Locating the centre of mass by mechanical means

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This article discusses *moment planimeters*, which are mechanical devices with which is it possible to locate the centre of mass of an irregular plane shape by mechanical and graphical methods. They are a type of analogue computing device. In addition to this they may be used to find the static moment (first moment) and moment of inertia (second moment) of a shape about a fixed line. Moment planimeters, sometimes called *integrometers* or *integrators*, are direct developments of the planimeter which is a mechanical device used to directly measure the area of a plane shape. While planimeters are reasonably well known, linear planimeters are less common than the polar planimeters of Amsler. Hence in this article we explain how planimeters work through the example of a linear planimeter, and then consider how these may be adapted to find the centre of mass. More detailed comparisons between other types of area measuring planimeters may be found in the comprehensive survey article of [2].

1 Area and centre of mass

Consider the region enclosed by the closed curve in Figure 1, through which we have drawn the x-axis. We consider the area to be split into two regions by this axis, and these regions are described by the the two functions, $f_1(x)$ and $f_2(x)$. Since a general plane shape cannot be described in this way the assumption represents a considerable loss of generality, hence we shall provide alternative explanation in a moment. The area of this shape will be

$$\int_a^b f_1(x) - f_2(x)\,\mathrm{d}x.$$

In the linear planimeter a rigid straight line of length l is constrained to move so that one end, P, traces around the boundary of the region. The other end, Q, is constrained to move along the x-axis. This is shown in Figure 1. Note that

$$y = f_1(x) = l\sin(\theta). \tag{1}$$

As is usual for a planimeter, we fix a freely rotating disc using this line as an axel, an example of which is shown in Figure 3. In this arrangement the roll of the disc will be the component of the motion perpendicular to the line. If we consider an infinitesimally thin vertical strip of width $\mathrm{d}x$ and height $f_1(x)$, then during the horizontal motion from x to $x + \mathrm{d}x$, the wheel will record a roll of

$$\mathrm{d}w = \sin(\theta)\,\mathrm{d}x. \tag{2}$$

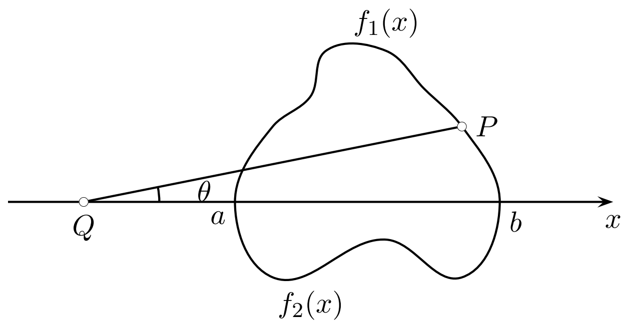

Figure 1: An irregular plane region, R

Hence

$$\int_a^b f_1(x)\ \mathrm{d}x = l \int_a^b \sin(\theta)\mathrm{d}x = l \int_a^b \mathrm{d}w.$$

If we denote the the total roll recorded on the wheel around the boundary of the region R from $x = a$ to $x = b$ along f_1, and back along f_2 by $\oint_{\delta R}$ we have that

$$\int_a^b f_1(x) - f_2(x)\ \mathrm{d}x = l \oint_{\delta R} \mathrm{d}w.$$

This illustrates that l multiplied by the total roll recorded while P traces around the boundary will be equal to the area of the shape. This is the fundamental property of planimeters.

Next we turn attention to the centre of mass. Imagine a thin uniform strip of width $\mathrm{d}x$, and hight y. The contribution this strip makes to the distance of the centre of mass of the whole shape from the x-axis will be

$$y\frac{y}{2}\mathrm{d}x.$$

And hence, $\bar{y}$, the distance of the centre of mass from the x-axis will be given by

$$\bar{y} = \frac{1}{2}\frac{\int y^2\ \mathrm{d}x}{\int y\ \mathrm{d}x} = \frac{\int\int y\ \mathrm{d}y\,\mathrm{d}x}{\int\int \mathrm{d}y\,\mathrm{d}x}. \tag{3}$$

From this it apparent that it will be sufficient to contrive a planimeter capable of being able to measure $\int y^2\ \mathrm{d}x$, since we are already capable of measuring the area.

Let us assume that we can attach another wheel at P which is at an angle of $\frac{\pi}{2} - 2\theta$ to the x-axis. Then the roll recorded will be

$$\sin(\tfrac{\pi}{2} - 2\theta) = \cos(2\theta) = 1 - 2\sin^2(\theta).$$

Considering the motion from $x = a$ to $x = b$ along the function $y = f_1(x)$ we have,

$$l^2 \int_a^b \mathrm{d}w = \int_a^b l^2 - 2l^2\sin^2(\theta)\ \mathrm{d}x = \int_a^b l^2 - 2y^2\ \mathrm{d}x.$$

If we the integrate back along f_2, the l^2 terms in the two integrals cancel, so that

$$\frac{-l^2}{2}\oint_{\delta R} \mathrm{d}w = \int_R y^2\ \mathrm{d}x,$$

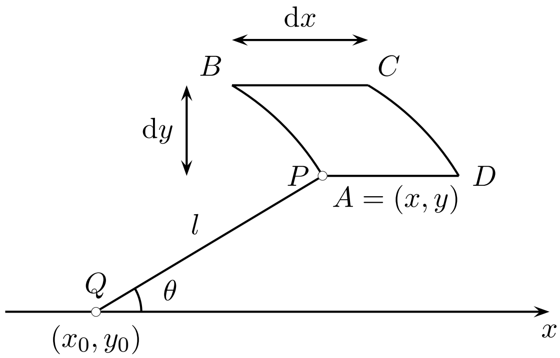

Figure 2: A small element

and so

$$\bar{y} = \frac{\frac{-l^2}{4} \oint_{\delta R} \mathrm{d}w}{\int y \,\mathrm{d}x}.$$

By this procedure we have calculated $\bar{y}$ and hence the line parallel to the x-axis on which the centre of mass lies. We choose another line for the x-axis, not parallel to the original, and repeat this procedure. The intersection of the two lines thus obtained locates the centre of mass.

2 Small elements

In this section we take a slightly different approach and, instead of considering integration of functions representing the boundary of the curve, we assume that the plane region has been decomposed into small curvy-parallelograms such as $R = ABCD$ shown in Figure 2. Here, the line PQ is of fixed length l, the point P moves around the boundary of the region R and the other end Q runs along the x-axis and so is constrained to move along $(x_0, y_0) = (x_0, 0)$. We note that for $ABCD$, the area equals $\mathrm{d}x\mathrm{d}y$ and the distance of the centre of mass of $ABCD$ from the x-axis is

$$\bar{y} = \frac{2y + \mathrm{d}y}{2}.$$

The point P moves around the perimeter from A, which has coordinates (x, y), to B, C, D and back to A. In each portion of this movement we examine the roll recorded by the two wheels considered in the previous Section and relate these to the area and centre of mass.

We consider first a wheel using the line PQ as an axel. As this moves from A to B, the point Q is fixed and the roll recorded w_{AB} is a pure roll proportional to the arc length $l\mathrm{d}\theta$. This is equal and opposite to that as the line moves from C to D, ie $w_{AB} = -w_{CD}$. As P moves from B to C the angle PQ makes with the x-axis is constant at $\theta + \mathrm{d}\theta$ with the horizontal and

$$w_{BC} = \sin(\theta + \mathrm{d}\theta)\mathrm{d}x$$

so that

$$lw_{BC} = l\sin(\theta + \mathrm{d}\theta)\mathrm{d}x = (y + \mathrm{d}y)\mathrm{d}x.$$

Similarly

$$lw_{DA} = -l\sin(\theta)\mathrm{d}x = -y\mathrm{d}x.$$

If we define the roll around the perimeter of this small element to be $\mathrm{d}w := w_{AB} + w_{BC} + w_{CD} + w_{DA}$ then

$$l\mathrm{d}w = \mathrm{d}x\,\mathrm{d}y.$$

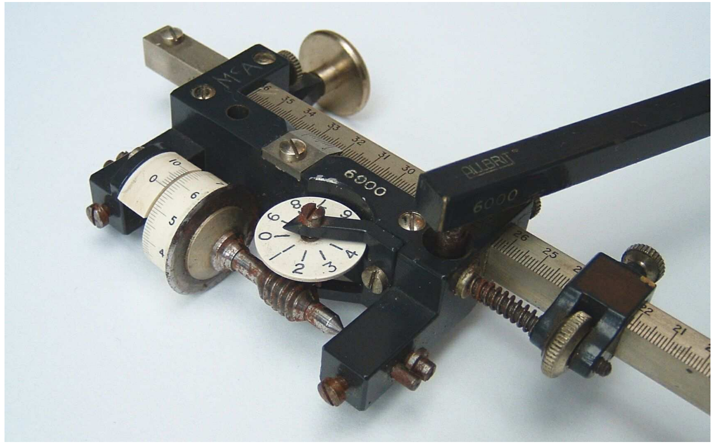

Figure 3: Details of the roll recording wheel on a planimeter

Every reasonable plane region R can be decomposed into small elements consisting of such curvy parallelograms. When doing this the rolls along internal edges of this decomposition cancel leaving only the roll around the outside perimeter to consider. Hence we have that

$$l \oint_{\delta R} \mathrm{d}w = \int\int_R \mathrm{d}x\,\mathrm{d}y,$$

where $\oint_{\delta R} \mathrm{d}w$ is the total roll as P moves around the (piecewise smooth) boundary of the region R, and the right hand side is nothing but the area.

The second wheel is at P on an axel at an angle $\frac{\pi}{2} - 2\theta$ to the horizonal. As before, $w_{AB} = -w_{CD}$. Furthermore,

$$l^2 w_{BC} = l^2 \sin(\tfrac{\pi}{2} - 2\theta - 2\mathrm{d}\theta)\mathrm{d}x = l^2\mathrm{d}x - 2(y + \mathrm{d}y)^2\mathrm{d}x,$$

and

$$l^2 w_{DA} = -l^2 \sin(\tfrac{\pi}{2} - 2\theta)\mathrm{d}x = -l^2\mathrm{d}x + 2y^2\mathrm{d}x,$$

Define, as before, $\mathrm{d}w := w_{AB} + w_{BC} + w_{CD} + w_{DA}$ then

$$l^2\mathrm{d}w = -4\frac{2y + \mathrm{d}y}{2}\mathrm{d}x\,\mathrm{d}y.$$

Again,

$$\frac{-l^2}{4}\mathrm{d}w = \frac{2y + \mathrm{d}y}{2}\mathrm{d}x\mathrm{d}y = \bar{y}\mathrm{d}x\,\mathrm{d}y.$$

Hence,

$$\bar{y} = \frac{\frac{-l^2}{4}\oint_R \mathrm{d}w}{\int\int_R \mathrm{d}x\,\mathrm{d}y}.$$

3 Green's Theorem for the plane

A justification of the polar planimeter of Amsler was given using Green's Theorem in [1]. We justify the results of the informal arguments in the previous sections using a similar approach. Assume we have a vector field $V(x,y) = (V_x(x,y), V_y(x,y))$. Green's Theorem states that

$$\oint_{\delta R} V_x \,\mathrm{d}y + V_y \,\mathrm{d}x = \int\int_R \mathrm{curl}(V)\,\mathrm{d}x\,\mathrm{d}y,$$

where $\oint_{\delta R}$ is the line integral around the (piecewise smooth) boundary of the region R. Imagine a vector field of unit vectors in the plane, which we denote by V. If a wheel is attached at P which is constrained to always point in the direction of this field, the roll of the wheel will record the total component of the vector field in the direction of the motion, effectively measuring this integral. If we denote the roll of the wheel by $\mathrm{d}w$ we have

$$\oint_{\delta R} \mathrm{d}w = \oint_{\delta R} V_x \,\mathrm{d}y + V_y \,\mathrm{d}x = \int\int_R \mathrm{curl}(V)\,\mathrm{d}x\,\mathrm{d}y.$$

For the linear planimeter we imagine a vector field generated by attaching a unit vector perpendicular to the end of the line PQ, of fixed length l, at P. It remains to find this vector field, and the corresponding curl.

As before in Figure 2, assume that when P is at A it has coordinates (x,y) and the other end Q runs along the x-axis and so is constrained to move along $(x_0, y_0) = (x_0, 0)$. Then we have

$$l^2 = (x-x_0)^2 + (y-y_0)^2,$$

so that

$$x - x_0 = \sqrt{l^2 - y^2}$$

and furthermore,

$$\sin(\theta) = \frac{y-y_0}{l} = \frac{y}{l} \quad \text{and} \quad \cos(\theta) = \frac{x-x_0}{l} = \sqrt{1-\frac{y^2}{l^2}}.$$

The planimeter vector field, which of course does not depend on the x-coordinate, is then

$$V = \begin{pmatrix} V_x(x,y) \\ V_y(x,y) \end{pmatrix} = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} = \begin{pmatrix} -\frac{y}{l} \\ \sqrt{1-\frac{y^2}{l^2}} \end{pmatrix}.$$

Since

$$\mathrm{curl}(V) = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}, \tag{4}$$

a trivial calculation shows that

$$\mathrm{curl}(V) = \frac{1}{l}.$$

Hence

$$l\oint_{\delta R} \mathrm{d}w = \int\int_R \mathrm{d}x\,\mathrm{d}y.$$

Turning attention to the centre of mass we have the vector field where the unit vector points points at an angle of $\frac{\pi}{2} - 2\theta$ to the horizontal. Hence,

$$V_x(x,y) = -\sin(\tfrac{\pi}{2} - 2\theta) = 2\sin^2(\theta) - 1 = 2\frac{y^2}{l^2} - 1,$$

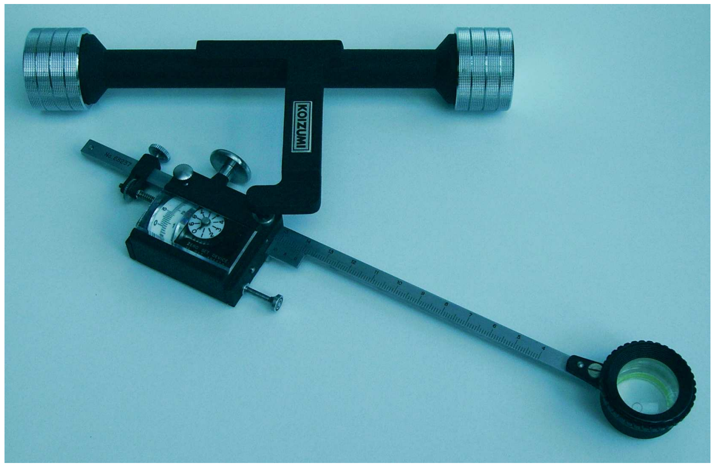

Figure 4: An example of a Koizumi linear roller planimeter

and

$$V_y(x, y) = \cos(\tfrac{\pi}{2} - 2\theta) = 2\cos(\theta)\sin(\theta) = 2\frac{y}{l}\sqrt{1 - \frac{y^2}{l^2}}.$$

By (4) we have that

$$\mathrm{curl}(V) = -\frac{4y}{l^2}.$$

Hence, by Green's Theorem we have that

$$\frac{-l^2}{4}\oint_{\delta R} \mathrm{d}w = \int\int_R y \,\mathrm{d}x \,\mathrm{d}y,$$

as required by (3) to find the centre of mass.

These are both rather trivial applications of Green's Theorem.

4 Further generalizations

In the previous sections we have considered how to calculate $\int y \,\mathrm{d}x$ and $\int y^2 \,\mathrm{d}x$. Further generalizations naturally occur with $\int y^3 \,\mathrm{d}x$, $\int y^4 \,\mathrm{d}x$ and so on. To calculate $\int y^3 \,\mathrm{d}x$, for example, we note that

$$\sin^3(\theta) = \frac{3}{4}\sin(\theta) - \frac{1}{4}\sin(3\theta),$$

and so it will be sufficient to have an instrument with wheels capable of recording the motion of a wheel at an angle 3θ. Further generalizations are possible, and devices along these lines were indeed made and used. It is the practical considerations we turn to in the next section.

5 Practical implementations

The most popular practical implementation of a planimeter is the polar planimeter of Amsler. The essential difference between this and the linear planimeter is that the point Q is constrained to move in

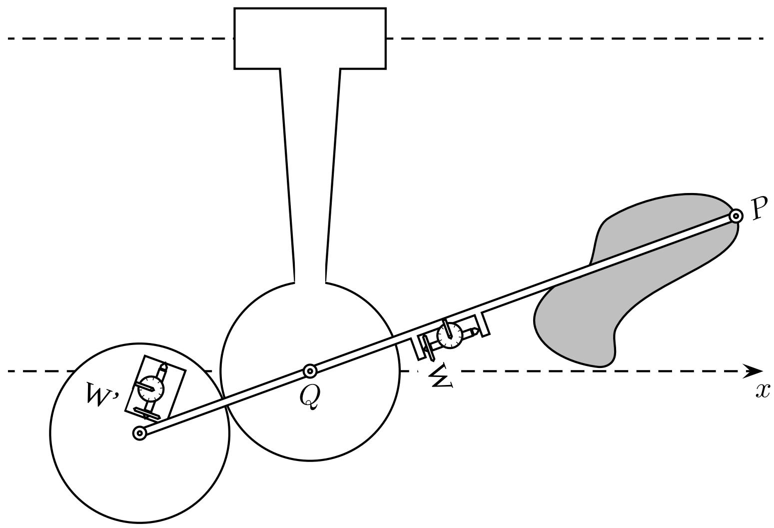

Figure 5: A moment planimeter schematic

a circular arc rather than a straight line. Linear planimeters were produced commercially, an example of which is shown in Figure 4. The point P can be located in the circular magnifying glass, and Q is constrained to move in a vertical straight line by the trolley, rather than along the x-axis as in our examples. Notice that the wheel need not actually be at P, but may be at any convenient position using an axel offset from, but parallel to, the line PQ.

Perhaps the simplest moment planimeter is an extension of the linear planimeter, and a schematic of such a device is given in Figure 5. The point Q is constrained to move in a straight line, marked as the x-axis, by an arm which is mounted upon a trolley. The wheel W shown is used to measure the area of the shape around which P traces. At the point Q, fixed to the trolley is a gear wheel, which acts on the second gear wheel attached to the line PQ in such a way as to ensure that the angle of the recording wheel W' is at $\frac{\pi}{2} - 2\theta$ to the horizontal as required by the theory. A direct reading of the moment can be obtained if the wheel W' is calibrated to take account of the factor $\frac{-l^2}{4}$.

Such a device is shown in Figure 6. The whole instrument is shown to the left of the figure. The top of the figure comprises a trolley, constraining the device to move horizontally. The point P is below the arm to the bottom right, and the ability to move this point effectively changes the length l. Notice the three wheels, together with their Vernier scales from which a reading is taken. One marked a is for area, the other m for centre of mass and the third i for moments of inertia. The details of the gear wheels are shown in the figure to the right which shows the reverse of the instrument. Other configurations were possible, such as the Hele-Shaw Integrator which employs three glass spheres upon which the roll recording wheels run, thus eliminating inaccuracies caused by inconsistencies of the contact of the paper with the wheels.

In mechanical engineering, it is common to want to find the work done in each stroke of an engine. It is relatively easy to measure the instantaneous pressure in the cylinder, and by finding the area under the graph of pressure against time the work done can be calculated. A linear planimeter specifically for this task is that of [4]. Finding the centre of mass was a problem of particular importance to navel architects, who needed to ensure that the centre of mass of a ship was below the water line. An interesting essay on this topic is given by Robb, A. M. in [3, pg 206–217].

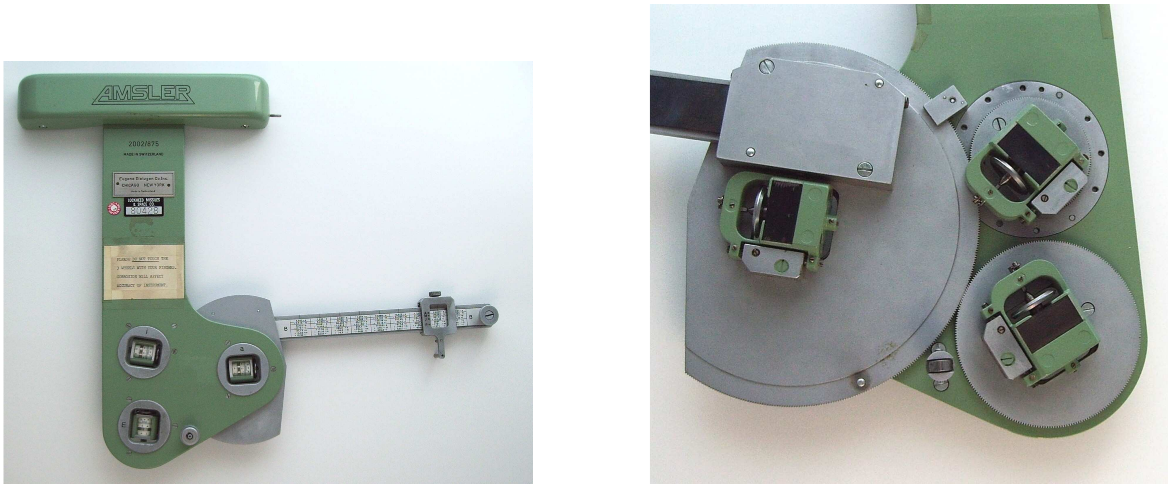

Figure 6: An Amsler moment planimeter

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