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CONTRIBUTIONS TO THE THEORY OF THE RIEMANN ZETA-FUNCTION AND THE THEORY OF THE DISTRIBUTION OF PRIMES

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I.

Introduction and summary.

1.1.[2] We have united in this paper a series of contributions towards the solution of various outstanding questions in the Analytic Theory of Numbers.

[1] Some of the results of which this memoir contains the first full account have already been stated shortly and incompletely in the following notes and abstracts.

G. H. HARDY: (1) 'On the zeros of RIEMANN's Zeta-function', *Proc. London Math. Soc.* (records of proceedings at meetings), ser. 2, vol. 13, 12 March 1914, p. XXIX; (2) 'Sur les zéros de la fonction $\zeta(s)$ de RIEMANN', *Comptes Rendus*, 6 April 1914.

J. E. LITTLEWOOD: 'Sur la distribution des nombres premiers', *Comptes Rendus*, 22 June 1914.

G. H. HARDY and J. E. LITTLEWOOD: (1) 'New proofs of the prime-number theorem and similar theorems', *Quarterly Journal*, vol. 46, 1915, pp. 215—219; (2) 'On the zeros of the RIEMANN Zeta-function' and (3) 'On an assertion of TSCHEBYSCHEF', *Proc. London Math. Soc.* (records etc.), ser. 2, vol. 14, 1915, p. XIV.

[2] The sections, paragraphs, and formulae contained in this memoir are numbered according to the decimal system of PEANO, the aggregate of numbers employed forming a selection of the rational numbers arranged in order of magnitude. Thus every number occurring in the first section begins with 1; the first paragraph is 1.1 and the first formula of the first paragraph 1.11. The second would naturally be 1.12; but here four formulae occur which are parallel for the purposes of our argument, and so these are numbered 1.121, 1.122, 1.123 and 1.124.

In a long and complicated memoir such as this, PEANO's system has many advantages. It enables the author, in the process of revision of his work, to delete or insert formulae without serious interference with the numbering of the remainder; and it enables the reader to discover any formula referred to with the minimum of trouble.

Our answers to these questions are naturally tentative and fragmentary. The importance and difficulty of the problems dealt with should be a sufficient apology for the incompleteness and miscellaneous character of the results.

We begin, in section 2, by considering some applications to the theory of primes of the formula

$$(1.11)\qquad \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\Gamma(s)y^{-s}\,ds = e^{-y}, \quad (\kappa>0, \Re(y)>0),$$

of CAHEN and MELLIN,[1] a formula which seems not unlikely to play a more prominent part in the Theory of Numbers than has been assigned to it hitherto. Using this formula in combination with some of the 'Tauberian' theorems which we have proved in a series of recent papers in the *Proceedings of the London Mathematical Society* and elsewhere, we are able (in 2.1) to deduce new theorems as to the convergence of Dirichlet's series of the most general type, from which follow as corollaries such results as

$$(1.121)\qquad \psi(x)\sim x,$$

$$(1.122)\qquad M(x)=o(x),$$

$$(1.123)\qquad \sum\frac{\mu(n)}{n}=0,$$

all of which are known to be equivalent[2] to the 'Prime Number Theorem'[3]

[1] CAHEN, *Thèse*, Paris, 1894, and *Annales de l'École Normale Supérieure*, ser. 3, vol. 11, 1894, pp. 75—164 (p. 99); MELLIN, *Acta Societatis Fennicae*, vol. 20, 1895, no 7, pp. 1—39 (p. 6), and *Math. Annalen*, vol. 68, 1910, pp. 305—337.

[2] By this we mean that, from any one of these results, all the rest can be deduced by elementary reasoning which involves no appeal to the theory of functions of a complex variable. That (1.121), and (1.124) are equivalent in this sense was shown by DE LA VALLÉE-POUSSIN (*Annales de la Société Scientifique de Bruxelles*, vol. 20, part 2, 1896, pp 360—361). The deduction of (1.122) from (1.123) is of a very simple character: that of (1.123) from (1.122) was first made by AXER (*Prace Matematyczno-Fizyczne*, vol. 21, 1910, pp. 65—93). That (1.123) follows from (1.121) was shown by LANDAU (*Dissertation*, Berlin, 1899), and the converse deduction is also due to him (*Wiener Sitzungsberichte*, vol. 115, 1906, pp. 589—632).

[3] We append the following definitions for the benefit of readers who may not be familiar with the notations usual in the Analytic Theory of Numbers.

(1) $f(x)=O(\varphi(x))$ means that a constant K exists such that $|f|<K\varphi$.

(2) $f(x)=o(\varphi(x))$ means that

$$\lim\frac{f(x)}{\varphi(x)}=0$$

when x tends to ∞, or to whatever limit may be in question.

(1.124) $$\Pi(x) \sim \frac{x}{\log x}.$$

In 2.2 we obtain an explicit formula for the function

(1.13) $$F(y) = \sum_{1}^{\infty} (\Lambda(n) - 1) e^{-ny} \quad (\Re(y) > 0),$$

from which we deduce that, assuming the hypothesis of RIEMANN as to the zeros of $\zeta(s)$,

(1.141) $$F(y) = O\sqrt{\frac{1}{y}}$$

as $y \to 0$, while a positive constant K exists, such that each of the inequalities

(1.142) $$F(y) < -K\sqrt{\frac{1}{y}}, \quad F(y) > K\sqrt{\frac{1}{y}}$$

is satisfied for an infinity of values of y tending to zero. From this follows as a corollary the theorem of SCHMIDT[1] which asserts the existence of a K such that each of the inequalities

(1.143) $$\psi(x) - x < -K\sqrt{x}, \quad \psi(x) - x > K\sqrt{x}$$

is satisfied for an infinity of values of x tending to infinity.

It should be observed, however, that our method does not enable us to prove the wider inequalities

(1.15) $$\psi(x) - x < -x^{\Theta-\delta}, \quad \psi(x) - x > x^{\Theta-\delta},$$

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(3) $\mu(n) = (-1)^q$ if n is a product of q different primes, and is otherwise zero.
(4) $\Lambda(n) = \log p$ if $n = p^m$, and is otherwise zero.

(5) $$M(x) = \sum_{n \leq x} \mu(n)$$

(6) $$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

(7) $\Pi(x)$ is the number of primes less than or equal to x.

[1] *Math. Annalen*, vol. 57, 1903, pp. 195—204; LANDAU, *Handbuch*, pp. 711 *et seq.* Naturally our argument does not give so large a value of K as SCHMIDT's. The actual inequalities proved by SCHMIDT are not the inequalities (1.143) but the substantially equivalent inequalities (1.51).

which hold when the upper limit Θ of the real parts of the zeros of $\zeta(s)$ is greater than $\frac{1}{2}$. Nor does it seem possible, in the present state of our knowledge of the properties of $\zeta(s)$, to give a satisfactory proof of the explicit formula for

$$f(y) = \sum_{1}^{\infty} \mu(n) e^{-ny}$$

which corresponds to that which we find for the function (1.13).

1.2. In 2.3 we are concerned with a statement made by TSCHEBYSCHEF[1] in 1853, of which no proof of any kind has yet been published. TSCHEBYSCHEF asserts that the function

$$F(y) = e^{-3y} - e^{-5y} + e^{-7y} + e^{-11y} - \cdots = \sum (-)^{\frac{p+1}{2}} e^{-py}$$

tends to infinity as $y \to 0$. We prove that this result is true if all the complex zeros of the function

(1.21) $$L(s) = 1^{-s} - 3^{-s} + 5^{-s} - \cdots \quad (\sigma > 0)$$

have their real part equal to $\frac{1}{2}$. There seems to be little doubt that, if this assumption is false, then TSCHEBYSCHEF's assertion is also false, but this we have not succeeded in proving rigorously. The difficulties which have debarred us from a proof are of the same nature as those which have prevented us from deducing the inequalities (1.15) from our explicit formula for the function (1.13).

In 2.4 we prove that

(1.22) $$\int_{-T}^{T} |\zeta(\tfrac{1}{2} + it)|^2 dt \sim 2T \log T$$

as $T \to \infty$. The method used may be adapted to show that

(1.23) $$\int_{-T}^{T} |\zeta(\beta + it)|^2 dt \sim (2\pi)^{2\beta-1} \zeta(2 - 2\beta) \frac{T^{2-2\beta}}{2 - 2\beta},$$

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[1] TSCHEBYSCHEF, *Bulletin de l'Académie Impériale des Sciences de St. Petersbourg*, vol. 11, 1853, p. 208, and *Oeuvres*, vol. 1, p. 697; LANDAU, *Rendiconti di Palermo*, vol. 24, 1907, pp. 155—156.

if $\beta < \frac{1}{2}$: but there is nothing essentially new in this last formula, as it follows from the functional equation satisfied by $\zeta(s)$ and the known result

$$\int_{-T}^{T} |\zeta(\beta + it)|^2 dt \sim 2\zeta(2\beta)T,$$

where $\beta > \frac{1}{2}$.[1]

We conclude this section by noticing a remarkable formula, the form of which was suggested to us by an observation of Mr S. Ramanujan. We are unable to give a satisfactory proof of this formula, but it seems to us well worthy of attention. It is intimately connected with an expression of the function $\frac{1}{\zeta(s)}$ as a definite integral, which is due to Marcel Riesz.[2]

1.3. In section 3 we are concerned with the series

(1.31) $$\sum e^{a\varrho \log(-i\varrho)} x^{\varrho} \varrho^{-\varkappa},$$

where a, x, and $\varkappa$ are real, and ϱ is a complex zero of $\zeta(s)$. Our object is to obtain results for this series similar to those obtained by Landau[3] for the simpler series

$$\sum x^{\varrho},$$

and our main argument is an adaptation of his.[4] The results of this section are simplified in form if we assume the truth of the Riemann hypothesis. Writing $\frac{1}{2} + i\gamma$ for ϱ, and confining ourselves to the zeros for which $\gamma > 0$, series of the type (1.31) are found to be substantially equivalent to series of the type

(1.32) $$\sum \gamma^{-\omega} e^{a i \gamma \log(\gamma\theta)},$$

where a, θ, and ω are real and the first two positive. Our principal result is that

[1] Landau, *Handbuch*, p. 816.
[2] *Acta Mathematica*, vol. 40, 1916, pp. 185—190.
[3] *Math. Annalen*, vol. 71, 1912, pp. 548—564.
[4] The idea which dominates the critical stage of the argument is also Landau's, but is to be found in another of his papers ('Über die Anzahl der Gitterpunkte in gewissen Bereichen', *Göttinger Nachrichten*, 1912, pp. 687—771, especially p. 707, Hilfsatz 10).

(1.33) $$\sum_{0<\gamma<T} e^{a i \gamma \log(\gamma\theta)} = O\left(T^{\frac{1+a}{2}}\right);$$

this result is trivial if $a > 1$, but otherwise significant. The apparent dependence of the order on a is curious, and we are disposed to believe that it does not really correspond to the truth, and that the order is really $O\left(T^{\frac{1}{2}+\delta}\right)$ for all values of a and all positive values of δ. But this we are unable to prove.

1.4. Section 4 is devoted to a closer study than has yet been published of the zeros of the Zeta-function which lie on the line $\sigma = \frac{1}{2}$. That *some* such zeros exist was first shown by Gram,[1] and the later investigations of de la Vallée-Poussin, Gram,[1] Lindelöf,[1] and Backlund[1] have shown that there are exactly 58 on the line

$$\left(\frac{1}{2} - 100i, \frac{1}{2} + 100i\right),$$

and no other complex zeros between the lines $t = -100$, $t = 100$. In other words the function $\Xi(t)$ of Riemann has exactly 58 real zeros between -100 and 100, and no complex zeros whose real part lies between these limits.

It was shown recently by Hardy[2] that $\Xi(t)$ has an infinity of real zeros. The method of proof depended on the use of (i) the Cahen-Mellin integral and (ii) a lemma relating to the behaviour of the series

$$\vartheta_3(0, \tau) = 1 + 2\sum q^{n^2}$$

when q tends in a certain manner to the point -1 on the circle of convergence. The proof given by Hardy was materially simplified by Landau,[3] who showed that no property of the ϑ-function was needed for the purpose of the proof except the obvious one expressed by the equation

$$\vartheta_3(0, \tau) = O\sqrt{\frac{1}{1-|q|}}.$$

[1] See Gram, *Acta Mathematica*, vol. 27, 1903, pp. 289—304; Lindelöf, *Acta Societatis Fennicae*, vol. 31, 1903, no. 3; Backlund, *Översigt af Finska Vetenskaps-Societetens Förhandlingar*, vol. 54, 1911—12, A, no. 3; and further entries under these names in Landau's bibliography.
[2] *Comptes Rendus*, 6 April, 1914.
[3] *Math. Annalen*, vol. 76, 1915, pp. 212—243.

LANDAU also extended the proof so as to apply to the functions defined by the series

$$\sum \frac{\chi(n)}{n^s},$$

where $\chi(n)$ is a 'character to modulus k',[1] and in particular to the function (1.21). He also proved that there is a zero of $\Xi(t)$ between T and $T^{1+\delta}$, for all positive values of δ and all sufficiently large values of T. From this it follows that the number $N_0(T)$ of zeros between 1 and T is of the form $\Omega(\log\log T)$.[2]

The original proof given by HARDY made use of two parameters α and p; and our first idea, for obtaining a more precise result, was to treat α and p as functions of one another. The result indicated by our investigations was that of the existence of a zero between T and $T+T^{\frac{1}{2}+\delta}$ for any positive δ and all sufficiently large values of T. This would prove that

$$N_0(T) = \Omega\left(T^{\frac{1}{2}-\delta}\right).$$

But this proof has never been completed, as we are now able to prove, by an entirely different method, that there is a zero between T and $T^{\frac{1}{4}+\delta}$ for any positive δ and all sufficiently large values of T. This shows that

(1.41) $$N_0(T) = \Omega\left(T^{\frac{3}{4}-\delta}\right).$$

Our proof of this result is now free from any reference either to the CAHEN-MELLIN integral or to the theory of elliptic functions.

We have entertained hopes of showing, by a modification of our argument, that

$$N_0(T) = \Omega(T^{1-\delta}).$$

But our attempts in this direction have so far been unsuccessful.

1.5. Finally, Section 5 contains a full demonstration of a result given still more recently, with an outline of the proof, by LITTLEWOOD.[3] It follows from the investigations of SCHMIDT, already referred to in 1.1, that the inequalities (1.143), or the substantially equivalent inequalities

[1] See LANDAU, *Handbuch*, pp. 401 *et seq.*

[2] For an explanation of this notation see our paper 'Some Problems of Diophantine Approximation (II)', *Acta Mathematica*, vol. 37, pp. 193—238 (p. 225).

[3] *Comptes Rendus*, 22 June 1914.

(1.51) $$\Pi(x) - Li\,x + \frac{1}{2} Li\sqrt{x} < -K\frac{\sqrt{x}}{\log x},\quad \Pi(x) - Li\,x + \frac{1}{2} Li\sqrt{x} > K\frac{\sqrt{x}}{\log x},$$

are each satisfied by values of x which surpass all limit. It is shown here that these last inequalities may be replaced by

(1.52) $$\Pi(x) - Li\,x < -K\frac{\sqrt{x}\log\log\log x}{\log x},\quad \Pi(x) - Li\,x > K\frac{\sqrt{x}\log\log\log x}{\log x}.$$

From the second of these inequalities it follows, in particular, that the relation

(1.53) $$\Pi(x) < Li\,x,$$

which has been regarded, for empirical reasons, as probably true, is certainly false.

The supposed inequality (1.53) is, as has been shown by GAUSS, GOLDSCHMIDT, GRAM, PHRAGMÉN and MEISSEL,[1] supported by evidence drawn from the distribution of the prime numbers less than 1,000,000,000. The difference $\Pi(x) - Li\,x$ contains (to put the matter roughly) a term $-\frac{1}{2}Li\sqrt{x}$ and an oscillating term of order not less than $\frac{\sqrt{x}\log\log\log x}{\log x}$, which is of course of higher order than the former term. But the increase of $\log\log\log x$ is exceedingly slow; thus

$$\log\log\log 10{,}000{,}000{,}000 = 1\cdot 143\cdots;$$

and it is not surprising, therefore, that the term of constant sign should exert a preponderating influence throughout the limits within which calculation is feasible.

The question arises as to whether the function $\log\log\log x$ can be replaced by any more rapidly increasing function. The method which we use, depending as it does on KRONECKER's theorems concerning Diophantine Approximation, has a certain analogy with that by which BOHR proved that $\zeta(1+ti)$ is not bounded for $t>1$.[2] In that case the conclusion is that $\zeta(1+ti)$ is sometimes of order as great as $\log\log t$; and LITTLEWOOD[3] has shown that (on the RIEMANN hypothesis)

$$\zeta(1+ti) = O(\log\log t \log\log\log t)$$

[1] See the references in LANDAU's bibliography, and LEHMER's *List of prime numbers from 1 to 10,006,721* (Washington, 1914).

[2] BOHR and LANDAU, *Göttinger Nachrichten*, 1910, pp. 303—330.

[3] *Comptes Rendus*, 29 Jan. 1912.

so that the conclusion is certainly very nearly the best possible of its kind. It is quite possible that this may be true also of the inequalities (1.52); but we are naturally not prepared to express any very definite opinion on the point. It may be remarked in this connexion that BOHR and LANDAU[1] have shown that, on the RIEMANN hypothesis, the true maximum order of

$$\frac{\zeta'(1+ti)}{\zeta(1+ti)}$$

is exactly $\log\log t$.

The method used in this section is capable of application to other important problems. It may be used, for example, to show that if

$$\psi_1(x) = \sum_{p^m \leq x} (-1)^{\frac{m(p-1)}{2}} \log p$$

then sequences of values of x exist for which $\psi_1(x)$ tends either to ∞ or to $-\infty$, and indeed as rapidly as

$$\sqrt{x} \log\log\log x;$$

and that, if $\Pi_1(x)$ denotes the excess of primes not greater than x and of the form $4n+3$ over those not greater than x and of the form $4n+1$, then sequences of x exist for which $\Pi_1(x)$ tends either to ∞ or to $-\infty$, and indeed as rapidly as

$$\frac{\sqrt{x} \log\log\log x}{\log x}.$$

This result is of particular interest when considered in connection with those of 2.3. It is known that (to put the matter roughly) the distribution of primes $4n+3$ is *in some senses* denser than that of primes $4n+1$. Our results confirm and elucidate this vague statement, and show in what senses it is true and in what senses false.[2]

2.

Some applications of the integral of Cahen and Mellin.

2.1.

The prime number theorem and allied theorems.

2.11. The investigations of this part of the paper will be based upon certain known results which we state in the form of lemmas.

[1] *Math. Annalen*, vol. 74, 1913, pp. 3—30.

[2] Compare LANDAU, *Math. Annalen*, vol. 61, 1905, pp. 527—550.

Lemma 2.111. If $\varkappa > 0$, $\Re(y) > 0$, and y^{-s} has its principal value, then

$$e^{-y} = \frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty} \Gamma(s) y^{-s} ds.$$

This is the CAHEN-MELLIN integral.

Lemma 2.112. If (i) *$F(\sigma + ti)$ is a continuous function of the real variable t and* (ii) *the integral*

$$\int_{-\infty}^{\infty} |F(\sigma + ti)|\, dt$$

is convergent, then

$$\int_{-\infty}^{\infty} x^{ti} F(\sigma + ti)\, dt \to 0$$

as $x \to 0$ or $x \to \infty$.

This result is due to WEYL; it is a generalized form of a theorem of LANDAU.[1]

Lemma 2.113. Let a be a positive number (or zero), and (λ_n) an increasing sequence such that $\lambda_n \to \infty$, $\frac{\lambda_n}{\lambda_{n-1}} \to 1$; and suppose that

(i) *a_n is real and satisfies one or other of the inequalities*

$$a_n > -K\lambda_n^{a-1}(\lambda_n - \lambda_{n-1}),\ a_n < K\lambda_n^{a-1}(\lambda_n - \lambda_{n-1}),$$

or is complex and of the form

$$O\{\lambda_n^{a-1}(\lambda_n - \lambda_{n-1})\};$$

(ii) *the series*

$$f(y) = \sum a_n e^{-\lambda_n y}$$

is convergent for $y > 0$, and

$$f(y) \sim A y^{-a}$$

as $y \to 0$. Then

$$A_n = a_1 + a_2 + \cdots\cdots + a_n \sim \frac{A\lambda_n^a}{\Gamma(1+a)}$$

as $n \to \infty$.

[1] See LANDAU, *Prace Matematyczno-Fizyczne*, vol. 21, p. 170.

This lemma is equivalent to Theorems D, E, and F of our paper 'Some theorems concerning DIRICHLET's series', recently published in the *Messenger of Mathematics*.[1]

2.12. **Theorem 2.12.** *Suppose that*

(i) *the series* $\Sigma a_n \lambda_n^{-s}$ *is absolutely convergent for* $\sigma > \sigma_0 > 0$,

(ii) *the function* $F(s)$ *defined by the series is regular for* $\sigma > c$, *where* $0 < c \leq \sigma_0$, *and continuous for* $\sigma \geq c$,

(iii) $$F(s) = O(e^{C|t|}),$$

where $C < \frac{1}{2}\pi$, *uniformly for* $\sigma \geq c$. *Then the series*

$$f(y) = \sum a_n e^{-\lambda_n y}$$

is convergent for all positive values of y, *and*

$$f(y) = o(y^{-c})$$

as $y \to 0$.

We have

$$e^{-\lambda_n y} = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \Gamma(s)(\lambda_n y)^{-s} ds \tag{2.121}$$

if $y > 0$, $x > 0$, and so

$$f(y) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \Gamma(s) y^{-s} F(s) ds \tag{2.122}$$

[1] Vol. 43, 1914, pp. 134—147. If a_n satisfies the second form of condition (i), the series $f(y)$ is necessarily convergent (absolutely) for $y > 0$, so that the first clause of condition (ii) is then unnecessary.

There are more general forms of this theorem, involving functions such as

$$y^{-a}\left\{\log\left(\frac{1}{y}\right)\right\}^{a_1}\left\{\log\log\left(\frac{1}{y}\right)\right\}^{a_2}\cdots\cdots,$$

which we have not troubled to work out in detail.

The relation $f(y) \backsim Ay^{-a}$ in condition (ii) must be interpreted, in the special case when $A = 0$, as meaning $f(y) = o(y^{-a})$; and a corresponding change must be made in the conclusion.

if $y > 0$, $x > \sigma_0$, the term by term integration presenting no difficulty. In virtue of the conditions (ii) and (iii) we may replace (2.122) by

$$f(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s) y^{-s} F(s) ds.^1 \tag{2.1221}$$

The result of the theorem now follows at once from Lemma *2.112*.

Theorem 2.121. *If the conditions* (i), (ii), *and* (iii) *of Theorem* **2.12** *are satisfied,* (iv) $\frac{\lambda_n}{\lambda_{n-1}} \to 1$, *and* (v) a_n *is real, and satisfies one or other of the inequalities*

$$a_n > -K\lambda_n^{c-1}(\lambda_n - \lambda_{n-1}),\quad a_n < K\lambda_n^{c-1}(\lambda_n - \lambda_{n-1}),$$

or is complex and of the form

$$O\{\lambda_n^{c-1}(\lambda_n - \lambda_{n-1})\};$$

then

$$A_n = a_1 + a_2 + \cdots\cdots + a_n = o(\lambda_n^c).$$

This theorem is obviously a direct corollary of Theorem **2.12** and Lemma *2.113*.

Suppose in particular that $\lambda_n = n$, $a_n = \mu(n)$, and $c = 1$. Then

$$F(s) = \sum \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

and all the conditions of Theorem **2.121** are satisfied. Hence we obtain the well-known formula

$$\sum_{\nu \leq n} \mu(\nu) = o(n),$$

which is (1.122).

[1] The argument is so much like that of LANDAU (*Prace Matematyczno-Fizyczne*, vol. 21, pp. 173 *et seq.*) that it is hardly worth while to set it out in detail. We apply CAUCHY's Theorem to the rectangle

$$c - iT,\ x - iT,\ x + iT,\ c + iT,$$

and then suppose that $T \to \infty$.

2.13. The theoreme of 2.12 do not furnish a direct proof of (1.121) or (1.123). In order to obtain such a proof of (1.123) we must frame analogues of Theorems **2.12** and **2.121** which are applicable when $c = 0$.

Theorem 2.13. *Suppose that* (i) *the conditions* (i), (ii), *and* (iii) *of Theorem* **2.12**, *and the conditions* (iv) *and* (v) *of Theorem* **2.121**, *are satisfied, with* $c = 0$; (ii) *that the function* $F(s)$ *is regular for* $s = 0$. *Then the series* Σa_n *is convergent and has the sum* $F(0)$.

The proof differs but slightly from that of Theorem **2.121**. Instead of (2.1221) we have the equation

$$f(y) = F(0) + \frac{1}{2\pi i}\int \Gamma(s) y^{-s} F(s)\, ds, \tag{2.131}$$

where the path of integration consists of (a) the imaginary axis from $-i\infty$ to $-i\delta$, (b) a semicircle described to the left on the segment of the axis from $-i\delta$ to $i\delta$, and (c) the axis from $i\delta$ to $i\infty$. That the rectilinear part of the integral tends to zero follows substantially as before. Also

$$\int_\gamma \Gamma(s) y^{-s} F(s)\, ds = \frac{y^{-i\delta}\Gamma(i\delta)F(i\delta) - y^{i\delta}\Gamma(-i\delta)F(-i\delta)}{\log\left(\frac{1}{y}\right)}$$

$$- \frac{1}{\log\left(\frac{1}{y}\right)}\int_\gamma y^{-s}\frac{d}{ds}\{\Gamma(s)F(s)\}\, ds = O\left\{\frac{1}{\log\left(\frac{1}{y}\right)}\right\} = o(1).$$

Thus $f(y) \to F(0)$ as $y \to 0$, and so, by Lemma *2.113*, $\Sigma a_n = F(0)$.

The conditions of the theorem are satisfied, for example, when

$$\lambda_n = n,\ a_n = \frac{\mu(n)}{n},\ c = 0,\ F(s) = \frac{1}{\zeta(s+1)}.$$

Hence the equation (1.123) follows as a corollary.

2.14. In order to obtain the equation (1.121), and so the prime number theorem, we require a slightly different modification of Theorem **2.121**.

Theorem 2.14. *Suppose that the conditions of Theorems* **2.12** *and* **2.121** *are satisfied, except that* $F(s)$ *has a simple pole at the point* $s = c$, *and that the residue at the pole is* g. *Then*

$$A_n = a_1 + a_2 + \cdots + a_n \sim \frac{g\lambda_n^c}{c}.$$

The formula (2.131) is in this case replaced by

$$f(y) = g\Gamma(c)y^{-c} + \frac{1}{2\pi i}\int \Gamma(s) y^{-s} F(s)\, ds, \tag{2.141}$$

where the path of integration is of a kind similar to that used in the preceding proof. Practically the same argument gives the result

$$f(y) \sim g\Gamma(c)y^{-c},$$

and from this, and Lemma *2.113*, the theorem follows at once.

If we take

$$\lambda_n = n,\ a_n = \Lambda(n),\ c = 1,\ F(s) = -\frac{\zeta'(s)}{\zeta(s)},$$

we obtain (1.121).

2.15. We add some further remarks in connection with these theorems.

(i) Theorem **2.14** may be regarded as a generalisation of a theorem of Landau,[1] to which it reduces if we suppose that $a_n \geq 0$, that $F(s)$ is regular on the line $\sigma = c$, and that the equation $F(s) = O(e^{C|t|})$ is replaced by $F(s) = O(|t|^K)$.

In his more recent paper already referred to[2] Landau generalizes the second of these hypotheses in the case in which the series for $F(s)$ is an ordinary Dirichlet's series, showing that it is enough to suppose that

$$\lim_{\sigma \to c+0}\left\{F(\sigma + ti) - \frac{g}{\sigma + ti - c}\right\}$$

should exist, uniformly in any finite interval of values of t. This hypothesis is more general than ours, and our result is naturally capable of a corresponding generalization, which may be effected without difficulty by any one who compares Landau's argument and ours.

(ii) Theorem **2.121** breaks down when the increase of λ_n is too rapid, for example when $\lambda_n = e^n$. It is interesting to observe that in this last case the result is still true but is an obvious corollary of familiar theorems. The series

[1] *Handbuch*, p. 874.
[2] *l. c.* pp. 128, 130 (pp. 173 *et seq.*).

$F(s)$ is now a power-series in e^{-s}; condition (iii) is satisfied *ipso facto;* and the continuity of $F(s)$ for $\sigma > c$ involves

$$a_n e^{-nc} = o(1),\ a_n = o(e^{nc}),\ s_n = o(e^{nc})$$

(iii) It is a natural conjecture that the occurrence, in Theorems **2.121**, etc., of the condition $C < \frac{1}{2}\pi$ (which seems somewhat artificial), is due merely to some limitation of the method of proof employed. It is easy to show, by modifying our argument a little, that this is so.

Theorem 2.15. *In Theorems* **2.121**, **2.13**, *and* **2.14**, *it is unnecessary to suppose that* $C < \frac{1}{2}\pi$.

Choose a so that $\frac{1}{2}\pi a > C$. Then we have instead of equations (2.121), etc.,

(2.151)
$$\frac{1}{a} e^{-(\lambda_n y)^{1/a}} = \frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty} \Gamma(as)(\lambda_n y)^{-s} ds,$$

(2.152)
$$f(y) = \frac{1}{a}\sum a_n e^{-(\lambda_n y)^{1/a}} = \frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty} \Gamma(as) y^{-s} F(s) ds =$$
$$= \frac{1}{2\pi i}\int_{c - i\infty}^{c + i\infty} \Gamma(as) y^{-s} F(s) ds,$$

(2.153)
$$f(y) = o(y^{-c});$$

or, if $y^{1/a} = \eta$ and $\lambda_n^{1/a} = \mu_n$,

(2.154)
$$\Phi(\eta) = \frac{1}{a}\sum a_n e^{-\mu_n \eta} = o(\eta^{-ac}).$$

Now

$$\mu_n^{ac-1}(\mu_n - \mu_{n-1}) = \frac{1}{a}\lambda^{c-\frac{1}{a}}(\lambda_n - \lambda_{n-1})\Lambda^{\frac{1}{a}-1}$$

where $\lambda_{n-1} < \Lambda < \lambda_n$. Thus the ratio

$$\frac{\mu_n^{ac-1}(\mu_n - \mu_{n-1})}{\lambda_n^{c-1}(\lambda_n - \lambda_{n-1})}$$

lies between fixed positive limits. Thus (*e. g.*) $a_n = O\{\lambda_n^{c-1}(\lambda_n - \lambda_{n-1})\}$ implies $a_n = O\{\mu_n^{ac-1}(\mu_n - \mu_{n-1})\}$. Hence we can deduce from (2.154) that

(2.155)
$$A_n = o(\mu_n^{ac}) = o(\lambda_n^c).$$

It follows that the truth of Theorem **2.121** is independent of the condition in question; and similar arguments apply to the later theorems.

2.2.

The function $\sum\{\Lambda(n) - 1\}e^{-ny}$.

2.21. If $\Re(y) > 0$ and $\varkappa > 1$, we have

(2.211)
$$f(y) = \sum \Lambda(n) e^{-ny} = -\frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty} \Gamma(s) y^{-s} \frac{\zeta'(s)}{\zeta(s)} ds.$$

Let $q = -m - \frac{1}{2}$, where m is a positive integer; and let us apply CAUCHY's Theorem to the integral

$$\int \Gamma(s) y^{-s} \frac{\zeta'(s)}{\zeta(s)} ds,$$

taking the contour of integration to be the rectangle

$$(q - iT,\ \varkappa - iT,\ \varkappa + iT,\ q + iT),$$

T having such a value that no zero of $\zeta(s)$ lies on the contour. When we make T tend to infinity, we obtain the formula

(2.212)
$$f(y) = -\frac{1}{2\pi i}\int_{q - i\infty}^{q + i\infty} \Gamma(s) y^{-s} \frac{\zeta'(s)}{\zeta(s)} ds - \sum R,$$

where R denotes a residue at a pole inside the contour of integration.[1]

---

[1] The passage from (2.211) to (2.212) requires in reality a difficult and delicate discussion. If we suppress this part of the proof, it is because no arguments are required which involve the slightest novelty of idea. All the materials for the proof are to be found in LANDAU's *Handbuch* (pp. 333—368). But the problem treated there is considerably more difficult than

If now $m \to \infty$, $q \to -\infty$, it is easy to prove that the integral in (2.212) tends to zero. For in the first place

$$\frac{\zeta'(s)}{\zeta(s)} = O(\log |t|)$$

uniformly for $\sigma \leq -1$.[1] On the other hand, if $y = re^{i\theta}$, where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, we have

$$y^{m+\frac{1}{2}-ti}\Gamma\left(-m-\frac{1}{2}+ti\right) = \frac{\Gamma\left(\frac{1}{2}+ti\right)}{\left(-m-\frac{1}{2}+ti\right)\cdots\cdots\left(-\frac{1}{2}+ti\right)} e^{\left(m+\frac{1}{2}-ti\right)(\log r + i\theta)} =$$

$$= O\left\{\frac{|y|^m}{m!} e^{-\left(\frac{1}{2}\pi-\theta\right)|t|}\right\},$$

$$\left|\int_{q-i\infty}^{q+i\infty} \Gamma(s) y^{-s} \frac{\zeta'(s)}{\zeta(s)} ds\right| = O\left\{\frac{|y|^m}{m!}\int_{-\infty}^{\infty} e^{-\left(\frac{1}{2}\pi-\theta\right)|t|} \log |t|\, dt\right\} \to 0.$$

Hence

(2.213) $$f(y) = -\sum R$$

where the summation now applies to all the poles of the subject of integration. These poles are

(i) a simple pole at $s = 1$, with residue $-\frac{1}{y}$;

(ii) a simple pole at $s = 0$, with residue $\frac{\zeta'(0)}{\zeta(0)}$;

(iii) simple poles at the points $s = \varrho$, the residue at $s = \varrho$ being $\Gamma(\varrho) y^{-\varrho}$;

(iv) simple poles at the points $s = -1, -3, -5, \cdots$, the residue at $s = -2p-1$ being

$$-\frac{y^{2p+1}}{(2p+1)!} \frac{\zeta'(-2p-1)}{\zeta(-2p-1)};$$

this one, inasmuch as the integrals and series dealt with are not absolutely convergent. Here everything is absolutely convergent, since $|\Gamma(\sigma+ti) y^{\sigma+ti}|$, where $\Re(y) > 0$, tends to zero like an exponential when $t \to \infty$.

[1] Landau, *Handbuch*, p. 336.

(v) double poles at the points $s = -2, -4, -6, \cdots\cdots$, the residue at $s = -2p$ being

$$\frac{y^{2p}}{(2p)!}\left\{\log\left(\frac{1}{y}\right) + 1 + \frac{1}{2} + \cdots + \frac{1}{2p} - A + \frac{1}{2}\frac{\zeta''(-2p)}{\zeta'(-2p)}\right\},$$

where A is Euler's constant.

Thus finally

(2.214) $$f(y) = \frac{1}{y} - \sum \Gamma(\varrho) y^{-\varrho} + \Phi(y),$$

where

(2.2141) $$\Phi(y) = \Phi_1(y) + y^2 \log\left(\frac{1}{y}\right)\Phi_2(y),$$

and $\Phi_1(y)$ and $\Phi_2(y)$ are integral functions of y.

2.22. On the other hand we have

$$\frac{1}{e^y - 1} = \sum e^{-ny} = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \Gamma(s) y^{-s} \zeta(s)\, ds$$

$$= \frac{1}{2\pi i}\int_{q-i\infty}^{q+i\infty} \Gamma(s) y^{-s}\zeta(s)\, ds + \frac{1}{y} + \sum_0^m \frac{(-y)^n}{n!}\zeta(-n).$$

The integral on the right hand side tends to zero, when $m \to \infty$, if $|y| < 2\pi$. For

$$\Gamma(s)\zeta(s) = \frac{1}{2}(2\pi)^s \sec\frac{1}{2}s\pi\, \zeta(1-s) = O\left\{(2\pi)^{-m} e^{-\frac{1}{2}\pi|t|}\right\},$$

and so

$$\int_{q-i\infty}^{q+i\infty} \Gamma(s) y^{-s}\zeta(s)\, ds = O\left\{\left(\frac{|y|}{2\pi}\right)^m \int_{-\infty}^{\infty} e^{-\left(\frac{1}{2}\pi-\theta\right)|t|} dt\right\}.$$

Thus

(2.221) $$\sum e^{-ny} = \frac{1}{y} + \sum_0^\infty \frac{(-y)^n}{n!}\zeta(-n).$$[1]

[1] This is merely another form of the ordinary formula which defines Bernoulli's numbers. That

$$\sum e^{-ny} = \frac{1}{y} + \phi(y),$$

where $\phi(y)$ is a power-series convergent for $|y| < 2\pi$, is of course evident.

Subtracting (2.221) from (2.214) we obtain

(2.222) $$F(y)=\sum(\Lambda(n)-1)e^{-ny}=-\sum\Gamma(\varrho)y^{-\varrho}+\Psi(y),$$

where

(2.2221) $$\Psi(y)=\Psi_1(y)+y^2\log\left(\frac{1}{y}\right)\Psi_2(y),$$

and $\Psi_1(y)$ and $\Psi_2(y)$ are power-series convergent for $|y|<2\pi$.

2.23. We shall now assume the truth of the Riemann hypothesis, and apply the formula (2.222) to the study of $F(y)$ when $y\to 0$ by positive values. We denote the complex zeros of $\zeta(s)$ whose imaginary part is positive by $\frac{1}{2}+i\gamma_1, \frac{1}{2}+i\gamma_2, \cdots\cdots$, where $\gamma_1\leq\gamma_2\leq\cdots\cdot$. It is known[1] that

$$\gamma_1=14\cdot 1\cdots,\ \gamma_2=21\cdot 0\cdots,\ \gamma_3=25\cdot 0\cdots\cdot.$$

We shall require some definite upper limit for

$$N(T+1)-N(T)$$

where $N(T)$ is the number of zeros for which $T<\gamma\leq T+1$. It is well-known that $N(T+1)-N(T)=O(\log T)$, and it is easy to replace this relation by a numerical inequality, such as

(2.2311) $$N(T+1)-N(T)<2\cdot 5\log T;$$

all that is necessary is to introduce numerical values for the constants in the argument given by Landau.[2] In order to prove the relation (2.2311), however, comparatively careful numerical calculations are needed; and a much cruder inequality is sufficient for our purpose. We shall use the inequality

(2.2312) $$N(T+1)-N(T)<2T,$$

in the proof of which only the roughest approximations are necessary.

[1] Gram, *l. c.*

[2] *Handbuch*, pp. 337 *et seq.* It is known that, on the Riemann hypothesis,

$$N(T+1)-N(T)\sim\frac{\log T}{2\pi}$$

(Bohr, Landau, Littlewood, *Bulletins de l'Académie Royale de Belgique*, 1913, no. 12, pp. 1—35).

We now write

(2.232) $$\sum\Gamma(\varrho)y^{-\varrho}=\frac{1}{\sqrt{y}}(u_1+u_1'+R),$$

where

$$u_1=\Gamma\left(\frac{1}{2}+i\gamma_1\right)y^{-i\gamma_1},\ u_1'=\Gamma\left(\frac{1}{2}-i\gamma_1\right)y^{i\gamma_1},\ |u_1|=|u_1'|=\sqrt{\frac{\pi}{\cosh\gamma_1\pi}}.$$

Then

$$\frac{|R|}{|u_1|}<2\sum_2^\infty\sqrt{\frac{\cosh\gamma_1\pi}{\cosh\gamma_\nu\pi}}<4e^{\frac{1}{2}\gamma_1\pi}\sum_2^\infty e^{-\frac{1}{2}\gamma_\nu\pi}$$

$$=4e^{\frac{1}{2}\gamma_1\pi}\sum_{r=21}^\infty\sum_{r<\gamma\leq r+1}e^{-\frac{1}{2}\gamma\pi}$$

$$<8e^{\frac{1}{2}\gamma_1\pi}\sum_{21}^\infty re^{-\frac{1}{2}r\pi}$$

(2.233) $$<240e^{-3\cdot 4\pi}<\frac{1}{50}.$$

2.24. From (2.222), (2.232) and (2.233) we can at once deduce

Theorem 2.24. *Suppose that $y\to 0$ by positive values. Further, suppose the* Riemann *hypothesis true. Then*

$$F(y)=\sum(\Lambda(n)-1)e^{-ny}=O\sqrt{\frac{1}{y}};$$

and there is a constant K such that each of the inequalities

$$F(y)<-\frac{K}{\sqrt{y}},\ F(y)>\frac{K}{\sqrt{y}}$$

is satisfied for an infinity of values of y tending to zero.

We can express this by writing[1]

(2.241) $$F(y)=O\sqrt{\frac{1}{y}},\ F(y)=\Omega_L\sqrt{\frac{1}{y}},\ F(y)=\Omega_R\sqrt{\frac{1}{y}}.$$

From the second assertion in Theorem **2.24** we can of course deduce as a corollary

[1] In our paper 'Some Problems of Diophantine Approximation', *Acta Mathematica*, vol. 37, p. 225, we defined $f=\Omega(\varphi)$ as meaning $f\neq o(\varphi)$. The notation adopted here is a natural extension.

Theorem 2.241. *There is a constant K such that each of the inequalities*

$$\psi(x) - x < -K\sqrt{x},\ \psi(x) - x > K\sqrt{x}$$

is satisfied for values of x surpassing all limit; that is to say

$$\psi(x) - x = \Omega_L(\sqrt{x}),\ \psi(x) - x = \Omega_R(\sqrt{x}).$$

This is substantially the well-known result of SCHMIDT. In Section 5 we shall show that it is possible to prove more.

It is known that, if the RIEMANN hypothesis is false, then more is true than is asserted by Theorem 2.241. In fact, if Θ is the upper limit of the real parts of the zeros of $\zeta(s)$, and δ is any positive number, then[1]

$$\psi(x) - x = \Omega_L(x^{\Theta-\delta}),\ \psi(x) - x = \Omega_R(x^{\Theta-\delta}).$$

It seems to be highly probable that in these circumstances we have also

$$F(y) = \Omega_L(y^{-\Theta+\delta}),\ F(y) = \Omega_R(y^{-\Theta+\delta}),$$

but we have not been able to find a rigorous proof.

2.25. The equations (2.241) show that, if the RIEMANN hypothesis is true, the function $F(y)$ behaves, as $y \to 0$, precisely as might be expected, that is to say with as much regularity as is consistent with the existence of the complex zeroes of $\zeta(s)$. The results which will be proved in Section 5 will show that this is not the case with the corresponding 'sum-function' $\psi(x) - x$. It might reasonably be expected that

$$\psi(x) - x = O(\sqrt{x}),\ \psi(x) - x = \Omega_L(\sqrt{x}),\ \psi(x) - x = \Omega_R(\sqrt{x});$$

but the first of these equations is untrue. This being so, an interesting question arises as to the behaviour of the corresponding CESÀRO means formed from the series $\Sigma(\Lambda(n) - 1)$. The analogy of the theory of FOURIER's series suggests that they are likely to behave with as much regularity as the function $F(y)$; and this conjecture proves to be correct.

[1] SCHMIDT, *Math. Annalen*, vol. 57, 1903, pp. 195—204; see also LANDAU, *Handbuch*, pp. 712 *et seq.* The inequalities are stated by SCHMIDT and LANDAU in terms of $\Pi(x)$.

We shall consider not CESÀRO's means but the 'arithmetic means' introduced by MARCEL RIESZ.[1] It has been shown by RIESZ[2] that these means are in all substantial respects equivalent to CESÀRO's; and they have many formal advantages over the latter. If

$$\Lambda(n) = a_n,\ f(y) = \sum a_n e^{-ny},$$

then RIESZ's mean of order δ is

$$s^\delta(\omega) = \sum_{n \leq \omega} \left(1 - \frac{n}{\omega}\right)^\delta a_n.$$

And[3], if $x > 1$,

$$(2.251)\qquad s^\delta(\omega) = -\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{\Gamma(\delta+1)\Gamma(s)}{\Gamma(\delta+1+s)} \frac{\zeta'(s)}{\zeta(s)} \omega^s ds.$$

If we perform on this integral transformations similar to those of 2.21,[4] we are led to the formula

$$(2.252)\qquad s^\delta(\omega) = \frac{\omega}{\delta+1} - \sum \frac{\Gamma(\delta+1)\Gamma(\varrho)}{\Gamma(\delta+1+\varrho)} \omega^\varrho + S\left(\frac{1}{\omega}\right),$$

where $S\left(\frac{1}{\omega}\right)$ is in general a power-series[5] in $\frac{1}{\omega}$ convergent for $\omega > 1$.

Similarly, if

$$1 = b_n,\ \frac{1}{e^y - 1} = \sum b_n e^{-ny},$$

and we denote RIESZ's mean of order δ, formed from the b's, by $t^\delta(\omega)$, we have

[1] M. RIESZ, *Comptes Rendus*, 5 July and 22 Nov. 1909.

[2] M. RIESZ, *Comptes Rendus*, 12 June 1911.

[3] This formula is a special case of a general formula, due to RIESZ and included as Theorem 40 in the Tract 'The general theory of Dirichlet's series' (*Cambridge Tracts in Mathematics*, no. 18, 1915) by G. H. HARDY and M. RIESZ.

[4] See 2.21 for our justification of the omission of the details of the proof. Here again the integrals which occur are absolutely convergent.

[5] If δ is an integer, then $S\left(\frac{1}{\omega}\right)$ is a finite series which may include logarithms. It is in any case without importance.

(2.253) $$t^{\delta}(\omega) = \frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty} \frac{\Gamma(\delta+1)\Gamma(s)}{\Gamma(\delta+1+s)}\zeta(s)\omega^{s}ds = \frac{\omega}{\delta+1} + T\left(\frac{1}{\omega}\right),$$

where $T\left(\frac{1}{\omega}\right)$ also is in general a power-series in $\frac{1}{\omega}$ convergent for $\omega > 1$. Finally, subtracting (2.253) from (2.232), we obtain

(2.254) $$\sum_{n<\omega}(\Lambda(n)-1)\left(1-\frac{n}{\omega}\right)^{\delta} = -\sum \frac{\Gamma(\delta+1)\Gamma(\varrho)}{\Gamma(\delta+1+\varrho)}\omega^{\varrho} + P\left(\frac{1}{\omega}\right),$$

where $P\left(\frac{1}{\omega}\right)$ is in general a power-series in $\frac{1}{\omega}$ convergent for $\omega > 1$. The series involving the ϱ's being absolutely convergent, it follows at once that the left hand side of (2.254) is (on the RIEMANN hypothesis) of the form $O(\sqrt{\omega})$. That it is of the forms $\Omega_L(\sqrt{\omega})$, $\Omega_R(\sqrt{\omega})$ requires no special proof; for this is a corollary of Theorem **2.24**. We have therefore

Theorem 2.25. *All* RIESZ'*s means (and so all* CESÀRO'*s means), formed from the series* $\Sigma\{\Lambda(n)-1\}$, *are, on the* RIEMANN *hypothesis, of the forms*

$$O(\sqrt{\omega}),\ \Omega_L(\sqrt{\omega}),\ \Omega_R(\sqrt{\omega}).$$

This theorem is in part deeper, in part less deep, than Theorem **2.24**. The O result of Theorem **2.24** is a corollary from that of Theorem **2.25**, and the Ω result of Theorem **2.25** a corollary from that of Theorem **2.24**, the deduction in each case being of an ordinary 'Abelian' type, *i. e.* of the kind used in the proofs of ABEL's fundamental theorem and its extensions.

2.3.

On an assertion of Tschebyschef.

2.31. It was asserted by TSCHEBYSCHEF[1] that the function

(2.311) $$F(y) = e^{-3y} - e^{-5y} + e^{-7y} + e^{-11y} - \cdots = \sum_{p>2}(-1)^{\frac{p+1}{2}}e^{-py}$$

tends to infinity as $y \to 0$.

[1] See 1.2.

We shall now prove that TSCHEBYSCHEF's assertion is correct if all the complex zeros of the function $L(s)$, defined for $\sigma > 0$ by the series $1^{-s} - 3^{-s} + 5^{-s} - \cdots$, have their real part equal to $\frac{1}{2}$.[1]

We have, if $\sigma > 0$,

$$L(s) = 1^{-s} - 3^{-s} + 5^{-s} - \cdots = \prod_{p>2}\left(\frac{1}{1-(-1)^{(p-1)/2}p^{-s}}\right),$$

$$\log L(s) = \sum_{p,m}\frac{(-1)^{m(p-1)/2}}{mp^{ms}},$$

$$-\frac{L'(s)}{L(s)} = \sum_{p,m}(-1)^{\frac{m(p-1)}{2}}\frac{\log p}{p^{ms}}.$$

Hence

(2.312) $$f(y) = \sum_{p,m}(-1)^{\frac{m(p-1)}{2}}\log p\ e^{-p^m y} = -\frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty}\Gamma(s)y^{-s}\frac{L'(s)}{L(s)}ds,$$

if $\varkappa > 1$.

We now transform this integral by CAUCHY's Theorem as in 2.21, and obtain the formula[2]

(2.313) $$f(y) = \sum\Gamma(\varrho)y^{-\varrho} + \Phi(y),$$

where ϱ is a complex zero of $L(s)$ and $\Phi(y)$ is a function of y of much the same form as the function $\Phi(y)$ of (2.214).[2]

2.32. We now require an upper limit for the sum $\Sigma|\Gamma(\varrho)|$. We could

[1] The evidence for the truth of this hypothesis is substantially the same as that for the truth of the RIEMANN hypothesis. LANDAU (*Math. Ann.*, vol. 76, 1915, pp. 212—243) has proved that there are infinitely many zeros on the line $\sigma = \frac{1}{2}$.

[2] The 'trivial' zeros of $L(s)$ are $s = -1, -3, -5, \cdots$: see LANDAU, *Handbuch*, p. 498.

$$\Phi(y) = \Phi_1(y) + y\log\left(\frac{1}{y}\right)\Phi_2(y).$$

obtain such a limit by an argument similar to that of 2.23: but it is simpler to proceed as follows.[1]

The function $L(s)$ satisfies the equation

$$L(1-s) = 2^s \pi^{-s} \Gamma(s) \sin \frac{1}{2} s\pi \, L(s).$$

We write

$$2^s \pi^{-\frac{1}{2}s} \Gamma\left(\frac{1+s}{2}\right) L(s) = \xi(s) = \xi\left(\frac{1}{2} + ti\right) = \Xi(t).$$

Then $\Xi(t)$ is real when t is real, and an even function of t. And if we write $\varrho = \frac{1}{2} + i\gamma$, then the zeros of $\Xi(t)$ are given by $t = \gamma$. We are supposing that all these zeros are real.

We have now

$$\Xi(t) = \Xi(0) \Pi\left(1 - \frac{t^2}{\gamma^2}\right),$$

(2.321)
$$\xi(s) = \Xi(0) \Pi\left(\frac{\frac{1}{4}+\gamma^2}{\gamma^2}\right) \Pi\left\{1 + \frac{s(s-1)}{\frac{1}{4}+\gamma^2}\right\},$$

where only the positive γ's occur in the products. Putting $s = 1$ we obtain

(2.322)
$$\Xi(0) \Pi\left(\frac{\frac{1}{4}+\gamma^2}{\gamma^2}\right) = \xi(1) = \frac{2L(1)}{\sqrt{\pi}} = \frac{1}{2}\sqrt{\pi},$$

and so

$$\Pi\left\{1 + \frac{s(s-1)}{\frac{1}{4}+\gamma^2}\right\} = \frac{2\xi(s)}{\sqrt{\pi}} = 2^{1+s} \pi^{-\frac{1}{2}(1+s)} \Gamma\left(\frac{1+s}{2}\right) L(s);$$

or, if $s = 1 + x$,

[1] Our argument is modelled on one applied to the Zeta-function by Jensen, *Comptes Rendus*, 25 april 1887.

(2.323)
$$\Pi\left\{1 + \frac{x(x+1)}{\frac{1}{4}+\gamma^2}\right\} = 2^{2+x} \pi^{-1-\frac{1}{2}x} \Gamma\left(1 + \frac{1}{2}x\right) L(1+x).$$

Finally, expanding each side of (2.323) in ascending powers of x, and equating the coefficients of x, we have

(2.324)
$$\sum \frac{1}{\frac{1}{4}+\gamma^2} = \log 2 - \frac{1}{2}\log \pi - \frac{1}{2} A + \frac{4}{\pi} L'(1),$$

where A is Euler's constant. From this it follows easily that, if γ_1 is the least of the positive γ's, then

(2.325)
$$\frac{1}{\frac{1}{4}+\gamma_1^2} < \sum \frac{1}{\frac{1}{4}+\gamma^2} < \cdot 1,$$

$$\gamma_1 > 3.^{1}$$

2.33. Now, as in 2.23, we have

$$\left|\Gamma\left(\frac{1}{2} + i\gamma\right)\right| = \sqrt{\frac{\pi}{\cosh \gamma\pi}},$$

and the ratio

$$\sqrt{\frac{\pi}{\cosh \gamma\pi}} : \frac{1}{\frac{1}{4}+\gamma^2}$$

decreases steadily as γ increases, for $\gamma > 3$. Moreover, the value of the ratio for $\gamma = 3$ is less than

$$25 e^{-\frac{3}{2}\pi} < \frac{1}{4}.$$

[1] It is in fact true that $\gamma_1 > 6$: see Grossmann, *Dissertation*, Göttingen, 1913.

Hence

$$\sum_{\gamma>0}\left|\Gamma\left(\frac{1}{2}+i\gamma\right)\right| < \frac{1}{4}\sum_{\gamma>0}\frac{1}{\frac{1}{4}+\gamma^2} < \frac{1}{40},$$

and so

(2.331) $$|\Sigma\Gamma(\varrho)y^{-\varrho}| < \frac{1}{20}\sqrt{\frac{1}{y}}.$$

If now we write

$$f(y) = f_1(y) + f_2(y) + f_3(y),$$

where $f_1(y)$ contains the terms of $f(y)$ for which $m = 1$, $f_2(y)$ those for which $m = 2$, and $f_3(y)$ the remainder, we have

(2.332) $$f_2(y) = \sum_p \log p\ e^{-p^2 y} \sim \frac{1}{2}\sqrt{\frac{\pi}{y}},$$

(2.333) $$f_3(y) = \sum_{p,\, m\geq 3}(-1)^{\frac{m(p-1)}{2}}\log p\ e^{-p^m y} = O\sqrt[3]{\frac{1}{y}}.$$

Hence, by (2.331), (2.232), and (2.333), we have

$$f_1(y) = \sum_p (-1)^{\frac{p-1}{2}}\log p\ e^{-py}$$

$$< -\frac{1}{4}\sqrt{\frac{\pi}{y}} + \frac{1}{20}\sqrt{\frac{1}{y}} + O\sqrt[3]{\frac{1}{y}}$$

$$< -\frac{1}{6}\sqrt{\frac{\pi}{y}},$$

for all sufficiently small values of y. We have thus proved

Theorem 2.33. *There is a constant K such that*

$$f_1(y) = \sum(-1)^{\frac{p-1}{2}}\log p\, e^{-py} < -K\sqrt{\frac{1}{y}}$$

for all sufficiently small positive values of y.

Suppose now that

(2.334) $$\varphi(y) = \sum a_n e^{-ny}$$

is a power-series in e^{-y}, convergent for $y > 0$, and that

$$\varphi(y) > Ky^{-\alpha}$$

for $0 < y \leq y_0$. Suppose also that $0 < s < \alpha$. Then

$$\sum n^{-s}a_n e^{-ny} = \frac{1}{\Gamma(s)}\int_0^\infty \varphi(t+y)t^{s-1}dt$$

$$= \frac{1}{\Gamma(s)}\int_0^{\frac{1}{2}y_0}\varphi(t+y)t^{s-1}dt + \frac{1}{\Gamma(s)}\int_{\frac{1}{2}y_0}^\infty \varphi(t+y)t^{s-1}dt = J_1 + J_2,$$

say. The second integral tends to a finite limit as $y \to 0$. If $0 < y \leq \frac{1}{2}y_0$, the first integral is greater than

$$\frac{K}{\Gamma(s)}\int_0^{\frac{1}{2}y_0}\frac{t^{s-1}dt}{(t+y)^\alpha} = \frac{Ky^{s-\alpha}}{\Gamma(s)}\int_0^{\frac{1}{2}y_0/y}\frac{u^{s-1}du}{(u+1)^\alpha} \sim \frac{K\Gamma(\alpha-s)}{\Gamma(\alpha)}y^{s-\alpha}.$$

Hence there is a constant H such that

$$\sum n^{-s}a_n e^{-ny} > Hy^{s-\alpha}$$

for all sufficiently small values of y. In particular we have

Theorem 2.331. *If $0 < s < \frac{1}{2}$, there is a constant H such that*

$$\sum(-1)^{\frac{p-1}{2}}\frac{\log p}{p^s}e^{-py} < -Hy^{s-\frac{1}{2}}$$

for all sufficiently small values of y.

2.34. In order to prove the actual assertion made by TSCHEBYSCHEF we have to introduce a convergence factor $\frac{1}{\log n}$ into the series (2.334).

It is not difficult to prove that

$$\frac{1}{\log n} = \int_0^\infty e^{-nt}\psi(t)\,dt,$$

where

$$\psi(t) = e^t + \int_0^\infty \frac{e^{-wt}\,dw}{\pi^2 + (\log w)^2};$$ [1]

so that

$$\sum_2^\infty \frac{a_n e^{-ny}}{\log n} = \int_0^\infty \psi(t)\varphi(t+y)\,dt$$ [2]

$$= \int_0^{\eta} \psi(t)\varphi(t+y)\,dt + \int_{\eta}^\infty \psi(t)\varphi(t+y)\,dt = J_1 + J_2,$$

say. As before, J_2 tends to a finite limit as $y \to 0$. It is moreover easy to see that

$$\psi(t) \sim \int_0^\infty \frac{e^{-wt}\,dw}{\pi^2 + (\log w)^2} \sim \int_0^{1/t} \frac{dw}{(\log w)^2} \sim \frac{1}{t\left(\log \frac{1}{t}\right)^2}$$

as $t \to 0$. We can therefore choose η so that, if $0 < y \leq \eta$,

[1] Cf. W. H. YOUNG, *Proc. London Math. Soc.*, ser. 2, vol. 12, pp. 41—70.

[2] We suppose that $a_0 = 0$, $a_1 = 0$, as evidently we may do without loss of generality.

$$J_1 > K\int_0^\eta \frac{1}{t\left(\log \frac{1}{t}\right)^2} \frac{dt}{(t+y)^\alpha}$$

$$= O(1) + \alpha K \int_0^\eta \frac{1}{\log \frac{1}{t}} \frac{dt}{(t+y)^{\alpha+1}}$$

$$= O(1) + \alpha K y^{-\alpha} \int_0^{\eta/y} \frac{1}{\log \frac{1}{u} + \log \frac{1}{y}} \frac{du}{(u+1)^{\alpha+1}}$$

$$> \frac{1}{2}\alpha K \frac{y^{-\alpha}}{\log \frac{1}{y}} \int_1^2 \frac{du}{(u+1)^{\alpha+1}}$$

$$> \frac{Hy^{-\alpha}}{\log \frac{1}{y}}.$$

Applying this result to Theorem 2.33, we obtain

Theorem 2.34. *There is a constant H such that*

$$F(y) = \sum (-1)^{\frac{p-1}{2}} e^{-py} < -\frac{H}{\sqrt{y}\log(1/y)}$$

for all sufficiently small positive values of y.

We have thus established the truth of TSCHEBYSCHEF's assertion, under the assumption of the truth of the analogue of the RIEMANN hypothesis. The nature of the proof makes it seem almost certain that the assertion must be false if the hypothesis is false, as the term $\frac{1}{2}\sqrt{\frac{\pi}{y}}$ of (2.332) must then be overwhelmed by oscillatory terms of higher order. But, as we explained in 1.2, we have been unable to find a rigorous proof.

2.35. We have proved that

$$\sum (-1)^{\frac{p-1}{2}} \log p\, e^{-py} \to -\infty$$

as $y \to 0$; and, when we remember the results of 2.25, we are naturally led to enquire whether a similar result holds for the Cesàro means formed from the series

$$\sum (-1)^{\frac{p-1}{2}} \log p.$$

If we denote Riesz's mean of order δ, formed from the series

$$\sum_{p,m} (-1)^{\frac{m(p-1)}{2}} \log p,$$

by $s^\delta(\omega)$, we have

(2. 351) $$s^\delta(\omega) = -\frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty} \frac{\Gamma(\delta+1)\Gamma(s)}{\Gamma(\delta+1+s)} \frac{L'(s)}{L(s)} \omega^s \, ds$$

$$= -\sum \frac{\Gamma(\delta+1)\Gamma(\varrho)}{\Gamma(\delta+1+\varrho)} \omega^\varrho + P\left(\frac{1}{\omega}\right),$$

where $P\left(\frac{1}{\omega}\right)$ is in general[1] a power-series convergent for $\omega > 1$.

From (2. 351) it follows at once that

(2. 352) $$s^\delta(\omega) = O(\sqrt{\omega}),$$

a result which says the more the smaller is δ.

Let us consider in particular the case in which $\delta = 1$. We have then

(2. 353) $$s^1(\omega) = -\sum \frac{\omega^\varrho}{\varrho(\varrho+1)} + P\left(\frac{1}{\omega}\right).$$

But

$$\left|\sum \frac{\omega^{i\gamma}}{\varrho(\varrho+1)}\right| \leq \sum \frac{1}{|\varrho(\varrho+1)|} < \sum \frac{1}{|\varrho|^2} = \sum \frac{1}{\frac{1}{4}+\gamma^2} < \frac{1}{5},$$

[1] See the footnote to p. 140.

by (2. 325). Hence

(2. 354) $$|s^1(\omega)| < \frac{1}{4}\sqrt{\omega}$$

for all sufficiently large values of ω.

Let us now write

(2. 355) $$s^1(\omega) = s_1^1(\omega) + s_2^1(\omega) + s_3^1(\omega),$$

where $s_1^1(\omega)$, $s_2^1(\omega)$, and $s_3^1(\omega)$ are formed respectively from the terms of the series for which $m = 1$, $m = 2$, and $m \geq 3$. Then

(2. 3561) $$s_2^1(\omega) = \sum_{p^2 < \omega} \log p \left(1 - \frac{p^2}{\omega}\right) > \frac{1}{2} \sum_{p^2 < \frac{1}{2}\omega} \log p > \frac{1}{3}\sqrt{\omega},$$

if ω is large enough. Also

(2. 3562) $$s_3^1(\omega) = \sum_{m \geq 3,\, p^m < \omega} (-1)^{\frac{m(p-1)}{2}} \log p \left(1 - \frac{p^m}{\omega}\right) = O\left(\sum_{m \geq 3,\, p^m < \omega} \log p\right) = O(\sqrt[3]{\omega}).$$

From (2. 354), (2. 355), (2. 3561), and (2. 3562) it follows that

(2. 3563) $$s_1^1(\omega) = \sum_{p < \omega} (-1)^{\frac{p-1}{2}} \log p \left(1 - \frac{p}{\omega}\right) < -\frac{1}{13}\sqrt{\omega}$$

for all sufficiently large values of ω.

We have thus proved

Theorem 2. 35. *Riesz's or Cesàro's mean of the first order, formed from the series*

$$\sum (-1)^{\frac{p-1}{2}} \log p,$$

tends to $-\infty$ as $\omega \to \infty$, at least as rapidly as a constant multiple of $-\sqrt{\omega}$.

From this we can deduce without difficulty

Theorem 2. 351. *The corresponding means, formed from the series*

$$\sum (-1)^{\frac{p-1}{2}}$$

tend to $-\infty$ *at least as rapidly as a constant multiple of* $-\frac{\sqrt{\omega}}{\log \omega}$.

Theorem **2. 33** is a corollary of theorem **2. 35**, and Theorem **2. 34** of Theorem **2. 351**: the deduction being in either case of the ordinary 'Abelian' type.

In concluding this sub-section we may repeat that, as has already been pointed out in 1. 5, the theorems here proved gain greatly in interest when considered in conjunction with those which may be established by the methods of Section 5.

2. 4.

The mean value of $\left|\zeta\left(\frac{1}{2}+it\right)\right|^2$

2. 41. LANDAU and SCHNEE[1] have shown that

$$\int_{-T}^{T} |\zeta(\beta+it)|^2 dt \sim 2\zeta(2\beta)T \tag{2. 411}$$

when $\beta > \frac{1}{2}$, and it is an easy deduction[2] that

$$\int_{-T}^{T} |\zeta(\beta+it)|^2 dt \sim (2\pi)^{2\beta-1}\zeta(2-2\beta)\frac{T^{2-2\beta}}{2-2\beta} \tag{2. 4111}$$

when $\beta < \frac{1}{2}$. We propose now to complete these results by proving

Theorem 2. 41. *We have*

$$\int_{-T}^{T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt \sim 2T\log T.$$

[1] See LANDAU, *Handbuch*, p. 816.

[2] Using the functional equation.

We shall require some preliminary lemmas. We write

$$K_0(y) = \int_1^\infty \frac{e^{-yu}\,du}{\sqrt{u^2-1}} \quad (\Re(y) > 0).$$

Lemma 2. 411. If $\varkappa$ *and the real part of* y *are positive, then*

$$H(y) = \frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty} \{\Gamma(s)y^{-s}\}^2 ds = 2K_0(2y),$$

y^{-s} *having its principal value.*

It is unnecessary to give the details of the proof of this formula which depends (like that of the 'CAHEN-MELLIN' formula) merely on a straightforward application of CAUCHY's Theorem.

Lemma 2. 412. If $y = re^{i\vartheta}$, *where* $|\vartheta| < \frac{3}{2}\pi - \delta < \frac{3}{2}\pi$, *and* $r \to \infty$, *then*

$$H(y) = e^{-2y}\sqrt{\frac{\pi}{y}}\left\{1 + O\left(\frac{1}{y}\right)\right\},$$

uniformly in ϑ.

This is a known result.[1]

Lemma 2. 413. If $f(x)$ *is positive and continuous, and*

$$f(x) = O(e^{\delta x})$$

for all positive values of δ; *and if*

$$\int_0^\infty f(x)e^{-\varepsilon x}dx \sim A\varepsilon^{-a}L\left(\frac{1}{\varepsilon}\right),$$

where $a > 0$ *and* $L(x)$ *is a finite product of logarithmic factors*

$$(\log x)^{a_1}(\log\log x)^{a_2}\ldots\ldots,$$

as $\varepsilon \to 0$; *then*

$$\int_0^T f(t)dt \sim \frac{AT^a L(T)}{\Gamma(1+a)}.$$

This is the analogue for integrals of a theorem first proved by us in the *Proc. London Math. Soc.*, ser. 2, vol. 13, pp. 180 *et seq.* This latter theorem redu-

[1] WHITTAKER and WATSON, *Modern Analysis*, ed. 2, pp. 367, 377.

ces, when $\alpha_1 = \alpha_2 = \ldots\ldots = 0$, to a special case of Lemma *2. 113*. The proofs for series and for integrals are in all important respects the same.

2. 42. If in Lemma *2. 411* we suppose $\varkappa > \frac{1}{2}$, write ny for y, multiply by $d(n)$, the number of divisors of n, and sum, we obtain

$$(2.\ 421)\qquad \frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty}\{\Gamma(s)\zeta(2s)y^{-s}\}^2 ds = \sum_1^\infty d(n)H(ny).$$

We now use Cauchy's Theorem to replace the integral by one taken along the line $\sigma = \frac{1}{4}$. There is a pole at $s = \frac{1}{2}$ of order 2, and the residue is

$$\frac{\pi}{2y}(A - \log y - 2\log 2),$$

where A is Euler's constant. Thus

$$(2.\ 422)\qquad \frac{1}{2\pi i}\int_{\frac14-i\infty}^{\frac14+i\infty}\{\Gamma(s)\zeta(2s)y^{-s}\}^2 ds = \sum_1^\infty d(n)H(ny) - \frac{\pi}{2y}(A-\log y-2\log 2) = S + S',$$

say. In this formula we write

$$\Gamma(s)\zeta(2s) = \frac{2}{2s(2s-1)}\pi^s\xi(2s),$$

$$\xi(2s) = \xi\left(\frac{1}{2}+2it\right) = \Xi(2t),$$

and we obtain

$$\frac{2}{\sqrt{\pi y}}\int_{-\infty}^{\infty}\left\{\frac{\Xi(2t)}{\frac14+4t^2}\right\}^2\left(\frac{\pi}{y}\right)^{2ti}dt = S + S'.$$

Finally we write[1]

$$y = \pi e^{i\alpha},$$

where $0 \leq \alpha < \frac{1}{2}\pi$, and we have

[1] These transformations are the same as those used by Hardy, *Comptes Rendus*, 6 April 1914.

$$(2.\ 423)\qquad \frac{2}{\pi}\int_{-\infty}^{\infty}\left\{\frac{\Xi(2t)}{\frac14+4t^2}\right\}^2 e^{2\alpha t}dt = e^{\frac12 i\alpha}(S+S'),$$

where

$$(2.\ 4241)\qquad S = \sum_1^\infty d(n)H(n\pi e^{i\alpha}),$$

$$(2.\ 4242)\qquad S' = -\frac{1}{2}e^{-i\alpha}(A - 2\log 2 - \log\pi - i\alpha).$$

2. 43. We now suppose that $\alpha = \frac{1}{2}\pi - \varepsilon$ and that $\varepsilon \to 0$. In the first place it is obvious that

$$(2.\ 431)\qquad S' = O(1).$$

Further, by Lemma *2. 412*, we hawe

$$H(n\pi e^{i\alpha}) = \frac{1}{\sqrt{n}}e^{-\frac12 i\alpha - 2n\pi(\cos\alpha+i\sin\alpha)} + O\left(\frac{e^{-2n\pi\cos\alpha}}{n^{3/2}}\right),$$

and it is plain that the contribution of the last term to S is of the form $O(1)$.

Hence we may write

$$(2.\ 432)\qquad \frac{2}{\pi}\int_{-\infty}^{\infty}\left\{\frac{\Xi(2t)}{\frac14+4t^2}\right\}^2 e^{2\alpha t}dt = \sum_1^\infty\frac{d(n)}{\sqrt{n}}e^{-2n\pi(\cos\alpha+i\sin\alpha)} + O(1).$$

But

$$\cos\alpha + i\sin\alpha = i + \varepsilon + O(\varepsilon^2),$$

$$e^{-2n\pi(\cos\alpha+i\sin\alpha)} = e^{-2n\pi\varepsilon+O(n\varepsilon^2)}$$

$$= e^{-2n\pi\varepsilon}\{1 + n\varepsilon^2 e^{O(n\varepsilon^2)}\},$$

and

$$\varepsilon^2\sum\sqrt{n}\,d(n)e^{-2n\pi\varepsilon+O(n\varepsilon^2)} = O\left\{\varepsilon^2\sum\sqrt{n}\,d(n)e^{-n\pi\varepsilon}\right\} = o(1).$$

We may therefore replace the series on the right hand side of (2. 432) by

$$\sum_1^\infty\frac{d(n)}{\sqrt{n}}e^{-2n\pi\varepsilon}.$$

But

$$\sum_1^{\nu} d(n) \sim \nu \log \nu,$$

$$\sum_1^{\nu} \frac{d(n)}{\sqrt{n}} \sim 2\sqrt{\nu} \log \nu;$$

and so

$$\sum_1^{\infty} \frac{d(n)}{\sqrt{n}} e^{-2n\pi\varepsilon} \sim \sqrt{\frac{1}{2\varepsilon}} \log\left(\frac{1}{\varepsilon}\right).$$

Henc we have

(2. 433) $$\int_{-\infty}^{\infty} \left\{\frac{\Xi(2t)}{\frac{1}{4}+4t^2}\right\}^2 e^{(\pi-2\varepsilon)t} dt \sim \frac{\pi}{2} \sqrt{\frac{1}{2\varepsilon}} \log\left(\frac{1}{\varepsilon}\right).$$

We may replace the lower limit by 0, since the part of the integral for which $t < 0$ is plainly of no importance. Doing this, and putting $2t = u$, we obtain

(2. 434) $$\int_0^{\infty} \left\{\frac{\Xi(u)}{\frac{1}{4}+u^2}\right\}^2 e^{\left(\frac{1}{2}\pi-\varepsilon\right)u} du \sim \pi \sqrt{\frac{1}{2\varepsilon}} \log\left(\frac{1}{\varepsilon}\right).$$

2. 44. It follows from (2. 434) and Lemma *2. 413*, that

(2. 441) $$\int_0^{T} \left\{\frac{\Xi(u)}{\frac{1}{4}+u^2}\right\}^2 e^{\frac{1}{2}\pi u} du \sim \sqrt{2\pi T} \log T.$$

But

$$\left\{\frac{\Xi(u)}{\frac{1}{4}+u^2}\right\}^2 \sim \sqrt{\frac{\pi}{2u}} e^{-\frac{1}{2}\pi u} \left|\zeta\left(\frac{1}{2}+iu\right)\right|^2,$$

so that

$$\int_0^T \left|\zeta\left(\frac{1}{2}+iu\right)\right|^2 \frac{du}{\sqrt{u}} \sim 2\sqrt{T} \log T.$$

And if we write

$$\Phi(u) = \int_0^u \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \frac{dt}{\sqrt{t}},$$

we have

$$\int_0^T \left|\zeta\left(\frac{1}{2}+iu\right)\right|^2 du = \int_0^T \sqrt{u}\, \Phi'(u) du$$

$$= \sqrt{T}\Phi(T) - \frac{1}{2}\int_0^T \frac{\Phi(u)}{\sqrt{u}} du$$

$$\sim 2T\log T - \int_0^T \log u\, du$$

$$\sim T \log T,$$

which is equivalent to the result of Theorem 2. 41.

2. 5.

The series

$$\sum \frac{\mu(n)}{n} e^{-(\alpha/n)^2}$$

and other similar series.

2. 51. In this sub-section we shall be concerned with some formulae which were suggested to us by some work of Mr S. Ramanujan. We have no satisfactory proof of the truth of the formulae, though this is highly probable; but they are so curious that it seems worth while to mention them.

If $-1 < \varkappa < 0$ and $\alpha > 0$, then

(2. 511) $$1 - e^{-(\alpha/n)^2} = -\frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty} \left(\frac{n}{\alpha}\right)^{2s} \Gamma(s) ds.$$

Hence

$$(2.512)\quad \sum \frac{\mu(n)}{n} e^{-(\alpha/n)^2} = -\sum_1^\infty \frac{\mu(n)}{n}\{1 - e^{-(\alpha/n)^2}\} = \frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty} \alpha^{-2s} \frac{\Gamma(s)}{\zeta(1-2s)}\,ds.$$

But

$$\frac{\Gamma(s)}{\zeta(1-2s)} = \pi^{2s-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}-s\right)}{\zeta(2s)};$$

and so

$$(2.513)\quad \sum_1^\infty \frac{\mu(n)}{n} e^{-(\alpha/n)^2} = \frac{1}{2i\pi\sqrt{\pi}} \int_{\varkappa - i\infty}^{\varkappa + i\infty} \left(\frac{\pi}{\alpha}\right)^{2s} \frac{\Gamma\left(\frac{1}{2}-s\right)}{\zeta(2s)}\,ds.$$

If now we assume that we may transform the last integral by moving the path of integration parallel to itself across the line $\sigma = \frac{1}{4}$, and introducing the obvious correction due to the poles of the subject of integration,[1] we obtain

$$(2.514)\quad \sum_1^\infty \frac{\mu(n)}{n} e^{-(\alpha/n)^2} = \frac{1}{2i\pi\sqrt{\pi}} \int_{\lambda - i\infty}^{\lambda + i\infty} \left(\frac{\pi}{\alpha}\right)^{2s} \frac{\Gamma\left(\frac{1}{2}-s\right)}{\zeta(2s)}\,ds - \frac{1}{2\sqrt{\pi}} \sum \left(\frac{\pi}{\alpha}\right)^{\varrho} \frac{\Gamma\left(\frac{1-\varrho}{2}\right)}{\zeta'(\varrho)},$$

where $\frac{1}{2} < \lambda < \frac{3}{2}$. This assumption of course includes that of the convergence of the series last written.

But

$$\frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \left(\frac{\pi}{\alpha}\right)^{2s} \frac{\Gamma\left(\frac{1}{2}-s\right)}{\zeta(2s)}\,ds = \frac{1}{2\pi i} \sum_1^\infty \mu(n) \int_{\lambda - i\infty}^{\lambda + i\infty} \left(\frac{\beta}{n}\right)^{2s} \Gamma\left(\frac{1}{2}-s\right) ds,$$

if $\alpha\beta = \pi$. Also, transforming the last integral by the substitution

$$\frac{1}{2} - s = S,$$

[1] In forming the series of residues we have assumed, for simplicity, that the poles are all simple.

we obtain

$$\frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \left(\frac{\beta}{n}\right)^{2s} \Gamma\left(\frac{1}{2}-s\right) ds = \frac{\beta}{2\pi i n} \int_{\varkappa - i\infty}^{\varkappa + i\infty} \left(\frac{\beta}{n}\right)^{-2S} \Gamma(S)\,dS,$$

where $-1 < \varkappa < 0$; and the last expression is equal to

$$\frac{\beta}{n} e^{-(\beta/n)^2}.$$

Hence

$$(2.515)\quad \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \left(\frac{\pi}{\alpha}\right)^{2s} \frac{\Gamma\left(\frac{1}{2}-s\right)}{\zeta(2s)}\,ds = \beta \sum_1^\infty \frac{\mu(n)}{n} e^{-(\beta/n)^2}.$$

Substituting in (2.514), and multiplying by $\sqrt{\alpha}$, we obtain

$$(2.516)\quad \sqrt{\alpha} \sum_1^\infty \frac{\mu(n)}{n} e^{-(\alpha/n)^2} - \sqrt{\beta} \sum_1^\infty \frac{\mu(n)}{n} e^{-(\beta/n)^2} = -\frac{1}{2\sqrt{\beta}} \sum \frac{\Gamma\left(\frac{1-\varrho}{2}\right)}{\zeta'(\varrho)} \beta^{\varrho}.$$

It follows from symmetry that we must have

$$\frac{1}{\sqrt{\alpha}} \sum \frac{\Gamma\left(\frac{1-\varrho}{2}\right)}{\zeta'(\varrho)} \alpha^{\varrho} + \frac{1}{\sqrt{\beta}} \sum \frac{\Gamma\left(\frac{1-\varrho}{2}\right)}{\zeta'(\varrho)} \beta^{\varrho} = 0,$$

and this relation may be verified without difficulty.

2.52. In order to obtain a satisfactory proof of (2.516), it would be enough to show that

$$\int_{\varkappa + iT}^{\lambda + iT} \beta^{2s} \frac{\Gamma\left(\frac{1}{2}-s\right)}{\zeta(2s)}\,ds \to 0$$

when $T \to \infty$ through an appropriately chosen sequence of values. It would certainly be enough, for example, to show that there is such a sequence (T_ν) for which

$$(2.521)\quad |\zeta(s)| > e^{-\left(\frac{1}{4}\pi - \delta\right)t}$$

$$(\delta > 0,\ t = T_\nu,\ \varkappa \le \sigma \le \lambda).$$

It would even be enough to show that the inequality (2. 521) holds on an appropriate sequence of curves each stretching from $\sigma = \varkappa$ to $\sigma = \lambda$. The existence of such a sequence seems highly probable: it is highly probable, in fact, that the series on the right hand side of (2. 516) is not merely convergent but very rapidly convergent. But we are quite unable to prove this, even when we assume the RIEMANN hypothesis.[1]

2. 53. Mr RAMANUJAN has indicated to us a generalisation of the formula (2. 516). Suppose that $\varphi(x)$ and $\psi(x)$ are a pair of 'reciprocal functions' connected by the relations

(2. 531) $$\int_0^\infty \varphi(x)\cos 2ux\,dx = \frac{1}{2}\sqrt{\pi}\,\psi(u),\quad \int_0^\infty \psi(x)\cos 2ux\,dx = \frac{1}{2}\sqrt{\pi}\,\varphi(u),$$

and let us write

(2. 532) $$\int_0^\infty x^{s-1}\varphi(x)\,dx = \Gamma(s)Z_1(s),\quad \int_0^\infty x^{s-1}\psi(x)\,dx = \Gamma(s)Z_2(s).$$

The simplest case is that in which

$$\varphi(x) = \psi(x) = e^{-x^2},\ Z_1(s) = Z_2(s) = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}s\right)}{\Gamma(s)}:$$

in this case the formulae reduce to those of 2. 51. Then it can be shown that, if φ and ψ satisfy certain conditions,[2]

(2. 533) $$Z(1-s) = \pi^{-\frac{1}{2}} 2^s \Gamma(s) \sin\frac{1}{2}s\pi\, Z_2(s).$$

We have also (again of course subject to certain restrictions on φ and ψ)

(2. 534) $$\varphi(x) = \frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty} \Gamma(s)Z_1(s)x^{-s}\,ds,\quad \psi(x) = \frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty} \Gamma(s)Z_2(s)x^{-s}\,ds,$$

for an appropriate value of $\varkappa$.[3]

---

[1] We can prove that *some* such sequence of curves as is referred to above exists, and that our series can be rendered convergent by *some* process of bracketing terms: but we can prove nothing about the distribution of the curves or the size of the brackets.

[2] As we do not profess to be able to give rigorous proofs of the main formulae of this sub-section, it seems hardly worth which to state such conditions in detail.

[3] MELLIN, *Acta mathematica*, vol. 25, 1902, pp. 139—164, 165—184 (p. 159): see also NIELSEN, *Handbuch der Theorie der Gamma-Funktion*, pp. 221 *et seq.*

We can use the first formula (2. 534) to express the series

$$\sum \frac{\mu(n)}{n}\varphi\left(\frac{\beta}{n}\right)$$

in the form of a definite integral. Carrying out a series of transformations analogous to those of 2. 51, with the aid of (2. 533) and the functional equations satisfied by the Gamma and Zeta-Functions, we are led finally to the formulae

(2. 535) $$\sqrt{\alpha}\sum\frac{\mu(n)}{n}\varphi\left(\frac{\alpha}{n}\right) - \sqrt{\beta}\sum\frac{\mu(n)}{n}\varphi\left(\frac{\beta}{n}\right)$$
$$= \frac{1}{\sqrt{\alpha}}\sum\frac{\Gamma(1-\varrho)Z_1(1-\varrho)\alpha^\varrho}{\zeta'(\varrho)}$$
$$= -\frac{1}{\sqrt{\beta}}\sum\frac{\Gamma(1-\varrho)Z_2(1-\varrho)\beta^\varrho}{\zeta'(\varrho)}$$

where $\alpha\beta = \pi$.

2. 54. Let us return for a moment to the formula (2. 516). We have

(2. 541) $$F(\alpha) = \sum_1^\infty \frac{\mu(n)}{n}e^{-(\alpha/n)^2} = \sum_{p=1}^\infty \frac{(-1)^p\alpha^{2p}}{p!}\sum_{\mu=1}^\infty\frac{\mu(n)}{n^{2p+1}} = \sum_{p=1}^\infty \frac{(-1)^p\alpha^{2p}}{p!\,\zeta(2p+1)},$$

an integral function of α. And

(2. 542) $$\sqrt{\alpha}F(\alpha) - \sqrt{\beta}F(\beta) = -\frac{1}{2}\sum\frac{\Gamma\left(\frac{1-\varrho}{2}\right)}{\zeta'(\varrho)}\beta^{\varrho-\frac{1}{2}}$$

when $\alpha\beta = \pi$. If we assume the RIEMANN hypothesis, and the absolute convergence of

$$\sum\frac{\Gamma\left(\frac{1-p}{2}\right)}{\zeta'(\varrho)}$$

then the right hand side of (2. 542) is of the form

$$O(1)$$

when $\alpha\to 0$ and $\beta\to\infty$. Writing y for β^2, and observing that $F(\alpha)\to 0$ as $\alpha\to 0$ we see that

(2. 543) $$P(y)=\sum_{p=1}^{\infty}\frac{(-y)^p}{p!\,\zeta(2p+1)}=O\left(y^{-\frac{1}{4}}\right)$$

when $y\to\infty$.

Now it has been proved by MARCEL RIESZ that[1]

(2. 544) $$\int_0^{\infty} y^{-s-1}P(y)\,dy=\frac{\Gamma(-s)}{\zeta(2s+1)}.$$

This formula certainly holds if $0<s<1$. If it could be proved to hold for $-\frac{1}{4}<s\leq 0$, the truth of the RIEMANN hypothesis would follow. The hypothesis is therefore certainly true if

(2. 545) $$P(y)=O\left(y^{-\frac{1}{4}+\delta}\right)$$

for all positive values of δ. The result of our previous analysis is therefore to suggest that the truth of (2. 545) is a necessary and sufficient condition for the truth of the RIEMANN hypothesis. It is not difficult to prove that the result thus suggested is in fact true. For LITTLEWOOD[2] has shown that, if the RIEMANN hypothesis is true, the series

$$\sum\frac{\mu(n)}{n^{\frac{1}{2}+\epsilon}}$$

is convergent for all positive values of ϵ, so that

(2. 546) $$M(\nu,n)=\sum_{\nu}^{n}\frac{\mu(m)}{m}=o\left(\nu^{\frac{1}{2}+\epsilon}\right)$$

uniformly in n. Hence

(2. 547) $$P(y)=\sum_1^{\infty}\frac{(-1)^p\beta^{2p}}{p!\,\zeta(2p+1)}=\sum_1^{\infty}\frac{\mu(n)}{n}e^{-(\beta/n)^2}=\sum_1^{\nu-1}+\sum_{\nu}^{\infty}=P_1+P_2,$$

[1] See RIESZ, *Acta mathematica*, vol. 40, 1916, pp. 185—190. The actual formula communicated to us by RIESZ (in 1912) was not this one, nor the formula for $\frac{1}{\zeta(s)}$ contained in his memoir, but the analogous formula for $\frac{1}{\zeta(s+1)}$. All of these formulae may be deduced from MELLIN's inversion formula already referred to in 2. 53. The idea of obtaining a necessary and sufficient condition of this character for the truth of the RIEMANN hypothesis is of course RIESZ's and not ours.

[2] *Comptes Rendus*, 29 Jan. 1912.

say, where $\nu=[\beta^{1-\epsilon}]$. Now

(2. 5471) $$P_2=\sum_{\nu}^{\infty}\frac{\mu(n)}{n}e^{-(\beta/n)^2}=\sum_{\nu}^{\infty}M(\nu,n).\Delta e^{-(\beta/n)^2}=o\left(\nu^{-\frac{1}{2}+\epsilon}\right)=o\left(\beta^{-\frac{1}{2}+2\delta}\right),$$

where $2\delta=\frac{3}{2}\epsilon-\epsilon^2$; and

(2. 5472) $$P_1=O\left(\nu e^{-\beta^{2\epsilon}}\right)=o\left(\beta^{-\frac{1}{2}+2\delta}\right).$$

From (2. 547), (2. 5471) and (2. 5472) it follows that

(2. 548) $$P(y)=o\left(\beta^{-\frac{1}{2}+2\delta}\right)=o\left(y^{-\frac{1}{4}+\delta}\right).$$

3.

The series $\sum\gamma^{-\omega}e^{ai\gamma\log(\gamma\theta)}$

3. 1. The results of this section will be stated on the assumption that the RIEMANN hypothesis is true. The truth of the hypothesis is not essential to our argument, and our results remain significant without it. But their interest depends to a considerable extent on the truth of the hypothesis, and the assumption that it is true enables us to state them in a simpler form than would be otherwise attainable.

We shall then denote the complex zeros of $\zeta(s)$ by $\varrho=\frac{1}{2}+i\gamma$, where γ is real. It has been proved by LANDAU[1] that

(3. 111) $$\sum_{0<\gamma<T}x^{\varrho}=O(\log T)$$

if x is real and not of the form p^m, and

(3. 112) $$\sum_{0<\gamma<T}x^{\varrho}=-\frac{T}{2\pi}\log p+O(\log T)$$

if $x=p^m$. If we assume the truth of the RIEMANN hypothesis, these results may be stated in the form

[1] *Math. Annalen*, vol. 71, 1912, pp. 548—564.

(3.121)
$$\sum_{0<\gamma<T} e^{i\gamma\log x} = O(\log T),$$

(3.122)
$$\sum_{0<\gamma<T} e^{i\gamma\log x} = -\frac{T}{2\pi}\log p + O(\log T).$$

In this section we shall apply an argument similar in principle to LANDAU's, but of a rather more intricate character, to the series

$$\sum \gamma^{-\omega} e^{ai\gamma\log(\gamma\theta)}$$

where a, θ, and ω are real and the first two positive. The principal result is

Theorem 3. 1. *If a, θ, and ω are real, and a and θ positive, then*

$$\sum_{0<\gamma<T} \gamma^{-\omega} e^{ai\gamma\log(\gamma\theta)} = O\left(T^{\frac{1+a}{2}-\omega}\right)$$

if $\omega < \frac{1+a}{2}$, and

$$\sum_{0<\gamma<T} \gamma^{-\omega} e^{ai\gamma\log(\gamma\theta)} = O(\log T)$$

if $\omega = \frac{1+a}{2}$. If $\omega < \frac{1+a}{2}$ then the series is convergent when continued to infinity. These results hold uniformly in any interval $0 < \theta_0 \leq \theta \leq \theta_1$.

The result is trivial when $a > 1$. We may therefore suppose that $a \leq 1$.

3. 2. Suppose that the theorem has been proved in the special case in which $\omega = 0$. Let

$$b_\gamma = e^{ai\gamma\log(\gamma\theta)}$$

and

$$\Phi(\tau) = \sum_{0<\gamma\leq\tau} b_\gamma,$$

so that

$$\Phi(\tau) = O\left(\tau^{\frac{1+a}{2}}\right).$$

Then

$$\sum_{0<\gamma\leq T} \gamma^{-\omega} b_\gamma = \sum_1^N \frac{\Phi(\gamma_n) - \Phi(\gamma_{n-1})}{\gamma_n^{\omega}}$$

where N is the largest number such that $\gamma_N < T$. Applying the method of partial summation, we obtain

$$\sum_{0<\gamma<T} \gamma^{-\omega} b_\gamma = \sum_1^{N-1} \Phi(\gamma_n)\left(\gamma_n^{-\omega} - \gamma_{n+1}^{-\omega}\right) + \Phi(\gamma_N)\gamma_N^{-\omega}$$

$$= \omega \sum_1^{N-1} \Phi(\gamma_n) \int_{\gamma_n}^{\gamma_{n+1}} u^{-\omega-1} du + \Phi(\gamma_N)\gamma_N^{-\omega}$$

$$= \omega \int_{\gamma_1}^{\gamma_N} \Phi(u) u^{-\omega-1} du + \Phi(\gamma_N)\gamma_N^{-\omega}$$

$$= O\int_{\gamma_1}^{T} u^{\frac{1+a}{2}-\omega-1} du + O\left(T^{\frac{1+a}{2}-\omega}\right)$$

and the general result of Theorem **3. 1** follows immediately. It is therefore sufficient to prove the *theorem* when $\omega = 0$.

It should be observed that the O's which occur in this argument are uniform in θ, when $0 < \theta_0 < \theta < \theta_1$; that is to say, the constants which they imply are independent of θ. This remark applies to the whole discussion which follows.

3. 31. We choose numbers α and δ such that

(3.311)
$$\alpha > \frac{3}{8}, \delta > 0, a\delta + \alpha < \frac{1}{2};$$

and we denote by C the rectangle

$$(1 + \delta + i, 1 + \delta + Ti, -2p - 1 + Ti, -2p - 1 + i),$$

where p is a large positive integer, and T a large positive number differing from any γ. This being so, we consider the integral

(3.312)
$$\int_C e^{as\log(-is)} x^s s^{-\frac{1}{2}a} \frac{\zeta'(s)}{\zeta(s)} ds,$$

where $x = \theta^a$ and $\log(-is)$, x^s and $s^{-\frac{1}{2}a}$ have their principal values. An application of CAUCHY's Theorem gives the formula

(3. 313)
$$2\pi i \sum_{0<\gamma<T} e^{a\varrho \log(-i\varrho)} x^{\varrho} \varrho^{-\frac{1}{2}a} = \int_{1+\delta+i}^{1+\delta+Ti} + \int_{1+\delta+Ti}^{-2p-1+Ti} + \int_{-2p-1+Ti}^{-2p-1+i} + \int_{-2p-1+i}^{1+\delta+i}$$
$$= I_1 + I_2 + I_3 + I_4,$$

say. When t is fixed and $\sigma \to -\infty$, $e^{as\log(-is)}$ tends to zero like $e^{a\sigma\log|\sigma|}$. It follows that $I_3 \to 0$ when $p \to \infty$, and that I_2 and I_4 tend to limits $\bar{I}_2$ and $\bar{I}_4$, the latter being independent of T. Thus

(3. 314)
$$2\pi i \sum_{0<\gamma<T} e^{a\varrho\log(-i\varrho)} x^{\varrho} \varrho^{-\frac{1}{2}a} = I_1 + \bar{I}_2 + O(1) = I_1 + \bar{I}_2 + O\left(T^{\frac{1+a}{2}}\right),$$

where

(3. 315)
$$\bar{I}_2 = \int_{1+\delta+Ti}^{-\infty+Ti} e^{as\log(-is)} x^{-s} s^{-\frac{1}{2}a} \frac{\zeta'(s)}{\zeta(s)}\, ds.$$

3. 32. We shall now prove that the term $\bar{I}_2$ in (3. 314) may be omitted. We write

(3. 321)
$$\bar{I}_2 = \int_{-1+Ti}^{-\infty+Ti} + \int_{1+\delta+Ti}^{-1+Ti} = \bar{I}_{2,1} + \bar{I}_{2,2}$$

The discussion of $\bar{I}_{2,1}$ is simple. If $s = \sigma + Ti$ and $\sigma < -1$, we have

$$|e^{as\log(-is)}| = e^{\frac{1}{2}a\sigma\log(\sigma^2+T^2) + aT\arctan(\sigma/T)} < T^{a\sigma},$$

$$|x^s| = x^{\sigma}$$

$$\left|s^{-\frac{1}{2}a}\right| < 1,$$

and

$$\frac{\zeta'(s)}{\zeta(s)} = O(\log T),$$

uniformly for $\sigma \leq -1$.[1] Hence

[1] Landau, *Handbuch*, p. 336.

(3. 322)
$$\bar{I}_{2,1} = O\left\{\log T \int_{-1}^{-\infty} (xT^a)^{\sigma} d\sigma\right\}$$
$$= O\left\{\frac{\log T}{xT^a \log(xT^a)}\right\}$$
$$= O(T^{-a})^1 = O\left(T^{\frac{1+a}{2}}\right).$$

Thus the integral $\bar{I}_{2,1}$ is without importance. The discussion of $\bar{I}_{2,2}$ is somewhat more difficult.

3. 33. We may write

(3. 331)
$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\gamma-T|<1} \frac{1}{s-\varrho} + \left\{\frac{\zeta'(s)}{\zeta(s)} - \sum_{|\gamma-T|<1} \frac{1}{s-\varrho}\right\} = Z_1(s) + Z_2(s),$$

say. Then

(3. 332)
$$Z_2(s) = O(\log T)$$

uniformly for $-1 < \sigma < 1 + \delta$.[2]

We now write

(3. 333)
$$\bar{I}_{2,2} = \int_{1+\delta+Ti}^{-1+Ti} e^{as\log(-is)} x^s s^{-\frac{1}{2}a} (Z_1(s) + Z_2(s))\, ds = \bar{I}_{2,2,1} + \bar{I}_{2,2,2},$$

say. It follows from (3. 332) that

(3. 334)
$$\bar{I}_{2,2,2} = O\left(T^{-\frac{1}{2}a} \log T \int_{-1}^{1+\delta} T^{a\sigma} d\sigma\right) = O(T^p \log T),$$

where

(3. 335)
$$p = \left(\frac{1}{2} + \delta\right) a < \frac{1+a}{2}.$$

Thus the integral $\bar{I}_{2,2,2}$ is of no importance.

[1] Observing that $\frac{1}{x} < \frac{1}{x_0}$, where $x_0 = \theta_0^a$, and that $\log(xT^a) > a\log T + \log x_0$.

[2] Landau, *Handbuch*, p. 339.

3. 34. On the other hand

(3. 341) $$I_{2,2,1} = \sum_{|\gamma - T| < 1} \int_{1+\delta+Ti}^{-1+Ti} e^{as\log(-is)} x^s s^{-\frac{1}{2}a} \frac{ds}{s-\varrho}.$$

We can transform each of these integrals, by CAUCHY's Theorem, into an integral along a semicircle described on the line $(-1+Ti,\ 1+\delta+Ti)$, taking the semicircle above or below the line according as $\gamma < T$ or $\gamma > T$.[1] Each integral is of the form $O(T^p)$ and their number is of the form $O(\log T)$. Hence

(3. 342) $$I_{2,2,1} = O(T^p \log T) = o\left(T^{\frac{1+a}{2}}\right).$$

From (3. 321), (3. 322), (3. 333), (3. 334), (3. 335), and (3. 342) it follows that

(3. 343) $$I_2 = o\left(T^{\frac{1+a}{2}}\right);$$

and from (3. 314) and (3. 343) that

(3. 344) $$2\pi i \sum_{0<\gamma<T} e^{a\varrho\log(-i\varrho)} x^\varrho \varrho^{-\frac{1}{2}a} = I_1 + o\left(T^{\frac{1+a}{2}}\right).$$

Thus the result of (3. 3) is to reduce the problem to the discussion of I_1.

3. 4. The main difficulty in the proof of the theorem lies in the discussion of the integral

(3. 41) $$I_1 = \int_{1+\delta+i}^{1+\delta+Ti} e^{as\log(-is)} x^s s^{-\frac{1}{2}a} \frac{\zeta'(s)}{\zeta(s)} ds.$$

We observe first that, when σ is fixed,

$$e^{as\log(-is)} = \left\{A + O\left(\frac{1}{t}\right)\right\} e^{ait\log t} t^{a\sigma},$$

where A is a constant. The contribution to I_1 of the term $O\left(\frac{1}{t}\right)$ is of the form

(3. 42) $$O\int_1^T t^{\left(\frac{1}{2}+\delta\right)a-1} dt = O(T^p) = o\left(T^{\frac{1+a}{2}}\right)$$

[1] Cf. LANDAU, *Math. Annalen*, vol. 71, 1912, p. 557.

and is therefore without importance. For similar reasons we may replace $s^{-\frac{1}{2}a}$, in I_1, by $(it)^{-\frac{1}{2}a}$ The problem is then reduced to the study of the integral

(3. 43) $$J = \int_1^T e^{ait\log t} x^{it} t^p \frac{\zeta'(1+\delta+it)}{\zeta(1+\delta+it)} dt.$$

Replacing ζ'/ζ by the DIRICHLET's series which represents it, and making an obvious formal transformation, we obtain

(3. 44) $$J = -\sum \frac{\Lambda(n)}{n^{1+\delta}} j(\xi),$$

where

(3. 45) $$\xi = \sqrt[a]{\frac{n}{x}}$$

and

(3. 46) $$j(\xi) = \int_1^T t^p e^{ait\log(t/\xi)} dt.$$

We write

(3. 47) $$J = -\left(\sum_1 + \sum_2 + \sum_3\right) \frac{\Lambda(n)}{n^{1+\delta}} j(\xi) = J_1 + J_2 + J_3,$$

say, where J_1 contains the terms (if any) for which

(3. 481) $$n + 1 \leq xe^a,$$

J_2 those for which

(3. 482) $$xe^a - 1 < n < x(eT)^a + 1,$$

and J_3 those for which

(3. 483) $$n \geq x(eT)^a + 1.$$

3. 51. The discussion of J_1 and J_3 is simple, and depends on a lemma which will be useful to us later in the paper.

Lemma 3. 51. *There is a number K, independent of τ_1, τ_2, and ξ, such that*

$$\left|\int_{\tau_1}^{\tau_2} e^{ait\log(t/\xi)} dt\right| < \frac{K}{\log(e\tau_1/\xi)}$$

when $0 < \xi < e\tau_1 < e\tau_2$, *and*

$$\left| \int_{\tau_1}^{\tau_2} e^{ait \log(t/\xi)} dt \right| < \frac{K}{\log(\xi / e\tau_2)}$$

when $\xi > e\tau_2 > e\tau_1$.

Suppose, for example, that $\xi < e\tau_1$. It is plain that we may consider the real and imaginary parts of the integral separately. If we put

$$w = t \log(t/\xi),$$

so that

$$\frac{dw}{dt} = \log\left(\frac{et}{\xi}\right),$$

and observe that w increases steadily, say from w_1 to w_2, as t increases from τ_1 to τ_2, we obtain

$$\int_{\tau_1}^{\tau_2} \cos aw \, dt = \int_{w_1}^{w_2} \cos aw \frac{dw}{\log(et/\xi)}.$$

But $\log(et/\xi)$ is positive, and increases as t increases. Hence

$$\int_{\tau_1}^{\tau_2} \cos aw \, dt = \frac{1}{\log(e\tau_1/\xi)} \int_{w_1}^{w_3} \cos aw \, dw,$$

where $w_1 < w_3 < w_2$. The truth of the lemma follows immediately.

3.52. We are now in a position to discuss J_1 and J_2. We begin with J_1, which exists only if $xe^a > 2$.

The real part of $j(\xi)$ is

(3.521) $$\int_1^T t^p \cos\left(at \log \frac{t}{\xi}\right) dt = T^p \int_{T_1}^T \cos\left(at \log \frac{t}{\xi}\right) dt,$$

where $1 < T_1 < T$. Since

(3.522) $$\xi = \sqrt[a]{\frac{n}{x}} < e < eT_1,$$

the right hand side of (3.521) is less in absolute value than a constant multiple of

$$\frac{T^p}{\log(eT_1/\xi)} < \frac{T^p}{\log(e/\xi)}.$$

The same argument may be applied to the imaginary part of $j(\xi)$, so that we may write

(3.523) $$j(\xi) = O\left\{\frac{T^p}{\log(e/\xi)}\right\}.$$

Also

(3.524) $$\left(\frac{e}{\xi}\right)^a = \frac{xe^a}{n} = \frac{\nu}{n},$$

(3.525) $$\log\left(\frac{e}{\xi}\right) = \frac{1}{a}\log\left(\frac{\nu}{n}\right),$$

where $\nu \geq n + 1$. Hence

(3.526) $$J_1 = O\left\{T^p \sum_{n \leq \nu - 1} \frac{\Lambda(n)}{n^{1+\delta}} \frac{1}{\log(\nu/n)}\right\} = O(T^p)[1] = O\left(T^{\frac{1+a}{2}}\right)$$

3.53. The discussion of J_2 is similar. It will be sufficient to write down the formulae which correspond to the formulae (3.522), etc. They are

(3.532) $$\xi = \sqrt[a]{\frac{n}{x}} > eT,$$

(3.533) $$j(\xi) = O\left\{\frac{T^p}{\log(\xi/eT)}\right\},$$

(3.534) $$\left(\frac{\xi}{eT}\right)^a = \frac{n}{x(eT)^a} = \frac{n}{\nu},$$

(3.535) $$\log\left(\frac{\xi}{eT}\right) = \frac{1}{a}\log\left(\frac{n}{\nu}\right)$$

(where $\nu \leq n - 1$),

(3.536) $$J_2 = O\left\{T^p \sum_{n \geq \nu+1} \frac{\Lambda(n)}{n^{1+\delta}} \frac{1}{\log(n/\nu)}\right\} = O(T^p)[1] = O\left(T^{\frac{1+a}{2}}\right).$$

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[1] Landau, *Handbuch*, p. 806.

3. 61. The discussion of J is accordingly reduced to that of J_2. In order to discuss J_2 we observe[1] that $\tau \log(\tau/\xi)$ is stationary when $\tau = \xi/e$. This point is the critical point in the integral $j(\xi)$. It falls in the range $(1, T)$ if

$$e \leq \xi \leq eT$$

or

$$xe^a < n < x(eT)^a.$$

This condition is certainly satisfied by every term of J_2 except possibly the first and last, and no serious modification is required, for these two possibly exceptional terms, in the analysis which follows.[2]

We write

(3. 611) $$j(\xi) = \left(\int_1^{\xi/e} + \int_{\xi/e}^{T}\right) t^p e^{ait \log(t/\xi)} dt = j_1(\xi) + j_2(\xi).$$

Then

(3. 6121) $$j_1(\xi) = \left(\frac{\xi}{e}\right)^{p+1} \int_{e/\xi}^{1} u^p e^{ai\xi w/e} du = \left(\frac{\xi}{e}\right)^{p+1} k_1(\xi),$$

(3. 6122) $$j_2(\xi) = \left(\frac{\xi}{e}\right)^{p+1} \int_1^{eT/\xi} u^p e^{ai\xi w/e} du = \left(\frac{\xi}{e}\right)^{p+1} k_2(\xi),$$

where

(3. 613) $$w = u \log u - u.$$

In general $e/\xi \leq 1 \leq eT/\xi$, and we write further

(3. 6141) $$k_1 = \int_{1-\varepsilon}^{1} + \int_{e/\xi}^{1-\varepsilon} = k_{1,1} + k_{1,2},$$

(3. 6142) $$k_2 = \int_1^{1+\varepsilon} + \int_{1+\varepsilon}^{eT/\xi} = k_{2,1} + k_{2,2},$$

[1] The fundamental idea in the analysis which follows is the same as that of Landau's memoir 'Über die Anzahl der Gitterpunkte in gewissen Bereichen' (*Göttinger Nachrichten*, 1912, pp. 687—771).

[2] The terms have to be retained in J_2 because ξ/e, though outside the range of integration, may be very near to one of the limits.

where

$$\varepsilon = T^{-a},$$

a being the number defined at the beginning of 3. 31. If, however, $e/\xi > 1$ or $eT/\xi < 1$, as may happen, each with one term only, we must write

(3. 6151) $$k_1 = -\int_1^{1+\varepsilon} - \int_{1+\varepsilon}^{e/\xi} = k_{1,1} + k_{1,2},$$

or

(3. 6152) $$k_2 = -\int_{1-\varepsilon}^{1} - \int_{eT/\xi}^{1-\varepsilon} = k_{2,1} + k_{2,2}.$$

These exceptional cases need not detain us further, as the treatment of k_2 in the general case covers *a fortiori* that of k_1 in the special case, and *vice versa*. Each of the formulae (3. 6141)—(3. 6152), however, may in certain cases require to be interpreted in the light of a further convention. It may happen, for example, that $1-\varepsilon < e/\xi < 1$. In this case, in the formula (3. 6142), we must regard $k_{1,2}$ as non-existent, and the subject of integration in as having the value zero for $1-\varepsilon < u < e/\xi$, and a similar understanding may be necessary in the other formulae. If regard is paid to these conventions, the analysis needed in every case is included in that which refers explicitly to the normal case in which

$$e/\xi \leq 1-\varepsilon < 1 < 1+\varepsilon < eT/\xi.$$

We may therefore conduct our argument as if these conditions were always satisfied. And we have

(3. 616) $$j(\xi) = \left(\frac{\xi}{e}\right)^{p+1} (k_{1,1}(\xi) + k_{1,2}(\xi) + k_{2,1}(\xi) + k_{2,2}(\xi)).$$

3. 62. The really important terms on the right hand side of (3. 616) are $k_{1,1}$ and $k_{2,1}$. We shall discuss $k_{1,2}$ and $k_{2,2}$ first.

The real part of $k_{1,2}$ is

(3. 621) $$\int_{e/\xi}^{1-\varepsilon} u^p \cos\left(\frac{a\xi w}{e}\right) du = \int_{u=e/\xi}^{1-\varepsilon} \frac{u^p}{\log u} \cos\left(\frac{a\xi w}{e}\right) dw = \frac{(1-\varepsilon)^p}{\log(1-\varepsilon)} \int_{u=\lambda}^{1-\varepsilon} \cos\left(\frac{a\xi w}{e}\right) dw$$

where $e/\xi < \lambda < 1-\varepsilon$. A similar argument may be applied to the imaginary part. Also

$$\log\frac{1}{1-\varepsilon} = O\left(\frac{1}{\varepsilon}\right),$$

and

$$\int_{\lambda'}^{\lambda} \cos\left(\frac{a\xi w}{e}\right) dw = O\left(\frac{1}{\xi}\right),$$

for all values of λ and λ'. It follows that

(3. 622) $$k_{1,2} = O\left(\frac{1}{\varepsilon\xi}\right) = O\left(\frac{T^a}{\xi}\right).$$

Similarly, the real part of $k_{2,2}$ is

(3. 623) $$\int_{1+\varepsilon}^{eT/\xi} u^p \cos\left(\frac{a\xi w}{e}\right) du = \int_{u=1+\varepsilon}^{eT/\xi} \frac{u^p}{\log u} \cos\left(\frac{a\xi w}{e}\right) dw = \left(\frac{eT}{\xi}\right)^p \int_{u=\lambda}^{eT/\xi} \cos\left(\frac{a\xi w}{e}\right) \frac{dw}{\log u}$$

$$= \frac{(eT/\xi)^p}{\log \lambda} \int_{u=\lambda}^{\lambda'} \cos\left(\frac{a\xi w}{e}\right) dw,$$

where

$$1+\varepsilon < \lambda < \lambda' < eT/\xi,$$

and a similar argument may be applied to the imaginary part. Hence

(3. 624) $$k_{2,2} = O\left(\frac{T^p}{\varepsilon\xi^{p+1}}\right) = O\left(\frac{T^{p+a}}{\xi^{p+1}}\right).$$

3. 63. We shall now consider $k_{1,1}$ and $k_{2,1}$. It is here that we are for the first time in touch with the real kernel of the problem. The two integrals are amenable to the same treatment, and we may confine our attention to one of them, say $k_{2,1}$.

We write

$$u = 1 + \mu,$$

so that $0 \leq \mu \leq \varepsilon$ and

$$u^p = 1 + O(\mu),$$

$$w = u \log u - u = -1 + \frac{1}{2}\mu^2 + O(\mu^3),$$

$$e^{ai\xi w/e} = e^{-(ai\xi/e)+(ai\xi\mu^2/2e)+O(\xi\mu^3)}$$

$$= e^{-(ai\xi/e)+(ai\xi\mu^2/2e)}(1 + O(\xi\mu^3)).$$

Then

(3. 631) $$k_{2,1} = e^{-ai\xi/e} \int_0^{\varepsilon} (1 + O(\mu) + O(\xi\mu^3))\, e^{ai\xi\mu^2/2e} d\mu$$

$$= O\left(\xi^{-\frac{1}{2}}\right) + O(\varepsilon^2) + O(\xi\varepsilon^4)$$

$$= O\left(\xi^{-\frac{1}{2}}\right) + O(T^{-2a}) + O(\xi T^{-4a}).$$

Combining (3. 631) with (3. 622) and (3. 624), and substituting in (3. 616), we obtain

(3. 632) $$j(\xi) = O\left(\xi^{p+\frac{1}{2}}\right) + O(\xi^{p+1}T^{-2a}) + O(\xi^{p+2}T^{-4a}) + O(\xi^p T^a) + O(T^{p+a}).$$

3. 64. We can now complete our discussion of J_2. We have

$$J_2 = -\sum_{xe^a - 1 < n < x(eT)^a + 1} \frac{\Lambda(n)}{n^{1+\delta}} j(\xi),$$

$$|J_2| \leq \sum_{n < O(T^a)} \frac{\Lambda(n)}{n^{1+\delta}} |j(\xi)|,$$

the symbolism used last implying that the summation is extended to all positive integers n less than some fixed multiple of T^a. We have now to estimate the five sums S_1, S_2, S_3, S_4, S_5, obtained by substituting in turn for $|j(\xi)|$ the five terms on the right hand side of (3. 632).

In the first place

(3. 641) $$S_1 = \sum_{n < O(T^a)} \frac{\Lambda(n)}{n^{1+\delta}} O\left(\xi^{p+\frac{1}{2}}\right)$$

$$= O \sum_{n < O(T^a)} \Lambda(n) n^{\frac{2p+1}{2a} - 1 - \delta}$$

$$= O \sum_{n < O(T^a)} \Lambda(n) n^{\frac{1+a}{2a} - 1}$$

$$= O\left\{(T^a)^{\frac{1+a}{2a}}\right\}$$

$$= O\left(T^{\frac{1+a}{2}}\right).$$

In the second place

$$S_2 = \sum_{n < O(T^a)} \frac{\Lambda(n)}{n^{1+\delta}} O(\xi^{p+1} T^{-2a}) \tag{3.642}$$

$$= O\left\{T^{-2a} \sum_{n < O(T^a)} \Lambda(n) n^{\frac{p+1}{a} - 1 - \delta}\right\}$$

$$= O\left\{T^{-2a} \sum_{n < O(T^a)} \Lambda(n) n^{\frac{2+a}{2a} - 1}\right\}$$

$$= O\left\{T^{-2a} (T^a)^{\frac{2+a}{2a}}\right\}$$

$$= O\left\{T^{\frac{2+a}{2} - 2a}\right\}$$

$$= o\left(T^{\frac{1+a}{2}}\right),$$

since $2a > \frac{3}{4} > \frac{1}{2}$.

Thirdly

$$S_3 = \sum_{n < O(T^a)} \frac{\Lambda(n)}{n^{1+\delta}} O(\xi^{p+2} T^{-4a}) \tag{3.643}$$

$$= O\left\{T^{-4a} \sum_{n < O(T^a)} \Lambda(n) n^{\frac{p+2}{a} - 1 - \delta}\right\}$$

$$= O\left\{T^{-4a} \sum_{n < O(T^a)} \Lambda(n) n^{\frac{4+a}{2a} - 2}\right\}$$

$$= O\left\{T^{-4a} (T^a)^{\frac{4+a}{2a}}\right\}$$

$$= o\left(T^{\frac{4+a}{2} - 4a}\right)$$

$$= o\left(T^{\frac{1+a}{2}}\right)$$

since $4a > \frac{3}{2}$.

Fourthly,

$$S_4 = \sum_{n < O(T^a)} \frac{\Lambda(n)}{n^{1+\delta}} O(\xi^p T^a) \tag{3.644}$$

$$= O\left\{T^a \sum_{n < O(T^a)} \Lambda(n) n^{\frac{p}{a} - 1 - \delta}\right\}$$

$$= O\left\{T^a \sum_{n < O(T^a)} \Lambda(n) n^{-\frac{1}{2}}\right\}$$

$$= O\left(T^{\frac{1}{2}a + a}\right)$$

$$= o\left(T^{\frac{1+a}{2}}\right),$$

since $a < \frac{1}{2}$.

Finally

$$S_5 = \sum_{n < O(T^a)} \frac{\Lambda(n)}{n^{1+\delta}} O(T^{p+a}) \tag{3.645}$$

$$= O(T^{p+a})$$

$$= O\left(T^{\frac{1}{2}a + a\delta + a}\right)$$

$$= o\left(T^{\frac{1+a}{2}}\right),$$

since $a\delta + a < \frac{1}{2}$. Combining (3.641)—(3.645), we see that

$$J_2 = o\left(T^{\frac{1+a}{2}}\right). \tag{3.646}$$

3.7. We have thus proved that

$$\sum_{0 < \gamma < T} e^{a\varrho \log(-i\varrho)} x^\varrho \varrho^{-\frac{1}{2}a} = o\left(T^{\frac{1+a}{2}}\right), \tag{3.71}$$

uniformly in any positive interval of values of x. We now assume the truth of the Riemann hypothesis, and write $\frac{1}{2} + i\gamma$ for ϱ and θ^a for x. We have

$$e^{a\varrho \log(-i\varrho)} x^\varrho \varrho^{-\frac{1}{2}a} = e^{a\varrho \log(-i\varrho) + a\varrho \log\theta - \frac{1}{2}a \log\varrho},$$

and

$$a\varrho\log(-i\varrho) = ai\gamma\log\gamma + \frac{1}{2}a\log\gamma + \frac{1}{2}a + O\left(\frac{1}{\gamma}\right),$$

$$a\varrho\log\theta = ai\gamma\log\theta + \frac{1}{2}a\log\theta,$$

$$-\frac{1}{2}a\log\varrho = -\frac{1}{2}a\log\gamma - \frac{1}{4}a\pi i + O\left(\frac{1}{\gamma}\right),$$

and so

$$e^{a\varrho\log(-i\varrho)}x^{\varrho}\varrho^{-\frac{1}{2}a} = A\,e^{ai\gamma\log(\gamma\theta)}\left\{1 + O\left(\frac{1}{\gamma}\right)\right\},$$

where A is a constant. It follows from (3. 71) that

(3. 72) $$\sum_{0<\gamma<T} e^{ai\gamma\log(\gamma\theta)} = O\left(T^{\frac{1+a}{2}}\right);$$

and the proof of Theorem **3. 1** is completed.

4.

The zeros of $\zeta(s)$ on the line $\sigma = \frac{1}{2}$.

4. 1. In this section we prove that the number $N_0(T)$ of zeros of $\zeta(s)$ on the line $\sigma = \frac{1}{2}$, between the points $\frac{1}{2}$ and $\frac{1}{2} + Ti$, is of the form $\Omega\left(T^{\frac{3}{4}-\epsilon}\right)$ for all positive values of ϵ. As a matter of fact we prove rather more than this, viz. that there is at least one zero of odd order between T and $T + T^{\frac{1}{4}+\epsilon}$ for all sufficiently large values of T.[1]

4. 2. We write

(4. 21) $$\pi^{-s}e^{-\frac{1}{2}\left(s-\frac{1}{4}\right)\pi i}\,\Gamma(s)\zeta(2s) = f(s) = f\left(\frac{1}{4} + ti\right) = X(t).$$

[1] See section 1 for a summary of previous results.

Then

$$X(t) = -\frac{2e^{\frac{1}{2}\pi t}\,\Xi(2t)}{\frac{1}{4} + 4t^2}$$

is real when t is real. Let ϵ be any positive number. We shall prove that, if $T > T_0(\epsilon)$, then $X(t)$ changes its sign at least once in the interval $(T, T+H)$, where $H = T^{\frac{1}{4}+\epsilon}$. We may obviously suppose, without loss of generality, that $\epsilon < \frac{1}{5}$.

There are two stages in the proof. The first consists in showing that

(4. 22) $$\int_T^{T+H} X(t)\,dt = O(T^{\delta})$$

for all positive values of δ; and the second in showing that, if $\delta < \epsilon$, and T is large enough, then the equation (4. 22) contradicts the assumption that $X(t)$ is of constant sign throughout the range of integration. The second stage of the proof is the easier, and we shall discuss it first.[1]

Suppose then that (4. 22) is true, with $\delta < \epsilon$, and that $X(t)$ is of constant sign throughout $(T, T+H)$. Then

$$\int_T^{T+H} |X(t)|\,dt = O(T^{\delta}),$$

$$\int_T^{T+H} e^{\frac{1}{2}\pi t}\left|\Gamma\left(\frac{1}{4} + ti\right)\right|\left|\zeta\left(\frac{1}{2} + 2ti\right)\right| dt = O(T^{\delta}).$$

Now

$$\left|\Gamma\left(\frac{1}{4} + ti\right)\right| \sim \sqrt{2\pi}\,\frac{e^{-\frac{1}{2}\pi t}}{t^{\frac{1}{4}}}$$

as $t \to \infty$. Hence

[1] The general idea used in this part of the proof is identical with that introduced by Landau in his simplification of Hardy's proof of the existence of an infinity of roots (see Landau, *Math. Annalen*, vol. 76, 1915, pp. 212—243).

$$\int_T^{T+H} \frac{\left|\zeta\left(\frac{1}{2}+2ti\right)\right|}{t^{\frac{1}{4}}}\, dt = O(T^{\delta}),$$

$$\int_T^{T+H} \left|\zeta\left(\frac{1}{2}+2ti\right)\right| dt = O\left(T^{\frac{1}{4}+\delta}\right),$$

$$\int_{\frac{1}{2}+2Ti}^{\frac{1}{2}+2(T+H)i} |\zeta(s)|\, ds = O\left(T^{\frac{1}{4}+\delta}\right),$$

(4. 23)
$$\int_{\frac{1}{2}+2Ti}^{\frac{1}{2}+2(T+H)i} \zeta(s)\, ds = O\left(T^{\frac{1}{4}+\delta}\right).$$

Applying Cauchy's Theorem to the rectangle whose vertices are $\frac{1}{2}+2Ti$, $2+2Ti$, $2+2(T+H)i$, and $\frac{1}{2}+2(T+H)i$, we obtain

(4. 24)
$$J_1+J_2+J_3 = \int_{\frac{1}{2}+2Ti}^{2+2Ti} \zeta(s)\, ds + \int_{2+2Ti}^{2+2(T+H)i} \zeta(s)\, ds + \int_{2+2(T+H)i}^{\frac{1}{2}+2(T+H)i} \zeta(s)\, ds = O\left(T^{\frac{1}{4}+\delta}\right).$$

Now

$$\zeta(s) = O\left(t^{\frac{1}{4}+\delta}\right)$$

uniformly for $\frac{1}{2} < \sigma < 2$.[1] Hence

(4. 25)
$$J_1 = O\left(T^{\frac{1}{4}+\delta}\right), \quad J_3 = O\left(T^{\frac{1}{4}+\delta}\right);$$

and from (4. 24) and (4. 25) it follows that

[1] Landau, *Handbuch*, p. 868.

(4. 26)
$$J_2 = O\left(T^{\frac{1}{4}+\delta}\right).$$

But

$$J_2 = \int_{2+2Ti}^{2+2(T+H)i} \sum \frac{1}{n^s}\, ds = 2Hi + \int_{2+2Ti}^{2+2(T+H)i} \sum_2^{\infty} \frac{1}{n^s}\, ds$$

$$= 2Hi + \sum_2^{\infty} \frac{n^{-2-2Ti}}{\log n} - \sum_2^{\infty} \frac{n^{-2-2(T+H)i}}{\log n}$$

$$= 2Hi + O(1);$$

which contradicts (4. 26), since $H = T^{\frac{1}{4}+\epsilon}$ and $\delta < \epsilon$.

4. 3. The problem is therefore reduced to the proof of (4. 22). Using Cauchy's Theorem in a manner very similar to that of 4. 2, we obtain

(4. 31)
$$\int_T^{T+H} X(t)\, dt = -i\left(\int_{\frac{1}{4}+Ti}^{\frac{1}{2}+\frac{1}{2}\delta+Ti} + \int_{\frac{1}{2}+\frac{1}{2}\delta+Ti}^{\frac{1}{2}+\frac{1}{2}\delta+(T+H)i} + \int_{\frac{1}{2}+\frac{1}{2}\delta+(T+H)i}^{\frac{1}{4}+(T+H)i}\right) f(s)\, ds$$

$$= J_1 + J_2 + J_3,$$

say. Now[1]

$$\zeta(2s) = O\left(t^{\frac{1}{2}-\sigma} \log t\right)$$

uniformly for $\frac{1}{4} \leq \sigma \leq \frac{1}{2}$, and

$$\zeta(2s) = O(\log t)$$

uniformly for $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{1}{2}\delta$. Also

$$e^{-\frac{1}{2}s\pi i}\, \Gamma(s) = O\left(t^{\sigma-\frac{1}{2}}\right)$$

uniformly for $\frac{1}{4} \leq \sigma \leq \frac{1}{2} + \frac{1}{2}\delta$. Hence

$$f(s) = O(\log t)$$

[1] Landau, *l. c. supra.*

uniformly for $\frac{1}{4} \leq \sigma \leq \frac{1}{2}$, and

$$f(s) = O\left(t^{\frac{1}{2}\delta}\right)$$

uniformly for $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{1}{2}\delta$; and so

(4. 32) $$f(s) = O(t^{\delta})$$

uniformly for $\frac{1}{4} \leq \sigma < \frac{1}{2} + \frac{1}{2}\delta$. It follows that

(4. 33) $$J_1 = O(T^{\delta}),\quad J_3 = O(T^{\delta});$$

and the problem is accordingly reduced to that of proving that

(4. 34) $$I = iJ_2 = \int_{\frac{1}{2}+\frac{1}{2}\delta+Ti}^{\frac{1}{2}+\frac{1}{2}\delta+(T+H)i} f(s)\,ds = O(T^{\delta}).$$

4. 4. Now, when $\sigma = \frac{1}{2} + \frac{1}{2}\delta$, we have

(4. 41) $$f(s) = \pi^{-s} e^{-\frac{1}{2}\left(s-\frac{1}{4}\right)\pi i}\,\Gamma(s) \sum \frac{1}{n^{2s}}.$$

We have also, by a straightforward application of STIRLING's Theorem,

(4. 42) $$\pi^{-s} e^{-\frac{1}{2}\left(s-\frac{1}{4}\right)\pi i}\,\Gamma(s) = t^{\frac{1}{2}\delta} e^{it\log(t/e\pi)}\left\{A + O\left(\frac{1}{t}\right)\right\}$$

where A is a constant. The term $O\left(\frac{1}{t}\right)$ in this equation may be neglected, for its contribution to I is of the form

$$O\left(T^{\frac{1}{2}\delta-1}H\right) = o(1).$$

We have also

$$t^{\frac{1}{2}\delta} = T^{\frac{1}{2}\delta} + O\left(T^{\frac{1}{2}\delta-1}H\right),\quad (T \leq t \leq T+H),$$

and the second term's contribution is of the form

$$O\left(T^{\frac{1}{2}\delta-1}H^2\right) = o(1)^1.$$

Thus we are at liberty to replace $t^{\frac{1}{2}\delta}$ by $T^{\frac{1}{2}\delta}$ in (4. 42), and to replace I in our argument by

(4. 43) $$T^{\frac{1}{2}\delta}\int_T^{T+H} e^{it\log(t/e\pi)} \sum \frac{1}{n^{1+\delta+2ti}}\,dt = T^{\frac{1}{2}\delta}\sum \frac{1}{n^{1+\delta}}\int_T^{T+H} e^{it\log(t/e\pi n^2)}\,dt$$

$$= T^{\frac{1}{2}\delta}\sum \frac{\Phi(e\pi n^2)}{n^{1+\delta}} = T^{\frac{1}{2}\delta} S,$$

say.

The integral $\Phi(e\pi n^2)$ belongs to a type considered in 3. 4 and the following sections;[2] and its behaviour depends on the position of the point $\tau = \pi n^2$ with reference to the interval $(T, T+H)$. At most one value of n can satisfy the inequalities

$$T < \pi n^2 < T + T^{\frac{1}{4}+\varepsilon};$$

so that πn^2 can fall inside $(T, T+H)$ for at most one value of n. We denote this value of n, if it exists, by ν; if there be no such value, we denote by ν the largest value of n for which $\pi n^2 < T$. And we write

(4. 44) $$S = \sum_1^{\nu-2} + \sum_{\nu-1}^{\nu+1} + \sum_{\nu+2}^{\infty} = S_1 + S_2 + S_3,$$

say.

---

[1] For $\varepsilon < \frac{1}{5}$, $4\varepsilon < 1 - \varepsilon$, and *a fortiori* $4\varepsilon < 1 - \delta$. Hence

$$\tfrac{1}{2}\delta - 1 + 2\left(\tfrac{1}{4} + \varepsilon\right) = 2\varepsilon - \tfrac{1}{2}(1-\delta) < 0.$$

[2] We have

$$\Phi(e\pi n^2) = j_{1,0}(e\pi n^2, T+H) - j_{1,0}(e\pi n^2, T),$$

where

$$j_{a,p}(\xi, T) = \int_1^T t^p e^{ait\log(t/\xi)}\,dt.$$

Then, in the first place, we have

$$(4.45)\qquad T^{\frac{1}{2}\delta} S_2 = O\left(T^{\frac{1}{2}\delta} T^{-\frac{1}{2}-\frac{1}{2}\delta}\right) = o(1).$$

Secondly, if $n \leq \nu - 2$, we have, by Lemma 2.432,

$$(4.46)\qquad \Phi(e\pi n^2) = O\left\{\frac{1}{\log\left(\frac{T}{\pi n^2}\right)}\right\}$$

But

$$\log\left(\frac{T}{\pi n^2}\right) = 2\left(\log\sqrt{\frac{T}{\pi}} - \log n\right)$$

and

$$T > \pi(\nu-1)^2, \quad \log\sqrt{\frac{T}{\pi}} > \log(\nu-1).$$

Hence

$$S_1 = O\left\{\sum_1^{\nu-2} \frac{1}{n^{1+\delta}\log\left(\frac{\nu-1}{n}\right)}\right\} = O(1)^1,$$

$$(4.47)\qquad T^{\frac{1}{2}\delta} S_1 = O\left(T^{\frac{1}{2}\delta}\right).$$

Similarly, if $n > \nu + 2$, we have

$$(4.48)\qquad \Phi(e\pi n^2) = O\left\{\frac{1}{\log\left(\frac{\pi n^2}{T+H}\right)}\right\},$$

$$\log\left(\frac{\pi n^2}{T+H}\right) = 2\left(\log n - \log\sqrt{\frac{T+H}{\pi}}\right),$$

$$T + H < \pi(\nu+1)^2, \log\sqrt{\frac{T+H}{\pi}} < \log(\nu+1),$$

$$S_2 = O\left\{\sum_{\nu+2}^{\infty} \frac{1}{n^{1+\delta}\log\left(\frac{n}{\nu+1}\right)}\right\} = O(1)^2,$$

[1] Landau, *Handbuch*, p. 806.
[2] Landau, *l. c. supra.*

$$(4.49)\qquad T^{\frac{1}{2}\delta} S_3 = O\left(T^{\frac{1}{2}\delta}\right).$$

From (4.45), (4.47) and (4.49) it follows that

$$(4.491)\qquad I = O\left(T^{\frac{1}{2}\delta}\right) = O(T^{\delta});$$

and our proof is therefore completed.

Theorem 4.41. *Let ε be any positive number. Then there is a number $T_0(\varepsilon)$ such that the segment*

$$\frac{1}{2} + Ti, \frac{1}{2} + \left(T + T^{\frac{1}{4}+\varepsilon}\right)i,$$

where $T > T_0(\varepsilon)$, contains at least one zero of $\zeta(s)$ of odd order.

As a corollary we have

Theorem 4.42. *The number $N_0(T)$ of zeros of $\zeta(s)$, on the line $\frac{1}{2}, \frac{1}{2} + Ti$, is of the form*

$$\Omega\left(T^{\frac{3}{4}-\delta}\right)$$

for every positive value of δ.

5.

On the order of $\psi(x) - x$ and of $\Pi(x) - Li\,x$.

5.1. In this section we shall prove that

$$(5.111)\qquad \psi(x) - x = \Omega_R(\sqrt{x}\log\log\log x), \quad \psi(x) - x = \Omega_L(\sqrt{x}\log\log\log x),$$

i. e. that there exists a constant K such that each of the inequalities

$$(5.112)\qquad \psi(x) - x > K\sqrt{x}\log\log\log x, \quad \psi(x) - x < -K\sqrt{x}\log\log\log x,$$

is satisfied for arbitrarily large values of x; and from these inequalities we shall deduce the inequalities (1.52). It is clear that we may base our proof on the assumption that the Riemann hypothesis is true. If it is false, then more is true than our inequalities assert.[1]

[1] Landau, *Handbuch*, pp. 712 *et seq.*

We shall found our proof on the formulae

(5. 121) $$\gamma_n = g(n) + O(1),$$

where $t = g(z)$ is the function inverse to

(5. 122) $$z = \frac{1}{2\pi} t \log t - \frac{1 + \log 2\pi}{2\pi} t;$$

and

(5. 131) $$\frac{\psi(x) - x}{\sqrt{x}} = -2 \sum_{\gamma_n \leq T} \frac{\sin \gamma_n \eta}{\gamma_n} + O(1),$$

where $\eta = \log x$, uniformly for $T > x^2$. Of these two formulae (5. 121) and (5. 131), the first is an immediate corollary of Von Mangoldt's formula

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + O(\log T)^1;$$

and the second is an immediate corollary of known formulae to be found in Landau's *Handbuch*.[2]

If we make T tend to infinity in (5. 131), we obtain

(5. 132) $$\frac{\psi(x) - x}{\sqrt{x}} = -2 \sum_1^\infty \frac{\sin \gamma_n \eta}{\gamma_n} + O(1),$$

since the series is known to be convergent.

5. 2. Let $z = \xi + i\eta$, and let $F(z)$ be the function of z defined by the series

(5. 21) $$F(z) = \sum_1^\infty \frac{e^{-\gamma_n z}}{\gamma_n} = \sum_1^\infty \frac{e^{-\gamma_n(\xi + i\eta)}}{\gamma_n},$$

convergent for $\xi > 0$. We shall consider the behaviour of this function in the semi-infinite strip defined by the inequalities $0 < \xi \leq 1, \eta \geq 1$. Our object will be to prove

Theorem 5. 2. *If $\Im F(z)$ is the imaginary part of $F(z)$ then*

[1] It has been shown by Bohr, Landau, and Littlewood (»Sur la fonction $\zeta(s)$ dans le voisinage de la droite $\sigma = \frac{1}{2}$«, *Bulletins de l'Académie Royale de Belgique*, 1913, pp. 1144—1175) that, on the Riemann hypothesis (which we are now assuming), the O in this formula and the corresponding O in (5. 121) can each be replaced by o.

[2] See pp. 387, 351.

$$-\Im F(z) = \sum_1^\infty e^{-\gamma_n \xi} \frac{\sin \gamma_n \eta}{\gamma_n} = \Omega_R(\log \log \eta),$$

$$-\Im F(z) = \Omega_L (\log \log \eta),$$

in the semi-infinite strip $0 < \xi \leq 1, \eta \geq 1$[1].

We shall consider the first of these relations: the second can of course be proved in a similar manner. And we shall begin by proving the following lemma.

Lemma 5. 21. We have

$$-\Im F(\xi + i\xi) = \sum \frac{e^{-\gamma_n \xi} \sin \gamma_n \xi}{\gamma_n} \sim \frac{1}{8} \log\left(\frac{1}{\xi}\right),$$

as $\xi \to 0$.

Suppose that $n < u \leq n + 1$. Then

(5. 22) $$g'(u) = \frac{2\pi}{\log\left(\frac{g}{2\pi}\right)},$$

and so

$$g(u) - g(n) = (u - n) g'(v) \quad (n \leq v \leq u)$$

$$= O\left(\frac{1}{\log n}\right) = O(1).$$

Hence

$$\frac{e^{-(\xi + i\xi) g(u)}}{g(u)} = \frac{e^{-(\xi + i\xi)\gamma_n + O(\xi)}}{\gamma_n} \left\{1 + O\left(\frac{\log n}{n}\right)\right\}$$

$$= \frac{e^{-\gamma_n(\xi + i\xi)}}{\gamma_n} \left\{1 + O(\xi) + O\left(\frac{\log n}{n}\right)\right\}.$$

It should be observed that the constants implied by these O's, and by those which occur in the argument which follows, are independent of both u (or n) and ξ.

Let u_0 be a fixed positive number, and let $g_0 = g(u_0)$. Then

[1] To write

(1) $$-\Im F(z) = \Omega_R (\log \log \eta),$$

for a fixed value of ξ, would be to assert the existence of a positive K such that

(2) $$-\Im F(z) > K \log \log \eta$$

for this value of ξ and arbitrarily large values of η. To assert (1) *in the strip* $0 < \xi \leq 1, \eta \geq 1$ is to assert the existence of a positive K such that (2) holds for arbitrarily large values of η (corresponding in general to different values of ξ).

(5. 23) $$F(\xi+i\xi)=\int_{u_0}^{\infty}\frac{e^{-(\xi+i\xi)g(u)}}{g(u)}\,du+O\int_{u_0}^{\infty}\frac{e^{-\xi g(u)}}{g(u)}\left(\xi+\frac{\log u}{u}\right)du+O(1)$$

$$=\int_{g_0}^{\infty}\frac{e^{-(\xi+i\xi)g}}{g}\frac{dg}{g'(u)}+O\left(\xi\int_{g_0}^{\infty}\frac{e^{-\xi g}}{g}\frac{dg}{g'(u)}\right)+O(1).$$

But, by (5. 22),

$$\frac{1}{g'(u)}=\frac{\log g-\log 2\pi}{2\pi}.$$

Hence

$$\int_{g_0}^{\infty}\frac{e^{-\xi g}}{g}\frac{dg}{g'(u)}=O\int_{g_0}^{\infty}\frac{e^{-\xi g}\log g}{g}\,dg=O\left(\log\frac{1}{\xi}\right)^2,$$

and

(5. 24) $$F(\xi+i\xi)=\frac{1}{2\pi}\int_{g_0}^{\infty}\frac{e^{-(\xi+i\xi)g}}{g}(\log g-\log 2\pi)\,dg+O(1).$$

Thus

(5. 25) $$-\Im F(\xi+i\xi)=\frac{1}{2\pi}\int_{g_0}^{\infty}\frac{e^{-\xi g}\sin\xi g}{g}\log g\,dg-\frac{\log 2\pi}{2\pi}\int_{g_0}^{\infty}\frac{e^{-\xi g}\sin\xi g}{g}\,dg+O(1)$$

$$=J_1+J_2+O(1),$$

say. But

(5. 26) $$J_1=\frac{1}{2\pi}\int_0^{\infty}\frac{e^{-\xi g}\sin\xi g}{g}\log g\,dg+O(1)$$

$$=\frac{1}{2\pi}\int_0^{\infty}\frac{e^{-w}\sin w}{w}(\log w-\log\xi)\,dw+O(1)$$

$$=\frac{1}{8}\log\left(\frac{1}{\xi}\right)+O(1);$$

and

(5. 27) $$J_2=-\frac{\log 2\pi}{2\pi}\int_{\xi g_0}^{\infty}\frac{e^{-w}\sin w}{w}\,dw=O(1).$$

From (5. 25), (5. 26), and (5. 27) it follows that

(5. 28) $$-\Im F(\xi+i\xi)\sim\frac{1}{8}\log\left(\frac{1}{\xi}\right)$$

as $\xi\to 0$.

5. 3. *Lemma* 5. 3. *There is a constant a such that*

(5. 31) $$\sum_{\gamma_n\xi>a}\frac{e^{-\gamma_n\xi}}{\gamma_n}<\frac{1}{32}\log\left(\frac{1}{\xi}\right)$$

for all sufficiently small values of ξ.

The number of γ's which lie between ν and $\nu+1$ is of the form $O(\log\nu)$. Hence

$$\sum_{\gamma_n\xi>a}\frac{e^{-\gamma_n\xi}}{\gamma_n}=O\sum_{\nu>\frac{a}{\xi}-1}\frac{e^{-\nu\xi}\log\nu}{\nu}$$

$$=O\int_{\frac{a}{\xi}}^{\infty}\frac{e^{-\xi u}\log u}{u}\,du$$

$$=O\left\{\int_a^{\infty}\frac{e^{-w}\log w}{w}\,dw+\log\left(\frac{1}{\xi}\right)\int_a^{\infty}\frac{e^{-w}}{w}\,dw\right\};$$

and (5. 31) follows immediately.

5. 4. We shall now make use of a well-known theorem of Dirichlet, the fundamental importance of which in the theory of Dirichlet's series was first recognised by Bohr.[1] Let us denote by $\bar{x}$ [2] the number which differs from x by an integer and satisfies the inequalities $-\frac{1}{2}<\bar{x}\leq\frac{1}{2}$. Then Dirichlet's theorem asserts that, given any positive numbers τ_0 (large), ξ (small), and N (integral), there exists a τ such that

(5. 41) $$\tau_0<\tau<\tau_0\left(\frac{1}{\xi}+1\right)^N,$$

and

(5. 42) $$\left|\overline{\frac{\gamma_n\tau}{2\pi}}\right|<\xi$$

[1] See Bohr and Landau, *Göttinger Nachrichten*, 1910, pp. 303—330, and a number of later papers by Bohr.

[2] The notation is that of our first paper, 'Some problems of Diophantine Approximation', *Acta Mathematica*, vol. 37, pp. 155—193.

for
$$n = 1, 2, \ldots N.$$

Let
$$\eta = \tau + \xi, N = \frac{a}{\xi}.$$

Then
$$|-\mathfrak{J}F(\xi + i\eta) + \mathfrak{J}F(\xi + i\xi)|$$
$$= \left| \sum_1^{\infty} \frac{\sin \gamma_n \eta}{\gamma_n} e^{-\gamma_n \xi} - \sum_1^{\infty} \frac{\sin \gamma_n \xi}{\gamma_n} e^{-\gamma_n \xi} \right|$$
$$< \sum_1^{N} \left| \frac{\sin \gamma_n \eta - \sin \gamma_n \xi}{\gamma_n} \right| + 2 \sum_{N+1}^{\infty} \frac{e^{-\gamma_n \xi}}{\gamma_n}.$$

But, by (5. 42),
$$\sin \gamma_n \eta = \sin(\gamma_n \xi + \gamma_n \tau) = \sin(\gamma_n \xi + \omega_n),$$
where
$$|\omega_n| < 2\pi \xi;$$
and so
$$|\sin \gamma_n \eta - \sin \gamma_n \xi| < 2\pi \xi.$$

Hence

(5. 43)
$$|-\mathfrak{J}F(\xi + i\eta) + \mathfrak{J}F(\xi + i\xi)| < 2\pi\xi \sum_1^N \frac{1}{\gamma_n} + \frac{1}{16} \log\left(\frac{1}{\xi}\right)$$
$$= \frac{1}{16} \log\left(\frac{1}{\xi}\right) + O(1) < \frac{1}{12} \log\left(\frac{1}{\xi}\right),$$

if ξ is small enough. It follows that the inequality

(5. 44)
$$-\mathfrak{J}F(\xi + i\eta) > \left(\frac{1}{8} - \frac{1}{12}\right) \log\left(\frac{1}{\xi}\right) = \frac{1}{24} \log\left(\frac{1}{\xi}\right)$$

holds for every sufficiently small value of ξ and a corresponding value of η satisfying

(5. 45)
$$\tau_0 < \eta < \xi + \tau_0 \left(\frac{1}{\xi} + 1\right)^{\frac{a}{\xi}}.$$

5. 5 We are now in a position to prove Theorem 5. 2. Suppose that the formula
$$-\mathfrak{J}F(\xi + i\eta) = \Omega_R(\log \log \eta)$$
is false. Then, given any positive number ε, we have

(5. 51)
$$-\mathfrak{J}F(\xi + i\eta) < \varepsilon \log \log \eta,$$

provided only
$$\eta > \tau_1 = \tau_1(\varepsilon).$$

Let us take $\tau_0 = \tau_1$: then (5. 51) holds for all values of η which satisfy (5. 45). We have therefore

(5. 52)
$$-\mathfrak{J}F(\xi + i\eta) < \varepsilon \log \log \left\{\xi + \tau_1 \left(\frac{1}{\xi} + 1\right)^{\frac{a}{\xi}}\right\}.$$

But
$$\log \log \left\{\xi + \tau_1 \left(\frac{1}{\xi} + 1\right)^{\frac{a}{\xi}}\right\} \sim \log\left(\frac{1}{\xi}\right)$$

as $\xi \to 0$. Thus (5. 52) contradicts (5. 44). Therefore (5. 51) must be false, and the theorem is proved.

5. 6. Our next object is to prove

Theorem 5. 6. *If we denote by $\mathfrak{J}F(i\eta)$ the limit of $\mathfrak{J}F(\xi + i\eta)$ as $\xi \to 0$, so that*
$$-\mathfrak{J}F(i\eta) = \sum_1^{\infty} \frac{\sin \gamma_n \eta}{\gamma_n},$$
then
$$-\mathfrak{J}F(i\eta) = \Omega_R(\log \log \eta), \quad -\mathfrak{J}F(i\eta) = \Omega_L(\log \log \eta).$$

If $F(z)$ were regular for $\xi > 0$, or regular for $\xi > 0$ and continuous for $\xi \geq 0$, we could deduce Theorem 5. 6 from Theorem 5. 2 by means of a well-known theorem of Lindelöf. Our argument would in fact be much the same as that used by Bohr and Landau[1] in deducing
$$\zeta(1 + it) = \Omega(\log \log t)$$
from
$$\zeta(s) = \Omega(\log \log t) \quad (\sigma \geq 1).$$

In the present case, however, $F(z)$ is not continuous for $\xi \geq 0$. We proceed therefore to frame a modification of Lindelöf's theorem adapted for our purpose.

Lemma 5. 61. Suppose that

(i) *$f(z)$ is regular in the open semi-infinite strip*
$$0 < \xi < 1, \eta \geq \eta_0 > 0;$$

(ii) *$f(\xi + i\eta)$ tends to a limit $f(i\eta)$ as $\xi \to 0$, for every such value of η; and that positive constants A_1, A_2, and p exist such that*

[1] *Göttinger Nachrichten*, 1910, p. 316.

(iii) *given any number y greater than η_0, we can find a positive number $\delta = \delta(y)$ such that*

$$\left|\frac{f(\xi + i\eta)}{f(i\eta)}\right| < A_1$$

for

$$0 \leq \xi < \delta, \eta_0 \leq \eta < y;[1]$$

(iv)
$$|f(z)| < A_2$$

on the boundary of the strip;

(v)
$$|f(z)| = O(e^{\eta^p})$$

in the interior.

Then there is a constant A such that

$$|f(z)| \leq A$$

in the interior and on the boundary of the strip.

There is plainly no real loss of generality in supposing that η_0 is greater than any number fixed beforehand. Let us then choose a number q greater than p, and suppose that

$$q \arc\tan\left(\frac{1}{\eta_0}\right) < \frac{1}{2}\pi.$$

If $z = Re^{i\Theta}$, then

$$\frac{1}{2}\pi - \arc\tan\left(\frac{1}{\eta_0}\right) \leq \Theta \leq \frac{1}{2}\pi$$

for all points of the strip, so that

$$\cos q\left(\Theta - \frac{1}{2}\pi\right) > 0$$

and

$$\Re(-iz)^q > 0.$$

If now

$$\Phi(z) = f(z)e^{-\epsilon(-iz)^q},$$

where ϵ is positive, then

(5. 61)
$$|\Phi(z)| < A_2$$

at all points of the boundary. Also $\Phi(z) \to 0$ as $\eta \to \infty$, uniformly for $0 \leq \xi \leq 1$. We can therefore choose a value of y, as large as we please, and such that

$$|\Phi(\xi + iy)| < A_2 \quad (0 \leq \xi < 1).$$

The inequality (5. 61) is then satisfied at all points of the boundary of the rectangle R whose corners are $(0, \eta_0)$, $(1, \eta_0)$, $(1, y)$ and $(0, y)$.

Now let δ be the number $\delta(y)$ of condition (iii), and let R' be the left-hand, and R'' the right-hand, of the two rectangles into which R is divided by the line $\xi = \delta$. It follows from condition (iii) that

(5. 62)
$$\Phi(z) < A_1 A_2$$

at all points in or on the boundary of R'. It is moreover evident that $A_1 > 1$. Hence (5. 62) holds also on the boundary of R'', and therefore, since $\Phi(z)$ is regular in and on the boundary of R'', inside R'' also. Thus (5. 62) holds inside and on the boundary of the whole rectangle R. Making ϵ tend to zero, as in the proof of LINDELÖF's theorem, we see that

$$|f(z)| \leq A = A_1 A_2$$

inside and on the boundary of R. Thus the lemma is proved, with $A = A_1 A_2$.

5. 7. We can now prove Theorem 5. 6. Let us suppose that the first proposition asserted in the theorem is false. Then, given any positive number δ, there is an η_0 such that

(5. 71)
$$-3F(i\eta) < \delta \log\log\eta$$

for
$$\eta > \eta_0.$$

Let

(5. 72)
$$f(z) = e^{iF(z)}(\log z)^{-K}$$

where $K > \delta$. We shall show that $f(z)$ satisfies all the conditions of Lemma *5. 61* in the strip $0 < \xi \leq 1, \eta \geq 2$. That conditions (i) and (ii) are satisfied is evident, and (iv) is satisfied in virtue of (5. 71). It remains to verify (iii) and (v).

It follows from (5. 131) that

$$\sum_{\gamma_n > T} \frac{\sin \gamma_n \eta}{\gamma_n} = O(1),$$

uniformly for $T > x^2 = e^{2\eta}$. If then we choose N so that $\gamma_{N+1} > e^{2y}$, we have

$$(5.\ 73) \qquad \sum_{\nu}^{\infty} \frac{\sin \gamma_n \eta}{\gamma_n} = O(1)$$

uniformly for $\nu > N$, $2 \leq \eta \leq y$. It follows by partial summation that

$$(5.\ 74) \qquad \sum_{N+1}^{\infty} \frac{e^{-\gamma_n \xi} \sin \gamma_n \eta}{\gamma_n} = O(1),$$

uniformly for $\xi \geq 0$, $2 \leq \eta \leq y$. Thus

$$|-3F(\xi + i\eta) + 3F(i\eta)| = \left| \sum_{1}^{\infty} (1 - e^{-\gamma_n \xi}) \frac{\sin \gamma_n \eta}{\gamma_n} \right|$$

$$\leq N\xi + \left| \sum_{N+1}^{\infty} \frac{\sin \gamma_n \eta}{\gamma_n} \right| + \left| \sum_{N+1}^{\infty} \frac{e^{-\gamma_n \xi} \sin \gamma_n \eta}{\gamma_n} \right|$$

$$= N\xi + O(1);$$

$$\left| \frac{f(\xi + i\eta)}{f(i\eta)} \right| = e^{-3F(\xi + i\eta) + 3F(i\eta)} \left| \frac{\log (i\eta)}{\log (\xi + i\eta)} \right|^K$$

$$< K_1 e^{N\xi + K_2},$$

where K_1 and K_2 are constants; so that condition (iii) is satisfied if we take $\delta = \frac{1}{N}$.

We have finally to verify that $f(z)$ satisfies condition (v). It is known that

$$\frac{\psi(x) - x}{\sqrt{x}} = O(\log x)^2,$$

and it follows from (5. 131) that

$$(5.\ 75) \qquad \sum_{1}^{\nu} \frac{\sin \gamma_n \eta}{\gamma_n} = O(\eta^2),$$

uniformly for $\gamma_\nu > x^2 = e^{2\eta}$. But, if $\gamma_\nu \leq e^{2\eta}$, we have

$$\sum_{1}^{\nu} \frac{\sin \gamma_n \eta}{\gamma_n} = O \sum_{\gamma_n < e^{2\eta}} \frac{1}{\gamma_n} = O \sum_{k \leq e^{2\eta}} \frac{\log k}{k} = O(\eta^2).$$

Thus (5. 75) holds uniformly for all values of ν; and so, by partial summation we obtain

$$-3F(\xi + i\eta) = \sum_{1}^{\infty} \frac{e^{-\gamma_n \xi} \sin \gamma_n \eta}{\gamma_n} = O(\eta^2),$$

$$(5.\ 76) \qquad f(z) = e^{O(\eta^2)} o(1) = O(e^{\eta^3}).$$

Thus condition (v) is satisfied with $p = 3$.

The function $f(z)$ therefore satisfies all the conditions of Lemma 5. 61, and so

$$f(z) = O(1)$$

for $0 \leq \xi \leq 1$, $\eta \geq 2$. Hence

$$e^{iF(z)} = O|(\log z)^K|$$

and so

$$(5.\ 77) \qquad -3F(z) < 2K \log \log \eta$$

for all sufficiently large values of η. But K, being restricted only to be greater than δ, is arbitrarily small; and so (5. 77) is in contradiction with Theorem **5. 2** It follows that (5. 71) is false, and therefore Theorem **5. 6** is true.

5. 8. From (5. 132) and Theorem **5. 6** we can at once deduce the theorem which it is our main object to prove, viz.,

Theorem 5. 8. *We have*

$$\psi(x) - x = \Omega_R(\sqrt{x} \log \log \log x), \quad \psi(x) - x = \Omega_L(\sqrt{x} \log \log \log x).$$

All that remains is to deduce from these relations the corresponding relations which involve $\Pi(x)$. This deduction presents one point of interest. It might be anticipated that nothing more than a partial summation would be needed; and if the one-sided relations involving Ω_R and Ω_L are replaced, in premiss and conclusion, by a single relation involving Ω, this is actually so. But the argument now required is a little more subtle and involves an appeal to the results established in 2. 25 concerning the Cesàro means of $\psi(x) - x$.

We have to show that

$$(5.\ 81) \qquad \Pi(x) - Li\, x = \Omega_R\left(\frac{\sqrt{x} \log \log \log x}{\log x}\right), \quad \Pi(x) - Li\, x = \Omega_L\left(\frac{\sqrt{x} \log \log \log x}{\log x}\right).$$

It is plainly enough to establish similar relations for the function

$$(5.\ 82) \qquad F(x) = \Pi(x) + \frac{1}{2}\Pi(\sqrt{x}) + \frac{1}{3}\Pi(\sqrt[3]{x}) + \ldots.$$

It is of course this function, and not $\Pi(x)$, which can be connected with $\psi(x)$ by a partial summation. We have in fact

$$f(x) = \sum_2^x \frac{\psi(n) - \psi(n-1)}{\log n}$$

$$= \sum_2^x \frac{1}{\log n} + \sum_2^x \frac{\{\psi(n) - n\} - \{\psi(n-1) - (n-1)\}}{\log n},$$

$$f(x) = Li\,x + O(1) + \sum_2^x \frac{\{\psi(n) - n\} - \{\psi(n-1) - (n-1)\}}{\log n}$$

$$= Li\,x + O(1) + \sum_2^{x-1} \{\psi(n) - n\} \left\{\frac{1}{\log n} - \frac{1}{\log(n+1)}\right\} + $$

$$+ \frac{\psi(x) - [x]}{\log\{[x]+1\}},$$

(5. 83)
$$f(x) - Li\,x - \frac{\psi(x) - x}{\log x} = \sum_2^x \frac{\psi(n) - n}{n(\log n)^2}$$

$$+ O \sum_2^x \frac{|\psi(n) - n|}{n^2(\log n)^2} + O(1)$$

$$= \sum_2^x \frac{\psi(n) - n}{n(\log n)^2} + O(1).$$

Let

$$\chi(x) = \sum_2^x \{\psi(n) - n\}.$$

Then

$$\chi(n) = O\left(n^{\frac{3}{2}}\right),$$

by Theorem 2. 25. Hence

(5. 84)
$$\sum_2^x \frac{\psi(n) - n}{n(\log n)^2} = \sum_2^x \frac{\chi(n) - \chi(n-1)}{n(\log n)^2}$$

$$= \sum_2^x \chi(n) \left[\frac{1}{n(\log n)^2} - \frac{1}{(n+1)\{\log(n+1)\}^2}\right] + \frac{\chi[x]}{([x]+1)\{\log([x]+1)\}^2}$$

$$= O \sum_2^x \frac{1}{\sqrt{n}(\log n)^2} + O\left\{\frac{\sqrt{x}}{(\log x)^2}\right\}$$

$$= O\left\{\frac{\sqrt{x}}{(\log x)^2}\right\}.$$

From (5. 83) and (5. 84) it follows that

(5. 85)
$$f(x) - Li\,x - \frac{\psi(x) - x}{\log x} = O\left\{\frac{\sqrt{x}}{(\log x)^2}\right\};$$

and from (5. 85) and Theorem **5. 8** we deduce

Theorem 5. 81. *We have*

$$\Pi(x) - Li\,x = \Omega_R\left(\frac{\sqrt{x}\log\log\log x}{\log x}\right), \quad \Pi(x) - Li\,x = \Omega_L\left(\frac{\sqrt{x}\log\log\log x}{\log x}\right).$$

We refer in the introduction (1. 5) to the other important applications which may be made of this section.

Additional Note.

While we have been engaged on the final correction of the proofs of this memoir, which was presented to the *Acta Mathematica* in the summer of 1915, two very interesting notes by M. DE LA VALLÉE-POUSSIN entitled 'Sur les zéros de $\zeta(s)$ de RIEMANN' have appeared in the *Comptes Rendus* (23 Oct. and 30 Oct. 1916). M. DE LA VALLÉE-POUSSIN obtains, by methods quite unlike those which we use here, a considerable part of the results of section 4 (18 Nov. 1916).

CORRECTIONS

p. 123, *last line of footnote* 4. For 'Hilfsatz' read 'Hilfssatz'.
p. 135, *line* 3. Read $\log|s|$ for $\log|t|$, with consequential changes below. See E. Landau, *Math. Zeitschrift*, 1 (1918), 213–19, footnote 1 on pp. 213–14.
p. 155, *line* 6. For 'Henc' read 'Hence'.
p. 161 (2.546). Read $o(\nu^{-\frac{1}{2}+\epsilon})$.
p. 163 (3.122). Insert $p^{-\frac{1}{2}m}$ before $\log p$.
Theorem 3.1. In line 5 of the statement, replace $<$ by $>$.
p. 165, *line* 2 *from below*. Read $O(\log|s|)$, though this is unimportant since $s^{-\frac{1}{2}\sigma}\log|s| = O(1)$.
p. 172, *line* 12. For (3.6142) read (3.6141).
p. 172, *line* 13. Insert $k_{1,1}$ after 'in'.

COMMENTS

Two of the main results, namely Theorems 2.34 and 4.41, were communicated to the London Mathematical Society at its meeting on 11 February 1915. See *Proceedings* (2), 14, xiv–xvi.

§ 1. Comments on individual topics will appear under the appropriate sections.

§ 2.1. The hypothesis (iii) $F(s) = O(e^{C|t|})$ may be omitted from Theorems 2.121, 2.13, 2.14, 2.15. The basic form of this discovery, which clarifies the logical relationship between the prime-number theorem and the properties of $\zeta(s)$, is the Wiener–Ikehara theorem. This arose out of N. Wiener's general Tauberian theory, but, as a result of the work of S. Bochner, H. Heilbronn, and E. Landau, it can now be proved without reference to this theory. See, e.g. Widder, pp. 233–6; A. E. Ingham, *Proc. London Math. Soc.* (2), 38 (1935), 458–80, Theorems 3(l), $2_1(l)$. (See also the first paragraph of the comments on 1921, 6.)

In connexion with § 2.15 (ii) we note that, besides omitting (iii) from the above theorems, we may also omit the hypothesis (iv) $\lambda_n/\lambda_{n-1} \to 1$, provided that, in Theorems 2.121 and 2.13, we supplement the 'real' alternative under (v) by the condition

$$\underline{\lim}\,\lambda_n^{-\epsilon} a_n \geqslant 0 \quad \text{or} \quad \overline{\lim}\,\lambda_n^{-\epsilon} a_n \leqslant 0,$$

as the case may be (this supplement being redundant in Theorem 2.14 since (iv) is implied by the other hypotheses). See Ingham, loc. cit. (§ 7 and top of p. 477).

The 'equivalence' concept described in footnote 2 on p. 120 lost its significance in 1949 when an 'elementary' proof of the prime-number theorem was found by A. Selberg and P. Erdős. See e.g. Hardy and Wright, § 22.14–16.

§ 2.2. The conjecture at the end of § 2.24 (p. 139) may be proved by Landau's method (see below, under § 2.3).

In the statement of Theorem 2.25 the word 'so' raises a question of logic, since the equivalence theorem for Riesz and Cesàro means applies only to convergent means. In the present context, however, the inference is valid (although the means oscillate) since $\Lambda(n) - 1 = O(\log n)$, from which it easily follows that the Riesz and Cesàro means of order $\delta > 0$ differ by $O(\log \omega)$.

§ 2.3. The desired proof of the converse of Theorem 2.34 was supplied simultaneously, and without knowledge of the work of Hardy and Littlewood, by E. Landau, *Math. Zeitschrift*, 1 (1918), 1–24 (1–5, 24). Landau (ibid. 213–19) also gave a modified proof of the direct theorem.

§ 2.4. For further developments of mean-value theorems, see Titchmarsh, ch. 7.

In Lemma 2.413 the condition $f(x) = O(e^{\delta x})$ is superfluous. The special functions $L(x)$ may be replaced by any positive $L(x)$ for which $L(cx) \sim L(x)$ as $x \to \infty$ for every fixed $c > 0$; and other proofs are now available. See Hardy, *Divergent series* (Oxford, 1949), Theorem 108 and the notes on pp. 175–6 (which also include comments on the condition $\lambda_n/\lambda_{n-1} \to 1$ in Lemma 2.113, analogous to those made above for Theorems 2.121–2.15); A. E. Ingham, *Proc. London Math. Soc.* (3), 14A (1965), 157–73.

§ 2.5. For further comments on Ramanujan's formula see Titchmarsh, § 9.8.

§ 3. The Abel means of this series may be linked with the formulae of § 2.2. Thus, by an extension of (2.214) and an application of Stirling's theorem to $\Gamma(ap)$, we find that the functions

$$f(y) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n^{1/a} y},$$

$$s(\epsilon) = \sum_{\gamma > 0} \gamma^{-\omega} e^{a(i\gamma \log(\gamma\theta) - \epsilon\gamma)},$$

with

$$\omega = \tfrac{1}{2} - \tfrac{1}{2}a, \quad y = re^{i(\frac{1}{2}\pi - \epsilon)}, \quad e\theta r = a,$$

behave similarly (for our purpose) when $\epsilon \to 0+$. We note that, when $r = \pi$ and $1/a$ is a positive integer, $f(y) \sim -f(\pi\epsilon)$, so that $s(\epsilon)$ is exactly of order ϵ^{-a}. This seems to indicate that, for suitable a and θ, the index $\tfrac{1}{2}(1+a)$ in (1.33) cannot be reduced, contrary to what is suggested at the end of § 1.3.

§ 4. See comments following 1921, 2.

§ 5. $\Pi(x) - Li\,x$ is the function more usually denoted by $\pi(x) - \mathrm{li}\,x$.

By the use of other 'absolute convergence factors' in place of the $e^{-\gamma_n \xi}$ in (5.21), we can avoid the Phragmén–Lindelöf theorem, and thus establish a closer link between the Analysis and the Arithmetic. See S. Skewes, *J. London Math. Soc.* 8 (1933), 277–83; A. E. Ingham, *Acta Arith.* 1 (1936), 201–11. [The latter proof may be further simplified by the use of the function $\mathfrak{K}(y) = 1$ ($|y| \leqslant 1$), 0 ($|y| > 1$) in place of the $\mathfrak{K}(y)$ actually used.]

The arguments based on the Riemann Hypothesis (RH) can be made 'effective' by the insertion of explicit constants in the inequalities. Thus (on RH) we can find a numerical X such that $\pi(x) - \mathrm{li}\,x$ changes sign in $2 < x < X$. This is more difficult on NRH (the negation of RH), although the theoretical Ω-result is 'all the more true' on NRH. Using an idea supplied by Littlewood, however, Skewes (*Proc. London Math. Soc.* (3), 5 (1955), 48–70) worked out a possible X' on NRH (or rather NH, where H is an approximation to RH calling for a slight increase in X). Possible values (on H and NH) are $X = e_3(7{\cdot}703)$ and $X' = e_4(7{\cdot}705)$, where $e_n(\ \ )$ is an n-fold exponential. Thus X' is an unconditional 'Skewes number'.

The difficulty with NRH (or NH) is that the effect of one zero may be largely neutralized by interference from neighbouring zeros. Other methods of meeting this difficulty have been proposed by G. Kreisel, *J. Symbolic Logic*, 17 (1952), 43–58, and by P. Turán (see S. Knapowski, *J. London Math. Soc.* 36 (1961), 451–60). A manuscript left by A. M. Turing contains new ideas which it is reasonable to suppose will lead to an unconditional Skewes number of roughly the same form as the above X (an e_3 instead of e_4). The manuscript, however, is very rough, and it is not possible to summarize it here. We may mention, however, that Turing says that his method was inspired by Ingham's paper referred to above (he uses a modified form of Ingham's kernel $\mathfrak{K}$), and that he contemplated extensive calculation on the first 1000 or so zeros. It is to be hoped that someone will carry out a complete proof on Turing's lines. See also R. S. Lehman, *Acta Arith.* 11 (1966), 397–410.

The seeming inaccessibility of an explicit solution of $\pi(x) > \mathrm{li}\,x$ ($x > 2$) is in striking contrast to the situation in some analogous problems, such as the one mentioned at the end of § 1.5. Thus, if $\pi(x; k, l)$ denotes the number of primes $p \leqslant x$ with $p \equiv l \pmod k$, the inequality $\pi(x; 4, 1) > \pi(x; 4, 3)$ has known solutions, of which the least is $x = 26861$. See John Leech, *J. London Math. Soc.* 32 (1957), 56–58; D. Shanks, *Mathematical Tables and other aids to computation*, 13 (1959), 272–84. The general problem of changes of sign of $\pi(x; k, l_1) - \pi(x; k, l_2)$ and associated functions has been considered from various aspects, but methods at present available seem to lead in the main to partial or conditional solutions. See S. Knapowski and P. Turán, *Acta Math. Acad. Sci. Hung.* 13 (1962), 299–314; 315–42; 343–64; 14 (1963), 31–42; 43–63; 65–78; 241–50; 251–68; *Acta Arith.* 9 (1964), 23–40; 10 (1964), 293–313; 11 (1965), 115–27; 147–61; 193–202; *J. d'Analyse Mathématique* 14 (1965), 267–74.