ON LOCALLY DIVIDED DOMAINS OF THE FORM INT(D)

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Abstract. Let D be an integral domain which is not a field. If either D is Noetherian or D is a Prüfer domain, then $\mathrm{Int}(D)$ is a treed domain if and only if it is a going-down domain. Suppose henceforth that $(D, \mathfrak{m})$ is Noetherian local and one-dimensional, with $D/\mathfrak{m}$ finite. Then $\mathrm{Int}(D)$ is a going-down domain if and only if D is unibranched (inside its integral closure); and $\mathrm{Int}(D)$ is locally divided if and only if D is analytically irreducible. Thus, if D is unibranched but not analytically irreducible, then $\mathrm{Int}(D)$ provides an example of a two-dimensional going-down domain which is not locally divided. Also, $\mathrm{Int}(D)$ is a locally pseudo-valuation domain if and only if D is itself a pseudo-valuation domain. Thus, $\mathrm{Int}(D)$ also provides an example of a two-dimensional locally divided domain which is not an LPVD.

Introduction

Among the generalizations of Prüfer domains introduced in the decade 1974-1983, one finds the locally pseudovaluation domains or LPVDs (introduced in [11]), the locally divided domains [7, 9], the going-down domains [5, 12] and the treed domains [5].

Let us first recall the corresponding definitions.

— A *pseudovaluation domain* is a quasilocal domain D with maximal ideal $\mathfrak{m}$ and quotient field K such that, for all $x, y \in K$, $xy \in \mathfrak{m}$ implies either $x \in \mathfrak{m}$ or $y \in \mathfrak{m}$. Equivalently, D is a quasilocal domain sharing its maximal ideal $\mathfrak{m}$ with an overring V which is a valuation domain. (By an *overring* of a domain D, we mean a ring contained between D and the quotient field of D.)

— A *divided domain* is a domain D such that, for every prime ideal $\mathfrak{p}$ of D, one has $\mathfrak{p} = \mathfrak{p}D_{\mathfrak{p}}$.

— A *locally* pseudovaluation (resp., locally divided) domain is a domain D such that, for each maximal ideal $\mathfrak{m}$ of D, $D_{\mathfrak{m}}$ is a pseudovaluation (resp., divided) domain.

— A ring extension $R \subseteq T$ *satisfies going-down* if, whenever $\mathfrak{p}$, $\mathfrak{q} \in \mathit{Spec}(R)$ and $\mathfrak{q}_1 \in \mathit{Spec}(T)$ satisfy $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{q}_1 \cap R = \mathfrak{q}$, there is $\mathfrak{p}_1 \in \mathit{Spec}(T)$ such that $\mathfrak{p}_1 \cap R = \mathfrak{p}$ and $\mathfrak{p}_1 \subset \mathfrak{q}_1$.

— A *going-down domain* is a domain D such that, for each overring T of D, the extension $D \subset T$ satisfies going-down.

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— A *treed ring* is a ring R such that $Spec(R)$ as a partially ordered set under inclusion is a tree, that is, no maximal ideal of R contains incomparable prime ideals.

In general, one has the implications

$$\text{Prüfer} \Longrightarrow \text{LPVD} \Longrightarrow \text{locally divided} \Longrightarrow \text{going-down} \Longrightarrow \text{treed.}$$

For arbitrary integral domains, none of these implications can be reversed : see [15, Example 2.1]; [8, Remark 4.10 (b)]; [7, Example 2.9]; and [13, Example 4.4], [10, Example 2.3], respectively. Nevertheless, for some prominent classes of integral domains, some of these implications *are* reversible. For instance, any Noetherian treed domain is locally divided since, on the one hand, the dimension of a Noetherian treed domain is at most one (for instance, apply [16, Theorem 144]), and, on the other hand, a one-dimensional domain is obviously locally divided. One impetus for the present work is to provide new counterexamples; of course, we have to consider non-Noetherian domains with dimension at least two.

For an integral domain D with quotient field K, our focus is on the domain of integer-valued polynomials

$$\mathrm{Int}(D) = \{f \in K[X] : f(D) \subseteq D\}.$$

Of course, if $D = K$, then $\mathrm{Int}(D) = K[X]$ is a Prüfer domain and so has all the properties in question; for this reason, we suppose throughout that D is not a field.

In the first section, Propositions 1.2 and 1.4 show that if either D is a Prüfer domain or D is a Noetherian integral domain, then $\mathrm{Int}(D)$ is treed if *and only if* $\mathrm{Int}(D)$ is a going-down domain. A key tool, Proposition 1.1, establishes that if $\mathrm{Int}(D)$ is treed, then D is an interpolation domain (in the sense of [2]), and so D has (Krull) dimension one and finite residue fields. In the Noetherian case, as all the issues are local, we may suppose $(D, \mathfrak{m})$ is local. It follows in particular that $\mathrm{Int}(D)$ is treed (or going-down) if and only if $\dim(D) = 1$, $D/\mathfrak{m}$ is finite, and D is unibranched (that is, the integral closure D' of D is local). These results show that this class of domains supports again some converse implications; hence, they may serve to explain the complexity in the above-cited constructions in [13] and [10] of quasilocal treed domains which are not going-down domains.

In the next and last section, however, we show that domains of the form $\mathrm{Int}(D)$ do provide some interesting counterexamples. Our main result, Theorem 2.1, gives a characterization of the Noetherian (local) domains D such that $\mathrm{Int}(D)$ is locally divided: D is analytically irreducible, $\dim(D) = 1$, and $D/\mathfrak{m}$ is finite. Thus, if $\mathrm{Int}(D)$ is a going-down domain, it is locally divided if and only if D' is a finitely generated D-module. One consequence is an infinite family of two-dimensional quasilocal going-down domains which are not divided domains; such localizations of integral domains of the form $\mathrm{Int}(D)$ seem more natural constructions than the one resorted to in [7, Example 2.9]. Finally, Theorem 2.4 establishes that (for a Noetherian local domain D) $\mathrm{Int}(D)$ is an LPVD if and only if D is a pseudovaluation domain. By choosing $(D, \mathfrak{m})$ to be an analytically irreducible integral domain such that $D/\mathfrak{m}$ is finite and D is not a pseudovaluation domain, one thus has an example of a two-dimensional locally divided domain with infinitely many maximal ideals which *is not* an LPVD.

We adopt standard usage throughout. In particular, dim denotes Krull dimension. For background on the classes of integral domains being studied, we refer the reader to the references cited in this introduction. For additional background on integral domains of the form $\mathrm{Int}(D)$, see [1].

1. Interpolation domains

Throughout, D denotes an integral domain which is properly contained in its quotient field K. We begin by showing that significant conditions are imposed on D by the requirement that Int(D) be treed. It is convenient to recall the following from [2]:
— The domain D is called an *interpolation domain* if, for each finite set $(a_1, \ldots, a_n)$ of distinct elements of D, and each corresponding set of "values" $(c_1, \ldots, c_n)$ in D, there exists $f \in \mathrm{Int}(D)$ such that $f(a_i) = c_i$, for $1 \leq i \leq n$.
— D is an interpolation domain if and only if, for each pair of distinct elements $a \neq b$ in D, and for each maximal ideal $\mathfrak{m}$ of D, there exists $f \in \mathrm{Int}(D)$ such that $f(a) \in \mathfrak{m}$ and $f(b) \notin \mathfrak{m}$ [2, Proposition 1.1].

Now it is clear that, for each $a \in D$, and each maximal ideal $\mathfrak{m}$ of D,

$$\mathfrak{M}_{\mathfrak{m},a} = \{f \in \mathrm{Int}(D) \mid f(a) \in \mathfrak{m}\}$$

is a maximal ideal of Int(D) containing $\mathfrak{m}$ (with residue field isomorphic to $D/\mathfrak{m}$). Hence D is an interpolation domain if and only if, for each $a \neq b$ in D, and each maximal ideal $\mathfrak{m}$ of D, the ideals $\mathfrak{M}_{\mathfrak{m},a}$ and $\mathfrak{M}_{\mathfrak{m},b}$ of Int(D) are distinct (as noted in [2, Corollary 1.2]). On the other hand, the prime ideal

$$< X - a > = (X - a)K[X] \cap \mathrm{Int}(D) = \{f \in \mathrm{Int}(D) \mid f(a) = 0\}$$

is clearly contained in $\mathfrak{M}_{\mathfrak{m},a}$. We thus derive the following necessary conditions:

Proposition 1.1. *Suppose that* Int(D) *is a treed domain. Then D is an interpolation domain. In particular,* $\dim(D) = 1$ *and, for each maximal ideal* $\mathfrak{m}$ *of* D, $D/\mathfrak{m}$ *is finite and* $\bigcap_n \mathfrak{m}^n = (0)$.

Proof. Suppose that D is not an interpolation domain. Then there exist two distinct elements a and $b \in D$ such that $\mathfrak{M}_{\mathfrak{m},a} = \mathfrak{M}_{\mathfrak{m},b}$. On the other hand, the prime ideals $< X - a >$ and $< X - b >$ are incomparable (since $X - b \notin < X - a >$ and $X - a \notin < X - b >$), but they are both contained in the maximal ideal $\mathfrak{M}_{\mathfrak{m},a}$. Thus, Int(D) cannot be treed. Moreover, if D is an interpolation domain, it follows from [2, Proposition 1.7] that $\dim(D) = 1$ and, for each maximal ideal $\mathfrak{m}$ of D, $D/\mathfrak{m}$ is finite and $\bigcap_n \mathfrak{m}^n = (0)$. □

For many classes of integral domains, it is known that "treed" is in fact equivalent to "going-down". For instance, this is the case for GCD domains [4, Corollary 4.3] or, as indicated in the introduction, for Noetherian domains [5, Theorem 2.2]. We shall see that the same often holds for Int(D). We may immediately deal here with the case where D is a Prüfer domain. Indeed, we know from [2, Corollary 3.2] that, in this case, D is an interpolation domain if and only if Int(D) is itself a Prüfer domain. From the implications that we recalled in the introduction, it clearly follows that all the properties considered in this paper are then equivalent:

Proposition 1.2. *Suppose that D is a Prüfer domain. Then the following conditions are equivalent*:

(1) *D is an interpolation domain,*
(2) Int(D) *is a treed domain,*
(3) Int(D) *is a going-down domain,*
(4) Int(D) *is a locally divided domain,*
(5) Int(D) *is an LPVD,*

(6) $\mathrm{Int}(D)$ *is a Prüfer domain.*

For the rest of this note, we focus on the case of Noetherian D. The next result permits reduction to the local case.

Lemma 1.3. *Let $\mathcal{C}$ denote any of the four classes of integral domains: treed domains, going-down domains, locally divided domains, LPVDs. If D is Noetherian, then the following conditions are equivalent:*

(1) $\mathrm{Int}(D)$ *is in* $\mathcal{C}$,
(2) $\mathrm{Int}(D_{\mathfrak{m}})$ *is in* $\mathcal{C}$ *for each maximal ideal* $\mathfrak{m}$ *of* D.

Proof. It is straightforward from the definitions that each of the properties are local properties, that is, for each class $\mathcal{C}$ and each domain R:
– if $R \in \mathcal{C}$, then $S^{-1}R \in \mathcal{C}$ for each multiplicative subset S of R,
– if $R_{\mathfrak{m}} \in \mathcal{C}$ for each maximal ideal $\mathfrak{m}$ of R, then $R \in \mathcal{C}$.

Moreover, with the Noetherian hypothesis on D, $\mathrm{Int}(D)_{\mathfrak{m}} = \mathrm{Int}(D_{\mathfrak{m}})$ for each maximal ideal $\mathfrak{m}$ of D. Thus (1) $\Rightarrow$ (2).

Suppose now that (2) holds. It follows from Proposition 1.1 that D is one-dimensional. If $\mathfrak{M}$ is a maximal ideal of $\mathrm{Int}(D)$ lying over a maximal ideal $\mathfrak{m}$ of D, then $\mathrm{Int}(D)_{\mathfrak{M}}$ is a localization of $\mathrm{Int}(D)_{\mathfrak{m}} = \mathrm{Int}(D_{\mathfrak{m}})$ and thus, belongs to $\mathcal{C}$. On the other hand, if $\mathfrak{M}$ is a maximal ideal of $\mathrm{Int}(D)$ lying over the ideal (0) of D, then (in fact, without any hypothesis on D) $\mathfrak{M} = qK[X] \cap \mathrm{Int}(D)$, where q is irreducible in $K[X]$, and $(\mathrm{Int}(D))_{\mathfrak{M}} = K[X]_{(q)}$ is a discrete valuation domain. Thus $\mathrm{Int}(D)_{\mathfrak{M}}$ belongs again to $\mathcal{C}$. Hence, $\mathrm{Int}(D)$ itself belongs to $\mathcal{C}$. □

From now on, we thus suppose that $(D, \mathfrak{m})$ is a Noetherian local domain, with maximal ideal $\mathfrak{m}$. Moreover, if we want $\mathrm{Int}(D)$ to be treed, it follows from Proposition 1.1 that D must be one-dimensional. With such hypotheses, we note that the dimension of $\mathrm{Int}(D)$ is two.

Recall that an integral domain A with integral closure A' is said to be *unibranched* (in the extension $A \subseteq A'$) if the canonical map $Spec(A') \to Spec(A)$ is a bijection. Following [18], we also say that A is an *i-domain* if, for each overring B of A, the canonical map $Spec(B) \to Spec(A)$ is an injection, or equivalently, if $A'_{\mathfrak{m}}$ is a valuation domain for each maximal ideal $\mathfrak{m}$ of A [18, Corollary 2.15]. Both notions are local properties and it is known that:
— a domain A is an i-domain if and only if A is unibranched (in A') and A' is a Prüfer domain [18, Proposition 2.14];
— each i-domain is a going-down domain (cf. [6, Corollary 2.5], [18, Corollary 2.13]).

Proposition 1.4. *Let $(D, \mathfrak{m})$ be a Noetherian local integral domain. Then the following conditions are equivalent:*

(1) D *is an i-domain,*
(2) $\dim(D) = 1$, $D/\mathfrak{m}$ *is finite, and* D *is unibranched,*
(3) D *is an interpolation domain,*
(4) $\mathrm{Int}(D)$ *is a treed domain,*
(5) $\mathrm{Int}(D)$ *is a going-down domain,*
(6) $\mathrm{Int}(D)$ *is an i-domain.*

Proof. It follows from general principles that (6) $\Rightarrow$ (5) $\Rightarrow$ (4), and by Proposition 1.1, that (4) $\Rightarrow$ (3). It follows from [2, Theorem 2.4] that (3) $\Rightarrow$ (2). Finally, if

D is local, Noetherian, unibranched, and one-dimensional, then D' is a (rank-one discrete) valuation domain, thus (2) $\Rightarrow$ (1) follows from the above characterization of i-domains. It then remains to prove that (1) $\Rightarrow$ (6), using the same characterization.

Since D is Noetherian, the integral closure of $\mathrm{Int}(D)$ is the ring $\mathrm{Int}(D, D')$ of polynomials $f \in K[X]$ such that $f(D) \subseteq D'$ [1, Theorem IV.4.7]. Since D is unibranched, $\mathrm{Int}(D, D')$ is a radical extension of $\mathrm{Int}(D)$, that is, for each $f \in \mathrm{Int}(D, D')$ there is an integer n such that $f^n \in \mathrm{Int}(D)$ [1, Proposition IV.4.5]; and hence, $\mathrm{Int}(D)$ is unibranched. Moreover, $\mathrm{Int}(D')$ is a Prüfer domain, since D' is a discrete valuation domain with finite residue field [1, Lemma VI.1.4]. Consequently, $\mathrm{Int}(D, D')$, which is an overring of $\mathrm{Int}(D')$, is also a Prüfer domain. We may conclude that $\mathrm{Int}(D)$ is an i-domain. $\square$

In view of Propositions 1.2 and 1.4, we ask the following:

Question 1.5. Are the following conditions always equivalent?

(1) D is an interpolation domain,
(2) $\mathrm{Int}(D)$ is a treed domain,
(3) $\mathrm{Int}(D)$ is a going-down domain.

2. Locally divided and pseudovaluation domains

The next result addresses what happens when "unibranched" is sharpened to "analytically irreducible". Recall that the local domain $(D, \mathfrak{m})$ is said to be *analytically irreducible* if its completion $\widehat{D}$ in the $\mathfrak{m}$-adic topology is an integral domain. Recall also that this condition implies that D is unibranched [17, (43.20)] and that the integral closure D' of D is a finitely generated D-module [17, (32.2)]. In fact, when D is one-dimensional, the following statements are equivalent [1, Proposition III.5.2]:

— D is analytically irreducible,
— D is unibranched and D' is a finitely generated D-module,
— D is unibranched and, if $\mathfrak{m}'$ denotes the maximal ideal of D', the $\mathfrak{m}'$-adic topology on D' induces the $\mathfrak{m}$-adic topology on D.

Theorem 2.1. *Let* $(D, \mathfrak{m})$ *be a Noetherian local integral domain. Then the following conditions are equivalent*:

(1) $\mathrm{Int}(D)$ *is a locally divided domain.*
(2) $\dim(D) = 1$, $D/\mathfrak{m}$ *is finite, and* D *is analytically irreducible.*

Proof. (1) $\Rightarrow$ (2) If $\mathrm{Int}(D)$ is a locally divided domain, then it is treed. Thus, by Proposition 1.4, $\dim(D) = 1$, $D/\mathfrak{m}$ is finite, and D is unibranched. Under these conditions, we assume, by way of contradiction, that the $\mathfrak{m}'$-adic topology on D' does not induce the $\mathfrak{m}$-adic topology (that is, that D is not analytically irreducible); we then show that $\mathrm{Int}(D)$ is not locally divided. Clearly we have the containment $\mathfrak{m}^n \subseteq \mathfrak{m}'^n \cap D$, for each n; thus, if the $\mathfrak{m}'$-adic topology does not induce the $\mathfrak{m}$-adic topology, there is an integer k, such that, for each positive integer n, $\mathfrak{m}'^n \cap D$ is not contained in $\mathfrak{m}^k$. Considering a nonzero element $b \in \mathfrak{m}^k$, we can thus produce a sequence $\{\alpha_n\}$ in D such that, for each n, $\alpha_n \in \mathfrak{m}'^n \cap D$, but $(\alpha_n/b) \notin D$. We let $\mathfrak{M}$ be the maximal ideal

$$\mathfrak{M} = \{f \in \mathrm{Int}(D) \mid f(0) \in \mathfrak{m}\}$$

and $\mathfrak{P}$ be the prime ideal

$$\mathfrak{P} = < X > = \{f \in \mathrm{Int}(D) \mid f(0) = 0\}.$$

Clearly, $\mathfrak{P}$ is contained in $\mathfrak{M}$, and we complete the proof by showing that $\mathfrak{P}\mathrm{Int}(D)_{\mathfrak{P}}$ is not equal to $\mathfrak{P}\mathrm{Int}(D)_{\mathfrak{M}}$. More precisely, letting $f = X/b$, then $f \in \mathfrak{P}\mathrm{Int}(D)_{\mathfrak{P}}$, and we show that $f \notin \mathfrak{P}\mathrm{Int}(D)_{\mathfrak{M}}$. For this, we claim that, if $g \in \mathrm{Int}(D)$ is such that $gf \in \mathrm{Int}(D)$, then $g \in \mathfrak{M}$. Indeed, if $gf \in \mathrm{Int}(D)$, then, for each n, we have $gf(\alpha_n) = g(\alpha_n)(\alpha_n/b) \in D$. Since $(\alpha_n/b) \notin D$, we then have $g(\alpha_n) \in \mathfrak{m}$, and a fortiori, $g(\alpha_n) \in \mathfrak{m}'$. Now g is a polynomial and D' is a one-dimensional local Noetherian domain, hence g is a continuous function in the $\mathfrak{m}'$-adic topology [1, Proposition III.2.1]. The sequence $\{\alpha_n\}$ converges to 0 in this topology, therefore $g(0) \in \mathfrak{m}'$. Since $g \in \mathrm{Int}(D)$, we obtain $g(0) \in \mathfrak{m}' \cap D = \mathfrak{m}$, that is, $g \in \mathfrak{M}$.

(2) $\Rightarrow$ (1) Suppose that $\dim(D) = 1$, $D/\mathfrak{m}$ is finite, and D is analytically irreducible. Let us first describe the prime ideals of $\mathrm{Int}(D)$ [1, Corollary V.1.2, Theorem V.2.10, and Proposition V.3.3].

— The nonzero prime ideals of $\mathrm{Int}(D)$ above (0) are in one-to-one correspondence with the monic polynomials irreducible in $K[X]$: to the irreducible polynomial q corresponds the prime ideal

$$< q > = qK[X] \cap \mathrm{Int}(D).$$

— The prime ideals of $\mathrm{Int}(D)$ above $\mathfrak{m}$ are maximal and in one-to-one correspondence with the elements of the completion $\widehat{D}$ of D in the $\mathfrak{m}$-adic topology: to each $\alpha \in \widehat{D}$, corresponds the maximal ideal

$$\mathfrak{M}_{\mathfrak{m},\alpha} = \{f \in \mathrm{Int}(D) \mid f(\alpha) \in \widehat{\mathfrak{m}}\}.$$

— The prime ideal $< q >$ is contained in the maximal ideal $\mathfrak{M}_{\mathfrak{m},\alpha}$ if and only if $q(\alpha) = 0$.

Obviously a one-dimensional domain is a locally divided domain. The only case we then have to consider is that of a prime ideal $\mathfrak{P} = < q >$, contained in a maximal ideal $\mathfrak{M} = \mathfrak{M}_{\mathfrak{m},\alpha}$ (and hence, such that $q(\alpha) = 0$), for which we must show that $\mathfrak{P}\mathrm{Int}(D)_{\mathfrak{P}} = \mathfrak{P}\mathrm{Int}(D)_{\mathfrak{M}}$. In fact, it is enough to show that $\varphi \in \mathfrak{P}\mathrm{Int}(D)_{\mathfrak{P}}$ implies $\varphi \in \mathrm{Int}(D)_{\mathfrak{M}}$, since we have the equality $\mathrm{Int}(D)_{\mathfrak{M}} \cap \mathfrak{P}\mathrm{Int}(D)_{\mathfrak{P}} = \mathfrak{P}\mathrm{Int}(D)_{\mathfrak{M}}$. This follows immediately from the next lemma, since $\varphi \in \mathfrak{P}\mathrm{Int}(D)_{\mathfrak{P}}$ implies $\varphi(\alpha) = 0$. □

Lemma 2.2. *Assume $(D, \mathfrak{m})$ is a local Noetherian, one-dimensional, analytically irreducible domain with finite residue field. Let $\mathfrak{M}$ be a maximal ideal of the form $\mathfrak{M} = \mathfrak{M}_{\mathfrak{m},\alpha}$, where α is an element of the completion $\widehat{D}$ of D. We then have the following:*

1. *The localization $\mathrm{Int}(D)_{\mathfrak{M}}$ is the set of rational functions $\varphi \in K(X)$ such that $\varphi(\alpha) \in \widehat{D}$.*
2. *The ideal $\mathfrak{M}\mathrm{Int}(D)_{\mathfrak{M}}$ is the set of rational functions $\varphi \in K(X)$ such that $\varphi(\alpha) \in \widehat{\mathfrak{m}}$.*

Proof. 1. It is obvious that $\varphi \in \mathrm{Int}(D)_{\mathfrak{M}}$ implies $\varphi(\alpha) \in \widehat{D}$. Conversely, we suppose that $\varphi(\alpha) \in \widehat{D}$ and we show there is $h \in \mathrm{Int}(D)$, $h \notin \mathfrak{M}$, such that $h\varphi \in \mathrm{Int}(D)$. We may write $\varphi = f/g$, where f and g are integer-valued polynomials, $g(\alpha) \neq 0$, and $f(\alpha) \in g(\alpha)\widehat{D}$. Considering f and g as continuous functions from $\widehat{D}$ to $\widehat{D}$ in the $\mathfrak{m}$-adic topology, there is a clopen neighborhood U of α such that, for $x \in U$, we have $f(x) \in g(\alpha)\widehat{D}$, and $g(x) \in g(\alpha)(1+\widehat{\mathfrak{m}})$. Consider the continuous function ψ defined on $\widehat{D}$ by

$$\psi(t) = \begin{cases} 1 & \text{if } t \in U \\ \\ g(\alpha) & \text{otherwise} \end{cases}$$

Since D is analytically irreducible, we may apply the Stone-Weierstrass theorem [1, Theorem III.5.3]: ψ can be approximated by an integer-valued polynomial. Consequently, there exists $h_0 \in \operatorname{Int}(D)$, such that $h_0(t)$ is a unit if $t \in U$, and $h_0(t) \in g(\alpha)\widehat{D}$, otherwise. Choose $b \in D$ of the form $b = g(\alpha)(1+m)$, where $m \in \widehat{\mathfrak{m}}$ (that is, $b = g(\alpha)u$, where u is a unit in $\widehat{D}$). Then $(fh_0)/b$ and $(gh_0)/b$ are integer-valued polynomials (the values of fh_0 and gh_0 are always divisible by b), and $(gh_0/b)(\alpha)$ is a unit in $\widehat{D}$. Letting $h = (gh_0)/b$, it follows that $h \in \operatorname{Int}(D)$, $h \notin \mathfrak{M}$, and $h\varphi = (fh_0)/b \in \operatorname{Int}(D)$.

2. As above, $\varphi \in \mathfrak{M}_{\mathfrak{m},\alpha}\operatorname{Int}(D)_{\mathfrak{M}}$ obviously implies $\varphi(\alpha) \in \widehat{\mathfrak{m}}$. Assume conversely that $\varphi(\alpha) \in \widehat{\mathfrak{m}}$. Choose $b \in D$ so that $b = \varphi(\alpha)u$, where u is a unit in $\widehat{D}$ (of the form $u = 1+m$, where $m \in \widehat{\mathfrak{m}}$). Letting $\psi = \varphi/b$, it follows from 1 that $\psi \in \operatorname{Int}(D)_{\mathfrak{M}}$; hence $\varphi = b\psi$ belongs to $\mathfrak{m}\operatorname{Int}(D)_{\mathfrak{M}}$, and thus a fortiori to $\mathfrak{M}\operatorname{Int}(D)_{\mathfrak{M}}$. □

Remark 2.3. Let $(D,\mathfrak{m})$ be a one-dimensional Noetherian local integral domain, with finite residue field. If D is unibranched but not analytically irreducible, it follows from Proposition 1.4 and Theorem 2.1 that $\operatorname{Int}(D)$ is a two-dimensional going-down domain which is not locally divided. In particular, we could derive from the proof of Theorem 2.1 that, for each $a \in D$, $(\operatorname{Int}(D))_{\mathfrak{M}_{\mathfrak{m},a}}$ is a two-dimensional quasilocal going-down domain which is not a divided domain. Examples of unibranched but not analytically irreducible domains D, with arbitrary nonzero characteristic, appear in [3, pp. 54–55]). We thus obtain infinitely many pairwise nonisomorphic examples of the desired phenomenon. This construction should be contrasted with the arguably more complicated example in [7, Example 2.9] of the first quasilocal going-down domain which is not divided.

In closing, we determine the Noetherian domains D such that $\operatorname{Int}(D)$ is a locally pseudovaluation domain (or LPVD).

Theorem 2.4. *Let D be a Noetherian integral domain. Then the following conditions are equivalent*:

(1) $\operatorname{Int}(D)$ *is an LPVD,*
(2) *D is an LPVD with finite residue fields.*

Moreover, if the above conditions hold, then $\dim(D) = 1$ *and* $\dim\big(\operatorname{Int}(D)\big) = 2$.

Proof. In view of Lemma 1.3, we may assume that $(D,\mathfrak{m})$ is is a Noetherian local domain (and, as usual, not a field).
(1) ⇒ (2) Assume that $\operatorname{Int}(D)$ is an LPVD. It follows from Proposition 1.1 that $D/\mathfrak{m}$ is finite, and from [11, Proposition 2.8] that $D = K \cap (\operatorname{Int}(D))_{\mathfrak{M}_{\mathfrak{m},0}}$ is a pseudovaluation domain.
(2) ⇒ (1) Assume that D is a pseudovaluation domain. Then D shares its maximal ideal with a valuation overring V. If t is a nonzero element of $\mathfrak{m}$, we thus have $tV \subseteq D$. Since D is Noetherian, it follows that V is a finitely generated D-module. Hence, V is itself Noetherian. Therefore V is a discrete rank-one valuation domain, $\dim(D) = 1$, V is the integral closure of D, and in particular, D is analytically

irreducible. If we assume moreover that $D/\mathfrak{m}$ is finite, then $V/\mathfrak{m}$ is also finite. Note also that the $\mathfrak{m}$-adic topology of V induces the $\mathfrak{m}$-adic topology in D, and that the completion $\widehat{\mathfrak{m}}$ of $\mathfrak{m}$ is both the maximal ideal of the completion $\widehat{D}$ of D and of the completion $\widehat{V}$ of V. We proceed to conclude that $\mathrm{Int}(D)$ is an LPVD. We recalled above that the maximal ideals $\mathfrak{M}$ of $\mathrm{Int}(D)$ are either of the form $< q >$, where q is irreducible in $K[X]$, or $\mathfrak{M}_{\mathfrak{m},\alpha}$, where $\alpha \in \widehat{D}$. If $\mathfrak{M} =< q >$, then $\mathrm{Int}(D)_{\mathfrak{M}} = K[X]_{(q)}$ is a valuation domain. It remains to prove that, for $\mathfrak{M} = \mathfrak{M}_{\mathfrak{m},\alpha}$, $\mathrm{Int}(D)_{\mathfrak{M}}$ is a pseudovaluation domain. Let $f, g \in K(X)$ be such that $fg \in \mathfrak{M}(\mathrm{Int}(D))_{\mathfrak{M}}$, then $f(\alpha)g(\alpha) \in \widehat{\mathfrak{m}}$. Since the completion $\widehat{V}$ of V is a valuation domain, either $f(\alpha) \in \widehat{\mathfrak{m}}$, or $g(\alpha) \in \widehat{\mathfrak{m}}$. It follows from Lemma 2.2 that either f or g belongs to $\mathfrak{M}(\mathrm{Int}(D))_{\mathfrak{M}}$.

□

Remarks 2.5. 1. As stated in the introduction, by choosing $(D, \mathfrak{m})$ to be analytically irreducible, with finite residue field, but not a pseudovaluation domain, we obtain a two-dimensional locally divided domain with infinitely many maximal ideals which *is not* an LPVD. For instance, letting $D = k[[X^2, X^3]]$, where k is a finite field, we obtain such examples in every nonzero characteristic.
2. Recall from [11, Theorem 2.9] that each overring of a domain R is an LPVD if and only if R is an LPVD and an i-domain (or equivalently R is an LPVD and the integral closure R' of R is a Prüfer domain). In general it is not sufficient that R be an LPVD. However, note that in the case where D is Noetherian, if $\mathrm{Int}(D)$ is an LPVD, then each overring of $\mathrm{Int}(D)$ is also an LPVD. Indeed, if $\mathrm{Int}(D)$ is an LPVD, it is a fortiori a treed domain, and hence, from Proposition 1.4, an i-domain.

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