

TESTING THE CONVERSE OF WOLSTENHOLME'S THEOREM

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Abstract

A classical result of Wolstenholme in 1862 shows that if $p \geq 5$ is a prime number then

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

Its converse, stating that a natural p satisfying this congruence is necessarily a prime number, is commonly believed to be true, although no proof has been given so far. In this note, we present an elementary proof of a partial result, namely, that the converse is true for even numbers and for powers of 3. Further, we prove that if $n = p^l$ is a prime power then

$$\binom{2n-1}{n-1} \equiv \binom{2p-1}{p-1} \pmod{p^4},$$

producing a relatively inexpensive converse test for powers of odd prime numbers.

Resumo

Um resultado clássico de Wolstenholme mostra que se p é um número primo, então

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

A recíproca desse teorema, indicando que um número natural p satisfazendo a congruência é necessariamente um número primo, embora acredita-se verdadeira, ainda não tem uma prova. Nesta nota, apresentamos uma prova elementar de um resultado parcial; especificamente, que a recíproca é verdadeira para números pares e para potências de 3. Além disso, provamos que se $n = p^l$ é uma potência de um primo, então

$$\binom{2n-1}{n-1} \equiv \binom{2p-1}{p-1} \pmod{p^4},$$

que é um teste eficiente para testar a recíproca de potência de primos ímpares.

1 Introduction

A simple characterization of prime numbers is always an interesting goal, not only because this may be a hard problem but also because a simple characterization may lead to an efficient primality test, a task today sought with great interest by the scientific community, especially in the area of number theory.

A famous result for prime numbers, called the Theorem of Wolstenholme, is the following

Property 1 (Wolstenholme, 1862) *If $p \geq 5$ is a prime number then*

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

Apparently, it was James P. Jones [10, Problem B31, p. 84] who first conjectured that the converse of this theorem is true, namely that a natural number p satisfying the congruence of Property 1 is necessarily prime. The converse of Wolstenholme's Theorem is regarded as a very difficult problem.

In [9], Richard J. McIntosh obtains restrictive conditions on n for solutions of $\binom{2n-1}{n-1} \equiv 1 \pmod{n^r}$ and concludes that Wolstenholme's converse is probably true. For example, he shows that if p is a prime number and $n = p^2$ satisfies

$$\binom{2n-1}{n-1} \equiv 1 \pmod{n^3},$$

(which would be a counterexample to the converse), then p satisfies

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^6},$$

which is unlikely. McIntosh also reports that the converse is known to be true for all composite numbers $n < 10^9$.

No proof, however, has been obtained for the converse of Wolstenholme's property. In section 3 we partially fill this gap, by proving that Property 1 does not hold for positive even numbers. This result is probably known to other

authors who work on the subject, but we are unaware of a published proof. Moreover, the proof we present uses only elementary mathematics.

For a given composite number n , to show that $\binom{2n-1}{n-1} \not\equiv 1 \pmod{n^3}$, it suffices to show that $\binom{2n-1}{n-1} \not\equiv 1 \pmod{R}$, where $R > 1$ is any factor of n . Using this idea, we study the converse of Wolstenholme's theorem for powers of primes p , by determining the value of the binomial coefficient modulo p^3, p^4 and p^5 . In section 4 we prove that if n is a power of 3, than it does not satisfy property 1, proving that the converse is true for $n = 3^l$. Additionally, we prove that if p is a prime number and $n = p^l$, $l \geq 2$, then

$$\binom{2n-1}{n-1} \equiv \binom{2p-1}{p-1} \pmod{p^4}, \tag{1}$$

which reduces the size of the computational task for testing the converse.

Finally, we claim that the converse of Wolstenholme's theorem is true for all powers of primes $p < 2.5 \times 10^8$ (see section 5), by using the criteria given by equation (1).

2 Generalities

In this section, we review some well known results that will be used throughout this note.

First, we make use of a well known equation that is sometimes called *Vandermonde's convolution* $\binom{r+s}{i} = \sum_{j=0}^i \binom{r}{j} \binom{s}{i-j}$ [6, p. 169]. If we set $i = s = r = n$, where n is a positive integer, then we obtain

$$\binom{2n}{n} = \sum_{j=0}^n \binom{n}{j}^2. \tag{2}$$

Also well known is the following equation, true for any positive integers r and s :

$$\binom{r}{s} = \frac{r}{s} \binom{r-1}{s-1}. \tag{3}$$

Applying this identity, it follows that for any positive integer n ,

$$\binom{2n-1}{n-1} = \frac{1}{2} \sum_{j=0}^n \binom{n}{j}^2. \tag{4}$$

We see from this equation that

$$\binom{2p}{p} \equiv 2 \pmod{p^3},$$

true for primes $p \geq 5$ (see the paper by D. F. Bailey [4, lemma 1, p. 209]), is equivalent to the Wolstenholme's theorem.

Also widely used in this note is the following well known fact.

Lemma 1 *Let p be a prime number. If $n = p^r$ and $s \leq r$ is the highest power of p dividing m , then, the highest power of p dividing $\binom{n}{m}$ is $r - s$.*

3 Even Numbers

In this section we show the following

Theorem 1 *If $n > 0$ is even, then*

$$\binom{2n-1}{n-1} \not\equiv 1 \pmod{n^3}.$$

That is, the converse of Wolstenholme's Theorem is true for even positive integers.

We begin by noting

Fact 1 *The binomial coefficient*

$$\binom{2n-1}{n-1}$$

is odd if and only if n is a power of 2.

This is a trivial consequence of a classical result of E. Kummer [7], which states that the power r to which a prime p divides $\binom{a+b}{a}$ is equal to the number of carries in the addition of a and b in base p arithmetic. Writing $\binom{2n-1}{n-1} = \frac{1}{2} \binom{n+n}{n}$, it is easy to see from the binary representation of n that $\binom{2n-1}{n-1}$ is odd only when n is a power of 2.

Fact 1 also follows from Lucas' Theorem [8], which states that

$$\binom{a}{b} \equiv \binom{a_0}{b_0} \binom{a_1}{b_1} \cdots \binom{a_k}{b_k} \pmod{p},$$

where $a = a_0 + a_1p + \cdots + a_kp^k, b = b_0 + b_1p + \cdots + b_kp^k, 0 \leq a_i, b_i \leq p - 1$ are the base p representations of a and b . Employing the usual convention that $\binom{a}{b} = 0$ when $a < b$, it can be seen from the binary representation of $2n - 1$ and $n - 1$ that $\binom{2n-1}{n-1}$ is odd only when n is a power of 2. It is also possible to show fact 1 using only elementary manipulations with binomial coefficients.

Given n a natural number, there are unique integers, q and r such that

$$\binom{2n-1}{n-1} = qn^3 + r, \tag{5}$$

where either $r = 0$ or $0 < r < n^3$. The number r is, by definition, the modulo sought, that is $r \equiv \binom{2n-1}{n-1} \pmod{n^3}$.

For n even, not a power of two, fact 1 shows that the LHS of equation (5) is even, so that r must also be even and the converse of the Wolstenholme's property is true.

To complete the proof of theorem 1 it remains to show

Lemma 2 *If $n = 2^l, l \geq 1$, then*

$$\binom{2n-1}{n-1} \equiv 3 \pmod{2^4}.$$

Proof. For $l \geq 2$, using equation (4), we write

$$\binom{2n-1}{n-1} = 1 + \sum_{j=1}^{2^{l-1}-1} \binom{2^l}{j}^2 + \frac{1}{2} \binom{2^l}{2^{l-1}}^2.$$

It remains to show that $B = \sum_{j=1}^{2^{l-1}-1} \binom{2^l}{j}^2 + \frac{1}{2} \binom{2^l}{2^{l-1}}^2 \equiv 2 \pmod{2^4}$. By lemma 1, 4 divides $\binom{2^l}{j}$ for any $j = 1, \dots, 2^{l-1} - 1$, implying that $B \equiv \frac{1}{2} \binom{2^l}{2^{l-1}}^2 \pmod{2^4}$. Lemma 1 also says that $\binom{2^l}{2^{l-1}} = 2X$, with X odd. Hence

$$\frac{1}{2} \binom{2^l}{2^{l-1}}^2 = 2X^2, \text{ odd } X.$$

As $X \equiv 1 \pmod{2}$, it follows that $X \equiv \pm 1 \pmod{8}$ or $X \equiv \pm 3 \pmod{8}$. In any case we have $X^2 \equiv 1 \pmod{8}$ and therefore $2X^2 \equiv 2 \pmod{16}$. Thus

$$B \equiv 2 \pmod{2^4}.$$

□

4 Odd Prime Powers

In this section we study the binomial congruence $\binom{2n-1}{n-1} \equiv 1 \pmod{n^3}$ for n a power of an odd prime number. We begin by showing

Theorem 2 *If $n = 3^l$, $l \geq 1$, then the converse of Wolstenholme's Theorem is true.*

Proof. We show that if $n = 3^l$, $l \geq 1$, then

$$\binom{2n-1}{n-1} \equiv 10 \pmod{3^5}.$$

For $l \geq 2$, we may write

$$\binom{2n-1}{n-1} = 1 + \sum_{j=1}^{(n-1)/2} \binom{n}{j}^2.$$

Applying lemma 1, we see that $\binom{n}{j}$ is divisible by 9 for all j , but for $j = 3^{l-1}$, so that

$$C = \sum_{j=1}^{(n-1)/2} \binom{n}{j}^2 \equiv \left(\binom{3^l}{3^{l-1}} \right)^2 \pmod{3^4}.$$

We also know that $\binom{3^l}{3^{l-1}} = 3Y$, with Y and 3 relatively prime. That means $Y \equiv \pm 1 \pmod{3}$, implying that $Y^2 \equiv 1 \pmod{3}$. It follows that

$$C \equiv \left(\binom{3^l}{3^{l-1}} \right)^2 \equiv (3Y)^2 \equiv 9 \pmod{3^3},$$

and the theorem is proved.

□

We notice that the result of lemma 2 implies that $\binom{2n-1}{n-1} \not\equiv 1 \pmod{4}$, for $n = 2^l$ and the result of theorem 2 implies that $\binom{2n-1}{n-1} \not\equiv 1 \pmod{3^3}$, for $n = 3^l$. We conclude that the converse of Wolstenholme's theorem is true for powers of 2 and 3.

If $n = p^l$, where p is a prime number greater than 3, we need to compute the value of the binomial coefficient modulo a power of p higher than 3 because of the following

Theorem 3 *If $p \geq 5$ is prime and $n = p^l$ $l \geq 1$, then*

$$\binom{2n-1}{n-1} \equiv 1 \pmod{p^3}.$$

Proof. Theorem 4 of [4] states that for any nonnegative integers k and r , $\binom{kp}{rp} \equiv \binom{k}{r} \pmod{p^3}$. Applying this result l times, we have

$$\binom{2n}{n} \equiv \binom{2p^l}{p^l} \equiv 2 \pmod{p^3}.$$

As $\binom{2n-1}{n-1} = \frac{1}{2} \binom{2n}{n}$ and p is odd, the result follows.

□

By considering the binomial coefficient modulo p^4 , we obtain the following reduction theorem.

Theorem 4 *If $p \geq 5$ is a prime number, $l \geq 1$ is an integer and $n = p^l$, then*

$$\binom{2n-1}{n-1} \equiv \binom{2p-1}{p-1} \pmod{p^4}.$$

Proof. Since p is odd, it suffices to show that $\binom{2n}{n} \equiv \binom{2p}{p} \pmod{p^4}$. We write

$$\binom{2n}{n} = 2 + \sum_{j=1}^{n-1} \binom{n}{j}^2 = 2 + \sum_{j=1}^{p^l-1} \binom{p^l}{j}^2,$$

and notice that if p^{l-1} does not divide j then, applying lemma 1, we see that

p^2 divides $\binom{p^l}{j}$ or that $\binom{p^l}{j}^2 \equiv 0 \pmod{p^4}$ and

$$\binom{2n}{n} \equiv 2 + \sum_{j=1}^{p-1} \binom{p^l}{jp^{l-1}}^2 \pmod{p^4}.$$

We quote lemma A of [5] which states that

$$\binom{p^k a}{p^k b} \equiv \binom{p^{k-1} a}{p^{k-1} b} \pmod{p^{3k}}.$$

Assuming that $l \geq 3$ and setting $k = 2$, we apply the result and write

$$\binom{p^l}{jp^{l-1}} \equiv \binom{p^k p^{l-2}}{p^k jp^{l-3}} \equiv \binom{pp^{l-2}}{jp^{l-3}} \pmod{p^6}.$$

We repeat the argument $l - 3$ more times, following that

$$\binom{p^l}{jp^{l-1}} \equiv \binom{p^2}{jp} \pmod{p^6},$$

for all $0 < j < p$ and $l \geq 2$. So we can write

$$\binom{2n}{n} \equiv 2 + \sum_{j=1}^{p-1} \binom{p^2}{jp}^2 \pmod{p^4}.$$

We invoke now theorem 2.2 of [5] to claim that $\binom{p^2}{jp} \equiv \binom{p}{j} \pmod{p^4}$ and so it follows that

$$\binom{2n}{n} \equiv 2 + \sum_{j=1}^{p-1} \binom{p}{j}^2 \equiv \binom{2p}{p} \pmod{p^4},$$

completing the proof. □

This reduction is computationally useful since one may compute $\binom{2p-1}{p-1} \pmod{p^4}$. If this value is not 1, then the converse of Wolstenholme's Theorem is true for all powers of the prime p .

Studying criteria for solutions of $\binom{2n-1}{n-1} \equiv 1 \pmod{n^r}$, R. J. McIntosh considers primes p satisfying

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^4} \tag{6}$$

and calls them *Wolstenholme primes* [9].

Theorem 4 shows that if p is not a Wolstenholme prime, then the converse of Wolstenholme's theorem is true for all powers of p .

5 Computer Experiments

We performed the computation of $\binom{2p-1}{p-1} \pmod{p^4}$ for all primes $p < 2.5 \times 10^8$ which is, according to the reduction criterion of theorem 4, sufficient to show that the converse of Wolstenholme's Theorem is true for all powers of p , when p is not a Wolstenholme prime.

It is important to notice that computing the binomial coefficient $\binom{2p-1}{p-1} \pmod{p^4}$ is not a trivial task, since the arithmetic involves large numbers.

The same computation of $\binom{2p-1}{p-1} \pmod{p^4}$ was considered by McIntosh in the search for Wolstenholme primes and several statements equivalent to equation (6) are proven in [9]. The following is the most appealing for computational purposes (because it reduces to calculations modulo p):

Theorem 5 *For all primes $p \geq 11$, p is a Wolstenholme prime if and only if*

$$\sum_{j=\lfloor p/6 \rfloor + 1}^{\lfloor p/4 \rfloor} \frac{1}{j^3} \equiv 0 \pmod{p}.$$

Using this criterion, we performed the computation for primes $p < 2.5 \times 10^8$. This extends the search of [9], reporting that there are only two Wolstenholme primes $p < 2 \times 10^8$, namely, $p_1 = 16,843$ and $p_2 = 2,124,679$.

We confirmed this computation and found no other Wolstenholme prime up to 2.5×10^8 . This implies that for all prime powers p^l , $l > 1$, with $p < 2.5 \times 10^8$, and $p \neq p_1, p_2$, the converse of Wolstenholme’s Theorem is true.

For powers of Wolstenholme primes p , we need to work some more in order to find out whether the converse of Wolstenholme’s Theorem is true. It is possible to prove that theorem 4 also holds modulo p^5 . Hence, if $\binom{2p-1}{p-1} \not\equiv 1 \pmod{p^5}$, then the converse of Wolstenholme’s Theorem is true. We then executed the computation $\binom{2p-1}{p-1} \pmod{p^5}$. As $\binom{2p-1}{p-1} = 1 + \sum_{j=1}^{(p-1)/2} \binom{p}{j}^2$ and p divides $\binom{p}{j}$, it follows that

$$\binom{2p-1}{p-1} = 1 + p^2 \sum_{j=1}^{(p-1)/2} \left(\binom{p}{j}/p\right)^2 \pmod{p^5}, \tag{7}$$

and so this computation can be done modulo p^3 . Defining $L_1 = 1$, we see that $L_j = \binom{p}{j}/p$ satisfies

$$L_{j+1} = L_j \frac{p-j}{j+1},$$

for $1 \leq j < (p-1)/2$. Thus the computation induced by equation 7 is accomplished with $O(p)$ arithmetic operations with integer of size $O(p^3)$.

Using this method, we performed the computations for p_1 and p_2 , obtaining for $l \geq 1$

$$\begin{aligned} \binom{2p_1^l-1}{p_1^l-1} &\equiv 267428775549681894924 \pmod{p_1^5} \\ \binom{2p_2^l-1}{p_2^l-1} &\equiv 33102388482153131789208640376695 \pmod{p_2^5} \end{aligned}$$

implying that the converse of Wolstenholme’s theorem is also true for powers of p_1 and p_2 .

Table 1 summarizes what is known to the authors about the converse of Wolstenholme’s theorem at the time this paper was written.

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Integer Type	Status
Even	True
Prime powers $p^l, l \geq 2$	True if $p < 2.5 \times 10^8$ Unknown if $p > 2.5 \times 10^8$
Other n positive composite numbers	True if $n < 10^9$ Unknown if $n \geq 10^9$

Table 1: Status of the converse of Wolstenholme's theorem

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