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The purpose of this paper is to discuss the micro-economic foundations of the "q" theory of investment advanced by James Tobin, and by Tobin and William Brainard. This theory of investment can be summarized as follows:[1]

The neoclassical theory of corporate investment is based on the assumption that the management seeks to maximize the present net worth of the company, the market value of the outstanding common shares. An investment project should be undertaken if and only if it increased the value of the shares. The securities markets appraise the project, its expected contributions to the future earnings of the company and its risks. If the value of the project as appraised by investors exceeds the cost, then the company's shares will appreciate to the benefit of existing stockholders. That is, the market will value the project more than the cash used to pay for it. If new debt or equity securities are issued to raise the cash, the prospectus leads to an increase of share prices.
[Tobin and Brainard, p. 242]

[Thus] the rate of investment — the speed at which investors wish to increase the capital stock — should be related, if to anything — to q, the value of capital relative to its replacement cost. [Tobin, p. 330]

Economic logic indicates that a normal equilibrium value for q is 1 for reproducible assets which are in fact being reproduced, and less than 1 for others. Values of q above 1 should

stimulate investment, in excess of requirements for replacement and normal growth, and values of q below 1 discourage investment.
[Tobin and Brainard, p. 238]

The q theory of investment is now quite popular.[2] However, there seems to be considerable confusion about how the theory is to be interpreted. For example, Robert Hall argued that the q theory is basically a neoclassical theory à la Dale Jorgenson, and that only incomplete information and delivery lags can account for "disequilibrium" values of q and for their relation to investment. Otherwise, he argued, investment would keep q equal to one. On the other hand, Michael Lovell, "struck by similarities between q theory and the neoclassical approach of Dale Jorgenson, Robert Hall, and other (p. 399), criticized both approaches in th ollowing way:

Neither approach works out the dynamics of the adjustment process within the context of a carefully articulated optimization framework that would specifically incorporate the process of expectation formation and adjustment costs. In this respect, both theories are dominated by a number of contributions that have derived the optimal time path of adjustment simultaneously with the determination of the properties of the ultimate long-run equilibrium. [pp. 399–400]

The major point of this paper is that q theory can be derived from a choice-theoretic framework which explicitly takes account of adjustment costs associated with investment. The framework is similar to those of Robert Lucas, John Gould, and

[1] Elaborations and qualifications of the theory can be found in section 1 of Tobin and Brainard.

[2] For example, Robert Hall called it "the major competitor to Jorgenson's theoretical framework for investment" (p. 85).

Hirofumi Uzawa, which were in turn anticipated by Robert Eisner and Robert Strotz. These authors criticized Jorgensonian investment theory on the following grounds. In Jorgenson's theory, "desired" capital stock is first determined and then investment is derived as a process induced by the discrepancy between the desired capital stock and current actual capital stock. The theory assumes, of course, the existence of adjustment costs associated with investment. Otherwise, Jorgenson's theory can explain only optimal capital stock, but not optimal investment. However, if the existence of adjustment costs is important, it must be taken into account explicitly in firms' decisions about investment. In this case, the notion of desired capital stock derived apart from adjustment costs loses much of its meaning.

In what follows, I show that the q theory of investment can be derived following Lucas-Gould-Uzawa models of investment which emphasize the role of adjustment costs. For this purpose, let us consider a one-commodity model in which the money price of this single commodity is denoted by p. Suppose that the production technique of the firm under consideration is described by the following production function:

$$(1) \qquad Q = F(L, K)$$

where Q, L, and K are, respectively, output, labor employed, and an index of fixed capital such as machines, factories, and managerial resources. The function F is assumed to be homogeneous of degree one. Given K, the firms' labor employment decision can be characterized in the usual way:

$$(2) \qquad \underset{L}{\text{Max}} \int_0^\infty (pQ - wL)e^{-\rho t}\, dt$$

where w is the wage rate and ρ is the rate of return on capital equity required by stockholders. Given expected p and w, the necessary condition is[3]

[3] For simplicity, I assume static expectations about p and w. The basic analysis, however, remains unchanged even if I assume changing expected p and w in so far as they are expected to change at the common trend rate of inflation.

$$(3) \qquad p(\partial Q/\partial L) = w \qquad \text{for all t}$$

Under the homogenity assumption of F, this is equivalent to

$$(4) \qquad f(k) - f'(k)k = w/p$$

where $f(k) = F(K/L, 1)$ and $k = K/L$. Denoting the expected marginal product of capital, or the expected rate of profit as r, we have

$$(5) \qquad r = f'(k)$$

Thus, under the optimal employment decision (4) corresponding to a given expected w and p, the stock price of the firm (2) is reduced to the following:

$$(6) \qquad \int_0^\infty prKe^{-\rho t}\, dt = prK/\rho$$

Let us consider a current "once and for all" investment decision of the firm. The firm must decide how fast it should increase its capital stock. An increase in capital is denoted by ΔK. As for a decrease in capital, there is usually a lower limit set by natural depreciation, since physical capital is not malleable and the firm incurs prohibitive costs when decreasing the existing capital stock too fast. (Kenneth Arrow analyzes the case where investment is irreversible in an explicitly dynamic model.) I will assume $\Delta K \geqslant 0$ in what follows, and for simplicity, depreciation is ignored.

Now the fundamental factor required to eliminate the indeterminancy of the optimal level of investment is the adjustment cost associated with high investment, that is, with a rapid rate of capital increase. It should be clear that this point is related to the distinction between the marginal efficiency of capital and the marginal efficiency of investment. Let us assume that the effective price (after taking into account adjustment costs) of investment goods is not lower than the going market price of capital goods p,

and that it increases as the amount of investment increases. This is due to the existence of a particular kind of resource such as management which cannot be increased proportionally to physical capital and/or to an increasing supply price of investment goods. Specifically, I will introduce the following function:

(7) $\psi(\alpha)$

where $\psi(0)=1, \psi'>0, \psi''>0, \alpha=\Delta K/K$

$$\lim_{\alpha \to 0+} \psi'(\alpha)=0$$

Under this assumption, the effective price of investment ΔK when the growth rate of capital is α, is not $p\Delta K$ but $p\psi(\alpha)\Delta K$, which is larger than $p\Delta K$. Then the market value of this firm after the investment is the following:

(8) $pr(K+\Delta K)/\rho - p\psi(\alpha)\Delta K$

$$=pr(1+\alpha)K/\rho - p\psi(\alpha)\alpha K$$

The firm is assumed to maximize this value with respect to $\alpha \geqslant 0$. The first-order condition is

(9) $r/\rho - \psi'\alpha - \psi \leqslant 0$

where, if $\alpha>0$, the equality holds. The second-order condition $-2\psi' - \alpha\psi'' < 0$ is always satisfied. From this, we know that for $\Delta K>0$, $q \equiv r/\rho > 1$ is necessary. Note that in the present framework the value of capital stock evaluated in the financial market prK/ρ, relative to its replacement cost pK, is r/ρ. This is Tobin's q. If $r/\rho \leqq 1$, $\Delta K=0$ in this model. If we now define $f(\alpha)=\psi'(\alpha)\alpha + \psi(\alpha)$ where $f'>0$ and $f(0)=1$, it can be seen from Figure 1 that the optimal growth rate of capital stock α^*, and thereby the optimal level of investment $\Delta K=\alpha^*K$, are uniquely determined. Note that $f(\alpha)$ in Figure 1 indicates the *marginal* effective cost of capital increase expressed as a function of the relative rate of capital increase, and that the condition for optimal α^* is the equality of q
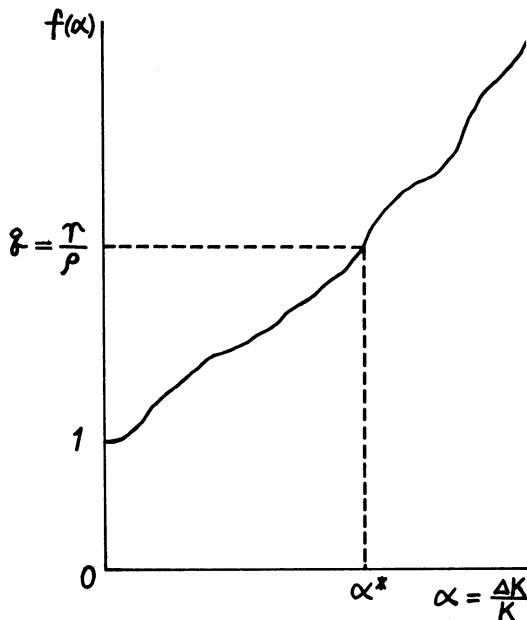

FIGURE 1

to this marginal effective cost, $f(\alpha)$. Thus given the marginal effective cost function $f(\alpha)$, investment is shown to be an increasing function of q in Figure 1.

Before proceeding to a more complicated case, some qualifications about the above analysis are in order. One is the specific nature of the adjustment cost function $\psi(\alpha)$ (equation (7)). Although a convex adjustment cost function is fairly standard in the literature (for example, see Lucas, Gould, and Uzawa), Michael Rothschild emphasized the possibility of a concave adjustment function. More general functions might affect the above analysis. Secondly, the fact that r is taken as given is valid only under our simplifying assumption of a homogeneous production function. Under decreasing returns to scale r decreases when K increases. This would not, however, affect the above analysis substantially. Finally, the expected return on new capital is assumed to be equal to that on existing capital r. If they are not equal, it is "the q ratio *on the margin* that matters for investment: the ratio of the increment of market valuation to the cost of the associated investment" (Tobin and Brainard, p. 243). This can be seen in

my analysis by assuming the following instead of (8):

(10) $\quad prK/\rho + pr'\Delta K/\rho - p\psi(\alpha)\Delta K$

$$= prK/\rho + pr'\alpha K/\rho - p\psi(\alpha)\alpha K$$

This is the market value of the firm when the expected return on new capital, r' is different from r. The maximization of (10) with respect to α leads to the optimality condition which requires the equality of q' $= r'/\rho$ to $f(\alpha)$. That is, the *marginal* q, q' must be equal to the *marginal* effective cost of capital increase, $f(\alpha)$. The reinterpretation of Figure 1 should be straightforward. These qualifications are also applied to a dynamic model to which I now turn.

The above static model should make clear my basic point but it assumed, among other things, that the investment decision considered is once and for all. More generally, however, investment decisions would be made successively, and in this case they must be made optimally from a dynamical viewpoint. Mathematically the problem is one of calculus of variations rather than differentiation.[4] Uzawa analyzes this more complicated problem. The simplified version of his model is

(11) $\quad \underset{z(t)}{\text{Max}} \int_0^\infty p(r - \Phi(z))K(t)e^{-\rho t}dt$

where $z(t) = DK(t)/K(t)$ and $r, \rho, K(0), p$ are all given. The term $\Phi(z)$ is the effective cost of adding capital after taking account of adjustment cost. This effective cost is greater the faster is the growth rate of capital. Specifically we assume $\Phi(z)$ has the following property:

(12)

$$\Phi'(z) > 0, \Phi''(z) > 0, \Phi(0) = 0, \Phi'(0) = 1$$

Uzawa's analysis shows that if a solution exists, then it is unique and the optimal z is constant over time. It is also shown that the optimal z, z^*, must satisfy the following relation:

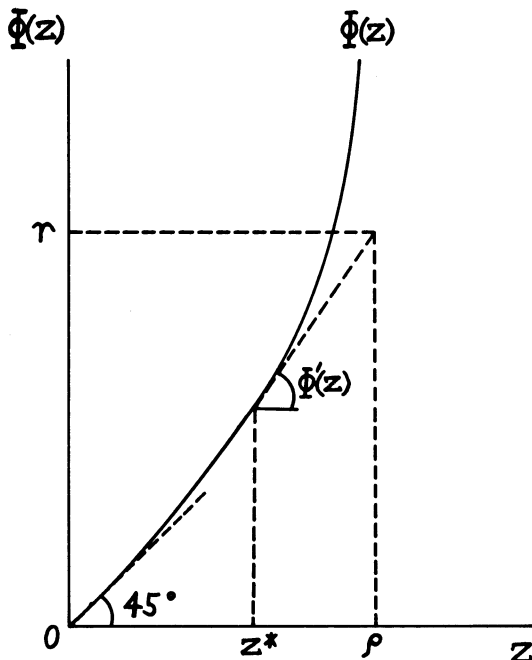[4] This is the problem analyzed by Lucas, Gould, and Uzawa.

FIGURE 2

(13) $\quad (r - \Phi(z^*))/(\rho - z^*) = \Phi'(z^*)$

Thus z^* can be found from Figure 2. Note that $(r - \Phi(z))/(\rho - z)$ is q in this dynamic context since on the optimum path z is constant and so from (11) the present value of the firm evaluated in the financial market is $p(r - \Phi(z))K(0)/(\rho - z)$, whereas its replacement cost is $pK(0)$. Therefore q is modified by the growth rate of the firm, z, and the steady investment payment $\Phi(z)$ in this dynamic model. The slope of $\Phi(z)$ is, on the other hand, the *marginal* effective cost of investment. It should be clear from Figure 2 that only at z^* is the optimality condition satisfied: $q = (r - \Phi(z))/(\rho - z)$ is equal to the *marginal* effective cost of investment, $\Phi'(z)$. This is a natural extension of the previous static result. It should also be clear that given the adjustment cost function $\Phi(z)$, and present capital stock K, investment (or z) is an increasing function of q.

One point to be noted here is the meaning of the assumption $\Phi'(0) = 1$. This assump-

tion means that the effective cost of adding one additional unit of capital is simply the market price of capital goods. If there exists some fixed cost associated with any investment, then we would have $\Phi'(0) > 1$ instead of $\Phi'(0) = 1$, and in this case the critical level of q would be larger than one.

Let me summarize my analysis. In a simple investment model emphasizing adjustment cost (this seems to be a reasonable one if we recognize that investment decisions concern *the speed of increase* in capital stock), it has been shown that the optimality condition requires the equality of q (the *marginal* q if the expected return on new capital is different from that on existing capital) to the *marginal* effective cost of investment. Aside from the price of capital goods, the latter is determined by the adjustment cost function. Therefore, the above optimality condition means that given the shape of the adjustment cost function, investment is an increasing function of q. Thus, contrary to Lovell's comment on the q theory of investment, this theory can be derived from the maximizing behavior of the firm explicitly taking into account adjustment cost. By implication, this paper has also shown that the q theory is different from Jorgenson's neoclassical theory. The q theory, allowing the divergence between the value of capital evaluated in the financial market and the price of capital goods, is a theory which explains how investment (change in capital stock) is motivated by this apparent short-run disequilibrium. No wonder, as observed above, adjustment cost plays a crucial role in this theory. Jorgenson's theory, on the other hand, basically concerns the long-run demand for capital stock.

One final remark is that in the q theory, as in John Maynard Keynes' exposition in chapter 12 of his *General Theory*, the rate of profit r is the one expected by stockholders. It seems worth future study to explore whether or not the divergence between expectations of r held by asset holders and by entrepreneurs has some implications for investment theory.[5]

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[5]One possible answer would be that if the two expectations are different, firms buy and sell their own

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stocks, so that the divergence between two expectations has not much significance. Cambridge economists, on the other hand, seem to emphasize the role of expectations of entrepreneurs distinguished from those of asset holders.