were presented at Teachers College, Columbia University, and sponsored by Prof. David Eugene Smith (the title pages of books based on these dissertations are reproduced below). Upcoming issues of the Journal will contain articles devoted to the analysis of these studies and their role in the further development of the discipline.



A History of the Teaching of Elementary Geometry

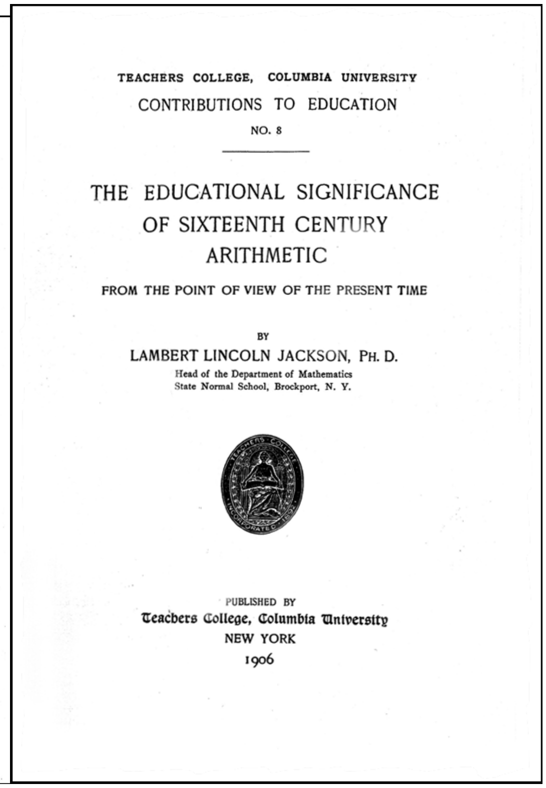
BY

Alva Walker Stamper, A. M.

Submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy, in the Faculty of Philosophy, Columbia University.

NEW YORK

1909



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CONTRIBUTIONS TO EDUCATION

NO. 8

THE EDUCATIONAL SIGNIFICANCE OF SIXTEENTH CENTURY ARITHMETIC

FROM THE POINT OF VIEW OF THE PRESENT TIME

BY

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The *International Journal for the History of Mathematics Education* publishes research articles, notes, and book reviews. All papers should be written in English, typed double-spaced, and must conform to the style specified in the *Publication Manual of the American Psychological Association* (5th ed). The following format is to be used for bibliographical information:

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