Simple models of competition

by Federico Etro
November 2006

Market competition and market equilibrium are the subjects of these notes. We will study them in a formal and general way and apply the results to different environments, characterized by competition in the market where firms choose their prices or their outputs or even other auxiliary strategies, and by competition for the market, where firms invest in R&D to innovate. The ultimate objective will be to employ our theoretical results to derive some insights on policy issues, and in particular on antitrust issues concerning abuse of dominance. For this purpose, we will pay a close attention to the behaviour of market leaders and to the interaction between these firms and the other firms, the followers.

Our analysis will focus on four general typologies of competition and their related equilibria. The first goes back to the early analysis of Cournot (1838) who was the real pioneer of modern economic analysis and the first to study market structures for homogeneous goods where firms choose output and the equilibrium between demand by consumers and supply by all firms determines the price. While the analysis of Cournot goes back to the first half of the XIX century, his equilibrium concept corresponds to the concept now associated with Nash (1950): each firm independently chooses its strategy to maximize profits taking as given the strategy of each other firm. This idea can be applied to more general market structures and when firms choose strategies different from their output, for instance when they choose their prices, or their investments. Hence, we will generally refer to a *Nash equilibrium* when an exogenous number of firms compete choosing simultaneously their strategies.

A second typology of competition extends these models of imperfect competition to endogenous entry of firms. A market is in equilibrium only when there are not further incentives for other firms to enter into it and make positive profits. This idea is often associated with the studies on competitive markets in partial equilibrium of the second half of the XIX century, and in particular by Marshall (1890), hence we will refer to this as to a *Marshall equilibrium.* In modern terms, the concept of Nash equilibrium with free entry characterizes this situation.

A third typology was introduced by Stackelberg (1934) who studied markets where a firm has a leadership over the others: a leadership is associated with the ability to choose strategies and commit to them before the other firms. Under Stackelberg competition, the leader can exploit its first mover advantage taking into account the reactions of the followers. Notice that the behaviour of a leader in a *Stackelberg equilibrium* requires a commitment power whose credibility is crucial but sometimes not realistic. However, recent formal analysis by Dixit (1980) and Fudenberg and Tirole (1985) has shown that proper strategic investments can be a valid substitute for this commitment: a firm can invest in cost reductions, in advertising, in R&D or in other investments to acquire a competitive advantage over the other firms. We will return to this possibility in the next chapter, while in this one we will analyze the simpler case in which a leader has indeed the ability to commit to strategies before the other firms.

The last typology of competition, *Stackelberg competition with free entry*, completes our taxonomy of the basic forms of competition combining the analysis of leadership and entry. In the second half the XX century there have been some attempts to model both these elements. The first is the literature on entry deterrence associated with Bain (1956), Sylos Labini (1956) and Modigliani (1958), who took in consideration the effects of entry on the predatory behaviour of market leaders mainly in the case of perfectly substitute goods and constant or decreasing marginal costs. The second is the dominant firm theory, which tried to explain the pricing decision of a market leader facing a competitive fringe of firms that take as given the same price of the leader. Assuming that the supply of this fringe is increasing in the price, the demand of the leader is total market demand net of this supply and its profit maximizing price is above marginal cost but constrained by the competitive fringe. The third attempt is the theory of contestable markets by Baumol, Panzar and Willig (1982), which shows that, in absence of sunk costs of entry, the possibility of "hit and run" strategies by potential entrants is compatible only with an equilibrium price equal to the average cost. One of the main implications of this result is that "one firm can be enough" for competition when there are aggressive potential entrants. As we will see, these theories can be developed in a more general game-theoretic framework based on Etro (2006).

Our initial focus will be on simple models of competition in quantities. After presenting the basic linear model which assumes constant marginal costs and homogenous goods, we will extend it to U-shaped cost functions and to product differentiation. Then we will present a simple model of competition in prices. Finally we will also discuss a simple model of competition for the market, that is a contest where firms compete investing to conquer a new market, and we will discuss the role of incumbent monopolists in such a contest.

1 A Simple Model of Competition in Quantities

Let us consider the simplest market one can think of. Imagine a homogenous good whose demand is linearly decreasing in the price, say $D(p) = a - p$ where $a > 0$ is a parameter representing the size of the market. If total production by all the firms is $X = \sum_{i=1}^{n} x_i$, where x_i is the production of each firm $i = 1, 2, ..., n$, in equilibrium between supply and demand we must have $X = D(p) =$

$a - p$, which provides the so called inverse demand function in equilibrium:

$$p = a - X = a - \sum_{i=1}^{n} x_i$$

Basically, the larger is production, the smaller must be the equilibrium price at which firms can sell the good. In this simple example, this relation is linear.

Imagine now that each firm can produce the good with the same standard technology. Producing x units requires a fixed cost of production $F \geq 0$ and a variable cost cx where $c \geq 0$ is a constant unitary cost, or marginal cost of production.[1] Notice that, while the average variable cost is constant (equal to c), the average total cost is decreasing in the output (equal to $(cx+F)/x = c+F/x$). Hence the general profit function of a firm i is the difference between revenue and costs:

$$\begin{aligned} \pi_i &= px_i - cx_i - F = \\ &= \left(a - \sum_{i=1}^{n} x_i\right) x_i - cx_i - F \end{aligned}$$

Before analyzing different forms of competition in this simple set up, let us investigate a simpler and extreme situation, that of a monopoly. Consider a single firm in this market producing x. Its profits must be given by:

$$\pi = px - cx - F = (a - x)x - cx -$$

F. Their maximization requires a monopolistic output satisfying the first order condition:

$$\partial \pi / \partial x = a - 2x^M - c = 0$$

that is:[2]

$$x = (a - c)/2$$

Notice that the monopolistic profits are $\pi = (a - c)^2/4 - F$, but of course there can be incentives for other firms to enter in this market and compete. That is why we will now turn to analyze different forms of competition, starting from the simplest possible, Nash competition between two or more firms.

1.1 Nash equilibrium

When two firms compete at the same level, it is natural to imagine that their strategic choices are taken independently but coherently. This is what happens

[1] We will assume that c is small enough to allow profitable entry in the market. In particular $c < a - 2\sqrt{F}$ is enough to allow that at least one firm in the market (a monopolist) can make positive profits.

[2] Hence, the monopolistic price can be derived from the inverse demand function as $p = a - x = (a + c)/2$.

in a Nash equilibrium, which, in case of market competition where quantities are the strategic choice, is usually called a Cournot equilibrium. If firms i and j compete choosing independently their outputs, firm i has the following profit function:

$$\pi_i = (a - x_i - x_j)x_i - cx_i - F$$

since total production is now $X = x_i + x_j$; of course the profit of firm j is the same after changing all indexes. Profit maximization by firm i requires $\partial\pi_i/\partial x_i = 0$ or $a - 2x_i - x_j = c$, from which we obtain the so called reaction function:

$$x_i(x_j) = \frac{a - c - x_j}{2}$$

This is a rule of behaviour for the firm which can be interpreted in terms of expectations: the larger is the expected production of firm j, the smaller should be the optimal production of firm i. Of course firm j will follow a similar rule. More exactly the other reaction function can be derived as:

$$x_j(x_i) = \frac{a - c - x_i}{2}$$

The geniality of Cournot's idea is that in equilibrium the two rules must be consistent with each other. In terms of expectations, the equilibrium production of each firm must be the optimal one given the expectation that the other firm adopts its equilibrium production. Mathematically, we can solve the system of the two reaction functions to find out the production of each firm in equilibrium. It is easy to verify that each firm will produce:

$$x = \frac{a - c}{3}$$

so that the price will be $p = (a + 2c)/3$ and the profit of each firm will be $\pi = (a - c)^2/9 - F$.

Competition increases total production, reduces the price and the total profits in the market compared to the monopolistic case. This outcome is due to the fact that each firm does not take in consideration the impact of its own production on the profits of the other firm, and hence tends to produce too much from the point of view of profit maximization. This externality leads to a price reduction and to a decline in total profits. This is way the two firms may try to collude and agree on limiting their production at a lower level, possibly at the monopolistic level. However, only a strong commitment could guarantee such a collusive behaviour, because in absence of a strong commitment each firm would have incentives to deviate and produce more.

Also in this case of competition with two firms profits are positive and there can be incentives for a third firm to enter in this market and compete against these two incumbents; and then for a fourth one, and so on. Imagine that there are n firms in the market. Each one will have profits:

$$\pi_i = \left(a - x_i - \sum_{j=1, j \neq i}^{n} x_j \right) x_i - c x_i - F$$

and will choose its production to satisfy the first order condition $a - 2x_i - \sum_{j=1,j\neq i}^{n} x_j = c$, which generates the reaction function:

$$x_i = \frac{a - \sum_{j=1,j\neq i}^{n} x_j - c}{2}$$

Notice that this is decreasing in the output of each other firm, $\partial x_i / \partial x_j < 0$. Hence, when a firm is expected to increase its own production, any other firm has an incentive to choose a lower production level. This is a typical property of models where firms compete in quantities.

The system of n conditions provides equilibrium outputs as in the duopoly case. However, its solution is immediate if we notice that all firms will produce the same output satisfying $a - 2x - (n-1)x = c$. This implies the following output per firm as a function of n:

$$x(n) = \frac{a - c}{n + 1}$$

with total production:

$$X(n) = \frac{n(a - c)}{n + 1}$$

and a price:

$$p(n) = \frac{a + nc}{n + 1}$$

Each firm earns the following profits:

$$\pi(n) = \left(\frac{a - c}{n + 1} \right)^2 - F$$

for each firm. Notice that total output is increasing in the number of firms,[3] hence the price is decreasing (and actually tending to the marginal cost). Meanwhile, the profits of each single firm are decreasing when the number of competitors increases.

[3] One can verify that both the cases of a monopoly and of the Cournot duopoly are particular cases for $n = 1$ and $n = 2$.

1.2 Marshall equilibrium

It is now extremely simple to extend the model to endogenize entry. In this simple symmetric model, we have seen that entry of a new firm enhances competition leading to a reduction in the profit of each single firm in the market. If we assume that entry takes place as long as profits can be made, an equilibrium should be characterized by a number of firms n satisfying a no entry condition:

$$\pi(n+1) = \left(\frac{a-c}{n+2}\right)^2 - F \leq 0$$

and a no exit condition:

$$\pi(n) = \left(\frac{a-c}{n+1}\right)^2 - F \geq 0$$

When the fixed cost of production is small enough, this equilibrium number is quite large. In this cases it is natural to take a short cut and approximate the endogenous number of firms with the real number satisfying the zero profit condition $\pi^C(n) = 0$, that is:

$$n = \frac{a-c}{\sqrt{F}} - 1$$

This allows to derive the equilibrium output per firm under Marshall competition:

$$x = \sqrt{F}$$

the total production:

$$X = a - c - \sqrt{F}$$

and the equilibrium price:

$$p = c + \sqrt{F}$$

which implies a mark up on the marginal cost to cover the fixed costs of production.[4]

[4]Adopting the standard definition of welfare which here corresponds to the consumer surplus (since all firms earn no profits), we have:

$$W = \frac{X^2}{2} = \frac{(a-c-\sqrt{F})^2}{2}$$

Notice that in this case the first best would require one single firm producing $X = a - c$ with a welfare $W^{FB} = (a-c)^2/2 - F$.

1.3 Stackelberg equilibrium

Let us now consider the case in which one of the firms has a first mover advantage and can choose its ouput in a first stage before the followers, while these choose their output in a second stage and independently from each other. Let us call x_L the production of the leader. In the second stage each follower decides how much to produce according to the first order condition $a - x_L - 2x_i - \sum_{j=1, j\neq i, j\neq L}^{n} x_j = c$. In a symmetric outcome each follower will produce:

$$x(x_L, n) = \frac{a - x_L - c}{n}$$

as long as positive profits can be made, as we will assume for now. As we noticed before, $\partial x(x_L, n)/\partial x_L < 0$: production by the leader crowds out production by other firms. Hence, in the first stage the leader perceives its profits as:

$$\pi_L = [a - x_L - (n-1)x(x_L, n)]\, x_L - cx_L - F$$

We can already realize that the indirect (or strategic) role of the impact of the leader's strategy on the followers'strategy is going to increase the production of the leader: such an aggressive strategy by the leader reduces the production of the followers shifting profits toward the same leader. Formally, we can rewrite profits as:

$$\begin{aligned} \pi_L &= \left[a - x_L - \frac{(n-1)(a - x_L - c)}{n}\right] x_L - cx_L - F = \\ &= \left(\frac{a - c - x_L}{n}\right) x_L - F \end{aligned}$$

which leads to the optimal commitment to the strategy:

$$x_L = \frac{a - c}{2}$$

which in this particular example corresponds to the monopolistic production. However, each one of the followers will end up producing:

$$x(n) = \frac{a - c}{2n}$$

for a total output:

$$X = (a - c)\left(1 - \frac{1}{2n}\right)$$

and a price:

$$p(n) = \frac{a}{2n} + c\left(1 - \frac{1}{2n}\right)$$

again tending to the marginal cost when the number of firms increases. The profits for the leader and for each follower are respectively:

$$\pi_L(n) = \frac{(a-c)^2}{4n} - F, \qquad \pi(n) = \frac{(a-c)^2}{4n^2} - F$$

Of course some followers have incentives to enter in the market only if $\pi(2) > 0$ or $F < (a-c)^2/16$, otherwise the leader supplies its monopolistic production and no one else enters. We assume away this possibility of a "natural monopoly" in what follows.

1.4 Stackelberg equilibrium with endogenous entry

Let us finally move to the last case, in which there is still a leadership in the market but this is facing endogenous entry of competitors. Formally, consider the following sequence of moves:

1) in the first stage, a leader chooses its own output, say x_L;

2) in the second stage, after knowing the strategy of the leader, all potential entrants simultaneously decide "in" or "out": if a firm decides "in", it pays the fixed cost F;

3) in the third stage all the followers that have entered choose their own strategy x_i (hence, the followers play Nash between themselves).

In this case, the leader has to take into account how its own commitment affects not only the strategy of the followers but also their entry decision. As we have seen, in the last stage, if there are $n \geq 2$ firms in the market and the leader produces x_L, each follower produces:

$$x(x_L, n) = \frac{a - x_L - c}{n}$$

This implies that the profits of each follower are:

$$\pi(x_L, n) = \left(\frac{a - c - x_L}{n}\right)^2 - F$$

which are clearly decreasing in the number of firms. This would imply that further entry or exit does not take place when $\pi(x_L, n+1) \leq 0$ and $\pi(x_L, n) \geq 0$. Moreover, no follower will find it optimal to enter in the market if $\pi(x_L, 2) \leq 0$, that is if not even a single follower can make positive profits given the output of the leader. This is equivalent to:

$$x_L \geq a - c - 2\sqrt{F}$$

If the leader adopts an aggressive strategy producing enough, entry will be deterred, otherwise, the number of entrants will be endogenous determined by a free entry condition. In this last case, ignoring the integer constraint on the number of firms, we can approximate the number of firms as a real number again setting $\pi(x_L, n) = 0$, which implies:

$$n = \frac{a - c - x_L}{\sqrt{F}}$$

When this is the endogenous number of firms, each one of the followers is producing:

$$x\left(x_L, \frac{a - c - x_L}{\sqrt{F}}\right) = \sqrt{F}$$

that is independent of the strategy of the leader. Hence, the higher the production of the leader, the lower the number of entrants, while the production of each one of them will be the same. This would imply also a constant level of total production $X = a - c - \sqrt{F}$ and a constant price $p = c + \sqrt{F}$, which would be equivalent to those emerging under Marshall competition.

After having derived the behaviour of the followers, it is now time to move to the first stage and examine the behaviour of the leader. Remembering that entry takes place only for a production level which is not too high, if this is the case, the profits of the leader must be:

$$\pi_L = px_L - cx_L - F = x_L\sqrt{F} - F \quad \text{if} \quad x_L < a - c - 2\sqrt{F}$$

In other words, when entry takes place, the market price is perceived as given from the leader, which is aware that any increase in production crowds out entry maintaining constant the equilibrium price. However, when the leader is producing enough to deter entry, its profits become:

$$\pi_L = x_L(a - x_L) - cx_L - F \quad \text{if} \quad x_L \geq a - c - 2\sqrt{F}$$

It can be immediately verified that the profit function is linearly increasing in the output of the leader for $x_L < a - c - 2\sqrt{F}$ and after this cut off it jumps upward and then decreases. Consequently the optimal strategy for the leader is to produce just enough to deter entry:

$$x_L = a - c - 2\sqrt{F}$$

which is equivalent to set the limit price:

$$p = c + \sqrt{F}$$

The profits of the leader are then:

$$\pi_L = 2\sqrt{F}\left(a - c - 2\sqrt{F}\right) - F$$

In conclusion, when the number of potential entrants is low enough, the market is characterized by all these firms being active, while when there are many potential entrants and a free entry equilibrium is achieved, there is just one firm in equilibrium, the leader. While the price is kept higher than in the Marshall equilibrium (the mark up $p-c$ is doubled from $\sqrt{F}$ to $2\sqrt{F}$), welfare as the sum of consumer surplus and profits is now always higher than in the Marshall equilibrium.[5]

It is now time to pause and summarize what we have learnt until now. The Cournot equilibrium has shown that, under competition in quantities, when the number of firms increases output expands and equilibrium prices are reduced. The positive rents of the firms are decreased by entry, while their individual size shrinks. When entry is endogenously constrained by the fixed costs of production a large number of firms supply the good at a price which equals its average cost.

When there is a leader in the market, as long as entry is exogenous, the leader tends to produce more than its followers and again entry reduces rents and size of the followers (while the output of the leader does not change with entry). However, when entry is endogenous and dissipates the profits of the followers, the behaviour of the leader changes radically. In equilibrium only the leader produces and obtains positive profits; nevertheless entry is free and welfare is ultimately improved compared to the corresponding Marshall equilibrium. The moral of this result is that a leadership can improve the allocation of resources when it is constrained by competitive pressure.

1.5 Endogenous costs of entry

The theory of Stackelberg competition with free entry can also be seen as depicting the way a market leader can extract rents from a competitive market in presence of costs of entry. These costs can be interpreted as technological costs which are taken as given by the firms. However, they can also be endogenized immagining that they characterize the market and that the same leader can choose them in a preliminary stage. For instance, by investing in R&D or paying for an advertising campaign, or even by establishing certain barriers to entry associated with a cost of entry, the leader can set a sort of benchmark: all the other firms have to undertake the same investment or pay the same advertising campaign or face the same costs of entry to be able to compete in the market.

Imagine that the leader can choose the investment F. The demand and cost characteristic of this market depend on this investment so that the parameters

[5]Indeed, it can be now calculated as:

$$\begin{aligned} W^S &= \frac{X^2}{2} + \pi_L = \frac{(a-c-2\sqrt{F})^2}{2} + 2\sqrt{F}\left(a-c-2\sqrt{F}\right) - F = \\ &= \frac{(a-c)^2}{2} - 3F \end{aligned}$$

It can be verified that welfare is higher under Stackelberg competition with endogenous entry for any $F < 4(a-c)^2/49$, which always holds under our regularity assumption $F < (a-c)^2/16$, which guarantees that the market is not a natural monopoly.

$a(F)$ and $c(F)$ are now functions of the endogenous investment. This will be chosen to maximize the expected profits of the leader:

$$\pi_L(F) = 2\sqrt{F}\left[a(F) - c(F) - 2\sqrt{F}\right] - F$$

In general the choice will imply a positive investment (otherwise the leader would expect zero profits). One can also show that from a welfare point of view, the leader will choose an excessive investment if this investment reduces its equilibrium production, but would choose a suboptimal investment in the opposite case.[6] In other words, leaders tend to do too little of good things and too much of bad things.

For instance, imagine that F serves no real purpose other than raising the cost of entry ($a'(F) = c'(F) = 0$), as what we usually caal a barrier to entry. Even in this case, the leader would choose a positive barrier to entry:

$$F^* = \frac{(a-c)^2}{25}$$

which delivers the net profit:

$$\pi_L = \frac{(a-5)^2}{5}$$

In other words the leader would create a completely useless barrier associated with a fixed cost (born by the leader as well) just to profit ex post from its entry deterring strategy. Of course, in this case the welfare maximizing barrier would be $F = 0$, which would lead to complete rent dissipation and marginal cost pricing with zero profits for everybody. The moral of this story is that the priority in industrial policy should be to create the conditions for free entry and hence to fight against endogenous barriers to entry, not to fight against leaders per se.

The extreme result on entry deterrence that we have found in a model of competition in quantities holds under more general conditions. For instance, as we will see in the next chapters, as long as goods are perfect subsitutes, any kind of demand function will generate entry deterrence by the leader when entry of firms is endogenous. However, when the cost function departs from a linear version that we used until now and when imperfect substitutability between goods is introduced, entry deterrence may not be the optimal strategy anymore. In these cases, nevertheless, the leader will always act in a very aggressive way, producing always more than the followers when their entry is endogenous. To show this and generalize our model, I will now turn to two related extensions of it.

[6]This is an immediate consequence of the definition of welfare as a sum of consumer surplus and profits. When profits of the leader are maximized the investment is excessive if the consumer surplus is decreasing in the investment, that is if output is decreasing. Of course this is still a second best comparison.

2 Extensions

The simple model of competition in quantities studied above can be extended in many ways. Here we will focus on the two most immediate generalizations: first we will depart from the assumption of constant marginal costs assuming a U-shaped cost function, and then we will depart from the assumption of homogenous goods introducing imperfect substitutability across goods.

2.1 U-shaped cost functions

In many markets, marginal costs of production are increasing at least beyond a certain level of output. Jointly with the presence of fixed costs of production, this leads to U-shaped average cost functions. Since technology often exhibits this pattern, it is important to analyse this case, and I will do it assuming a simple quadratic cost function.

In particular, the general profit for firm i becomes:

$$\pi_i = x_i \left(a - x_i - \sum_{j=1, j\neq i}^{n} x_j \right) - \frac{dx_i^2}{2} - F$$

where $d \in [0, 1]$ represents the degree of convexity of the cost function. When $d = 0$ we are back to the case of a constant marginal cost (zero in such a case). When $d > 0$ the average cost function is U-shaped. One can easily verify that the marginal cost is increasing and convex, and it crosses the average total cost at its bottom, that is at the efficient scale of production: the one that minimizes average costs. This efficient scale of production can be derived formally as:

$$\hat{x} = \arg\min \left(\frac{dx}{2} + \frac{F}{x} \right) = \sqrt{\frac{2F}{d}}$$

Let us look now at the different forms of competition. Our four main equilibria can be derived as before. In particular, Nash competition would generate the individual output:

$$x(n) = \frac{a}{n + d + 1}$$

for each firm. Under Marshall competition each firm would produce:

$$x = \sqrt{\frac{2F}{2 + d}} < \hat{x}$$

with a number of firms approximated by:

$$n = a\sqrt{\frac{2 + d}{2F}} - d - 1$$

Notice that the equilibrium production level is below the cost minimizing level. This is not surprising since imperfect competition requires a price above marginal cost and free entry requires a price equal to the average cost. Since the average cost is always decreasing when it is higher than the marginal cost, it must be that individual output is smaller than the efficient scale.

Under Stackelberg competition, the leader produces:

$$x_L(n) = \frac{a(1+d)}{[2(1+d)+d(n+d)]}$$

and each follower produces:

$$x(n) = \frac{a\,[1+d+d(n+d)]}{[2(1+d)+d(n+d)]\,(n+d)}$$

Finally consider Stackelberg competition with endogenous entry. In the last stage an entrant chooses $x(x_L, n) = (a - x_L)/(n+d)$, but the zero profit condition delivers a number of firms:

$$n = (a - x_L)\left(\sqrt{\frac{2+d}{2F}}\right) - d$$

each one producing:

$$x = \sqrt{\frac{2F}{2+d}}$$

which is the same output as with Marshall competition. Of course this happens when there is effective entry, that is when $n \geq 2$ or $x_L < a - (2+d)\sqrt{2F/(2+d)}$. In such a case, total production is:

$$X = a - (1+d)\sqrt{\frac{2F}{2+d}}$$

and the price becomes

$$p = (1+d)\sqrt{\frac{2F}{2+d}}$$

which are both independent from the leader's production. The gross profit function of the leader in the first stage, can be derived as:

$$\begin{aligned} \pi_L &= px_L - \frac{d}{2}x_L^2 - F = \\ &= (1+d)\sqrt{\frac{2F}{2+d}}x_L - \frac{d}{2}x_L^2 - F \end{aligned}$$

which is concave in x_L. As long as d is large enough, we have an interior optimum and in equilibrium the leader prefers to allow entry producing:

$$x_L = \frac{1+d}{d}\sqrt{\frac{2F}{2+d}} > \hat{x}$$

so that the equilibrium number of firms is:

$$n = a\sqrt{\frac{2+d}{2F}} - \left(\frac{1+d}{d} + d\right)$$

and total output and price are the same as in the Marshall equilibrium (but welfare must be higher since the leader makes positive profits).[7]

Notice that the leader is producing always more than each follower. While followers produce below the efficient scale, the leader produces more than the efficient scale. Again the intution is straightforward. Followers have to produce at a price where their marginal revenue equates their marginal cost, but free entry imples also that the price has to be equal to the average cost. Since marginal and average costs are the same at the efficient scale, the followers must be producing below the efficient scale. The equilibrium price represents the perceived marginal revenue for the leader, and the leader must produce where this perceived marginal revenue equates the marginal cost, which in this case can only be above the efficient scale.

2.2 Product Differentiation

We now move to another simple extension of the basic model introducing product differentiation and hence imperfect substitutability between the goods supplied by the firms. We retain the initial assumptions of constant marginal costs and competition in quantities.

For simplicity, consider the inverse demand function for firm i:

$$p_i = a - x_i - b\sum_{j\neq i} x_j$$

where $b \in (0, 1]$ is an index of sustituibility between goods. Of course, for $b = 0$ goods are perfectly independent and each firm sells its own good as a pure monopolist, while for $b = 1$ we are back to the case of homogeneous goods. In

[7] In general the profit of the leader in case of an interior solution is:

$$\pi_L = \frac{F(1+d)^2}{d(2+d)} - F$$

while in the alternative case entry deterrence, the leaders produces:

$$x_L = a - (2+d)\sqrt{\frac{2F}{2+d}}$$

and its profits is:

$$\pi_L = \left[a - \sqrt{2F(2+d)}\right]\left[(a+d/2)\sqrt{2F(2+d)} - 2F - \frac{da}{2}\right]$$

which is higher when d is high enough.

this more general framework the profit function for firm i is:

$$\pi_i = x_i \left(a - x_i - b \sum_{j=1, j \neq i}^{n} x_j \right) - cx_i - F$$

The four main equilibria can be derived as usual. In particular a Nash equilibrium would generate the individual output:

$$x(n) = \frac{a-c}{2+b(n-1)}$$

for each firm. In the Marshall equilibrium each firm would produce:

$$x = \sqrt{F}$$

with a number of firms:

$$n = 1 + \frac{a-c}{b\sqrt{F}} - \frac{2}{b}$$

Under Stackelberg competition, the leader produces:

$$x_L = \frac{(a-c)(2-b)}{2}$$

and each follower produces:

$$x(n) = \frac{(a-c)[2-b(2-b)]}{2[2+b(n-2)]}$$

Finally, under Stackelberg competition with free entry, as long as substitutability between goods is limited enough (b is small) there are entrants producing $x(x_L, n) = (a - bx_L - c)/[2 + b(n-2)]$. Setting their profits equal to zero the endogenous number of firms results in:

$$n = 2 + \frac{a - bx_L - c}{b\sqrt{F}} - \frac{2}{b}$$

implying once again a constant production:

$$x = \sqrt{F}$$

for each follower. Plugging everything into the profit function of the leader, we have:

$$\begin{aligned} \pi_L &= x_L \left[a - x_L - b(n-1)x \right] - cx_L - F = \\ &= x_L \left[(2-b)\sqrt{F} - (1-b)x_L \right] - F \end{aligned}$$

that is maximized when the leader produces:

$$x_L = \frac{2-b}{2(1-b)}\sqrt{F}$$

which is always higher than the production of the followers. This strategy leaves space to the endogenous entry of firms so that the total number of firms in the market is:

$$n = 2 + \frac{a-c}{b\sqrt{F}} - \frac{2}{b} - \frac{2-b}{2(1-b)}$$

Notice that the leader will offer its good at a lower price than the followers, namely:

$$p_L = c + \left(1 - \frac{b}{2}\right)\sqrt{F} < p = c + \sqrt{F}$$

but the leader will also produce more than each follower and so it will earn positive profits. Again one should remember that this outcome emerges if the degree of product differentiation is high enough, while for b large enough the only possible equilibrium implies entry deterrence, with the production of the leader $x_L = (a - c - 2\sqrt{F})/b$ and the limit price $p_L = [c - (1-b)a + 2\sqrt{F}]/b$.

As we have seen, product differentiation allows different prices to emerge in the market. This leads us to the need to explicitly consider the choice of prices, that is to models of price competition.

3 A Simple Model of Competition in Prices

In many markets, especially under relevant product differentiation, firms compete in prices rather than in quantities. The initial equilibrium concept for markets of this kind was the Bertrand equilibrium, which was however referring to the case of homogenous goods. If goods are perfect substitutes, indeed, the equilibrium is quite simple: it boils down to a price equal to the average cost for each firm, since any different strategy either would leave space for profitable deviations, or would lead to losses. Things are not that simple when products are differentiated.

One of the simplest cases emerges when the demand function is log-linear. A direct demand which is often used for empirical studies is the Logit demand, which in its simplest form is $D_i = e^{-\lambda p_i} / \left[\sum_{j=1}^{n} e^{-\lambda p_j}\right]$ where of course p_i is the price of firm i, while $\lambda > 0$ is a parameter governing the slope of the demand function.[8] Since we focus on substitute goods, such a demand for firm i is decreasing in the price of the same firm i and increasing in the price of any other firm j. The general profit function for a firm facing this demand and, once

[8]We assume the regularity condition $F < 1/\lambda$.

again, a constant marginal cost c is:

$$\pi_i = D_i(p_i - c) = \frac{e^{-\lambda p_i}}{\sum_{i=1}^{n} e^{-\lambda p_j}}(p_i - c)$$

In a Nash equilibrium each firm chooses its own price taking as given the prices of the other firms. The first order condition for the optimal price of a single firm i is:

$$-\frac{\lambda e^{-\lambda p_i}(p_i - c)}{\sum_{i=1}^{n} e^{-\lambda p_j}} + \frac{e^{-\lambda p_i}}{\sum_{i=1}^{n} e^{-\lambda p_j}} - \frac{\lambda e^{-\lambda p_i}}{\left(\sum_{i=1}^{n} e^{-\lambda p_j}\right)^2} = 0$$

and it simplifies to:

$$p_i = c + \frac{1}{\lambda(1 - D_i)}$$

While this is an implicit expression (on the right hand side the demand of the firm i depends on the price of the same firm), it emphasizes quite clearly that the price is set above marginal cost. Moreover, since an increase in the price of any other firm j, p_j, increases demand for firm i,D_i, it also increases the optimal price of firm i: formally, $\partial p_i / \partial p_j > 0$. This important property, which holds virtually in all models of competition in prices, suggests that a higher price by one firm induces other firms to increase their prices as well. In other words, an accomodating behaviour of one firm leads other firms to be accomodating too.

To conclude our analysis of the Nash equilibrium, notice that in a symmetric situation with a price p for each firm, demand boils down to $D = 1/n$ and the equilibrium price is decreasing in the number of firms:

$$p(n) = c + \frac{1}{\lambda\,(1 - 1/n)}$$

In a Marshall equilibrium one can easily derive that the number of active firms is:

$$n = 1 + \frac{1}{\lambda F}$$

and each one of these sells its product at the price:

$$p = c + \frac{1}{\lambda} + F$$

Let us now move to models of price leadership. Of course it can be even harder for a firm to commit to a price rather than to a different strategy as the quantity of production. Later on we will deal with this problem in a deeper way and we will suggests that there are realistic ways in which a strategic investment can be a good substitute for a commitment to a strategy, including a price strategy. However, for now we will assume that a firm can simply commit to a pricing strategy and analyze the consequence of this.

For the Stackelberg equilibrium we do not have analytical solutions. However, it is important to understand the nature of the incentives of the firms, which is now rather different from models with product differentiation and competition in quantities. Here the leader is aware that an increase in its own price will lead each other follower to increase its own price, which exerts a positive effect on the profits of the leader. Hence, the commitment possibility is generally used adopting an accomodating strategy: the leader chooses a high price to induce its followers to choose high prices as well.[9] The only case in which this does not happen is when fixed costs of production are high enough and the leader finds it profitable to deter entry, which can only be done adopting a low enough price: hence, the leader can only be aggressive for predatory purposes. However, this standard result emphasizes a possible inconsistency within this model, at least when applied to describe real markets. We have suggested leaders are accomodating when fixed costs of production (or entry) are small, because in such a case an exclusionary strategy would require to set a very low price and would be too costly. But this are exactly the conditions under which other firms may want to enter in the market: fixed costs are low and exclusionary strategies by incumbents are costly. Hence, the assumption that the number of firms, and in particular of followers, is exogenous becomes quite unrealistic. In these cases, it would be useful to endogenize entry.

Let us look at the Stackelberg equilibrium with endogenous entry. The solution in this case is slightly more complex, but it can be fully derived. First of all, as usual, let us look at the stage in which the leader as already chosen its price p_L and the followers enter and choose their prices. As before, their choice will follow the rule:

$$p_i = c + \frac{1}{\lambda(1 - D_i)}$$

where the demand on the right hand side depends on the price of the leader and all the other prices as well. However, under free entry we must have also that the markup of the followers exactly covers the fixed cost of production, hence:

$$D_i(p_i - c) = F$$

If the price of the leader is not too low or the fixed cost not to high, there is indeed entry in equilibrium and we can solve these two equations for the demand of the followers and their prices in symmetric equilibrium:

$$p = c + \frac{1}{\lambda} + F, \quad D = \frac{\lambda F}{1 + \lambda F}$$

Notice that neither the one or the other endogenous factors depend on the price chosen by the leader. Hence, it must be that the strategy of the leader is going

[9]Nevertheless, the followers will have incentives to choose a lower price than the leader, and each one of them will then have a larger demand and profits than the leader: there is a second-mover advantage rather than a first-mover advantage.

to affect only the number of followers entering in equilibrium, but not their prices or their equilibrium production.

The leader is going to perceive this because its demand can now be calculated as:

$$
\begin{aligned}
D_L &= \frac{e^{-\lambda p_L}}{\sum_{i=1}^{n} e^{-\lambda p_j}} = \\
&= \frac{e^{-\lambda p_L}}{e^{-\lambda p}} \frac{e^{-\lambda p}}{\sum_{i=1}^{n} e^{-\lambda p_j}} = \\
&= \frac{e^{-\lambda p_L}}{e^{-\lambda p}} D
\end{aligned}
$$

Since neither p or D depend on the price of the leader, its demand is a simple function of its own price, and the profits of the leader can be derived as:

$$
\begin{aligned}
\pi_L &= (p_L - c) D_L = \\
&= (p_L - c) e^{-\lambda p_L} \left[\frac{e^{\lambda(c+F)+1}}{1 + 1/\lambda F} \right]
\end{aligned}
$$

where we used our previous results for p or D. Profit maximization by the leader provides its equilibrium price:

$$
p_L = c + \frac{1}{\lambda}
$$

which is now lower than the price of each follower. Finally the number of firms active in the market is:

$$
n = 2 + \frac{1}{\lambda F} - e^{\lambda F}
$$

Rather than being accomodating as in the Stackelberg equilibrium, the behaviour of the leader in a Stackelberg equilibrium with endogenous entry is radically different: the leader is aggressive since it chooses a lower price and ends up selling more of its products. However, some followers enter in the market, and they have to choose a higher price than the leader without earning any profits.[10]

[10] Also in this case, if the fixed cost is high enough, it may be optimal for the leader to fully deter entry, choosing a price $p_L = c + 1/\lambda + F - (1/\lambda)\log(1/\lambda F)$.

4 A Simple Model of Competition for the Market

The last example we are going to consider in this chapter will introduce us to a topic that we will encounter later on in the book, the competition to innovate and hence conquer a market with better products. In many sectors of the New Economy and in general in high tech sectors, this is becoming a main form of competition, since the life of a product is quite short and R&D investment strategies to conquer future markets are much more important than price or production strategies in the current markets.

Competition for the market works as a sort of contest. Firms invest to innovate and especially to arrive first in the contest. It may be that the first innovator can obtain a patent on the invention and exploit monopolistic profits for a while on its innovation, it maybe that the same innovator just keeps it secret and exploits its leadership on the market until an imitator replaces it. Anyway, the expected gain from an innovation is what drives firms to invest in R&D. Also in this case we can study alternative market structure depending on the timing of moves and on the entry conditions.

Consider a simple contest between agents for the expected gain $V < 1$, in which each contestant i invests resources $z_i \in [0, 1)$ to win the contest and the associated "prize". This investment has a cost and we will assume that it is quadratic for simplicity, that is $z_i^2/2$. The investment provides the contestant with the probability z_i to innovate. The innovator wins the contest if no other contestant innovates, for instance because in case of multiple winners competition between them would drive profits away. Hence the probability to win the contest is $\Pr(i\ wins) = z_i \prod_{j=1,j\neq i}^{n} [1 - z_j]$, that is its probability to innovate multiplied by the probability that no one else innovates. In conclusion, the general profit function is:[11]

$$\pi_i = z_i \prod_{j=1,j\neq i}^{n} [1 - z_j] V - \frac{z_i^2}{2} - F$$

Consider first Nash equilibrium. The first order condition for the optimal investment by a firm i is:

$$z_i = \prod_{j=1,j\neq i}^{n} [1 - z_k] V$$

which shows that when the investment of a firm increases, the other firms have incentives to invest less: $\partial z_i / \partial z_j < 0$. In case of two firms, each one would invest

[11] We assume $V \in (\sqrt{2F}, 1)$, which guarantees profitable entry for at least one firm. Indeed, a single firm would invest $z = V < 1$ expecting $\pi = V^2/2 - F > 0$.

$z = V/(1+V)$ in equilibrium, while with n firms, the equilibrium investment would be implicitly given by:

$$z = (1-z)^{n-1}V$$

In a Marshall equilibrium we must also take into account the free entry condition:

$$z(1-z)^{n-1}V - z^2/2 = F$$

and solving the system of the two conditions we have the number of agents:

$$n = 1 + \frac{\log\left(V/\sqrt{2F}\right)}{\log\left[1/(1-\sqrt{2F})\right]}$$

and the investment:

$$z = \sqrt{2F}$$

Consider now a Stackelberg equilibrium. As already noticed, remember that when the investment by one firm is higher, the other firms have incentives to invest less: then in a Stackelberg equilibrium the leader exploits its first mover advantage by investing more than the followers, so as to reduce their investment and increase its relative probability of winning. For instance, in a Stackelberg duopoly the leader invests $z_L = V(1-V)/(1-2V^2)$ and the follower invests $z = V(1-V-V^2)/(1-2V^2)$.

In a Stackelberg equilibrium with endogenous entry, as long as the investment of the leader z_L is small enough to allow entry of at least one firm, the first order conditions and the free entry conditions are:

$$(1-z)^{n-2}(1-z_L)V = z$$
$$z(1-z)^{n-2}(1-z_L)V = z^2/2 + F$$

which deliver the same investment choice by each entrant as in the Marshall equilibrium, $z = \sqrt{2F}$, and the number or firms:

$$n(z_L) = 2 + \frac{\log\left[(1-z_L)V/\sqrt{2F}\right]}{\log\left[1/(1-\sqrt{2F})\right]}$$

Putting together these two equations and substituting in the profit function of the leader, we would have:

$$\begin{aligned}\pi_L &= z_L(1-z)^{n-1}V - \frac{z_L^2}{2} - F = \\ &= \frac{z_L}{1-z_L}\sqrt{2F}\left(1-\sqrt{2F}\right) - \frac{z_L^2}{2} - F\end{aligned}$$

which has not an interior optimum: indeed, it is always optimal for the leader to deter entry investing enough. This requires a slightly higher investment than the one for which the equilibrium number of firms would be $n = 2$. Since $n(z_L) = 2$ requires $\log\left[(1 - z_L)V/\sqrt{2F}\right] = 0$, we can conclude that the leader will invest:

$$z_L(V) = 1 - \frac{\sqrt{2F}}{V}$$

which is increasing in the value of innovations and decreasing in their fixed cost. Hence, in a contest with a leader and free entry of participants, the leader invests enough to deter investment by the other firms and is the only possible winner of the contest.

4.1 The Arrow's Paradox

Until now we investigated a form of competition for the market where all firms are at the same level. Often times, competition for the market is between an incumbent leader that is already in the market with the leading edge technology or with the best product and outsiders trying to replace this leadership. In such a case the incentives to invest in innovation may be different and it is important to understand how. Arrow (1962) was the first to examine this issue and he found that incumbent monopolists have lower incentives than outsiders to invest. His insight was simple but powerful: while the gains from an innovation for the incumbent monopolist are just the difference between profits obtained with the next innovation and those obtained with the current one, the gains for any outsiders are the full profits from the next innovation. Hence the incumbent has lower incentives to invest in R&D. The expected gains of the incumbent are even diminuished when the number of outsiders increases. And when the latter arrives to the point that expected profits for the outsiders are zero, the incumbent has no more incentives at all to participate to the competition. Such a strong theoretical result is of course too drastic to be realistic. Many technological leaders invest a lot in R&D and try to maintain their leadership, often managing: persistent leadership are not so unusual. However, before offering a theoretical explanation for this dilemma, we will extend the model to include an asymmetry between an incumbent monopolist and the outsiders.

Imagine a two period extension of the model. In the first period the incumbent monopolist is alone and can exploit its current patent to obtain profits $K \in (0, V]$. Meanwhile, all firms can invest to innovate and conquer the gain V from the next innovation to be exploited in the second period. If no one innovates, the monopolist retains its profits K also in the second period. This happens with probability $\Pr(\textit{no innovation}) = \prod_{j=1}^{n} [1 - z_j]$. Then the expected profits of the incumbent monopolist, that we now label with the index

M, are:

$$\pi_M = K + z_M \prod_{j=1,j\neq M}^{n} [1-z_j]\, V + (1-z_M) \prod_{j=1,j\neq M}^{n} [1-z_j]\, K - \frac{z_M^2}{2} - F$$

in case of positive investment in the competition for the market, otherwise expected profits are given only by the current profits plus the expected value of the current profits when noone innovates. The profits of the other firms are the same as before. Consider a Nash equilibrium. If the monopolist does not invest, the equilibrium is the same we studied before and the expected profit if the monopolist must be:

$$\pi_M(V) = K + \frac{\sqrt{2F}(1-\sqrt{2F})K}{V}$$

which is linearly increasing in the value of current profits K and equal to zero for $K = 0$, and decreasing in the value of the innovation V (since this increases the incentives of other firms to innovate and replace the monopolist).

If the monopolist was investing, however, the first order conditions for the monopolist and for the other firms in Nash equilibrium would be:

$$z = (1-z)^{n-2}(1-z_M)V$$

$$z_M = (1-z)^{n-1}V - (1-z)^{n-1}K$$

Even if we cannot solve analytically this system, it is easy to verify that there are two effects goin in opposite directions: the Arrow effects pushes toward a lower investment for the monopolist (because its marginal gain from the innovation is just the difference between the value of the new innovation V and that of the current one K), while the usual Stackelberg effect pushes toward a higher investment for the monopolist, and the first is prevailing when the current profits of the monopolist is large enough.[12]

However, the Arrow effect is When entry of firms is free, investors enter as long as expected profits are positive, that is until the following zero profit condition holds:

$$z(1-z_M)(1-z)^{n-2}V = z^2/2 + F$$

This implies that each one of the other firms invests again $z = \sqrt{2F}$, while the monopolist should invest less than this, according to the rule:

$$z_M(1-z_M) = \sqrt{2F}(1-\sqrt{2F})(V-K)/V$$

[12] For instance, with two firms we have:

$$z_L = \frac{VK + (1-V)(V-K)}{1-2V(V-K)} \qquad z = \frac{VK + (1-V)V - V^3}{1-2V(V-K)}$$

and the investment of the monopolist because the Arrow effect prevails whenever $K > V^3/(1-V)$.

which also implies that the optimal investment of the monopolist should decrease with K: from the same level as for the other firms $z_M = \sqrt{2F}$ when $K = 0$ toward zero investment $z_M = 0$ when approaching $K = V$. The profits of the monopolist in case of positive investment would be:

$$\pi_M(z_M) = K + \frac{\sqrt{2F}(1-\sqrt{2F})}{V}\left[\frac{(V-K)z_M + K}{1-z_M}\right] - \frac{z_M^2}{2} - F$$

where z_M should be at its optimal level derived above. Notice that for $K = 0$ these expected profits are zero so the monopolist is indifferent between investing or not, while for $K = V$ they tend to $K + \sqrt{2F}(1-\sqrt{2F}) - F$, which is lower than the expected profit in case the monopolist does not invest at all. It can be verified that this is always the case for $K \in (0, V]$,[13] hence the monopolist always prefers not to invest and decides to give up to any chance of innovation.

4.2 Innovation by leaders

It can be reasonable to imagine that an incumbent monopolist with the leading edge technology may invest to replace this same technology with a better one and may commit to such an investment even before other firms. In other words we can associate a strategic advantage in the competition for the market to the current leader (Etro, 2004).

Consider Stackelberg competition where the incumbent monopolist is the first mover. The reaction of the other firms to the investment of the leader is still governed by their equilibrium first order condition:

$$z = (1-z)^{n-2}(1-z_L)V$$

where now z_L is the known investment of the leader, which is known at the time of the choice of the other firms. The above rule cannot be solved analytically but it shows again that the investment of the outsider firms must be decreasing in that of the leader, $\partial z/\partial z_L < 0$: the higher is the investment of the leader, the smaller is the probability that noone innovates and hence the expected gain from the investment of the followers. This implies that the leader has an incentive to choose a higher investment to strategically reduce the investment of the followers. However, the investment of the leader does not need to be higher than the investment of the other firms, because the Arrow effect is still pushing in the opposite direction.

When entry is endogenous, however, things are simpler. As long as the investment of the leader is small enough to allow entry of at least one outsider, the free entry condition is:

$$z(1-z)^{n-2}(1-z_L)V = z^2/2 + F$$

[13] This immediate after comparing profits for the monopolist in case of zero and positive investment in Nash equilibrium as functions of K.

which delivers again the investment $z = \sqrt{2F}$ for each outsider. Putting together the two equilibrium conditions in the profit function of the leader, we would have:

$$\begin{aligned} \pi_L &= K + z_L(1-z)^{n-1}(V-K) - \frac{z_L^2}{2} - F = \\ &= K + \frac{z_L}{1-z_L}\sqrt{2F}\left(1-\sqrt{2F}\right) + \frac{K}{V}\sqrt{2F}\left(1-\sqrt{2F}\right) - \frac{z_L^2}{2} - F \end{aligned}$$

whose third element, the one associated with the current profits obtained in case no one innovates, is independent from the choice of the leader. Hence, the choice of the leader is taken exactly as in our earlier model (with $K = 0$) and requires an investment:

$$z_L(V) = 1 - \frac{\sqrt{2F}}{V}$$

such that no other firm invests in innovation. Consequently, the profits of the leader can be calculated as a function of the value of the innovation:

$$\pi_L(V) = K + z_L(V)V + [1 - z_L(V)]\,K - \frac{z_L(V)^2}{2} - F$$

Welfare comparisons are ambiguous: on one side the aggregate probability of innovation is lower under Stackelberg competition with free entry rather than in the Marshall equilibrium, on the other side expenditure in fixed and variable costs of research is lower in the first than in the second case. However, in a dynamic environment where the value of the innovation is endogenous, things would change. While without a leadership of the monopolist, the value of innovation would be just the value of expected profits from this innovation (the innovator will not invest further), with a leadership by the monopolist, the value of innovation should take into account the option value of future leadership and future innovations: this would endogenously increase the value of being an innovator and would increase the aggregate incentives to invest.

Moreover, notice that when the monopolist is leader in the competition for the innovation, the Arrow effect disappears, since the choice of the monopolist is independent from the current profits. The leadership in the competition for the market radically changes the behaviour of a monopolist: from zero investment to maximum investment!

CIAO