Real Face of János Bolyai

Tamás Dénes

On the 150th anniversary of the death of János Bolyai[1]

In Vol. 56, No. 11, of *Notices*, I saw a fascinating article with the title: *Changing Faces—The Mistaken Portrait of Legendre*. The author, Peter Duren, writes with understandable bewilderment: "*It seems incredible that such egregious error could have gone undetected for so many years*." This encouraged me to publish, in the same journal, the story—no less fascinating—of the real face of János Bolyai.

It should be explained that in the case of János Bolyai two different interpretations of the word "face" are justified: the "face" in terms of the portrait (painting, drawing), and the "face" as an abstract concept. The first part of the article introduces the surprising story of his only portrait, which, although universally accepted, turns out not to be of him at all. The second part explores his intellectual or "mind-face" (it is my own word formation) and outlines a new approach to Bolyai's creative life and work.

The town of Marosvásárhely—which lies in the heart of the Central European Transylvania—fulfills an important role in the history of the Bolyai family: János lived most of his life there, and the local library holds most of his manuscripts. The reader might find it strange that while János Bolyai is well known around the world as an eminent Hungarian, Transylvania (and so Marosvásárhely) can be found within the borders of Romania. This can be explained by the unsettled history of Transylvania. If we look back only to the nineteenth century, the part of Transylvania that was populated mainly by Hungarians was autonomous at times, whereas at other times it belonged to Hungary. In 1947, following World War II, the Treaty of Paris gave this area to Romania, so that is where it is found on today's maps.

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[1]*Dedicated to the memory of Professor Elemér Kiss (1929–2006).*

The article entitled "The real face of János Bolyai" originates from my conversations with Professor Elemér Kiss. Unfortunately his serious illness, followed by his death in 2006, thwarted our writing of a shared article. This piece of work is intended to fill this gap.

Only Two Pictures of János Bolyai Ever Existed, Neither of Which Has Survived

János Bolyai (December 15, 1802–January 29, 1860) emerges like a comet from the history of Hungarian mathematics.

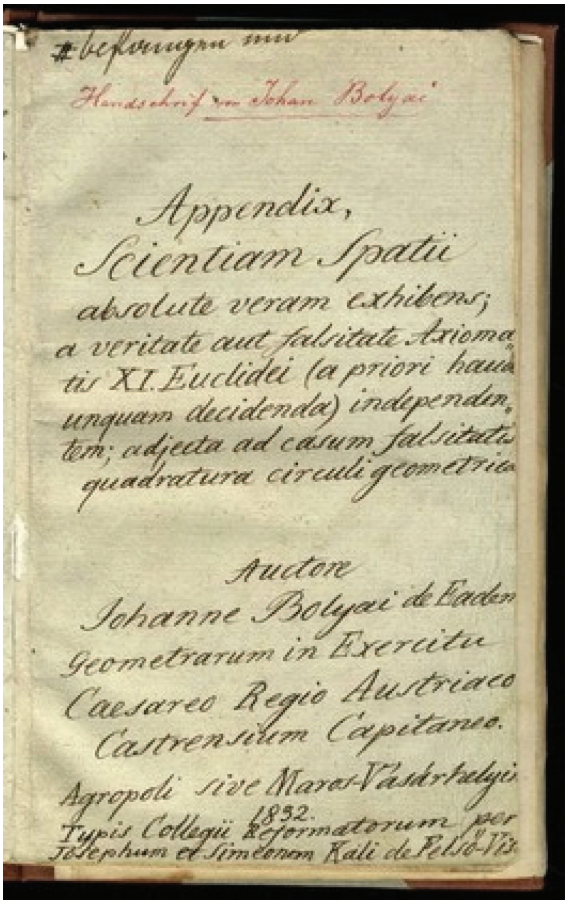
"*He was an illustrious mathematician with a great mind; the first amongst the first*"—read the record of his death in the book of the Reform Church of Marosvásárhely in Transylvania. On November 3, 1823, he had sent a letter to his father from Temesvar, including the words that would later become famous: "*I created a new, different world out of nothing*."

What he meant by this "*new world*" was the idea of hyperbolic geometry, which was outlined in 1832, as an appendix to the book *Tentamen* by Farkas Bolyai,[2] entitled "The absolute true science of space" (*Scientiam Spatii absolute veram exhibens*). This, under the name "Appendix", has become his best known piece of writing.

The theory outlined in this work has been named "Bolyai-Lobachevsky geometry", following a decision made in 1894 at the International Bibliographic Congress of Mathematical Sciences. In January 2009 the "Appendix" by János Bolyai was added to UNESCO's Memory of the World Register.

János Bolyai was the son of Farkas Bolyai—himself a defining figure of nineteenth-century Hungarian mathematics, who was in regular correspondence with Gauss. As a result, it is perhaps not surprising that Farkas Bolyai and his wife Zsuzsanna Árkosi Benkö were immortalized by

[2]*Father of János Bolyai.*


Handschrift von Johan Bolyai

Appendix,
Scientiam Spatii
absolute veram exhibens;
a veritate aut falsitate Axioma
tis XI. Euclidei (a priori haud
unquam decidenda) independen
tem; adjecta ad casum falsitatis
quadratura circuli geometrica.

Auctore
Johanne Bolyai de Eadem
Geometrarum in Exercitu
Caesareo Regio Austriaco
Castrensium Capitaneo.

Agropoli sive Maros-Vásárhelyini
1832.
Typis Collegii Reformatorum per
Josephum et Simeonem Kali de Felső-Vist

(Source: Hungarian Academy of Sciences Library)

Figure 1. Title page of the "Appendix" by János Bolyai.

contemporary artists in both drawings and oil paintings. It might therefore be assumed that their child, who had already become famous in his lifetime, would be immortalized in a similar manner.

But based on contemporary sources, only two pictures of János Bolyai ever existed, neither of which has survived. One of the "Vienna pictures" was mentioned by Farkas Bolyai himself in a letter written to his son on September 3, 1821. According to other sources, by 1837 this picture could no longer be found.

The other one was made while he was serving as a lieutenant. The destruction of this was accounted for by János Bolyai himself: "*I tore up this picture, which had been taken in a military parade, for I was not worthy of my father. I wasn't craving the outward immortality so wildly promoted by others*."

The most recent research on Bolyai, by professors Tibor Weszely[3] and Elemér Kiss,[4] also supports the idea that there is no surviving authentic image of János Bolyai.

[3]*Sapientia University, Tg-Mures, Romania.*

[4]*He had been a professor of mathematics until his death (2006) at the Sapientia University, Tg-Mures, Romania.*

The Portrait of János Bolyai That Isn't

The question is: how has this—supposedly authentic—portrait spread around the world with the name of János Bolyai?

Well, exactly fifty years ago, on the centenary of János Bolyai's death, Hungarian and Romanian stamps were published with Bolyai's name on them. Since then there has been an increased presence of this portrait everywhere: in books, on postcards, and most recently on the Internet, too. Today we know for certain that this portrait is not that of János Bolyai.

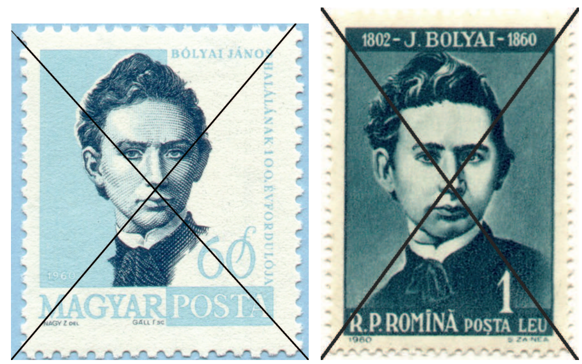


Figure 2. The portrait, not of János Bolyai (on Hungarian and Romanian stamps in 1960), that has been circulated around the world.

The year 2010 is the 150th jubilee of János Bolyai's death, so it's about time that—after a latent period of fifty years—we resolve this scandalous mystery and bring to the public the results of the latest Bolyai research. In order to do this, I first need to briefly introduce the reader to two contemporaries of Bolyai: two Hungarian painters and a painting that plays a key role in the story.

Mór Adler (1826–1902) was one of the pioneers of Hungarian painting. He stood out as a student of some merit at the Weisenberger School of Graphic Art, from which he went to the Vienna Academy. There he was taught by the then well-known historical and religious painters, between 1842 and 1845. He then traveled to Munich in 1845 to study the works of Zimneirmann and Schnorr von Carolsfeld and for further studies in 1846–1847 in Paris. Next he settled in Pest in 1848, where he would become a respected figure in the art world by the end of his career. He took part in the Pest Artists Group exhibition in 1851 and would take part in this annually for the next fifty-eight years. He was best known for his portraiture and still-life paintings, which he executed in a fine realistic manner.

Mór Adler created the large oil painting above (150 x 100cm) in 1864.

The name of the person depicted in this painting does not appear either on the front or the back of the painting, nor is it mentioned in any contemporary documents. One thing we know for

Figure 3. Oil painting by Mór Adler from 1864.

Figure 4. Károly Lühnsdorf's drawing of Mór Adler's painting. On it, his handwritten note that has, until now, misled the world.

certain is that a drawing by Károly Lühnsdorf (1893–1958)[5] was made, based on this painting. He wrote the name János Bolyai at the bottom of the drawing, accompanied by the following note: "*I have drawn this portrait based on the only remaining picture of János Bolyai, painted from life by Mór Adler (1826–1902) artist from Óbuda in 1864—Károly Lühnsdorf*." The original drawing by Károly Lühnsdorf is now owned by the Bolyai family, but the photo of it and Mór Adler's painting can be found on the walls of the János Bolyai Mathematical Society.

To sum up, taking account of Mór Adler's and János Bolyai's biographical data and the fact that the painting is of a twenty-year-old man, we can draw the following conclusion. If the painting was of János Bolyai, it would have to have been created around 1822, when Mór Adler wasn't yet born.

Lühnsdorf states that he drew his picture "based on Adler's original painting from life", a clear assumption that Adler painted his picture of Bolyai himself. However, we know from biographical data that Mór Adler traveled around Europe until 1848. Only then did he settle in Hungary—at which time Bolyai was already forty-six years old.

To assume that the painter did not paint from life but from memory would also be a mistake, as in 1826, when Mór Adler was born, Bolyai was already twenty-four years old. If they met toward the end of the 1840s, when Adler was beginning his artistic career, János Bolyai would have already been past forty.

Thus Mór Adler's painting cannot be of János Bolyai, and Károly Lühnsdorf must have written his note based on false information, thereby misleading future generations. This is how this portrait, that IS NOT OF JÁNOS BOLYAI, started its journey around the world, being mistakenly recognized by mathematicians, students, and institutions as the only original portrait of him.

The Real Face of János Bolyai

> "*He was the first Hungarian mathematician who (according to Loránd Eötvös)*[6] *created something world*

[5] *He studied at the Hungarian Academy of Arts between 1921 and 1928. His main interest was in painting portraits and biblical scenes; he acquired fame in the field of portrait painting. These depicted scientists, historical figures, and personalities from religious and public life.*

[6] *Loránd Eötvös (1848–1919) is remembered today for his experimental work on gravity, in particular his study of the equivalence of gravitational and inertial mass (the so-called weak equivalence principle) and his study of the gravitational gradient on the Earth's surface. Eötvös's law of capillarity (weak equivalence principle) served as a basis for Einstein's theory of relativity, and the Eötvös experiment was cited by Albert Einstein in his 1916 paper* "The foundation of the general theory of relativity"*. (Capillarity: the property or exertion of capillary attraction of repulsion, a force that is the resultant of adhesion, cohesion, and surface tension in liquids that are in contact with solids, causing the liquid surface to rise—or be depressed.) The Eötvös torsion balance, an important instrument of geodesy and geophysics throughout the whole world, studies the Earth's physical properties.*

Figure 5. Here is the picture of the only authentic relief of János Bolyai on the front of the Culture Palace in Marosvásárhely (Romania).

Figure 6. The only authentic relief of János Bolyai.

> *famous. Unfortunately, of this scientist-giant no picture survives, his features being forever hidden from future generations. The only source describing his appearance is his passport (made when he was forty-eight): he was of average build, blue-eyed and with a long face.*" (Elemér Kiss)

From contemporary descriptions we may learn what he looked like. We know that he sported a dark brown beard, that his hair was the same color, that his eyes were dark blue. According to József Koncz (historian of the College of Marosvásárhely), János Bolyai looked very much like General György Klapka.[7] Another important fact: his son, Dénes Bolyai, stated that there was a huge resemblance between himself and his father.

I took this thinking further. On the facade of the Culture Palace in Marosvásárhely, above the mirror room windows, there are six carved stone reliefs of nineteenth-century intellectual geniuses. Underneath them, faded but readable subtitles identify each figure.

The third one from the left is Farkas Bolyai, the fourth one is János Bolyai. With the exception of János Bolyai, we have authentic pictures of all of the others. I compared these pictures with the reliefs, and I found the features to be easily recognizable.

Then I looked at portraits of György Klapka and Dénes Bolyai and placed them beside the János Bolyai representation from the Culture Palace. I was fascinated by the likeness: as if they were showing the same person.

The Culture Palace in Marosvásárhely was built between 1911 and 1913. At that time there were people living in the town who knew or saw János Bolyai, including his son Dénes Bolyai. He was a retired judge who took part in the exhumation of his father and grandfather on June 7, 1911. The artist who set János Bolyai's features in stone at this time must have—naturally—consulted the son (Dénes Bolyai) and his acquaintances regarding his father's looks.

Help of Computer Graphics

Based on the above reasoning, we have to accept that there isn't any authentic portrait of János Bolyai. We have proved that Mór Adler's and Károly Lühnsdorf's pictures aren't of János Bolyai, and there is little likelihood of ever coming upon an authentic photo or painting in the back of the archives. It is very important, however, that we have authentic portraits of his father, Farkas Bolyai, his mother, Zsuzsanna Benkö, and his son, Dénes Bolyai.

From these data, with the help of computer graphics (Meesoft SmartMorph software), Róbert

It is used for mine exploration and also in the search for minerals, such as oil, coal, and ores.

[7] *György Klapka was a heroic general of the Hungarian freedom fight in 1848–1849.*

Figure 7. Computer transformation of a face: János Bolyai—Dénes Bolyai.

Figure 8. Computer transformation of a face: Dénes Bolyai—Farkas Bolyai (The likeness of grandson and grandfather can only be explained by the genetic mediation of János Bolyai between the two generations.)

Figure 9. Computer transformation of a face: György Klapka—János Bolyai.

Oláh-Gál[8] and Szilárd Máté[9] have created a virtual portrait of János Bolyai [20]. The aim of this experiment was to reduce the subjective element in deciding which portrait is more accurate, the picture painted by Mór Adler in 1864 or János Bolyai's half-relief on the building of the Culture Palace in Marosvásárhely.

After much experimentation, using the facial transformation technique on the computer, the following conclusion was drawn: in all probability, only one of the pictures comes close to János Bolyai's real likeness, and it is the half-relief on the Culture Palace.

After several decades of silence, attention needs to be drawn to the fact that the face of János Bolyai

[8] *Sapientia University, Department of Mathematics and Informatics, Miercurea-Ciuc, Romania.*
[9] *Sapientia University, Department of Mathematics and Informatics, Miercurea-Ciuc, Romania.*

in the public consciousness is not really his face. The only authentic source of his real portrait is the Culture Palace in Marosvásárhely (Figure 6). Being a defining figure of mathematical history, in the future he deserves to be associated with his real facial features.

I present gladly to the reader the next two works of art, which follow this mentality (see Figures 10 and 11).

Furthermore, this shocking statement, referring to his appearance, may equally be applied to the "well-known" facts about his professional activity ("mind-face").

János Bolyai's Real "*Mind-Face*" as a Mathematician (Based on E. Kiss: Mathematical Gems from the Bolyai Chests [16])

In his life János Bolyai's only published work was "The absolute true science of space", better known as the "Appendix". This was enough to make him world famous, but it also reduced his intellectual creation to this single piece of work.

When János Bolyai died the military governor seized all his manuscripts and had them put in chests and transported to the castle so that they could be examined for military secrets. Thus were his papers preserved for posterity, approximately 14,000 pages of manuscripts. The task facing researchers has not been easy. There are few dates, no numbered pages, pages missing, notes on envelopes and theater programs, idiosyncratic mathematical notations, and newly invented words.

However, János Bolyai didn't just leave us with the "Appendix" but with a heritage, consisting of 14,000 pages of letters to his father and manuscripts which are now kept in chests in the Teleki-Bolyai Library in Marosvásárhely. In these chests one can find mathematical theories—treasures in Bolyai's words—that have been hidden from the public for nearly 100 years. These pages convince us that János Bolyai, who was known purely as a geometer, was actually a universal mathematical genius who worked on many branches of mathematics, at times preceding significant inventions of other big names by decades.

The task Elemér Kiss took on, deciphering the contents of the "Bolyai chests", led to extraordinary results. The expression "deciphering" describes the tedious act of many decades by which it has been possible to reconstruct the contents of these materials. The contents, the grammar, the mathematical symbols, which differed significantly from those of present times, were often unreadable. Today we know that the results of this hard work have left us with *a brand new "mind-face" of János Bolyai.*

Elemér Kiss's book was published in 1999 in Hungarian and in English, followed by an extended second edition in 2005 by Typotex and Akadémia Publishers [16].

Figure 10. Reconstructed portrait drawing with India ink, made by Attila Zsigmond (a painter who lived in Marosvásárhely in 1927–1999), using Bolyai contemporary texts and other sources. The picture can be found in the Bolyai Museum, Marosvásárhely.

Figure 11. Bolyai Memorial Medal prepared for the Bolyai anniversary (in 2002) by Kinga Széchenyi, based on the relief on the Culture Palace in Marosvásárhely.

The first chapter, "The life of János Bolyai and the science of space", gives a brief account of the journey the scientist took in the creation of a new geometry. In addition, there is a real novelty in Chapter 1.6, considering the discovery of non-Euclidean geometry based on facts from Bolyai's correspondence. The author comes up with a convincing reasoning for the priority of Bolyai in the Bolyai-Gauss-Lobacsevszky relation.

In the second chapter we can read a systematic and comprehensive description of the "Bolyai chests". Parts of this chapter explain the language and symbols used by Bolyai, the result of meticulous research, as some of the original texts resemble complicated riddles.

In Chapter 4.3 we can read that in one of Bolyai's notes, he writes: "*My long nourished expectations and hopes had grown and mounted higher, namely, that I can devise primes based solely on their order in their series, independently or directly …, in other*

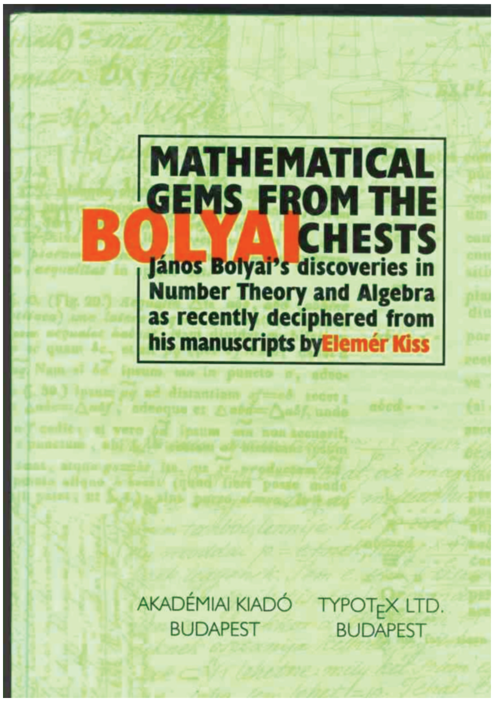


Figure 12. Elemér Kiss's book reveals a brand-new aspect of the mathematical work of János Bolyai.

words, that it is possible to give a formula which will define only primes."

He could find no formula for rational integer primes and neither could anyone else till this day, but his investigations led János Bolyai to an important discovery: he hit upon the first pseudoprime.[10]

Bolyai believed he had discovered the formula of primes in Fermat's Little Theorem.[11] Urged by his father, he attempted to prove the inverse of Fermat's Theorem, but a few attempts convinced him that proof was impossible and the inverse of Fermat's Little Theorem does not hold in general.[12] He did not find the prime formula, but he discovered the first pseudoprime. He communicated the discovery of the smallest pseudoprime (relative to 2), 341, in a letter to his father:

> "…*the most immediate and proper main question, namely that it may be the case that* $2^{\frac{m-1}{2}} \equiv 1 \pmod{m}$, *even though* m *is not a prime (which can of course be proven by even one example, such as the following which I happened to stumble on by chance, but not without theoretical considerations):* $2^{340} \equiv 1 \pmod{341}$ *is divisible by* $341 = 11 \cdot 31$, *which is infinitely easy to ascertain from* $2^{10} = 1024$, *which gives a remainder of* 1 *when divided by* 341, *therefore the remainder of* $2^{10^{17}} = 2^{170} = 2^{\frac{341-1}{2}}$ *and* $2^{341-1} \equiv 1 \pmod{341}$ *alike, thus neither the Fermat theorem nor the nice conjecture with regard to* $2^{\frac{m-1}{2}}$ *is valid (neither in the general case, nor in the particular one when* $a = 2$), *which is only to be regretted, since they could have supplied an excellent and very comfortable new means of identification (criterion) of primes* …"

From this letter, especially interesting is the fragment: "*but not without theoretical considerations*". What could those earlier notes include? János Bolyai examined the question under what conditions the congruence

$$a^{pq-1} \equiv 1 \pmod{pq} \tag{1}$$

is satisfied, where p and q are primes, and a is an integer not divisible by either p or q. His reasoning was as follows, according to Fermat's Little Theorem: $a^{p-1} \equiv 1 \pmod{p}$ and $a^{q-1} \equiv 1 \pmod{q}$. By raising both sides of the first congruence to the

[10] *Various composite numbers m for which the expression* $a^{m-1} \equiv 1 \pmod{m}$ *is valid. The fact that the number of pseudoprimes is infinite has been known since 1904 [5]. There are composite numbers m which satisfy this congruence for each a, whenever a is a relative prime to m. These numbers are called* Carmichael numbers *in honor of their discoverer [1]. One of the most recent results of number theory is the proof that Carmichael numbers exist beyond all boundaries (based on a 1956 idea of the magnificent Hungarian mathematician Paul Erdös (1913–1996)). J. Chernick proved a theorem in 1939 that can be used to construct a subset of Carmichael numbers. The number* $(6k+1)(12k+1)(18k+1)$ *is a Carmichael number if its three factors are all prime. In 1994 it was shown by W. R. Alford, Andrew Granville, and Carl Pomerance that there really do exist infinitely many Carmichael numbers. Specifically, they showed that for sufficiently large n, there are at least* $n^{2/7}$ *Carmichael numbers between* 1 *and n. The research on the pseudoprime numbers was completed in the twentieth century. and its more important application is in cryptography [7]. I would like to add at this point that Carmichael numbers have practical applications, namely to attack RSA cryptosystems [21].*

[11] *This theorem states that if p is a prime and a is an integer not divisible by p, then the difference* $a^{p-1} - 1$ *is divisible by p; a usual shorthand for this is:* $a^{p-1} \equiv 1 \pmod{p}$.

[12] *The inverse of Fermat's Little Theorem is: If* $a^{p-1} \equiv 1 \pmod{p}$ *holds, it does not necessarily follow that p is a prime.*

power of $q-1$ and those of the second one to the power of $p-1$, we obtain:

$$\begin{aligned} & a^{(p-1)(q-1)} \equiv 1 \pmod{p} \quad \text{and} \\ (2) \qquad & a^{(p-1)(q-1)} \equiv 1 \pmod{q} \\ \Rightarrow\ & a^{(p-1)(q-1)} \equiv 1 \pmod{pq}. \end{aligned}$$

Next Bolyai observes that if the congruence $a^{p+q-2} \equiv 1 \pmod{pq} = a^{p-1} \cdot a^{q-1} \equiv \pmod{pq}$ were true, then by multiplying the two expressions obtained earlier, one could arrive at the desired congruence (1).

The following step must be finding the conditions that ensure the validity of the latter congruence. Since $a^{p-1} \equiv 1 \pmod{p}$ and $a^{q-1} \equiv 1 \pmod{q}$, continues Bolyai, there must exist integers h and k such that $a^{p-1} = 1 + hp$ and $a^{q-1} = 1 + kq$. In other words the condition of the validity of (1) is that

$$(3) \quad hp + kq = (a^{p-1} - 1) + (a^{q-1} - 1) \equiv 0 \pmod{pq}.$$

It is satisfied if p is a divisor of k and q is a divisor of h, according to Bolyai, this means that $a^{pq-1} \equiv 1 \pmod{pq}$ is true of primes p and q for which $\frac{a^{p-1}-1}{pq}$ and $\frac{a^{q+1}-1}{pq}$ are integers, in which case

$$(4) \quad \frac{a^{p-1}-1}{q} \quad \text{and} \quad \frac{a^{q-1}-1}{p} \quad \text{are also integers.}$$

In the simple case when $a = 2$ Bolyai substitutes some primes satisfying (4) and arrives at $p = 11$ and $q = 31$. This is how János Bolyai discovered the smallest pseudoprime.

Although he emphasized in his letter quoted above that "*even one example*" suffices, various counterexamples emerge from the remaining manuscripts. He constructed more congruences:

$$(5) \quad \begin{aligned} & 2^{340} \equiv 1 \pmod{341}, \qquad 4^{14} \equiv 1 \pmod{15}, \\ & \qquad\qquad 2^{2^{32}} \equiv 1 \pmod{2^{32}+1}. \end{aligned}$$

Bolyai says that if in the congruence (1) is $a = 2$, then the congruence

$$(6) \qquad 2^{pq-1} \equiv 1 \pmod{pq} \text{ follows.}$$

This corresponds exactly to the theorem of James Hopwood Jeans (1877–1940), which he published in 1898 [15], decades after the death of János Bolyai. This is the case with the Jeans theorem as well. Bolyai's discovery, like many others apart from the "Appendix", was not communicated even to his father. This is why one of Bolyai's beautiful theorems does not bear the name of János Bolyai but that of its rediscoverer.

Bolyai aimed to extend his method (6) to the case in which n is a multiple of three prime numbers: "*…but it will be considerably more difficult with three factors.*" Such congruences were constructed by R. D. Carmichael in [1], [2]. Bolyai's attempt suggests the following idea of generalizing Jeans's theorem: Let $p_1, p_2, \ldots, p_n$ be primes $n \geq 1$ and let a be an integer not divisible by either of these primes.

$$(7) \quad \textit{If} \quad \left.\begin{aligned} & a^{p_1 p_2 \cdots p_{n-1}-1} \equiv 1 \pmod{p_n} \\ & a^{p_1 p_2 \cdots p_{n-2} p_n - 1} \equiv 1 \pmod{p_{n-1}} \\ & \cdot \\ & \cdot \\ & \cdot \\ & a^{p_2 p_3 \cdots p_{n-1} p_n - 1} \equiv 1 \pmod{p_1} \end{aligned}\right\}$$

$$\textit{then } a^{p_1 p_2 \cdots p_n - 1} \equiv 1 \pmod{p_1 p_2 \cdots p_n}.$$

Chapter 4.6 reveals that János Bolyai was also captivated by Fermat numbers.[13] In one of the letters written to his father, he alludes to his approach to Fermat numbers in two places: "*By the way, my previous demonstration of* numerus perfectus[14] *and that of concerning* $2^{2^m}+1$ *are good and nice ….*

"*I intended to show that any number of the form* $2^p - 1$ *is a prime number if p is prime, at the same time when I took pains over* $2^{2^m}+1$, *since as my writings show, I thought that* $2^p - 1$ *was always a prime for any prime p. …*"

This chapter represents a special value where Bolyai's theorem on Fermat numbers is introduced. According to this "*Fermat numbers are always of the form of* $6k-1$, *and therefore are never divisible by* 3." He proves the proposition as follows:

$2^{2m-1} + 1 = (2+1)(\ldots)$; consequently, $2^{2m-1} + 1 = 3n$, and so $2^{2m-1} = 3n - 1$, thus $2^{2m} = 6n - 2$, that is, 2 raised to an even power, is of the form $6n - 2$. Then $2^{2m} + 1$ and thus $2^{2^m} + 1$ is of the form $6n - 1$, with $m, n \geq 0$ being natural numbers.

The international significance of this theorem and the weight of Elemér Kiss's research have been supported by the publication of [18], in which Theorem 3.12 was called the "Bolyai Theorem".

This is the first highly reputable source in which János Bolyai's name is mentioned in the field of number theory as opposed to geometry, which is a real milestone on the way to revealing the real "mind-face" of János Bolyai.

In Chapter 4.7, Kiss pointed out that János Bolyai was able to prove the converse of Wilson's Theorem,[15] but he was unaware of the earlier proof

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[13] *Numbers of the form* $F_n = 2^{2^n} + 1$ *where n is a natural number. Fermat firmly believed that all such numbers were primes, even though he had only calculated* $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65637$. *His conjecture was disproven when Euler in 1732 showed that the next Fermat number* $F_5 = 641\text{x}6700417$ *is not a prime. By the early 1980s* F_n *was known to be composite for all* $5 \leq n \leq 32$.

[14] *Perfect number.*

[15] *In 1770 Edward Waring (1736–1798) announced the following theorem by his former student John Wilson (1741–1793): if p is prime, then* $(p-1)! \equiv -1 \pmod{p}$, *that is,* $(p-1)! + 1$ *is divisible by p. The theorem*

of Lagrange. Gauss discusses Wilson's theorem in *Disquisitiones Arithmeticae*, but on its converse he keeps silent. János Bolyai acquired the bulk of their number theoretic knowledge from the work of Gauss; thus Bolyai was unaware of the proof of the inverse to Wilson's theorem.

The inverse to the theorem was important for János Bolyai, who was interested in the prime or composite nature of various large numbers and who also searched for the formula of primes. He notes that "*I have proven the inverse of the very beautiful and significant Theorem of Wilson.*" For the reader's delight I present Bolyai's proof:

Suppose that

(8) $$(p-1)! \equiv -1 \bmod p.$$

Let q be a prime divisor of p, namely $p = q \cdot p_1$, then

(9) $$(p-1)! \equiv -1 \bmod q,$$

and according to Wilson's theorem

(10) $$(q-1)! \equiv -1 \bmod q.$$

It follows from (9) and (10) that

(11) $$(q-1)! \equiv (p-1)! \bmod q \Rightarrow 1 \equiv \frac{(p-1)!}{(q-1)!} \bmod q.$$

Assume now $q < p$, then q is a divisor of $(p-1)!$ but not of $(q-1)!$. Hence

(12) $$q < p \Rightarrow \frac{(p-1)!}{(q-1)!} \equiv 0 \bmod q.$$

The congruences (11) and (12) are contradictory; consequently the assumption $q < p$ is false. Thus $p = q$ is a prime.

From Chapter 4.8 we may find out that János Bolyai obtained results on the general construction of magic squares[16] of small orders.

Bolyai wrote on the sheet containing the magic square that $a = 3b$ (see Figure 13); consequently $b = 5$ is true.

At the end of his note Bolyai invites the reader to generalize his 3x3 magic square to *nxn* squares:

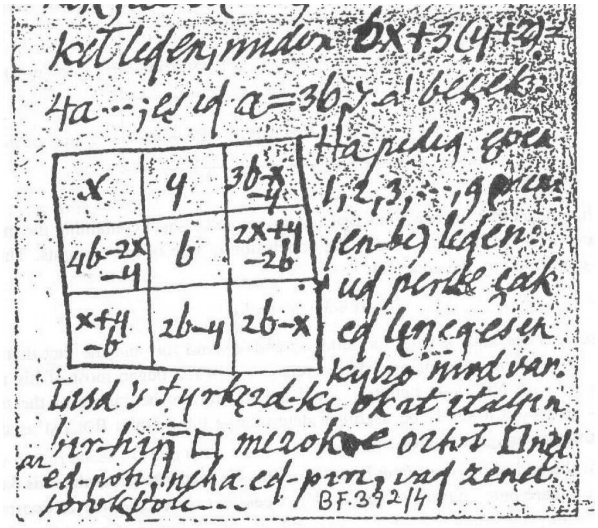


Figure 13. János Bolyai's general construction of a 3x3 magic square.

x	y	36-x-y	a=15
46-2x-y	6	2x+y-26	a=15
x+y-6	26-y	26-x	a=15
a=15	a=15	a=15	a=15 a=15

8	3	4
1	5	9
6	7	2

Figure 14. Concrete solution of János Bolyai's 3x3 magic square.

"*Seek and search the way to construct general magical squares from a □*[17] *divided into any number of equal □ – s, be it arithmetical, geometrical or harmonic progression…*" I would like to add that Bolyai's ideas were reinvented by Cayley [3] and Chernick [4], and a survey on the general construction of magic squares can be found in an encyclopedic book [6].

Chapter 5 talks about a few results of Bolyai's work that have never been seen before: about complex integers,[18] into the research of which Bolyai invested huge amounts of energy. These studies, which Bolyai called the "*theory of primes*" or "*imaginary number theory*", dealt with the arithmetic of complex integers.

The theory of divisibility of complex integers was founded and developed by Gauss [12], [13]. He proved the theorem corresponding to the fundamental theorem of number theory inside Gaussian integers and discussed congruences involving complex numbers. János Bolyai elaborated the arithmetic of complex integers independently of Gauss and approximately at the same time.

was published by E. Waring, but he acknowledged that it had first been formulated by J. Wilson without a proof. The theorem was first proved by Joseph Louis Lagrange (1736–1813) in 1771, and he proved the inverse to Wilson's theorem as well: if n is a divisor of $(n-1)!+1$, *then n is prime.*

[16] *A magic square of order n is an arrangement of* n^2 *numbers, usually distinct integers, in a square, such that the n numbers in all rows, all columns, and both diagonals sum to the same constant. A normal magic square contains the integers from* 1 *to* n^2. *The constant sum in every row, column, and diagonal is called the magic constant or magic sum, a. The magic constant of a normal magic square depends only on n and has the value* $a = \frac{n(n^2+1)}{2}$.

[17] *This is the original symbol on Bolyai's sheet (see Figure 13).*

[18] *The* complex *or* Gaussian integer *is a complex number whose real and imaginary parts are both integers. Formally, Gaussian integers are the set* $\mathbb{Z}[i] = \{a+bi \mid a,b \in \mathbb{Z}\}$, $i = \sqrt{-1}$.

The beginnings of Bolyai's investigation of complex numbers can be dated exactly, because in a letter to his father he pinpoints the date: "*I sought the theory of imaginary quantities in their proper place and fortunately found it in 1831*." Based on his statements it can be asserted that János Bolyai saw his own theory clearly at the beginning of the 1830s, even in the years prior to the publication of the "Appendix".

In his manuscripts it can be deciphered that he clearly identified primes in the ring of complex integers. He asserted that complex primes are either

(13) *the numbers* $1 + i$, $1 - i$, $-1 + i$, $-1 - i$,
(14) *rational primes of the form* $4m + 3$,
(15) *complex factors of rational primes of the form* $4m + 1$.

Bolyai merely enumerates the numbers (13) as "*perfect primes*", and he notes that $2 = (1+i)(1-i)$, but he clearly indicates that $1 + i$ cannot be written as the product of two complex integers.

About numbers (14) Bolyai presents various proofs that they are "*absolute primes*". One of his proofs: "*If a prime* p *is of the form* $4m + 3$*, then* $p = t^2 + u^2$ *is impossible, because if both* t *and* u *are even or odd at the same time, then the sum of their squares would yield an even number and such a number is not a prime. If one of* t *and* u *is even and the other is odd, then the sum of their squares is a number of the form* $4m+1$*. Then* p *is an absolute prime.*"

Bolyai showed that the complex integer $m + ni$ has no divisor different from its associates provided that $p = m^2+n^2$ is a prime. He relates this property of numbers (15) to Fermat's Christmas theorem, and he writes: "*Every prime* p *of the form* $4m+1$ *is the product of two imaginary primes, since all such numbers are the sum of two full squares*." For example: $13 = (2 + 3i)(2 - 3i)$.

János Bolyai also discussed unique factorization of complex integers, and he proved the following theorem: "*Every number of the form* $a + bi$ *can be uniquely (up to the order of the factors) decomposed into a product of finitely many primes*."

He not only elaborated the theory of complex numbers, but its applications were also of importance to him. He skillfully applied his conclusions on complex integers in proofs of various number theoretic theorems.

Chapter 6, "The theory of algebraic equations", reveals Bolyai's struggles in connection with the solvability of algebraic equations of fifth and higher order. His unpublished collection contains many notes pertaining to this subject. It has been summed up by Elemér Kiss at the end of this chapter: "*János Bolyai thought long about this important problem without knowing that it had been resolved before.*"

This is why the connection between Bolyai and the theory of algebraic equations is especially interesting. János Bolyai frequently mentions the two-volume *Vorlesungen über höhere Mathematik* by Andreas von Ettingshausen (1796–1878), which was published in Vienna in 1827 (see [11]). In this book the author devoted an entire chapter to the impossibility of solving equations of a degree higher than four and cited Paolo Ruffini's (1765–1822) proof of 1799[19] [22], [23]. Bolyai cites Joseph Louis Lagrange's (1736–1813) book [19], which addresses the fundamental problem of why the methods used for solving equations of a degree equal to or lower than four are inapplicable in equations of higher degree.[20] On his reading Bolyai writes: "*…to give a proof of this impossibility for degree 5 and for higher degrees as well: this proof being given by Ruffini (as the deserved Ettingshausen writes) wittily enough, but with a great many mistakes, in short, only in his fancies*."

This led him to conclude that the theorem was not valid and consequently at first, he searched for the solution of equations of degree higher than four with great enthusiasm, and he wrote in 1844: "*By refuting the demonstration of impossibility (by Ruffini) …it will be proven* eo ipso *(self-evidently) in a new way*."

János Bolyai thought long about this important problem without knowing that it had been resolved before. On the other hand, the world didn't know about this nineteenth-century Hungarian scientist who (perhaps late and only for its own sake) had put an end to a centuries-long debate.

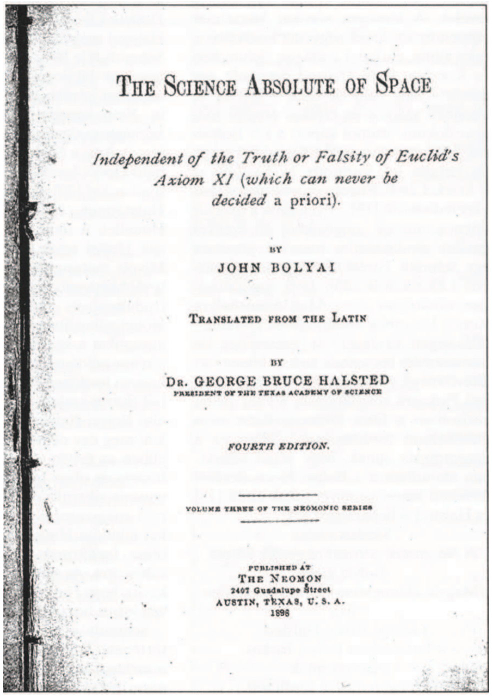
The above facts suggest the isolation Bolyai worked in all his life and the enormous creative power through which he was able to "*create a new, different world out of nothing*", not exclusively in the field of geometry.

Acknowledgment

It is hard to overestimate the value of the work of Elemér Kiss in the process of revealing János Bolyai's real face after 150 years of silence. What makes it even more meaningful is the fact that János Bolyai's face has been unknown so far, as the famous image that has been circulated around the world is certainly not his. To put an end to this misconception, and also to publicize essays and research in related areas, "The Real Face of János Bolyai" has been created at: `http://www.titoktan.hu/Bolyai_a.htm`.

[19] *The 1826 article by Niels Henrik Abel (1802–1829) could not have been included in this work published in 1827. The Abel-Ruffini theorem states that there is no general solution in radicals to polynomial equations of degree five or higher*.

[20] *This observation not only impelled Ruffini and Abel to continue research in this direction but also led to Galois's conception of group theory.*


THE SCIENCE ABSOLUTE OF SPACE

Independent of the Truth or Falsity of Euclid's Axiom XI (which can never be decided a priori).

BY

JOHN BOLYAI

TRANSLATED FROM THE LATIN

BY

DR. GEORGE BRUCE HALSTED

PRESIDENT OF THE TEXAS ACADEMY OF SCIENCE

FOURTH EDITION.

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George Bruce Halsted (1853–1922), American mathematician, translated the "Appendix" by János Bolyai into English in 1896.

At this point it is necessary to remember the American mathematician George Bruce Halsted (1853–1922), who in 1896—before any other foreign Bolyai researcher—visited Marosvásárhely and translated the main work of János Bolyai: the "Appendix" [14]. With his activity he contributed significantly to the international appreciation of the two Bolyais.

I would like to sincerely thank the referees who supported the publication of this essay, for continuing the tradition initiated by G. B. Halsted. By doing so they make a significant contribution to the introduction of János Bolyai's real face to the world.

This essay could not have been written without our personal discussions with Professor Elemér Kiss, the written accounts of Professor András Prékopa, and Róbert Oláh-Gál's support with the face animation.

I would like to say a special thank you to Ildikó Rákóczi, the director of the János Bolyai Mathematical Society, for making it possible for me to photograph and publish pictures of key importance for the purposes of this project.

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