Astrodynamic Coordinates

This ***Numerit*** program (`csystems`) can be used to calculate and convert different types of astrodynamic coordinates. The following is the "main" menu for this program.

```
            coordinate system menu

    <1>  Greenwich apparent sidereal time
    <2>  classical orbital elements to eci state vector
    <3>  eci state vector to classical orbital elements
    <4>  spherical (adbarv) coordinates to eci state vector
    <5>  eci state vector to spherical (adbarv) coordinates
    <6>  classical orbital elements to equinoctial elements
    <7>  equinoctial elements to classical orbital elements
    <8>  geocentric coordinates to geodetic coordinates
    <9>  geodetic coordinates to geocentric coordinates
    <10> osculating orbital elements to mean elements
    <11> mean orbital elements to osculating elements
    <12> eci state vector to ecf state vector
    <13> ecf state vector to eci state vector
```

For most these menu items the user can elect to either interactively input the data or use "internal" data already computed by the software in one or more previous calculations.

Classical orbital elements

The following diagram illustrates the geometry of classical orbital elements.

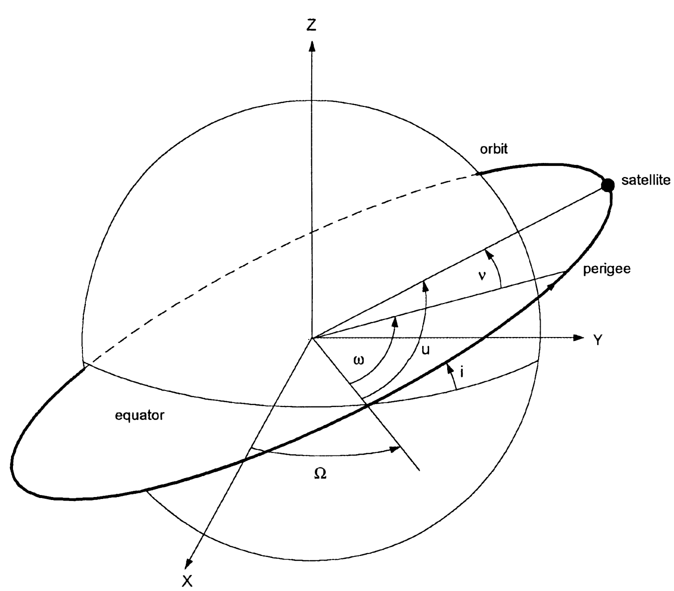

Figure 1. Classical Orbital Elements

The *semimajor axis* defines the size of the orbit and the *orbital eccentricity* defines the shape of the orbit. The angular orbital elements are defined with respect to a fundamental x-axis, the vernal equinox, and a fundamental plane, the equator. The z-axis of this system is collinear with the spin axis of the Earth, and the y-axis completes a right-handed coordinate system.

The *orbital inclination* is the angle between the equatorial plane and the orbit plane. Satellite orbits with inclinations between 0 and 90 degrees are called *direct* orbits and satellites with inclinations greater than 90 and less than 180 degrees are called *retrograde* orbits. The *right ascension of the ascending node* (RAAN) is the angle measured from the x-axis (vernal equinox) eastward along the equator to the ascending node. The *argument of perigee* is the angle from the ascending node, measured along the orbit plane in the direction of increasing true anomaly, to the argument of perigee. The *true anomaly* is the angle from the argument of perigee, measured along the orbit plane in the direction of motion, to the satellite's location.

Finally, the *argument of latitude* is the angle from the ascending node, measured in the orbit plane, to the satellite's location in the orbit. The argument of latitude is equal to $u = n + w$.

The *orbital eccentricity* is an indication of the type of orbit. For values of $0 \le e < 1$, the orbit is circular or elliptic. The orbit is parabolic when $e = 1$ and the orbit is hyperbolic if the condition $e > 1$ is true.

The semimajor axis a is calculated using the following expression:

$$a = \frac{1}{\frac{2}{r} - \frac{v^2}{m}} \tag{1}$$

where $r = |\boldsymbol{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2}$ is the scalar position and $v = |\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ is the scalar velocity or speed of the space object.

The angular orbital elements are calculated from the *equinoctial* orbital elements *h*, *k*, *p* and *q* which are in turn calculated from the rectangular components of the body-centered inertial position and velocity vectors. The equinoctial orbital elements are an alternative set of *non-singular* elements which avoid computational problems when working with orbits with small or zero values of eccentricity or inclination.

The mathematical relationship between equinoctial and classical orbital elements is given by the following expressions:

$$
\begin{aligned}
a &= a \\
h &= e \sin(w + \Omega) \\
k &= e \cos(w + \Omega) \\
l &= M + w + \Omega \\
p &= \tan(i/2) \sin\Omega \\
q &= \tan(i/2) \cos\Omega
\end{aligned}
\tag{2}
$$

In the fourth equation M is the mean anomaly and l is called the mean longitude.

The scalar orbital eccentricity e is determined from h and k as follows:

$$e = \sqrt{h^2 + k^2} \tag{3}$$

The orbital inclination i is determined from p and q using the following expression

$$i = 2\arctan\left(\sqrt{p^2 + q^2}\right) \tag{4}$$

For values of inclination greater than a small value e, the right ascension of the ascending node Ω is given by

$$\Omega = \arctan(p, q) \tag{5}$$

Otherwise, the orbit is equatorial and there is no RAAN.

If the value of orbital eccentricity is greater than e, the argument of perigee w is determined from

$$w = \arctan(h, k) - \Omega \tag{6}$$

Otherwise, the orbit is circular and there is no argument of perigee. In the ***Numerit*** code for these calculations, $e = 10^{-8}$.

Finally, the true anomaly n is found from the expression

$$n = l - \Omega - w \tag{7}$$

In this computer program, all two argument inverse tangent calculations use a four quadrant ***Numerit*** function called `atan3` to determine the correct quadrant for the angle. Angular orbital elements which can range from 0 to 360 degrees are also processed with a modulo $2p$ function named `modulo`. This utility function ensures that all angular elements are "range-reduced" to a value between 0 and $2p$.

Position and velocity vectors

The body-centered, inertial rectangular components of the position and velocity vectors can be determined from the classical orbital elements as follows:

$$r_x = p[\cos\Omega\cos(w+n) - \sin\Omega\cos i\sin(w+n)]$$

$$r_y = p[\sin\Omega\cos(w+n) + \cos\Omega\cos i\sin(w+n)]$$

$$r_z = p\sin i\sin(w+n)$$

$$v_x = -\sqrt{\frac{m}{p}}\left[\cos\Omega\{\sin(w+n) + e\sin w\} + \sin\Omega\cos i\{\cos(w+n) + e\cos w\}\right] \tag{8}$$

$$v_y = -\sqrt{\frac{m}{p}}\left[\sin\Omega\{\sin(w+n) + e\sin w\} - \cos\Omega\cos i\{\cos(w+n) + e\cos w\}\right]$$

$$v_z = -\sqrt{\frac{m}{p}}\left[\sin i\{\cos(w+n) + e\cos w\}\right]$$

In these equations p is called the *semiparameter* of the orbit and is calculated from $p = a(1-e^2)$. m is the gravitational constant of the *primary* or central body.

Geodetic and geocentric coordinates

The following diagram illustrates the geometric relationship between geocentric and geodetic coordinates of a satellite.

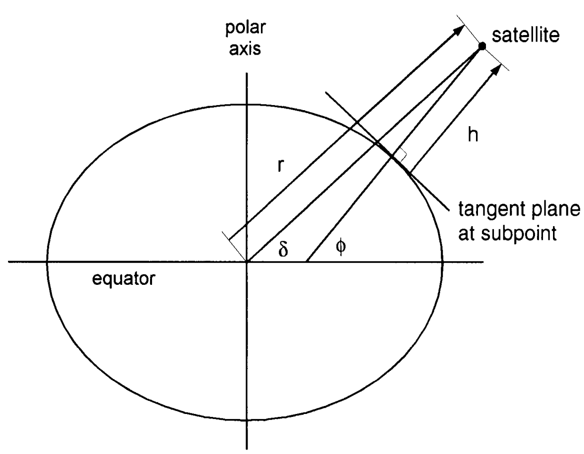

Figure 2. Geodetic and Geocentric Coordinates

In this diagram, d is the geocentric declination, f is the geodetic latitude, r is the geocentric distance, and h is the geodetic altitude.

The exact mathematical relationship between geocentric and geodetic coordinates is given by the following system of two nonlinear equations

$$
\begin{aligned}
(c+h)\cos f - r\cos d &= 0 \\
(s+h)\sin f - r\sin d &= 0
\end{aligned}
\tag{9}
$$

where the geodetic constants c and s are given by

$$c = \frac{r_{eq}}{\sqrt{1-(2f-f^2)\sin^2 f}}$$

$$s = c(1-f)^2$$

and r_{eq} is the Earth equatorial radius (6378.14 kilometers) and f is the flattening factor for the Earth (1/298.257).

The geodetic latitude is determined using the following expression:

$$f = d + \left(\frac{\sin 2d}{r}\right)f + \left[\left(\frac{1}{r^2} - \frac{1}{4r}\right)\sin 4d\right]f^2 \tag{10}$$

The geodetic altitude is calculated from

$$\hat{h} = (\hat{r}-1) + \left\{\left(\frac{1-\cos 2d}{2}\right)f + \left[\left(\frac{1}{4r} - \frac{1}{16}\right)(1-\cos 4d)\right]f^2\right\} \tag{11}$$

In these equations, r is the geocentric distance of the satellite, $\hat{h} = h/r_{eq}$ and $\hat{r} = r/r_{eq}$.

The equations for converting geodetic latitude and altitude to geocentric position magnitude and geocentric declination are as follows:

$$d = f + \left(\frac{-\sin 2f}{\hat{h}+1}\right)f + \left\{\frac{-\sin 2f}{2(\hat{h}+1)^2} + \left[\frac{1}{4(\hat{h}+1)^2} + \frac{1}{4(\hat{h}+1)}\right]\sin 4f\right\}f^2 \tag{12}$$

and

$$\hat{r} = (\hat{h}+1) + \left(\frac{\cos 2f - 1}{2}\right)f + \left\{\left[\frac{1}{4(\hat{h}+1)} + \frac{1}{16}\right](1-\cos 4f)\right\}f^2 \tag{13}$$

where the geocentric distance *r* and geodetic altitude *h* have been normalized by $\hat{r} = r/r_{eq}$ and $\hat{h} = h/r_{eq}$, respectively, and r_{eq} is the equatorial radius of the Earth.

Another useful coordinate transformation converts the geodetic latitude, longitude and altitude to an Earth-centered-fixed (ECF) position vector. The three components of this geocentric vector are given by

$$\boldsymbol{r}_{\text{geocentric}} = \begin{bmatrix} (N+h)\cos f \cos l_e \\ (N+h)\cos f \sin l_e \\ \left[N(1-e^2)+h\right]\sin f \end{bmatrix} \tag{14}$$

where

$$\begin{aligned} N &= \frac{r_{eq}}{\sqrt{1-e^2\sin^2 f}} \\ e^2 &= 2f - f^2 \\ f &= \text{ Earth flattening factor} \\ r_{eq} &= \text{ Earth equatorial radius} \\ f &= \text{ geodetic latitude} \\ l_e &= \text{ east longitude} \\ h &= \text{ geodetic altitude} \end{aligned}$$

The geocentric distance is determined from the components of the geocentric position vector as follows:

$$r = \sqrt{r_x^2 + r_y^2 + r_z^2} \tag{15}$$

The geocentric declination can be computed from the *z* component of the geocentric position vector with

$$d = \sin^{-1}\left(\frac{r_z}{r}\right) \tag{16}$$

ADBARV elements

The components of the ADBARV coordinate system are as follows:

Alpha = right ascension
Delta = geocentric declination
Beta = conjugate flight path angle
A = azimuth
R = position magnitude
V = velocity magnitude

The following diagram illustrates the geometry of the ADBARV coordinates. In this picture $\boldsymbol{a}$ is the right ascension, $\boldsymbol{d}$ is the geocentric declination and b is the conjugate flight path angle.

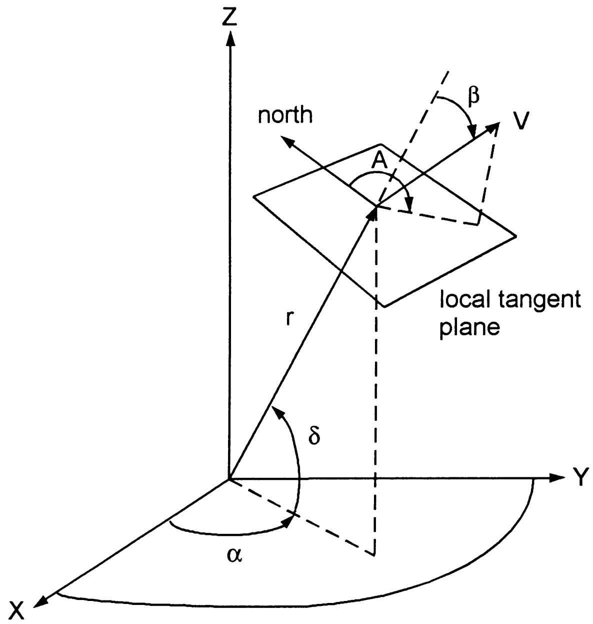

Figure 3. ADBARV elements

The mathematical relationships between ADBARV elements and the components of the ECI position and velocity vectors are as follows:

$$r = \sqrt{r_x^2 + r_y^2 + r_z^2}$$

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$\boldsymbol{a} = \tan^{-1}\left(r_y, r_x\right)$$

$$\boldsymbol{d} = \tan^{-1}\left(r_z, \sqrt{r_x^2 + r_y^2}\right) \tag{17}$$

$$b = \cos^{-1}\left(\frac{\boldsymbol{r} \bullet \boldsymbol{v}}{\left|\boldsymbol{r} \bullet \boldsymbol{v}\right|}\right)$$

$$A = \tan^{-1}\left[r\left(r_x v_y - r_y v_x\right), r_y\left(r_y v_z - r_z v_y\right) - r_x\left(r_z v_x - r_x v_z\right)\right]$$

The inertial position and velocity vectors can be determined from the ADBARV elements with this set of equations:

$$
\begin{aligned}
r_x &= r\cos \boldsymbol{d} \cos \boldsymbol{a} \\
r_y &= r\cos \boldsymbol{d} \sin \boldsymbol{a} \\
r_z &= r\sin \boldsymbol{d} \\
v_x &= v\left[\cos \boldsymbol{a} \left(-\cos A \sin \boldsymbol{b} \sin \boldsymbol{d} + \cos \boldsymbol{b} \cos \boldsymbol{d}\right) - \sin A \sin \boldsymbol{b} \sin \boldsymbol{a}\right] \\
v_y &= v\left[\sin \boldsymbol{a} \left(-\cos A \sin \boldsymbol{b} \sin \boldsymbol{d} + \cos \boldsymbol{b} \cos \boldsymbol{d}\right) + \sin A \sin \boldsymbol{b} \cos \boldsymbol{a}\right] \\
v_z &= v\left(\cos A \sin \boldsymbol{b} \cos \boldsymbol{d} + \cos \boldsymbol{b} \cos \boldsymbol{d}\right)
\end{aligned}
\tag{18}
$$

Equinoctial elements

The relationship between classical and equinoctial orbital elements is given by the following expressions:

$$
\begin{aligned}
a &= a \\
h &= e\sin(w + \Omega) \\
k &= e\cos(w + \Omega) \\
l &= M + w + \Omega \\
p &= \tan(i/2)\sin\Omega \\
q &= \tan(i/2)\cos\Omega
\end{aligned}
\tag{19}
$$

The *mean longitude* is defined by $l = M + w + \Omega$,

the *eccentric longitude* by $F = E + w + \Omega$

and the *true longitude* by $L = \boldsymbol{n} + w + \Omega$.

The equinoctial form of Kepler's equation is given by

$$l = F + h\cos F - k\sin F \tag{20}$$

We can solve for F using Newton's method as follows:

$$
\begin{aligned}
F_0 &= l \\
F_{i+1} &= F_i - \left(\frac{F_i + h\cos F_i - k\sin F_i - l}{1 - h\sin F_i - k\cos F_i}\right)
\end{aligned}
\tag{21}
$$

The position and velocity vectors in the equinoctial coordinate system are given by

$$
\begin{aligned}
x_1 &= a\left[\left(1-h^2 b\right)\cos F + hkb\sin F - k\right] \\
y_1 &= a\left[\left(1-k^2 b\right)\cos F + hkb\cos F - h\right] \\
\dot{x}_1 &= \frac{na^2}{r}\left[hkb\cos F - \left(1-h^2 b\right)\sin F\right] \\
\dot{y}_1 &= \frac{na^2}{r}\left[\left(1-k^2 b\right)\cos F - hkb\sin F\right]
\end{aligned}
\tag{22}
$$

where the geocentric scalar distance is calculated from

$$
r = a\left(1 - h\sin F - k\cos F\right) \tag{23}
$$

and n is the mean motion.

Finally, the ECI position and velocity vectors are determined from the expressions

$$
\begin{aligned}
\boldsymbol{r} &= x_1\mathbf{f} + y_1\mathbf{g} \\
\mathbf{v} &= \dot{x}_1\mathbf{f} + \dot{y}_1\mathbf{g}
\end{aligned}
\tag{24}
$$

where the components of the **f** and **g** unit vectors are as follows:

$$
\begin{aligned}
f_x &= b\left(1 - p^2 + q^2\right) \\
f_y &= b(2pq) \\
f_z &= -b(2p)
\end{aligned}
$$

and

$$
\begin{aligned}
g_x &= b(2pq) \\
g_y &= b\left(1 + p^2 - q^2\right) \\
g_z &= b(2p)
\end{aligned}
$$

The constant b is calculated from

$$
b = \frac{1}{1 + p^2 + q^2}
$$

Earth-centered-fixed (ECF) coordinates

The transformation of an ECI position vector $\mathbf{r}_{eci}$ to an ECF position vector $\mathbf{r}_{ecf}$ is given by the following vector-matrix operation

$$\mathbf{r}_{ecf} = [\mathbf{T}]\mathbf{r}_{eci} \tag{25}$$

where the elements of the transformation matrix $\mathbf{T}$ are given by

$$[\mathbf{T}] = \begin{bmatrix} \cos q & \sin q & 0 \\ -\sin q & \cos q & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{26}$$

where q is the Greenwich sidereal time at the moment of interest. Greenwich sidereal time is given by the following expression:

$$q = q_{g0} + w_e t \tag{27}$$

where q_{g0} is the Greenwich sidereal time at 0 hours UT, w_e is the inertial rotation rate of the Earth, and t is the elapsed time since 0 hours UT.

The ECF velocity vector is determined by differentiating the expression given by Equation (25) as follows:

$$\mathbf{v}_{ecf} = [\mathbf{T}]\dot{\mathbf{r}}_{eci} + [\dot{\mathbf{T}}]\mathbf{r}_{eci} = [\mathbf{T}]\mathbf{v}_{eci} + [\dot{\mathbf{T}}]\mathbf{r}_{eci} \tag{28}$$

The elements of the $[\dot{\mathbf{T}}]$ matrix are determined by differentiating the elements of the $[\mathbf{T}]$ matrix as follows:

$$[\dot{\mathbf{T}}] = \begin{bmatrix} -w_e \sin q & w_e \cos q & 0 \\ -w_e \cos q & -w_e \sin q & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{29}$$

The transformation from ECF to ECI coordinates involves the transpose of the ECI-to-ECF transformation matrices described above as follows:

$$\begin{aligned} \mathbf{r}_{eci} &= [\mathbf{T}]^T \mathbf{r}_{ecf} \\ \mathbf{v}_{eci} &= [\mathbf{T}]^T \dot{\mathbf{r}}_{ecf} + [\dot{\mathbf{T}}]^T \mathbf{r}_{ecf} = [\mathbf{T}]^T \mathbf{v}_{ecf} + [\dot{\mathbf{T}}]^T \mathbf{r}_{ecf} \end{aligned} \tag{30}$$

In the Earth-centered-fixed coordinate system the *x*-axis points in the direction of the Greenwich meridian. The *fundamental* plane of the ECF coordinate system is the equator of the Earth.

The following is a typical draft output created with this software.

```
convert eci state vector to classical orbital elements

eci state vector

rx        7475.226183658  kilometers
ry        1103.0128215013  kilometers
rz        2150.11864824741  kilometers

vx        -0.0490037505580695  km/sec
vy        6.62947126301278  km/sec
vz        -2.7744865902077  km/sec

classical orbital elements

     sma (km)         eccentricity    inclination (deg)    argper (deg)
         8000            0.025          28.5                 100

    raan (deg)     true anomaly (deg)   arglat (deg)       period (min)
          220              45                 145         118.6846843
```

Important Note

When electing main menu options (12) or (13) be sure to calculate the Greenwich sidereal time, main menu option (1), before selecting either one of these program options.