The History of Combinatorial Game Theory

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Abstract

A brief history of the people and the ideas that have contributed to Combinatorial Game Theory.

1 The Newcomer

Games have been recorded throughout history but the systematic application of mathematics to games is a relatively recent phenomonon. Gambling games gave rise to studies of probability in the 16th and 17th century. What has become known as *Combinatorial Game Theory* or *Combinatorial Game Theory à la Conway*—this to distinguish it from other forms of game theory found in economics and biology, for example—is a babe-in-arms in comparison. It has its roots in the paper [12] written in 1902, but the theory was not 'codified' until 1976-1982 with the publications of *On Numbers and Games* [15] by John H. Conway and *Winning Ways* [10] by Elwyn R. Berlekamp, John H. Conway and Richard K. Guy.

In the subject of *Impartial* games (essentially the theory as known before 1976), the first MSc thesis appears to be in 1967 by Jack C. Kenyon [31] and the first PhD by Yaacov Yesha [63] in 1978. Using the full theory is Laura J. Yedwab's MSc thesis in 1985 [62] and David Wolfe's 1991 PhD [60]. (Note: Richard B. Austin's MSc thesis [3] contains a little of the *Partizan* theory, but Yedwab's thesis is purely Partizan theory.)

In the sequel, there are several Mathematical Interludes that give a peek into the mathematics involved in the theory. Note that the names of games are given in small capitals such as CHESS. Also the notion of game is both general, like CHESS which refers to a a set of rules, and specific as in a specific CHESS position. Whenever I talk mathematically, as in the Interludes, it is this specific notion that should be invoked.

2 What is Combinatorial Game Theory?

This *Combinatorial Game Theory* has several important features that sets it apart. Primarily, these are games of pure strategy with no random elements. Specifically:

1. There are *Two Players* who *Alternate Moves*;
2. There are *No Chance Devices*—hence no dice or shuffling of cards;
3. There is *Perfect Information*—all possible moves are known to both players and, if needed, the whole history of the game as well;

4. *Play Ends, Regardless*—even if the players do not alternate moves, the game must reach a conclusion;
5. The *Last Move* determines the winner—**Normal** play: last player to move wins; **Misère** play last player to move loses!

The players are usually called *Left* and *Right* and the genders are easy to remember —Left for **L**ouise Guy and Right for **R**ichard Guy who is a important 'player' in the development of the subject. More on him later.

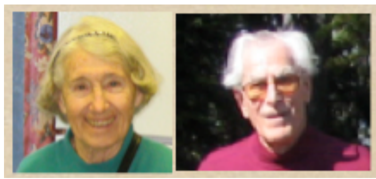

Figure 1: Louise and Richard Guy in Banff

Examples of games[1] NOT covered by these rules are: DOTS-&-BOXES and GO, since these are scoring games, the last person to move is not guaranteed to have either the highest or the lowest score; CHESS, since the game can end in a draw; BACKGAMMON, since there is a chance element (dice); BRIDGE, the only aspect that this game satisfies is that it ends.

Games which are covered by the conditions are: NIM[2]; AMAZONS, CLOBBER, DOMINEERING and HEX. In fact, NIM, AMAZONS, DOMINEERING and also, despite the coments of the previous paragraph, DOTS-&-BOXES and GO have a property that makes the theory extraordinarily useful for the analysis of these games. The board breaks up into separate components, a player has to choose a component in which to play. Moreover, his opponent does not have to reply in the same component. This is why condition (4) is important. The aspect is so important that it has its own name.

The **disjunctive sum** *of games G and H, written $G+H$, is the game where a player must choose to play in exactly one of G and H.*

The game of NIM with heaps of sizes 3, 4 and 5 is the disjunctive sum of three one-heap games of NIM. One could

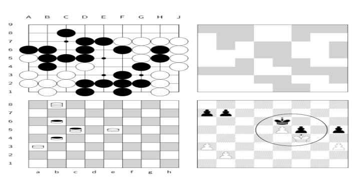

Figure 2: A diabolical disjunctive sum

also imagine playing the disjunctive sum of a game of CHESS with a game of CHECKERS and a game of GO. On a move, a player moves in only one of the games but the opponent does not have to reply in the same game. The winner will be the player making the last move over all. As a rule-of-thumb, if a position breaks up into components so that the resulting game is a disjunctive sum then this theory will be useful. If the game does not become a disjunctive sum, HEX for example, then the theory is less useful.

We still need a few more definitions. In an *Impartial* game both players have exactly the same moves—NIM for example. In a *Partizan* game the players have different moves—in CHESS a player can only move his own pieces and not those of his opponent; she would get rather upset if he did.

[1]For rules of these, and other, games see `http://en.wikipedia.org/wiki/` with *Game_of_the_Amazons*; *Clobber*; *Domineering*; *Dots_and_boxes*; or *Hex_(board_game)* at the end of the URL.

[2]NIM **Rules:** On a table there are several heaps of counters. A player chooses any heap and removes any number of counters from that heap. The person taking the last counter wins. For example, suppose there are three heaps with 3, 5 and 7 counters respectively, a player could choose the heap of 7 and remove any number from 1 through 7 counters.

A game belongs to one of four *outcome* classes. This was first noted, by Ernst Zermelo [64] in 1912, but phrased differently. A game can be won by:

- $\mathcal{L}$eft regardless of moving first or second;
- $\mathcal{R}$ight regardless of moving first or second;
- by the $\mathcal{N}$ext player regardless of whether this is Left or Right;
- or by the $\mathcal{P}$revious player regardless of whether this is Left or Right.

An outcome class is usually referred to by its initial. In an Impartial game such as NIM, since both players have the same moves thus the outcome of a position must be either $\mathcal{N}$ or $\mathcal{P}$.

A main aim of the theory is to give a value to each component: essentially how much of an advantage the position is to one of the players—positive value for Left and negative for Right. First, though, we have to deal with *equality*: two games should be the same if both players are indifferent to playing in one or the other. Or

Equality or the 'Axiom of Indistinguishability': *$G = H$ if, for all games X, the outcome for $G+X$ is the same as the outcome for $H+X$.*

Finally, we are ready to talk history! The history breaks up into three main threads and all threads are still very active:

- **Impartial** games under the *Normal* play ending condition which starts with Bouton and NIM [12] through Guy & Smith [26];
- **Partizan** games again under the *Normal* play rule starting with Milnor's [40] and Hanner's [27] work (from GO) through Berlekamp, Conway & Guy [15, 10];
- **Impartial** games under the *Misère* rules starting with Dawson in 1935. (See the Dover collection [18]).

What about the obvious fourth thread?

- **Partizan Misère** games: there are exactly two papers on the subject, [39, 48] both in 2007. This topic is hard!

Mathematical Interlude 1. *How to play and win at* NIM.

If there is one heap, take it all!

If there are two unequal heaps, remove from the larger to leave two the same size.

For three or more heaps, write each heap size as a sum of powers of 2, i.e., as sums of 1, 2, 4, 8, etc; pair off equal powers of 2; if all powers are paired off, invite your opponent to go first. He must disturb the pairings and your winning response is to remove enough counters to re-establish a pairing. Mathematically, write the numbers in binary and add without carrying.

For example: with heaps of size 1, 5 and 7 then as sums of powers of 2, $1 = 1$, $5 = 4+1$ *and* $7 = 4+2+1$, *the 4s pair off but not the 2s or the 1s. The winning move (there could be more than one but not in this situation) is to play to remove 3 (=2+1) from the 7 heap to leave the position 1, 5, 4 where* $1 = 1$, $5 = 4+1$ *and* $4 = 4$. *If the opponent were now to move to 1, 3, 4 then* $1 = 1$, $3 = 2+1$ *and* $4 = 4$ *and only the 1s are paired. No move will ever create another 4 so the 4 has to go but at the same time you should leave a 2 to pair off with the other 2, i.e. move to the position* $1 = 1$, $3 = 2+1$ *and* $2 = 2$.

3 The Winners at Normal Play

3.1 Impartial Normal Play

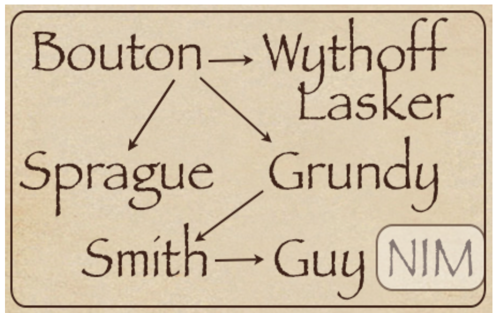

Figure 3: Genealogy of Early Impartial Players

The game of NIM is *Impartial* since both players have the same moves. Charles L. Bouton [12] analyzed NIM. (See [38] for Bouton's mathematical obituary.) There have been discussions over the origin of the the name *nim* with some references to a Chinese origin. However, Bouton did his PhD in Leipzig so it is likely that the name owes much to the German verb *nimm* meaning ‘ take’. It took three decades before it was realized, and proved, that each Impartial game is equivalent to a NIM position. Knowing how to win at NIM then allows a player to know how to win all Impartial games. Of course, calculating the equivalent NIM position is non-trivial. The Bouton paper sparked an interest in the area and several important papers resulted. From our vantage point, some of the games that were suggested were very interesting, even important, but went off the track of developing the general theory.

First was WYTHOFF'S GAME, introduced and solved by Willem A. Wythoff [61] in 1907. (See [32] for a very brief biography.) The rules are: *There are two heaps of counters on a table. On a turn, a player either chooses a heap and takes as many counters as they wish; or they may take an equal number from both. The player taking the last counter wins.* The game was also given, independently, by Rufus Issacs, see [5] p. 53: Play with a Chess Queen on a quarter infinite board and a move must move the Queen closer to the corner of the board. The heaps are the coordinates of the Queen's position. Hence the game is sometimes called WYTHOFF QUEENS. The game has interesting connections

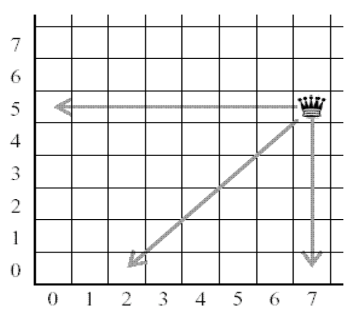

Figure 4: WYTHOFF'S GAME as WYTHOFF'S QUEENS

to the golden ratio and Fibonacci numbers, and has led to some very interesting and beautiful mathematics, see [17]. However, it turned out to be a wrong direction for the theory. The game does not break up into disjoint components. Several other authors followed including Eliakim H. Moore (MOORE'S NIM [41]), where a player make take from up to k heaps, k being fixed in advance, which is also off in the wrong direction since it merges rather than separates the heaps.

In the right direction, Emanuel Lasker [34] p.183 in 1931, introduced LASKER'S NIM—the same rules as NIM with the extra option of removing no counters but splitting a heap into two (non-empty) heaps. This clearly highlights the disjunctive sum aspect. According to Jörg Bewersdorff [11] (pp.174-176 in the English version), Lasker just missed developing the whole theory of Impartial games. He did understand the outcome classes $\mathcal{P}$ and $\mathcal{N}$ and how they interacted. And, according to Richard K. Guy [1], Michael Goldberg in the 1938 edition of W. W. Rouse Ball's *Mathematical Recreations and Essays* [4] solved much of KAYLES and "was unlucky not to have discovered the complete analysis and the S-G [Sprague-Grundy] theory". The theory was 'in the air' and it was left to Sprague and Grundy to find it.

Roland P. Sprague [55, 56] in 1935 and independently Patrick M. Grundy [20] in 1939 published complete solutions on how to solve Impartial games. (See [54] for a mathematical obituary of Grundy.) This became known as the *Sprague-Grundy Theory* and the value associated with an Impartial game was referred to as the *Grundy-value*. (Guy and Smith didn't learn about Sprague's work until after Grundy's death in 1959.) Since this value is equal to the size of the NIM-heap to which it is equivalent, and since it has been placed inside more encompassing theory, many authors now refer to the *nim-value* of a game and the set of values as the *nimbers*.

In 1949, R. K. Guy, in solving DAWSON'S CHESS[3], also re-discovered the Sprague-Grundy theory and in addition, an infinity of games to which the theory could be applied. (See Mathematical Interlude 2.) R. K. Guy was steered toward Cedric A. B. Smith who had worked with Grundy. (Smith was also a member of *Blanche Descarte*[4].) This led to the 1956 article [26] and to a career that is still active today. In the area of combinatorial game theory, R. K. Guy has: published over 20 articles; published two books (more on those in the Partizan Section); helped organize five major conferences; edited one Conference proceedings; and maintains an 'Unsolved Problems in Combinatorial Game Theory' column. This doesn't count the over two hundred and fifty other publications of his. One interesting aspect, despite all his achievements, Guy got the first solution wrong! Or rather, he solved the wrong problem. Dawson asked for the Misère version and Guy solved the Normal play version believing ([24, 1]), like so many others, that the winning strategy for Misère play is a slight tweak of the strategy for Normal play. Moreover, the original DAWSON'S CHESS is still unsolved today!

WYTHOFF'S GAME, GRUNDY'S GAME and MOORE'S GAME are associated with beautiful and sometimes surprising mathematics. One other game that should also be mentioned in the same vein is WELTER'S GAME [58, 59] from 1952[5]. Welter knew of the work of Sprague and generalized one his NIM games. Despite the apparent 'welter' of confusion about the equivalent nim-values, there is a very pretty way to decide on a good move [15] pp.153–165 and [10] pp.506–515, see also [9, 16].

In terms of the development of the theory the focus now shifts, but before moving on, one other person, *Aviezri Fraenkel* ([19, 25]) should also be noted as the one who has the largest number of publications in the area, with well over sixty papers on mainly, but not restricted to, Impartial games and complexity results. (This count does not include his numerous papers on other areas of mathematics and computer science.)

Figure 5: Aviezri Fraenkel

Mathematical Interlude 2. *The games of R. K. Guy are the innocuous sounding* SUBTRACTION *and* OCTAL *games.*

[3]DAWSON'S CHESS [18] *Given two equal lines of opposing Pawns, White on 3rd rank, Black on 5th, in adjacent files. White to play and capturing is mandatory. The player who makes the last move loses. Who wins?*

[4]See `http://www.squaring.net/history_theory/brooks_smith_stone_tutte.html`.

[5]WELTER'S GAME: is played with coins on a strip of squares numbered 1 through whatever. The coins can be moved to any smaller numbered, unoccupied square but no square can have more than one coin. The coins accumulate at one end of the strip and so the game finishes.

Given a finite set of numbers, say $S = \{2,3,5\}$, called the subtraction *set, and a heap of n counters, a player may take away 2, 3 or 5 counters. The outcome-sequence for the set S is the sequence of the outcome for a heap: of size 0, 1, 2; …. The outcome sequence for $S = \{1\}$, i.e. where a player is only allowed to take away 1 counter, is: $\mathcal{P}, \mathcal{N}, \mathcal{P}, \mathcal{N}, \mathcal{P}, \mathcal{N}, \ldots$. To see this, a heap of size 0 is a $\mathcal{P}$revious player win, since neither player has a move; 1 is a $\mathcal{N}$ext since the Next player can reduce the heap to 0 whereupon his opponent has no move and so loses. From a heap of 2, then you, as the next player, must move to a heap of 1 which is an $\mathcal{N}$-position, i.e. you give your opponent a good move, so the outcome of a 2-heap is $\mathcal{P}$. A little more thought gives that a heap of even size is a $\mathcal{P}$-position and an odd-sized heap is a $\mathcal{N}$-position giving the sequence sometimes referred to as* SHE-LOVES-ME-SHE-LOVES-ME-NOT. *The reader is encouraged to find the outcome sequences of: $S = \{1,2\}$; $S = \{1,2,3\}$; etc.*

In general, it is known that for any set S, eventually, the nim-sequence will be periodic, but no-one has discovered a relationship between S and the form of the period. Actually, researchers look for the nim-*sequence where instead of the outcomes, the size of the equivalent* NIM-*heap is recorded. Note that in this case, the $\mathcal{P}$-positions correspond exactly to heaps of size 0.*

OCTAL *games are like* LASKER'S GAME, *where a move, depending on the exact rules, may allow a player to take from a heap and possibly split the remaining heap into two heaps. In* GRUNDY'S *game, heaps are allowed only to be split into two non-equal, non-empty heaps. Despite having been analyzed to heap-sizes of many billions, no periodicity or other regularity has been discovered.*

For an in-depth discussion of these, and other heap games, see [10] Chapter 4.

3.2 Early Partizan Players

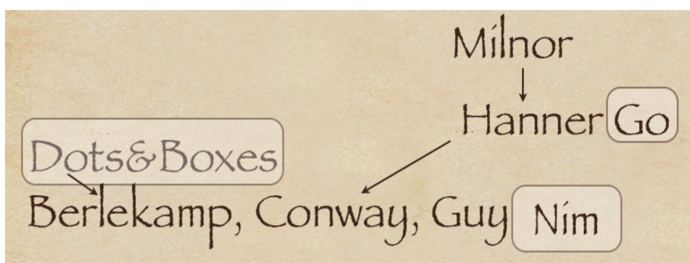

Figure 6: Genealogy of Early Partizan Players

In 1953, arising out of his research in *classical* game theory (the game theory used in economics and biology and other sciences), John Milnor [40] wrote the first theoretical paper on Partizan games. He recognized that there were *hot* positions, positions in which both players are eager to move because of the advantage gained. A mathematical approach to approximating such positions is to see what happens when there are many copies of the position. This gives an idea of the 'mean-value' of the position—on average, what advantage might the position be worth.

Olof Hanner was interested in GO and in 1957 whilst wandering around Stockholm he found a GO book with an annotated game. The author claimed that Black won by one point but Hanner found that Black should win by two points. (See [45] for more details about Hanner.) This led Hanner [27] to define his own version of the 'mean-value' of a game. To quote Yedwab, [62], "*One way to view Hanner's strategy, is that it addresses a basic weakness found in Milnor's strategy, i.e., tempo. In Milnor's strategy, the follower is a wimp that passively responds to the leader's move, even when it is obvious that the leader's move is not sente.*" That is the 'leader' takes advantage of the disjunctive sum by freely choosing which component to play in, but the follower is constrained to playing in the same component as the leader. Note that a move is *sente* if the opponent has to reply in order to prevent a large loss. Also, Milnor's definition of a mean-value is not robust. The approximation to the mean-value could get worse, not better, as more copies are added to the sum, but Hanner had hit upon the right idea.

Richard K. Guy now re-enters the scene as a unifying force. John H. Conway had been interested in games.

(Conway has many achievements including the GAME OF LIFE.) Conway knew Mike Guy, Richard's son. They met in 1960[6] in Cambridge when John Conway was a first year graduate student and Mike Guy a new undergraduate. Mike Guy passed along the Impartial theory developed by his father.

Elwyn R. Berlekamp met Richard at a conference in 1966. Berlekamp had just 'solved' the Impartial game of DOTS-&-BOXES[7] with help from the Guy-Smith paper [26]. According to legend and eyewitnesses, Elwyn Berlekamp has not lost a game of DOTS-&-BOXES in over 40 years. (See [7] for more on the game.) He suggested that they write a book and Guy suggested adding Conway. They started work soon after and the two volume set *Winning Ways* appeared in 1982. The present version is the four volume set [10]. During the 16-year production, John H. Conway realized that there is a wonderful mathematical theory, based on the 'disjunctive sum' concept, underpinning these games and first published *On Numbers and Games* in 1976 (re-published in 2001 [15]). Conway developed a new number system out of evaluations of games. This system, called *surreal numbers* (by Donald Knuth! [33]) extends the real numbers in a manner similar to that of *Dedekind cuts* which extends the rational numbers to the reals. Later, Elwyn Berlekamp observed that GO games frequently broke up into a disjunctive sum and a new area of games research was born, see [8].

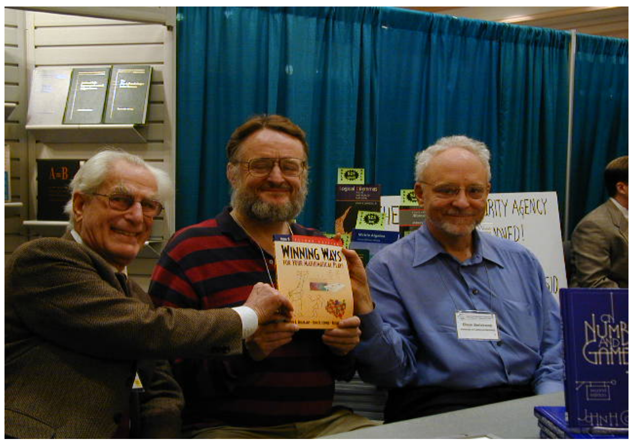

Figure 7: Guy, Conway and Berlekamp

These books still are the standard references and bibles of the subject. Admittedly, *On Numbers and Games* is a graduate level mathematics text but *Winning Ways* is a recreational mathematics book and is very accessible.

Mathematical Interlude 3. *Mathematical Structure of Games—Partially-Ordered Abelian Group.*

1. **Zero:** *If G is a 2nd player win then the outcome of $G+H$ is the same as that of H for all games H. The player who can win H plays this strategy and never plays in G except to respond to his opponent's moves in G. Thus, any 2nd player win game acts like 0 in that it changes nothing when added to another game.*

2. **Negative:** *Given a game G, $-G$ is G with the roles reversed. For example, in* CHESS *this is the same as turning the board around.*

3. **Equality:** *$G = H$ if $G+(-H)$ is a 2nd player win; i.e. neither player has an advantage when playing first. Note this is a 'definition' of equality and mathematically we can say $G+(-H) = 0$ is the same as $G = H$. [Note that this really defines an equivalence relation and the 'equality' is for the equivalence classes.]*

4. **Associativity:** *$G+(H+K) = (G+H)+K$ is straightforward from the definition of the disjunctive sum;*

5. **Commutativity:** *$G+H = H+G$, again straightforward;*

[6]Richard and Louise's daughter Anne, the first person to solve *Rubik's Cube*, also started in maths at Cambridge the same year.

[7]John C. Holladay in [29] partially solved DOTS-&-BOXES in the version where a player MUST take a box if one is present. Also, Holladay [28] rediscovered the Sprague-Grundy theory in 1957.

6. **Inverses:** *For any game G,* $G + -(G)$ *is a 2nd player win, (i.e.* $G + (-G) = 0$*) by 'Tweedledum-Tweedledee'—whatever you play in one, I play exactly the same in the other.*

7. **Inequality:** $G \geq H$ *if Left wins* $G + (-H)$*; i.e. there is a bigger advantage to Left in G than in H.*

The structure really is a partial order. For example, when playing NIM*, a heap of size 1 and a heap of size 2 are incomparable. Let's call these games* $*1$ *and* $*2$ *for easy reference. Note that* $-(*2)$ *is the same game as* $*2$ *since the Left moves are the same as the moves available to Right so interchanging them has no effect on the play of the game. We already know that* $*1 + *2$ *is a first player win—the winning move is to* $*1 + *1$*. By the definition of 'equality' then* $*1 \neq *2$*. Moreover, by the definition of inequality* $*1 \not\geq *2$ *and* $*1 \not\leq *2$*.*

3.3 Very Modern Normal History

With the publishing of *On Numbers and Games* and *Winning Ways* the full framework of the theory of Combinatorial Games was laid down. However, much work remains to be done within that framework. Indeed, the activity can be classified (roughly) into five main areas.

- **Algorithmic Game Theory:** Theory is fine, but most people want to know how to win in an actual game. When analyzing games, hand calculation can only go so far[8]. In the early 1990s, David Wolfe developed the software *Gamesman's Toolkit* which has been superseded in 2000 by Aaron Siegel's *CGSuite* [49].
- **Complexity:** long an interest of Computer scientists as well as mathematicians. Look to researchers such as Aviezri Fraenkel and Erik Demaine.
- **Hot Games:** Games in which there is a large advantage in moving first. These games are perhaps of the only ones of interest to real world games players. This area has the most overlap of computer scientists and mathematicians. The mathematicians want to know the exact values and the very best strategies by working backwards from the end. The computer scientist wants good heuristics that will allow good play from the beginning.
- **Impartial games:** Even though these games, such as NIM, started the area, much remains to be discovered.
- **All-small games:** Games like CLOBBER[9] in which either both players have a move or neither does. It is not possible to build a large advantage as in Hot games.

But, as is the wont of mathematicians, almost immediately, if not sooner, research started into pushing the envelope, relaxing the conditions that define combinatorial game theory. Briefly, the highlights are:

- **Scoring Games** such as GO and DOTS-&-BOXES. The Chinese scoring convention almost makes GO into a combinatorial game and the last player to move in DOTS-&-BOXES is usually the winner so the theory is applicable to these games. Indeed, research into GO is pushing the limits of the theory. A new avenue of research has been started by Elwyn R. Berlekamp [6]. He introduced the idea of an *enriched* environment—a stack of coupons in decreasing order. A player may make a move or take the top coupon of the stack, the value of which is added to the player's score. This is a useful analysis tool in all hot games and has even made its appearance in International GO events.

[8] Having said that, around 1950, R. K. Guy was calculating the nim-sequences for SUBTRACTION and OCTAL games up to heaps of size 600, whilst C. B. Haselgrove wrote a program that ran EDSAC out of memory at size 400. P. M. Grundy managed to get the nim-sequence, from a computer, for his game (i.e. GRUNDY'S GAME) up to heap size 1100 many of which were wrong because of overflow errors! E. R. Berlekamp discovered a structure within the nim-sequence of this game, called the *sparse space* phenomenon, that has allowed dedicated machines to extend the nim-sequence to roughly 17 billion.

[9] CLOBBER is played on a rectangular board, 8×6 for example, starting with alternating black (Left) and white (Right) pieces. A piece is moved one square horizontally or vertically provide the new square is occupied by an opponent's piece which is then removed from the board.

- **Loopy Games:** Games in which the play is not guaranteed to end. In 1966, C. A. B. Smith [53] first extended the Impartial theory (of Sprague and Grundy) to games with cycles (see also [47]) and, in 1978, John H. Conway [14] showed that canonical forms could be defined for some loopy games. A mistake in the analysis of FOX-&-GEESE in *Winning Ways* led to more, very recent, advances obtained by Aaron Siegel [50, 51, 52].

- Allowing a **random** element. In Richman games [36, 35], players bid for the right to play next in an otherwise combinatorial game.

- Allowing **three** or more players. The problem is that there could be off-the-board strategies; the players could form coalitions for all or some of the game. Also, player A on his last turn could make either of players B and C the winner but not himself. Li [37] (1977) and Straffin [57] (1985) considered the formation and behavior of coalitions. Propp [46] (2000) and Cincotti [13] (2000) considered the situations when one player has a winning strategy against a coalition of the other two.

4 Misère Play; or the Best Losers

Thomas R. Dawson ([30]) was a composer of chess problems (which is how he and R. K. Guy met). The solution to his 1935 Misère problem of DAWSON'S CHESS, of course, depends on the width of the board. This is the problem that Guy solved for Normal play. Many researchers believe that the strategy for a Misère game is to take the Normal play strategy and tweak it at the end of the game. While this is true for NIM, it is not true in general.

Results in Misère play have been few and far between. From 1935 up to 2001, there are only thirteen papers on Misère games, although they are also considered in both *On Numbers and Games* and *Winning Ways*. In *On Numbers and Games*, Conway shows that the theory for Normal play will not translate to Misère play. In Normal play, many games are equal to each other which means that a strategy for one works for all the others. In Misère play almost all games are equal to themselves and few, if any, others: the strategy for one game does not help with the strategy for any other game.

First, Patrick M. Grundy and Cedric A. B. Smith [21] considered Impartial games under Misère play rules. Essentially, all but two of these eleven papers, only add to human knowledge by dealing with specific games. However, in the last few years, Thane Plambeck, and now Aaron Siegel, have started a new and exciting chapter in this area. The history of Misère games is only now being written.

Figure 8: Thane Plambeck on the left and Aaron Siegel on the right

Mathematical Interlude 4. *Recall that, having decided which universe (Normal or Misère) of games we are playing in, the definition of equality of games is:*

$G = H$ if for all games X in this universe, the outcome for $G+X$ is the same as the outcome of $H+X$.

Plambeck's approach is to limit the size of the universe. Given a game position G, Plambeck's universe is restricted to only those games that can be reached from G. This universe is called the closure *of G. Equality is now defined as:*

Given a game G then for H and K in the closure of G, $H = K$ if for all games X in the closure, the outcome of $H+X$ is the same as the outcome of $K+X$. *Games can now be equal in one universe but unequal in another.*

5 Sources

Not everyone can have the good fortune of having talked to Richard Guy, and also Elwyn Berlekamp, John Conway, Aviezri Fraenkel, Thane Plambeck, Aaron Siegel and David Wolfe. For the less fortunate, some biographical details can be found on Wikipedia and the St. Andrews history website, These early 'players' were, and still are, proficient mathematicians who accomplished much across many fields. It is well worth reading the more personal recollections, stories and biographies that can be found in [1, 19, 24, 32, 38, 45, 54].

More mathematical papers are listed in the bibliography. Another great resource is Aviezri Fraenkel's bibliography of papers in [42, 43, 44] which also appear as a dynamic survey in the *Electronic Journal of Combinatorics* at `http://www.combinatorics.org/`.

For some mathematical survey papers see [23, 42, 43, 44]—the last will appear later this year. For an introduction to Impartial games see *Fair Game* [22], to the full theory see *Winning Ways* or *Lessons in Play* [2].

Acknowledgements

Figure 9: Richard J. Nowakowski

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