The Ghosts of Departed Quantities

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1 Introduction

Nonstandard analysis is the branch of mathematics concerned with constructing a number system that includes infinitesimals. These are numbers smaller than any real number yet still greater than zero. Although nonstandard analysis was developed in 1966 by Abraham Robinson, it is based off of Gottfried Wilhelm Leibniz's construction of the calculus, created some 300 years earlier. Nonstandard analysis differs from the standard approach to analysis in its use of infinitesimals. Infinitesimals in nonstandard analysis replace epsilon-delta technique in standard analysis; a method many would argue is far less intuitive than the nonstandard use of infinitesimals.

While Robinson may have been the first to arithmetically combine infinitesimals with the real numbers and rigorously define the rules that govern this new number system, he certainly was not the first to use infinitesimals. The basic idea of an infinitesimal dates back to antiquity to a period when the Greeks were afraid of the infinite. This was the period in Greek history shortly after Zeno presented his paradoxes which shook geometers until they banned the use of infinity in mathematics. It was during this time when infinitesimals gained firm ground in mathematics.

Archimedes incorporated infinitesimals into his "method of compression" which would later be formalized by Leibniz into his infinitesimal calculus. However, since neither Leibniz nor his followers could rigorously define the rules that govern the infinitesimals used in his construction of the Calculus, Leibniz's construction lost support to Newton's calculus of limits, which became the foundation for the modern epsilon-delta approach to analysis. However, many students find the epsilon-delta approach to be counter intuitive and hard to follow. It was this sentiment which allowed the infinitesimal calculus to persist through out centuries of attack by people like George Bishop Berkely, who criticized them for being "neither finite quantities, nor quantities infinitely small, nor yet nothing." Who mocking asked "May we not call them the ghosts of departed quantities?" (Hurd & Loeb 1985).

The Calculus of Leibniz is by no means the same as Robinson's nonstandard analysis. While nonstandard analysis is motivated by infinitesimal calculus, it differs mostly in Robinson's rigorous construction of a number system containing infinitesimals. Leibniz believed that the infinitesimals in his calculus could be considered to be well founded fictions. That is, they could be used to arrive at the correct answer, but would disappear once they were no longer needed. Robinson only disagreed with Leibniz's claim that his use of infinitesimals were well founded, however Robinson agreed that they were fictions of the mind. The fact that Robinson did not believe in infinitesimals is not surprising, as Robinson considered himself a formalist, who believed that infinity is theoretically meaningless but should be used in mathematics as though that was not the case.

Robinson's approach to nonstandard analysis employs heavy use of logic and model theory in order to rigorously define what Leibniz failed to. Robinson's approach to nonstandard analysis is only one of two approaches to nonstandard analysis. The other, which is beyond the scope of this paper, was developed by Edward Nelson about ten years after Robinson developed what is now called classical nonstandard analysis. Nelson's approach avoids the model theoretic approach that Robinson uses and follows a more axiomatic formulation.

In order to truly understand a difference between classical nonstandard analysis and the standard epsilon-delta approach, we will construct the nonstandard reals and show some basic results of nonstandard analysis in order to prove the familiar Intermediate Value Theorem using both approaches. This will allow the reader an idea of the differences between the standard and nonstandard approach to analysis and by way of the nonstandard approach, show the reader that "the widely held belief that one cannot get something for nothing is a superstition" (Nelson).

2 History

Infinitesimals have been used since the time of antiquity to approximate the area of geometric shapes. The ancient Greeks incorporated infinitesimals in the "method of exhaustion" in which they inscribed n-sided polygons, P_n, into a circle "exhausting" the area. This method was devised by Eudoxus to provide an alternative to taking "a vague and unexplained limit" of the area of P_n as $n \to \infty$. The methodology of the method of exhaustion is easily explained by the ancient Greek "horror of the infinite" (Edwards 1979).

The Greek "horror of the infinite" transcends mathematics and evidenced itself, long before it appeared in mathematics, in Greek mythology, as seen in the stories of Tantalus, Sisyphus, and Prometheus.

Tantalus, the mortal son of Zeus, attempted to trick the gods and expose the ease in which they are tricked. However, he was caught and as punishment was forever to reside in Hades by the arch-sinner pool; a pool with water that would drain into the ground whenever Tantalus knelt to drink from it. Above this pool there was a fruit tree which, whenever Tantalus attempted to take the fruit, the wind would blow the fruit out of reach. Tantalus was to be forever thirsty by a pool of water and forever hungry by a tree of fruit. Sisyphus and Prometheus, like Tantalus, were also punished by the gods each to be tormented forever. Sisyphus was punished by being forced forever attempt to complete a task which he could never accomplish. He was forced to push a boulder up a hill, but every time he reached the top of the hill, the rock would come tumbling back down, forcing him to start over. Prometheus, on the other hand, was bound to a rock while an eagle ate his liver, only to have his liver grow back the next day to again be eaten (Hamilton 1969).

The mathematical *horror infiniti* arose from the greek mathematical philosopher Zeno. At the time of Zeno, the popular Pythagorean idea was that a line is made up of beads, and time is made up of a series of discrete moments. Zeno pointed out the absurdities of "infinite divisibility" of space and time by his famous paradox involving Achilles and a tortoise (Decarli).

The paradox took Achilles, the fastest runner in Greece, and put him in a race against a tortoise who was given a head start; because of this head start, Achilles could never catch the tortoise. This, as Zeno said, was because by the time Achilles had reached the spot where the tortoise was, the tortoise had moved forward by a small amount, making it impossible for Achilles to catch it (Decarli).

This, along with Zeno's three other paradoxes, is related to the application of infinite processes to geometry. Because the Greek geometers were unable to answer the paradoxes, they banished the concept of infinity from the mathematics and made the "horror of the infinite" part of the Greek mathematical tradition (Decarli).

This insistence on absolute rigor and the rejection of infinite numbers prevented Greek mathematicians from developing a theory of limits. The theory that would later replace their *reducto ad absurdum* proofs, such as the method of exhaustion, in the Calculus (Edwards).

The method of exhaustion was then used by Archimedes to develop the "method of compression", in which Archimedes both inscribed and circumscribed n-sided polygons in circles. After developing this method, Archimedes applied it to find the volume of conoids (a paraboloid or hyperboloid of revolution) and spheroids (a ellipsoid of revolution), which had not been done before. He did this by inscribing the shape of revolution in a sphere then slicing the sphere into n equally thin pieces then both inscribing and circumscribing these sphere-slices into the shape of revolution. Archimedes may have been the first to find the volume of shapes of revolution, but he lacked a generalized algorithm for the calculation of areas and volumes (Edwards).

Some 1800 years later, Gottfried Wilhelm Leibniz and Isaac Newton each developed a generalized algorithm for determining the area and volume of shapes, and simultaneously yet independently developed the Calculus. While they each developed the Calculus, the two versions were wildly different. Leibniz' version involved infinitesimals which are smaller in absolute value than any ordinary real number, but according to him, still obey the laws of arithmetic (Hurd & Loeb). Newton's calculus involved the fluxion, as opposed to the infinitesimal, which he saw as a rate of change (Edwards).

Leibniz found inspiration for his Calculus as well as his famous "characteristic triangle" from Pascal's work. Pascal had proposed a challenge to find the area and centroid of an arbitrary segment of cycloid and to find the volumes and centroids of various solids of revolution. Although many mathematicians of the day proposed solutions, Pascal wasn't satisfied with any of the solutions, so he proposed his own. This solution involved a right triangle E_1E_2K with hypotenuse E_1E_2 tangent to the circle at a point D. He noticed that E_1E_2K and ADI were similar (Figure 1). Therefore, $\frac{AD}{E_1E_2} = \frac{DI}{E_2K}$.

Leibniz noticed that Pascal's infinitesimal triangle could be constructed on an arbitrary curve forming the similar triangles in figure 2.

While Leibniz could not formally construct infinitesimals into the real

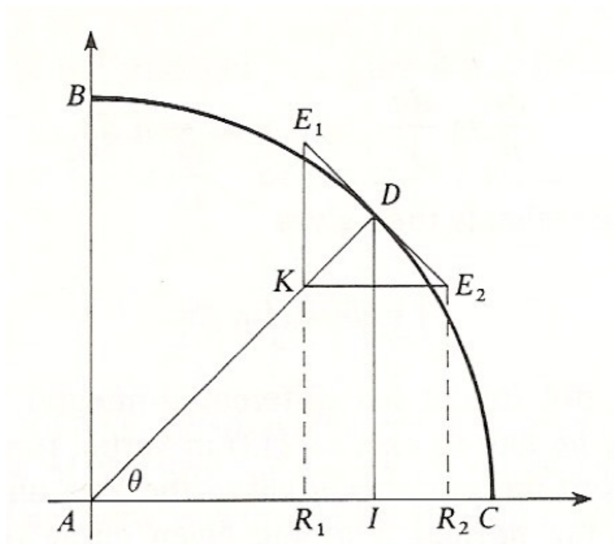

Figure 1: Pascal's similar triangles constructed to find the arclength of a circle (Edwards)

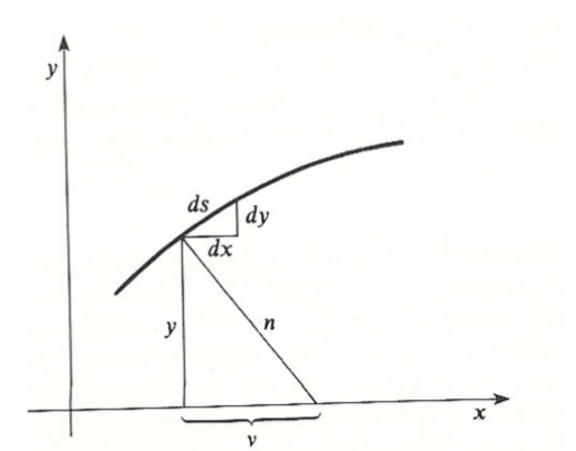

Figure 2: Leibniz's characteristic triangle, based off of Pascal's similar triangles, but used to find the slope of a line tangent to any arbitrary curve (Edwards).

number system, he stressed that proper application of his rules for calculating and manipulating differentials would invariably lead to correct results, despite the fact that uncertainty remained about the precise meaning of the infinitesimals that appeared in his calculations. Leibniz said that regardless of whether or not infinitesimals actually exist, they can serve as well founded "fictions useful to abbreviate and speak universally" (Edwards).

In late 1665, Newton attacked the problem of finding a line tangent to a curve. He did this by combining the velocity components of a moving point in a coordinate system. This provided the motivation for Newton's method of fluxions. Newton regarded a function as the intersection of two moving lines, one vertical and one horizontal, both varying with respect to time. The motion of a moving point on the function is the sum of the vertical component $\dot{y}$ and the horizontal component $\dot{x}$. The slope of the tangent line at any given point is $\frac{\dot{y}}{\dot{x}}$ (Edwards).

Newton defines $\dot{x}$ and $\dot{y}$ to be fluxions, which are rates of change with respect to time. However, Newton does not define "the fluxional speeds;" instead, he regards speed as "intuitively apparent on physical grounds" (Edwards 1979). In modern terms, the fluxions $\dot{x}$ and $\dot{y}$ are simply the derivatives of x and y with respect to time, $\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}$. The ratio of these derivatives, $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}$, is the slope of the tangent line (figure 3).

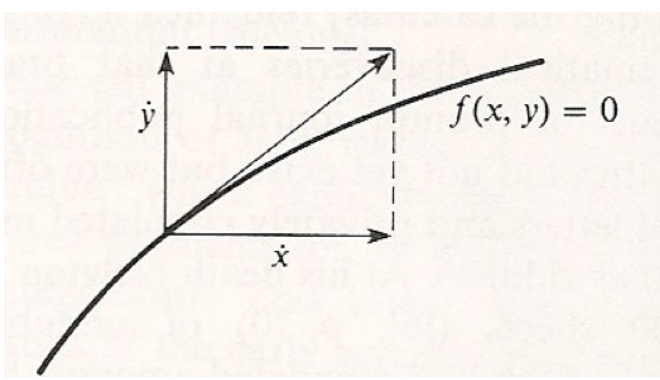

Figure 3: Newton's method for finding a line tangent to a curve. $\dot{x}$ and $\dot{y}$ are fluxions and their sum is the line tangent to the curve $f(x, y) = 0$

During their concurrent discoveries of the Calculus, Newton and Leibniz corresponded by letter. Newton rejected Leibniz's idea of an infinitesimal on the grounds that Leibniz had taken crucial suggestions from Newton during their correspondence. Newton viewed Leibniz's construction of the Calculus as, at best, a plagiarized version of his own construction (Edwards).

Leibniz's construction of the calculus was based on the classical atomist view which asserted that all objects in the universe were composed of very small indestructible building blocks. This view was shared by the Greek mathematicians who developed the idea of infinitesimals at the time of antiquity. Leibniz's construction of infinitesimals, which he described as "a grain of sand with respect to the earth, or the earth with respect to the distance between two fixed stars" (Meli 1993) embodies his atomist stance that infinitesimals are smaller than any real number, but greater than zero, making them the smallest indivisible part.

Newton on the other hand followed Descartes and the leading philosophy of the time, that matter is a continuum and that "in the division of the parts of matter there is really an endless series" (Nikulin 2002). Newton's construction of fluxions, like Leibniz's construction of infinitesimals, mirrors his philosophy on matter. This is evident in the way he views the fluxion as "an ultimate ratio of evanescent quantities" (Edwards).

Accusations of plagiarism and inconsistencies between Newton's version and Leibinz's version of the rules that govern differentials along with the feud that erupted over the construction of the Calculus resulted in attacks on infinitesimal calculus. When no one could arithmetically combine finite and infinite numbers, support for Leibniz' version of the Calculus declined and eventually the infinitesimal calculus was replaced by the Calculus of limits (Edwards).

Despite being criticized for lacking a formal proof, infinitesimals managed to persist for centuries as a tool to explain limits. This is arguably because infinitesimals are more intuitive and make more sense than limits. Many mathematicians argue that especially in teaching calculus, infinitesimals are easier for students to comprehend than limits.

In the 1960's, some 300 years after Leibniz first constructed infinitesimal calculus, Abraham Robinson rigorously constructed a number system that combined infinitesimals, infinite numbers and finite numbers. This number system is called the nonstandard reals and is the basis for the field of nonstandard analysis.

3 Basic Construction

Constructing the nonstandard reals requires taking the real numbers and adjoining to it infinitesimals. The complication of this merger of finite and infinitesimal numbers lies in trying to construct rules about adjoining these two classes of numbers in order to ensure that everything that can be done in the standard reals, can also be done in the nonstandard reals. In order to do this, a new equivalence relation must be introduced, one which relies on the use of an ultrafilter. So before continuing on to construct the nonstandard reals, we must first develop the notion of an ultrafilter.

The real number system is a complete, linearly ordered field $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$. We will define the nonstandard reals, $^*\mathbb{R}$ to also be a linearly ordered field, denoted by $^*\mathcal{R} = (^*\mathbb{R}, +, \cdot, <)$ that contains an isomorphic copy of the standard reals, but is strictly larger. This construction requires an introduced equivalence relation which employs the use of a free ultrafilter. Otherwise, we have that the product of two nonzero numbers can be zero.

3.1 Ultrafilter

The notion of an ultrafilter is necessary to introduce the equivalence relation ($\equiv$) which we use to define the nonstandard reals.

Definition: Let I be a nonempty set. A *filter* $\mathcal{F}$ on the set I is a nonempty collection $\mathcal{F} \subset \mathcal{P}(\mathcal{I})$ of subsets of I on which the following three things hold:

F1: $\mathcal{F}$ does not contain the empty set,

F2: If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, and

F3: If $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$

A filter is called an ultrafilter $\mathcal{U}$ if for each subset A of $\mathcal{I}$, either $A \in \mathcal{U}$ or its complement $A' = \mathcal{I} \setminus A \in \mathcal{U}$, but not both

A free ultrafilter in $\mathbb{N}$ is a collection $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ such that A, a finite subset of $\mathbb{N}$ implies that $\mathbb{N} \setminus A \in \mathcal{U}$. That is, a free ultrafilter $\mathcal{U}$ does not contain any finite sets A.

The Ultrafilter Axiom If $\mathcal{F}$ is a filter on I, then there is an ultrafilter $\mathcal{U}$ on I which contains $\mathcal{F}$.

We let $\widehat{R}$ denote the set of all sequences of real numbers of the form $r = \langle r_1, r_2, \ldots \rangle$ denoted by $\langle r_i \rangle$. We define addition and multiplication between $r = \langle r_i \rangle_{i=1}^{\infty}$ and $s = \langle s_i \rangle_{i=1}^{\infty}$ to be:

$$\begin{aligned} r \oplus s &= \langle r_i + s_i \rangle_i \\ r \odot s &= \langle r_i \cdot s_i \rangle_i \,. \end{aligned}$$

We then use this and the definition of a filter to define an equivalence relation $\equiv$ on $\widehat{R}$ which is dependent on a free ultrafilter, $\mathcal{U}$. Given $r = \langle r_i \rangle_i$ and $s = \langle s_i \rangle_i$ both in $\widehat{R}$ then $r \equiv s$ if and only if $\{i \in \mathbb{N} \mid r_i = s_i\}$ is an element of $\mathcal{U}$. We then say $\langle r_i \rangle_i \;=\; \langle s_i \rangle_i$ almost everywhere.

This equivalence relation distinguishes between two sequences that have the same limit as $n \to \infty$. That is $\langle 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \rangle \not\equiv \langle 0, 0, 0, \ldots \rangle$. This type of sequence $\langle 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \rangle$ will later help to define infinitesimals. This equivalence relation also helps in eliminating the problem of the product of two nonzero numbers equaling zero. That is $\langle 0, 0, 1, 1, 0, 0, \ldots \rangle \odot \langle 1, 1, 0, 0, 1, 1, \ldots \rangle = \langle 0, 0, 0, 0, \ldots \rangle$, but one of these, depending on the particular ultrafilter used to define $\equiv$, is equivalent to $\langle 0, 0, 0, 0, \ldots \rangle$.

Definition Fix a free ultrafilter. We define $^*\mathcal{R}$ to be the set of all equivalence classes of $\widehat{R}$ introduced by $\equiv$. We denote equivalence class that contains a particular sequence $\hat{s} = \langle s_i \rangle_{i=1}^{\infty}$ by [s] or $\mathbf{s}$. Thus, if $\mathbf{r} \equiv \mathbf{s}$ in $\widehat{R}$, then $\mathbf{r} = [r] = [s] = \mathbf{s}$.

3.2 Nonstandard Reals

We call elements of $^*\mathbb{R}$ nonstandard or hyperreal numbers. We have that $^*\mathcal{R}$ is a commutative ring with zero $\mathbf{0} = [\langle 0, 0, 0, \ldots \rangle]$ and unit $\mathbf{1} = [\langle 1, 1, \ldots \rangle]$. We also have that the sum of two positive elements is positive, the product of two positive elements is positive, and a given number $\mathbf{r}$, is either positive or **-r** is positive. Thus we have that $^*\mathcal{R}$ an ordered field. The notion of absolute value in the nonstandard reals is the obvious extension of the notion of absolute value in the standard reals.

Definition If $r \in \mathbb{R}$, we define $*(r) = {}^*r$, where $^*r = [\langle r, r, \ldots \rangle] \in \mathbb{R}$.

Definition If $A \subset \mathbb{R}$ then $(A)_*$ is the set of all elements $^*a$, where $a \in A$.

Naturally, $(R)_*$ is the set of all *standard* numbers in $^*\mathbb{R}$.

In order to show that ${}^*\mathbb{R}$ contains numbers other than standard numbers, we assume that $\mathcal{U}$ is a free ultrafilter. Consider $\omega = [\langle 1, 2, 4, 8, \ldots \rangle]$. This number cannot equal any standard number ${}^*r = [\langle r, r, r, \ldots \rangle]$, for the set $\{i \in \mathbb{N} \mid r = i\}$. Thus, ${}^*\mathbb{R} \supset (R)_*$ and ω is called an infinite number.

Similarly, if we look at $\omega^{-1} = [\langle 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \rangle] \notin (R)_*$ and we call ω^{-1} an infinitesimal.

3.3 Infinitesimals and unbounded numbers

Numbers in ${}^*\mathbb{R}$ can either be standard or nonstandard. A number is considered standard if it is in both $\mathbb{R}$ and in ${}^*\mathbb{R}$. A number is considered nonstandard if it is in ${}^*\mathbb{R}$, but not in $\mathbb{R}$.

Definition Standard or nonstandard numbers in ${}^*\mathbb{R}$ can be *infinite*, *finite*, or *infinitesimal*.

A number is considered to be *infinite* if its absolute value is greater than all standard natural numbers; *finite* if its absolute value is smaller than some standard natural number; *infinitesimal* if it is greater than zero but smaller than every positive standard real number.

Given the definition of a finite, infinite, and infinitesimal number, we can then define *near* and *finitely close*, which allows us to relate nonstandard real numbers to standard real numbers. We will use the definitions of near and finitely close to define a map from the nonstandard reals to the standard part of each nonstandard number.

Definition Given $x, y \in {}^*\mathbb{R}$, say that x and y are *near* if $x - y$ is infinitesimal. We write this as $x \simeq y$ and $x \not\simeq y$ if x and y are not near. We define the *monad* [1], $m(x)$ of x to be the set of points that are near x. Therefore, the set of infinitesimal are denoted by $m(0)$.

Definition Again given $x, y \in {}^*\mathbb{R}$, we say that x and y are *finitely close* if $[x - y]$ is finite. We write $x \sim y$ if x and y are finitely close and $x \not\sim y$ if x and y are not finitely close. We define the *galaxy* of x to be the set of points that are finitely close to x. The set of finite numbers is denoted by $G(0)$.

[1] Monad, from Greek $\mu o \nu \acute{\alpha} \varsigma$ meaning *the source* or *one without division* was, according to the Pythagoreans the term for God.

3.4 Standard-part map

For a finite ρ in the nonstandard reals, we define the standard-part map of ρ to be the unique number r, in the standard reals, such that r is near ρ. This defines a map $\mathsf{std} : G(0) \to \mathbb{R}$ from the finite numbers in the nonstandard reals to the reals; this map is called the standard part map.

It should be noted that each finite ρ is near a unique real number since, if z is near a and z is near b, then a and b are near, but $a - b$ is real therefore the standard part map is well defined.

Theorem The map std is an order-preserving homomorphism of G(0) onto $\mathbb{R}$:
i. $\mathsf{std}(x \pm y) = \mathsf{std}(x) \pm \mathsf{std}(y)$
ii. $\mathsf{std}(xy) = \mathsf{std}(x)\mathsf{std}(y)$
iii. $\mathsf{std}(\frac{x}{y}) = \frac{\mathsf{std}(x)}{\mathsf{std}(y)}$, given that $\mathsf{std}(y)$ is not zero, equivalently given that y is not infinitesimal and is in the galaxy of 0.
iv. $\mathsf{std}(x) \leq \mathsf{std}(y)$ if $x \leq y$

4 Transfer Principle for Simple Sentences

"Without mathematical logic, mathematicians tried in vain for 300 years to construct hyperreals. With it, Robinson was able to surmount the difficulties with astonishing ease" (Henle and Kleinberg 1979). Thus, in order to construct nonstandard analysis we must first start with a basic introduction to logic.

4.1 Basic Logic

The symbols $\wedge$ and $\rightarrow$, are interpreted as "and" and "implies". A symbol $\underline{s}$, a relational symbol $\underline{P}$, or a function symbol $\underline{f}$ is called the name of s for each $s \in S$, each relation $P \in S$ or function $f \in S$.

Definition *Terms* are defined inductively as follows:
i. Each constant and variable symbol is a term
ii. If $\underline{f}$ is the name of a function of n variables and $\tau^1, \ldots, \tau^n$ are terms, then $\underline{f}(\tau^1, \ldots, \tau^n)$ is a term. A term containing no variables is a constant term.

Definition given any structure $\mathcal{L}$, *A Simple Sentence* is a string of symbols in $L_{\mathcal{L}}$ which takes either of the following forms:

A. *Atomic Sentences.* Such sentences are of the form $\underline{P}\ \langle\tau^1,\ldots,\tau^n\rangle$, where $\underline{P}$ is the name of an n-ary relation and the $\tau^i(i=1,\ldots,n)$ are constant terms.
B. *Compound Sentences.* Such sentences are of the form:

$$(\forall x_1)\cdots(\forall x_n)\Big[\bigwedge_{i=1}^{k}\underline{P}_i\langle\hat{\tau}_i\rangle \rightarrow \bigwedge_{j=1}^{l}\underline{Q}_j\langle\hat{\sigma}_j\rangle\Big]$$

where $\bigwedge_{i=1}^{n}\underline{S}_i$ denotes $\underline{S}_1,\wedge\cdots\wedge,\underline{S}_n$, $\hat{\tau}_i$ and $\hat{\sigma}_j$ are n_i-tuples and n_j-tuples of terms involving no other variables than $x_1,\ldots,x_n,n_i$ and n_j.

Definition A constant term is *interpretable* in L if either:
i. it is a constant symbol $\underline{s}$ naming an element $s\in S$, in which case it is interpreted as s, or
ii. it is of the form $\underline{f}(\tau^1,\ldots,\tau^n)$, where the terms $\tau^1,\ldots,\tau^n$ are interpretable in $\mathcal{L}$ and hence can be interpreted as the elements $s^1,\ldots,s^n\in S$, and the n-tuple $\langle s^1,\ldots,s^n\rangle$is in the domain of the function f named by $\underline{f}$; in this case $\underline{f}(\tau^1,\ldots,\tau^n)$ is interpreted as $f(s^1,\ldots,s^n)$.

If r is a name in $L_{\mathcal{R}}$ of $r\in R$ then $\underline{r}$ is also the name in $L_{*\mathcal{R}}$ of $*r\in *R$. If $\underline{P}$ is a name in $L_{\mathcal{R}}$ of the relation P on R then $^*\underline{P}$ is a name in $L_{*\mathcal{R}}$ of the relation $^*P$ on $^*R$ in particular, If $\underline{f}$ is a name in $L_{\mathcal{R}}$ of the function f on R then $^*\underline{f}$ is a name in $L_{*\mathcal{R}}$ of the function $^*f$ on $^*R$. The symbols $<$,+, and $\cdot$ will denote the corresponding relation and functions in $\mathcal{R}$ and $^*\mathcal{R}$.

4.2 Skolem Function

A Skolem function replaces every existentially quantified variable, y, with a term $\psi(x_1,\ldots,x_n)$ whose functional symbol $\underline{\psi}$ does not occur anywhere else in the formula. The Skolem function introduces universally quantified variables $x_1,\ldots,x_n$ whose quantification precedes that of y and whose domain is dependent on the quantified variable it is replacing.

An example of a *Skolem Function* is the following:
Consider, "for each nonzero $x\in\mathbb{R}$, $\exists y\in\mathbb{R}$ so that $xy=1$." ψ is a Skolem function of one variable whose domain is the set of nonzero reals, that satisfies $x\cdot\psi(x)=1$.

4.3 Transfer Principle

We will define the transfer principle on simple sentences to allow us to take standard terms and transfer them into the nonstandard reals, making them nonstandard terms. This allows us to go back and forth between standard and nonstandard results.

Definition The $* - Transform$ of terms is defined by induction as follows:
i. If τ is a constant or variable symbol then $\tau = {}^*\tau$.
ii. If $\tau = \underline{f}\ (\tau^1, \ldots, \tau^n)$ then ${}^*\tau = {}^*\underline{f}({}^*\tau^1,\ \ldots,\ {}^*\tau^n)$.

Definition If Φ is a simple sentence in $L_{\mathcal{R}}$ we define the $* - Transform$ of ${}^*\Phi$ of Φ as follows:
a. If Φ is the atomic sentence $\underline{\mathrm{P}}\ \langle \tau^1, \ldots, \tau^n \rangle$ then ${}^*\Phi$ is ${}^*\underline{P}\ \langle *\tau^1, \ldots, *\tau^n \rangle$
b. If Φ is the sentence

$$(\forall x_1) \ldots (\forall x_n) \Big[\bigwedge_{i=1}^{k} \underline{P}_i \langle \hat{\tau}_i \rangle \to \bigwedge_{j=1}^{l} \underline{Q}_j \langle \hat{\sigma}_j \rangle \Big]$$

then ${}^*\Phi$ is the sentence

$$(\forall x_1) \ldots (\forall x_n) \Big[\bigwedge_{i=1}^{k} *\underline{P}_i \langle *\hat{\tau}_i \rangle \to \bigwedge_{j=1}^{l} *\underline{Q}_j \langle *\hat{\sigma}_j \rangle \Big]$$

where $*\hat{\tau} = \langle *\tau^1, \ldots, *\tau^n \rangle$ if $\hat{\tau} = \langle \tau^1, \ldots, \tau^n \rangle$.

Theorem The Transfer Principle If Φ is a simple sentence in $L_{\mathcal{R}}$ which is true in $\mathcal{R}$, then ${}^*\Phi$ is true in ${}^*\mathcal{R}$.

5 Nonstandard Calculus

Nonstandard calculus is the approach to calculus that is based on Leibniz's ideas, but formalized by Robinson; it allows the simplicity and intuitive power of infinitesimals, while still working in a mathematically rigorous number system (Henle & Kleinberg). We will construct basic results of nonstandard analysis only to prove the intermediate value theorem, which we will prove using both standard techniques and nonstandard techniques to give the reader the flavor of a nonstandard proof.

5.1 Topology

We say that a subset $A \in \mathbb{R}$ is open if and only if the monad of a is contained in $^*A$ for each $a \in A$. We say that A is closed if and only if $m(a) \cap {}^*A$ is empty for each a in the complement of A. Therefore, the closure, $\hat{A}$, of a set $A \in \mathbb{R}$ is the set of $x \in \mathbb{R}$ such that the monad of $x \cap {}^*A$ is not empty.

To obtain Robinson's characterization of compactness, we need the following standard Lemma.

Lemma Each cover of $A \subseteq \mathbb{R}$ by open sets $A_i, \forall i \in I$ contains a finite subcover if each cover of A by a collection of open intervals (a_n, b_n) with rational endpoints, contains a finite subcover.

Robinson's Theorem The set $A \subset \mathbb{R}$ is compact if and only if for each $y \in {}^*A$ there is an $x \in$ A with $x \simeq y$, that is, every point in $^*A$ is near a point in A.

Proof: Suppose that A is compact but y $\in {}^*A$ is not near any $x \in A$. Then for each $x \in A$ there is a $\delta_x > 0$ *in* $\mathbb{R}$ such that $\mid x - y \mid \geq \delta_x$. Since A is compact, we take a finite subcover

$$A_i = \{ \ z \ \in \ \mathbb{R} : \ \mid \ x_i \ - \ z \ \mid < \delta_{x_i} \}$$

for $i = 1, 2, \ldots, n$ from the cover of A by the sets

$$A_x \ = \ \{z \in \ \mathbb{R} : \ \mid \ x - z \ \mid < \delta_x\} \forall x \in A.$$

It follows that

$$(\forall \ y) \Big[\underline{A} \ \langle y \rangle \wedge \mid \ x_1 - y \ \mid \geq \delta_{x_1} \ \wedge \cdots \wedge \mid x_{n-1} - y \mid \geq \delta_{x_{n-1}} \rightarrow \mid x_n - y \mid \ < \ \delta_{x_n} \Big]$$

is true in $\mathcal{R}$. Transferring to $^*\mathcal{R}$, we obtain a contradiction with the fact that $y \in {}^*A$ and $\mid \ x_i \ - \ y \ \mid \ \geq \ \delta_{x_i}$ for i = 1, 2, ..., n.

Assume now that a cover $A_i, \forall i \in I$ contains no finite subcover. By the lemma, there exists a cover of A by a countable collection of

$$I_n = \{x \in \mathbb{R} : a_n < x < b_n\}, \forall n \in \mathbb{N}$$

of open intervals with rational end points which has no finite subcover. Thus there is a Skolem function $\psi : \mathbb{N} \to A$ so that

$$(\forall n)(\forall k)\Big[\underline{\mathbb{N}}\langle n\rangle \wedge \underline{\mathbb{N}}\langle k\rangle \wedge k \leq n \wedge a_k < \underline{\psi}(n) \to b_k \leq \underline{\psi}(n)\Big]$$

is true in $\mathcal{R}$. By transfer, we see that if ω is infinite, then ${}^*\psi(\omega) \notin {}^*(a_k, b_k)$ for any $k \in \mathbb{N}$. Thus, ${}^*\psi(\omega) \in {}^*A$ is not near a point $x \in A$. Since the monad of x is contained in ${}^*(a_k, b_k)$ for some $k \in \mathbb{N}$.

□

It should be noted that Robinson's theorem is analogous to the Heine Borel Theorem.

5.2 Limits and Continuity

Limits and continuity can be described using nonstandard analysis in much the same way that topological notions were.

Proposition Recall that $\hat{A}$ is the closure of the set A. Let f be defined on A and choose $a \in \hat{A}$. Then, the limit $\lim_{x \to a} f(x)$ exists if and only if ${}^*f(x) \simeq {}^*f(y)$ for all $x, y \in {}^*A$ with $x \simeq a$ and $y \simeq a$ but $x \neq a, y \neq a$.

Proposition Let f be defined on $A \subseteq \mathbb{R}$. Then, f is continuous at $a \in A$ if and only if ${}^*f(x) \simeq f(a)$ for all $x \in {}^*A$ with $x \simeq a$.

Theorem If f and g are defined on A then $\forall a \in A$ where f and g are continuous, so are $f + g$ and fg and, if $g(a) \neq 0$ then $\frac{f}{g}$ is as well.

Armed with the preceding four theorems, we can then move on to prove the Intermediate Value Theorem using both the standard epsilon-delta approach and the nonstandard infinitesimal approach.

Intermediate Value Theorem If f is continuous on the closed and bounded interval $[a, b]$ and $f(a) < d < f(b)$ for some d, then there exists a $c \in (a, b)$ with $f(c) = d$.

Proof using standard techniques Let X be the set of all $x \in [a, b]$ such that $f(x) \leq d$. We have that S is non-empty since $a \in X$. We also have that S is bounded above by b. Since the real numbers are complete, the supremum $c = \sup X$ exists. That is, c is the least upper bound for X.

Suppose that $f(c) > d$. Then we have that $f(c) - d > 0$, thus, $\exists \delta > 0$ such that

$$\mid f(x) - f(c) \mid < f(c) - d$$

for all x such that $\mid x - c \mid < \delta$. This is because f is continuous. But, if this were true, then

$$f(x) > f(c) - [f(c) - d] = d,$$

whenever $\mid x - c \mid < \delta$. Thus $c - \delta$ is an upper bound, which is a contradiction of c being the least upper bound.

Now suppose that $f(c) < d$. Again, $\exists \delta > 0$ such that

$$\mid f(x) - f(c) \mid < d - f(c)$$

for all x such that $\mid x - c \mid < \delta$. Then we have that

$$f(x) < f(c) + [d - f(c)] = d$$

for all $x \in (c - \delta, c + \delta)$. Thus there exist an $x > c$ such that $f(x) < d$. This again is a contradiction to c being the least upper bound. Thus we have that $f(c) = d$. □

Proof using nonstandard techniques Consider points $x_k = a + \frac{k[b-a]}{n}, 0 \leq k \leq n$. If we consider the values of f at x_k, we see that there exist a Skolem function $\psi : \mathbb{N} \to [a, b)$ satisfying $f(\psi(n)) < d$ and $f\big(\psi(n) + \frac{[b-a]}{n}\big) \geq d$. Hence the sentence

$$(\forall n)\Big[\underline{\mathbb{N}}\langle n \rangle \to a \leq \underline{\psi}(n) < b \wedge \underline{f}(\underline{\psi}(n)) + d \wedge \underline{f}\big(\underline{\psi}(n)\frac{[b-a]}{n}\big) \geq d\Big]$$

is true in $\mathcal{R}$.

Transferring to ${}^*\mathcal{R}$, and letting $n \in {}^*\mathbb{N}$, we have ${}^*f\big({}^*\psi(n)\big) < d$ and ${}^*f\big({}^*\psi(n) + \frac{[b-a]}{n}\big) \geq d$. Let $c = \mathsf{std}({}^*\psi(n)) = \mathsf{std}\big({}^*\psi(n) + \frac{[b-a]}{n}\big)$. By continuity, we have $f(c) \leq d$ and $f(c) \geq d$, hence $f(c) = d$. Note that c cannot equal a or b or else $f(c)$ would equal either $f(a)$ or $f(c)$. □

6 Robinson's Philosophy of Nonstandard Analysis

Throughout most of their existence, infinitesimals were regarded as well founded fictions that could be used to arrive at the correct answer, which would disappear once the correct answer had been obtained. Leibniz referred to them as "fictions of the mind which enable one to adopt a succinct manner of speaking and which can be eliminated after having been put to use in mathematical reasoning." Leibniz declared that for infinitesimals in particular, "everybody can substitute as small a quantity as he wishes" (Keisler et al. 1979).

In spite of mathematical differences between Robinson's nonstandard analysis and and Leibniz's infinitesimal calculus, Robinson suggests a similar philosophical conception of infinity as Leibniz. That is, they both agree in a distinction between a theological infinity and a mathematical infinity. Furthermore, Robinson agrees with Leibniz's assertion that infinitesimals are fictions of the mind. However, Robinson objects to the claim that Leibniz's use of infinitesimals was well founded. Robinson vehemently objects to Leibniz's inability to "state with *sufficient precision* just what rules were supposed to govern [his] extended number system." In fact, Robinson's rigorous definition of the rules governing infinitesimals is seen as perhaps Robinson's greatest contribution to mathematics (Keisler et al.).

Robinson's take on infinity is what is to be expected given that he considered himself to be a formalist. Thus his philosophy of mathematics is founded on the following three principles:

1. the principle of the *theoretical* meaninglessness of the notion of an infinite totality,
2. the principle of the usefulness of the notion of infinity as a *well-founded* fiction, and
3. the principle that there exists an *intuitive, nonconventional core* of mathematics and logic from which is presupposed in all mathematical thinking (Keisler et al.).

Thus, it makes sense that while he objects to Leibniz's construction of the rules governing infinitesimals for claiming to be well founded while lacking the appropriate rigor, he does not object to the use of infinitesimals once they were rigorously constructed into the nonstandard reals.

Nonstandard analysis, while initially created by Robinson to introduce a standard of rigor into Leibniz's construction of the Calculus, has become a tool which some would argue provides a better approach than the standard epsilon-delta, to calculus and analysis. C. S. Peirce declared in *The Law of Mind* "the idea of an infinitesimal involves no contradiction– as a mathematician, I prefer the method of infinitesimals to that of limits as far easier and less infested with snares".

Thus, the importance of nonstandard analysis exists on the grounds of using it as a rigorously defined tool to help understand the standard approach. Nonstandard analysis is seen as a tool which can provide a new perspective on the results obtained and one that in many cases produces a shorter more elegant, yet equivalent result to the results obtained by the standard approach.

The significance of nonstandard analysis surpasses just its use as a tool. The mere existence of nonstandard analysis and the consequence that it achieves the exact same results as the standard approach, implies the inherent truth in the results of the Calculus. The fact that regardless of the method the results of the Calculus are the same, implies a higher level of truth.

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