Author	Date	Time	Rev	No.	Reference
Randy Yates	August 15, 2009	21:41	PA2	n/a	white.tex

Mean of a White-Noise Process

Randy Yates

August 15, 2009

Digital Signal Labs
signal processing systems

http://www.digitalsignallabs.com

888-708-3698

Typeset using LaTeX 2_ε

Contents

1 Mean of a Continuous-Time White, Random Process

1.1 Problem Statement

If $X(t)$ is a real, continuous-time, stationary, white random process, can we conclude that $E[X(t)] = 0$?

1.2 Response

The term *white noise* is ambiguous. Brown [1] defines it as

> "... a stationary random process having a constant spectral density function."

Papoulis [2] defines it differently:

> "We shall say that a process $\mathbf{v}(t)$ is white noise if its values $\mathbf{v}(t_i)$ and $\mathbf{v}(t_j)$ are uncorrelated for every t_i and $t_j \neq t_i$: $C(t_i, t_j) = 0,\ t_i \neq t_j$"

The following subsections explore the implications of each definition with respect to the mean of the resulting process.

1.2.1 White Noise as a Constant Power Spectral Density

When $\Phi_{xx}(\omega) = \alpha$ for some constant α, then the corresponding autocorrelation function for the process is

$$\phi_{xx}(\tau) = \alpha\delta(\tau). \tag{1}$$

Assume that $X(t)$ is a zero-mean white-noise process and $Y(t) = X(t) + m$ is a non-zero-mean process. Then

$$\begin{aligned}
\phi_{yy}(\tau) &= \mathrm{E}[Y(t)Y(t+\tau)] && (2)\\
&= \mathrm{E}[(X(t)+m)(X(t+\tau)+m)] && (3)\\
&= \alpha\delta(\tau) + m^2, && (4)
\end{aligned}$$

and therefore

$$\Phi_{yy}(\omega) = \alpha + 2\pi m^2\delta(\omega), \tag{5}$$

which is clearly not constant, thus $Y(t)$ violates the requirement for a white-noise process by this definition.

Therefore we may conclude that Brown's definition implies a zero-mean process.

1.2.2 White Noise as Uncorrelated Random Variables

Let us restate Papoulis' definition of white noise using our own language and notation. The random process $X(t)$ is called white noise if any two distinct ensemble variables are uncorrelated:

$$\mathrm{E}[X(t)X(t+\tau)] = \mathrm{E}[X(t)]\mathrm{E}[X(t+\tau)], \ \tau \neq 0. \tag{6}$$

From this definition we see that the autocorrelation function of such a process is

$$\begin{aligned} \phi_{xx}(\tau) &= \mathrm{E}[X(t)X(t+\tau)] && (7)\\ &= \mathrm{E}[X(t)]\mathrm{E}[X(t+\tau)], \ \tau \neq 0 && (8)\\ &= m_x^2, \ \tau \neq 0, && (9) \end{aligned}$$

where we use the property of stationary processes that the mean is not a function of time.

The autocovariance function of such a process is

$$\begin{aligned} \mu_{xx}(\tau) &= \mathrm{E}[X(t)X(t+\tau)] - m_x^2 && (10)\\ &= \mathrm{E}[X(t)]\mathrm{E}[X(t+\tau)] - m_x^2, \ \tau \neq 0 && (11)\\ &= m_x^2 - m_x^2, \ \tau \neq 0 && (12)\\ &= 0, \ \tau \neq 0. && (13) \end{aligned}$$

Thus we **cannot** conclude that a white noise process has zero mean solely from Papoulis' definition of white noise. However, if we also knew that the power spectral density of the process was constant, then we would know that $\phi_{xx}(\tau) = 0, \tau \neq 0$, and thus that the mean is zero.

2 Revision History

Table 1 lists the revision history for this document.

Rev.	Date/Time	Person	Changes
PA1	16-Feb-2006	Randy Yates	Initial Version
PA2	15-Aug-2009	Randy Yates	1. Reversed the conclusion of section 1.2.1. The previous revision contained an unfounded assertion ("This constraint does **not** require that the mean be zero.") that I have shown in this version of the document to be false. 2. Changed wording slightly in section 1.2.2. 3. Removed the "real white noise" section—too informal and not really relevent. 4. Added this revision table.

Table 1: Revision History

3 References

[1] Robert Grover Brown, *Introduction to Random Signal Analysis and Kalman Filtering*. John Wiley and Sons, 1983.

[2] Athanasios Papoulis, *Probability, Random Variables, and Stochastic Processes*, 3rd ed. WCB/McGraw-Hill, 1991.