Gaussian Curvature[*]

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We have learned that the two principal curvatures (and vectors) determine the local shape of a point on a surface. One characterizes the rate of maximum bending of the surface and the tangent direction in which it occurs, while the other characterizes the rate and tangent direction of minimum bending. The rate of surface bending along any tangent direction at the same point is determined by the two principal curvatures according to Euler's formula. This lecture introduces two new measures of the curvature of a surface, namely, its Gaussian and mean curvatures, which turn out to have greater geometrical significance than the principal curvatures.

1 Gaussian and Mean Curvatures

Let κ_1 and κ_2 be the principal curvatures of a surface patch $\boldsymbol{\sigma}(u, v)$. The *Gaussian curvature* of $\boldsymbol{\sigma}$ is

$$K = \kappa_1 \kappa_2,$$

and its *mean curvature* is

$$H = \frac{1}{2}(\kappa_1 + \kappa_2).$$

To compute K and H, we use the first and second fundamental forms of the surface:

$$E du^2 + 2F du dv + G dv^2 \qquad \text{and} \qquad L du^2 + 2M du dv + N dv^2.$$

Again, we adopt the matrix notation:

$$\mathcal{F}_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \qquad \text{and} \qquad \mathcal{F}_2 = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

By definition, the principal curvatures are the eigenvalues of $\mathcal{F}_1^{-1}\mathcal{F}_2$. Hence the determinant of this matrix is the product $\kappa_1\kappa_2$, i.e., the Gaussian curvature K. So

$$K = \det({\mathcal{F}_1}^{-1}\mathcal{F}_2) = \det(\mathcal{F}_1)^{-1}\det(\mathcal{F}_2) = \frac{LN - M^2}{EG - F^2}.. \tag{1}$$

The trace of the matrix is the sum of its eigenvalues, thus, twice the mean curvature H. After some calculation, we obtain

$$H = \frac{1}{2}\mathrm{trace}({\mathcal{F}_1}^{-1}\mathcal{F}_2) = \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2}. \tag{2}$$

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[*] The material is adapted from the book *Elementary Differential Geometry* by Andrew Pressley, Springer-Verlag, 2001.

An equivalent way to obtain K and H uses the fact that the principal curvatures are also the roots of

$$\mathcal{F}_2 - \kappa\mathcal{F}_1 = 0,$$

which expands into a quadratic equation

$$(EG - F^2)\kappa^2 - (LG - 2MF + NE)\kappa + LN - M^2 = 0.$$

The product K and the sum $2H$ of the two roots, can be determined directly from the coefficients. The results are the same as in (1) and (2).

Conversely, given the Gaussian and mean curvatures K and H, we can easily find the principal curvatures κ_1 and κ_2, which are the roots of

$$\kappa^2 - 2H\kappa + K = 0,$$

i.e., $H \pm \sqrt{H^2 - K}$.

Example 1. We have considered the surface of revolution (see Example 1 in the notes titled "Surface Curvatures")

$$\boldsymbol{\sigma}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

where we can assume, without loss of generality, that $f > 0$ and $\dot{f}^2 + \dot{g}^2 = 1$ everywhere. Here a dot denotes d/du. The coefficients of the first and second fundamental forms were determined:

$$E = 1, \quad F = 0, \quad G = f^2, \quad L = \dot{f}\ddot{g} - \ddot{f}\dot{g}, \quad M = 0, \quad N = f\dot{g}.$$

So the Gaussian curvatures is

$$K = \frac{LN - M^2}{EG - F^2} = \frac{(\dot{f}\ddot{g} - \ddot{f}\dot{g})f\dot{g}}{f^2}.$$

Meanwhile, differentiate $\dot{f}^2 + \dot{g}^2 = 1$:

$$\dot{f}\ddot{f} + \dot{g}\ddot{g} = 0.$$

Thus,

$$\begin{aligned}(\dot{f}\ddot{g} - \ddot{f}\dot{g})\dot{g} &= -\dot{f}^2\ddot{f} - \ddot{f}\dot{g}^2 \\ &= -\ddot{f}(\dot{f}^2 + \dot{g}^2) \\ &= -\ddot{f}.\end{aligned}$$

So the Gaussian curvature gets simplified to

$$K = -\frac{\ddot{f}}{f}.$$

Example 2. Next, here we compute the Gaussian and mean curvatures of a Monge patch $z = f(x, y)$. Namely, the patch has description $\boldsymbol{\sigma}(x, y) = (x, y, f(x, y))$. First, we obtain the first and second derivatives:

$$\boldsymbol{\sigma}_x = (1, 0, f_x), \quad \boldsymbol{\sigma}_y = (0, 1, f_y), \quad \boldsymbol{\sigma}_{xx} = (0, 0, f_{xx}), \quad \boldsymbol{\sigma}_{xy} = (0, 0, f_{xy}), \quad \boldsymbol{\sigma}_{yy} = (0, 0, f_{yy}).$$

Immediately, the coefficients of the first fundamental form are determined

$$E = 1 + f_x^2, \qquad F = f_x f_y, \qquad G = 1 + f_y^2.$$

So is the unit normal to the patch:

$$\boldsymbol{n} = \frac{\boldsymbol{\sigma}_x \times \boldsymbol{\sigma}_y}{\|\boldsymbol{\sigma}_x \times \boldsymbol{\sigma}_y\|} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}.$$

With the normal $\boldsymbol{n}$, we obtain the coefficients of the second fundamental form:

$$\begin{aligned} L &= \boldsymbol{\sigma}_{xx} \cdot \boldsymbol{n} = \frac{f_{xx}}{\sqrt{1 + f_x^2 + f_y^2}}, \\ M &= \boldsymbol{\sigma}_{xy} \cdot \boldsymbol{n} = \frac{f_{xy}}{\sqrt{1 + f_x^2 + f_y^2}}, \\ N &= \boldsymbol{\sigma}_{yy} \cdot \boldsymbol{n} = \frac{f_{yy}}{\sqrt{1 + f_x^2 + f_y^2}}. \end{aligned}$$

Plug the expressions for E, F, G, L, M, N into (1) and (2). A few more steps of symbolic manipulation yield:

$$\begin{aligned} K &= \frac{LN - M^2}{EG - F^2} = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}, \\ H &= \frac{1}{2} \frac{LG - 2MF + NE}{EG - F^2} = \frac{f_{xx}(1 + f_y^2) - 2 f_{xy} f_x f_y + f_{yy}(1 + f_x^2)}{2(1 + f_x^2 + f_y^2)^{3/2}}. \end{aligned}$$

2 Classification of Surface Points

The Gaussian curvature is independent of the choice of the unit normal $\boldsymbol{n}$. To see why, suppose $\boldsymbol{n}$ is changed to $-\boldsymbol{n}$. Then the signs of the coefficients of L, M, N change, so do the signs of both principal curvatures κ_1 and κ_2, which are the roots of $\det(\mathcal{F}_2 - \kappa \mathcal{F}_1)$. Their product $K = \kappa_1 \kappa_2$ is unaffected. The mean curvature $H = (\kappa_1 + \kappa_2)/2$, nevertheless, has its sign depending on the choice of $\boldsymbol{n}$.

The sign of K at a point $\boldsymbol{p}$ on a surface $\mathcal{S}$ has an important geometric meaning, which is detailed below.

$K > 0$ The principal curvatures κ_1 and κ_2 have the same sign. The normal curvature κ in any tangent direction $\boldsymbol{t}$ is equal to $\kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta$, where θ is the angle between $\boldsymbol{t}$ and the principal vector corresponding to κ_1. So κ has the same sign as that of κ_1 and κ_2. The surface is bending *away* from its tangent plane in all tangent directions at $\boldsymbol{p}$. The quadratic approximation of the surface near $\boldsymbol{p}$ is the paraboloid

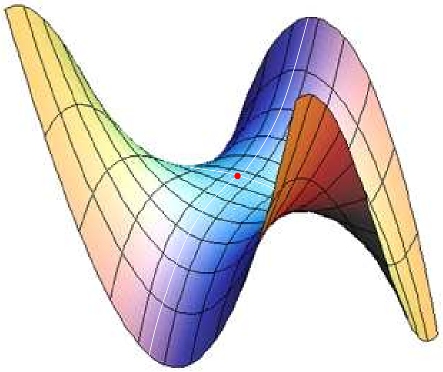
$$z = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2).$$

We call $\boldsymbol{p}$ an *elliptic point* of the surface.

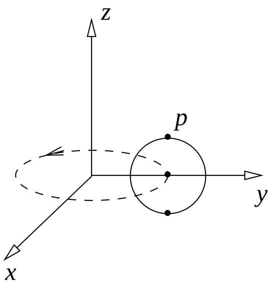
$K < 0$ The principal curvatures κ_1 and κ_2 have opposite signs at $\boldsymbol{p}$. The quadratic approximation of the surface near $\boldsymbol{p}$ is a hyperboloid. The point is said to be a *hyperbolic point* of the surface.

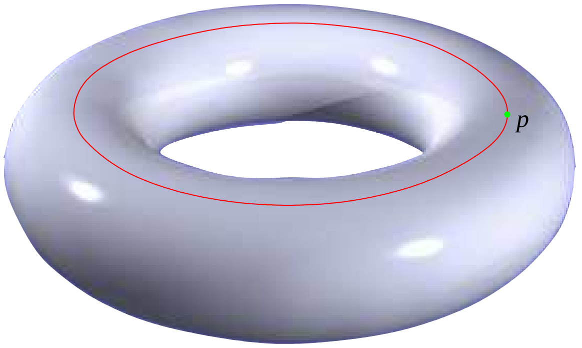
$K = 0$ There are two cases:

1. Only one principal curvature, say, κ_1, is zero. In this case, the quadratic approximation is the cylinder $z = \frac{1}{2}\kappa_2 y^2$. The point $\boldsymbol{p}$ is called a *parabolic point* of the surface.
2. Both principal curvatures are zero. The quadratic approximation is the plane $z = 0$. The point $\boldsymbol{p}$ is a *planar point* of the surface. One cannot determine the shape of the surface near $\boldsymbol{p}$ without examining the third or higher order derivatives. For example, a point in the plane and the origin of a monkey saddle $z = x^3 - 3xy^2$ (shown below) are both planar points, but they have quite different shape.



A torus is the surface swept by a circle originally in the yz-plane and centered on the y-axis at a distance greater than its radius from the origin, when the circle revolves about the z-axis. It is a good example which has all three types of points. At points on the outer half of the torus, the torus bends away from from its tangent plane; hence $K > 0$. At each point on the inner half, the torus bends toward its tangent plane in the horizontal direction, but away from it in the orthogonal direction; hence $K < 0$. On the two circles, swept respectively by the top and bottom points of the original circle, every point has $K = 0$.





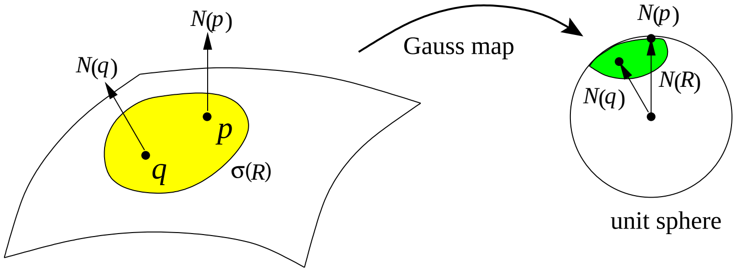
A surface $\mathcal{S}$ is *flat* if its Gaussian curvature is zero everywhere. A plane is flat. Let it be the xy-plane with the parametrization $(x, y, 0)$. We can easily show that the plane has zero Gaussian curvature. A circular cylinder, treated in Example 3 of the notes "Surface Curvatures", has one principal curvature equal to zero and the other equal to the inverse of the radius of its cross section. So a circular cylinder is also flat, even though it is so obviously curved.

A surface is *minimal* provided its mean curvature is zero everywhere. Minimal surfaces have Gaussian curvature $K \leq 0$. This is because $H = (\kappa_1 + \kappa_2)/2 = 0$ implies $\kappa_1 = -\kappa_2$.

3 The Gauss Map

The standard unit normal $\boldsymbol{n}$ to a surface patch $\boldsymbol{\sigma}$ measures the 'direction' of its tangent plane. The change rate of $\boldsymbol{n}$ in a tangent direction, i.e., the normal curvature, indicates the degree of variation of surface geometry in that direction at the point. To make the notion of change of geometry independent of any tangent direction, we can measure by the 'rate of change of $\boldsymbol{n}$ per unit area'.

Note that $\boldsymbol{n}$ is a point of the unit sphere S^2 centered at the origin. The *Gauss map* from a surface patch $\boldsymbol{\sigma}(u, v) : U \to \mathbb{R}^3$ to the unit sphere S^2 sends a point $\boldsymbol{p} = \boldsymbol{\sigma}(u, v)$ to the point $\boldsymbol{n}(u, v)$ of S^2. The Gauss map may be a many-to-one mapping since multiple points on the patch can have the same unit normal.



Let $R \subseteq U$ be a region. The amount by which $\boldsymbol{n}$ varies over the corresponding region $\boldsymbol{\sigma}(R)$ on the surface is measured by the area of the image region $N(R)$ on the unit sphere. The rate of change of $\boldsymbol{n}$ per unit area is the limit of the ratio of the area $\mathcal{A}_N(R)$ of $N(R)$ to the area $\mathcal{A}_{\boldsymbol{\sigma}}(R)$ of the surface region $\boldsymbol{\sigma}(R)$, as R shrinks to a point. To be more precise, we consider R to be a closed disk of radius δ centered at $(u, v) \in U$. This ratio is

$$\lim_{\delta \to 0} \frac{\mathcal{A}_N(R)}{\mathcal{A}_{\boldsymbol{\sigma}}(R)}.$$

It can be shown [2, pp. 166-168] that the above ratio is the absolute value of the Gaussian curvature at $\boldsymbol{p}$, i.e.,

$$\lim_{\delta \to 0} \frac{\mathcal{A}_N(R)}{\mathcal{A}_{\boldsymbol{\sigma}}(R)} = |K|.$$

The integral of the Gaussian curvature K over a surface $\mathcal{S}$,

$$\iint_{\mathcal{S}} K d\mathcal{S},$$

is called the *total Gaussian curvature* of $\mathcal{S}$. It is the *algebraic area* of the image of the region on the unit sphere under the Gauss map. Note the use of the word 'algebraic' since Gaussian curvature can be either positive or negative,

Suppose the patch $\mathcal{S} = \boldsymbol{\sigma}(u, v)$ is defined over the domain $[a, b] \times [c, d]$. Then the total Gaussian curvature is computed as

$$\int_c^d \int_a^b K(u, v) \sqrt{EG - F^2}\, du dv.$$

Example 3. If the Gaussian curvature K of a surface $\mathcal{S}$ is constant, then the total Gaussian curvature is $K\mathcal{A}(S)$, where $\mathcal{A}(S)$ is the area of the surface. Thus a sphere of radius r has total Gaussian curvature $\frac{1}{r^2} \cdot 4\pi r^2 = 4\pi$, which is independent of the radius r.

Example 4. Without any computation, we can determine that an ellipsoid also has total curvature 4π. The Gauss map is bijective (one-to-one and onto) since every point on the ellipsoid has a distinct normal. The image region covers the unit sphere. Because the Gaussian curvature is everywhere positive on the ellipsoid, the area of the unit sphere, 4π, is the total Gaussian curvature of the ellipsoid.

References

[1] B. O'Neill. *Elementary Differential Geometry.* Academic Press, Inc., 1966.

[2] A. Pressley. *Elementary Differential Geometry.* Springer-Verlag London, 2001.