THE BG-RANK OF A PARTITION AND ITS APPLICATIONS

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Abstract. Let π denote a partition into parts $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots$. In a 2006 paper we defined BG-rank(π) as

$$\text{BG-rank}(\pi) = \sum_{j\geq 1} (-1)^{j+1} \frac{1-(-1)^{\lambda_j}}{2}.$$

This statistic was employed to generalize and refine the famous Ramanujan modulo 5 partition congruence. Let $p_j(n)$ denote the number of partitions of n with BG-rank $= j$. Here, we provide a combinatorial proof that

$$p_j(5n+4) \equiv 0 \pmod 5, \quad j \in \mathbb{Z},$$

by showing that the residue of the 5-core crank mod 5 divides the partitions enumerated by $p_j(5n+4)$ into five equal classes. This proof uses the orbit construction from our previous paper and a new identity for the BG-rank. Let $a_{t,j}(n)$ denote the number of t-cores of n with BG-rank $= j$. We find eta-quotient representations for

$$\sum_{n\geq 0} a_{t,\lfloor \frac{t+1}{4} \rfloor}(n) q^n \quad \text{and} \quad \sum_{n\geq 0} a_{t,-\lfloor \frac{t-1}{4} \rfloor}(n) q^n,$$

when t is an odd, positive integer. Finally, we derive explicit formulas for the coefficients $a_{5,j}(n)$, $j = 0, \pm 1$.

1. Introduction

A partition π is a nonincreasing sequence

$$\pi = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$

of positive integers (parts) $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$. The norm of π, denoted $|\pi|$, is defined as

$$|\pi| = \sum_{i\geq 1} \lambda_i.$$

If $|\pi| = n$, we say that π is a partition of n. The (Young) diagram of π is a convenient way to represent π graphically: the parts of π are shown as rows of unit squares (cells). Given the diagram of π we label a cell in the i-th row and j-th column by the least nonnegative integer $\equiv j - i \pmod t$. The resulting diagram is called a t-residue diagram [7]. We can also label cells in the infinite column 0 and the infinite row 0 in the same fashion and call the resulting diagram the extended t-residue diagram [5]. And so with each partition π and positive integer t we can

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associate the t-dimensional vector

$$\vec{r}(\pi, t) = (r_0(\pi, t), r_1(\pi, t), \ldots, r_{t-1}(\pi, t))$$

with

$$r_i(\pi, t) = r_i, \quad 0 \le i \le t-1$$

being the number of cells colored i in the t-residue diagram of π. If some cell of π shares a vertex or edge with the rim of the diagram of π, we call this cell a rim cell of π. A connected collection of rim cells of π is called a rim hook if (diagram of π)$\backslash$(rim hook) represents a legitimate partition. We say that a partition is a t-core, denoted $\pi_{t\text{-core}}$, if its diagram has no rim hooks of length t [7].

The Durfee square of π is the largest square that fits inside the diagram of π. Reflecting the diagram of π about its main diagonal, one gets the diagram of π' (the conjugate of π). More formally,

$$\pi' = (\lambda'_1, \lambda'_2, \lambda'_3, \ldots)$$

with λ'_i being the number of parts of π that are $\ge i$. In [2] we defined a new partition statistic

$$\text{BG-rank}(\pi) := \sum_{j \ge 1} (-1)^j \frac{(-1)^{\lambda_j} - 1}{2}. \tag{1.1}$$

It is easy to verify that

$$\text{BG-rank}(\pi) = r_0(\pi, 2) - r_1(\pi, 2) \tag{1.2}$$

and

$$\text{BG-rank}(\pi) \equiv |\pi| \pmod 2. \tag{1.3}$$

In [2] we proved the following $\pmod 5$ congruences

$$p_j(5n) \equiv 0 \pmod 5 \quad \text{if } j \equiv 1, 2 \pmod 5, \tag{1.4}$$

$$p_j(5n+1) \equiv 0 \pmod 5 \quad \text{if } j \not\equiv 1, 2 \pmod 5, \tag{1.5}$$

$$p_j(5n+2) \equiv 0 \pmod 5 \quad \text{if } j \not\equiv 0, 3 \pmod 5, \tag{1.6}$$

$$p_j(5n+3) \equiv 0 \pmod 5 \quad \text{if } j \equiv 0, 3 \pmod 5, \tag{1.7}$$

$$p_j(5n+4) \equiv 0 \pmod 5 \quad \text{for all } j \in \mathbb{Z}. \tag{1.8}$$

Here $p_j(n)$ denotes the number of partitions of n with BG-rank $= j$. Clearly,

$$p(5n+4) = \sum_j p_j(5n+4)$$

with $p(n)$ denoting the number of unrestricted partitions of n. And so (1.8) implies the famous Ramanujan congruence [11]

$$p(5n+4) \equiv 0 \pmod 5.$$

In this paper, we build on the developments in [2] to provide a combinatorial proof of (1.8).

For t-odd it is surprising that the BG-rank($\pi_{t-\text{core}}$) assumes only finitely many values. In fact, we will show that if t is an odd, positive integer, then

$$-\left\lfloor \frac{t-1}{4} \right\rfloor \le \text{BG-rank}(\pi_{t-\text{core}}) \le \left\lfloor \frac{t+1}{4} \right\rfloor. \tag{1.9}$$

Here $\lfloor x \rfloor$ denotes the integer part of x.

We will establish the following identities. For odd $t > 1$

$$C_{t,(-1)^{\frac{t-1}{2}}\lfloor\frac{t-1}{4}\rfloor}(q) = q^{\frac{(t-1)(t-3)}{8}} F(t, q^2), \tag{1.10}$$

$$C_{t,(-1)^{\frac{t+1}{2}}\lfloor\frac{t+1}{4}\rfloor}(q) = q^{\frac{t^2-1}{8}} \frac{E^t(q^{4t})}{E(q^4)}, \tag{1.11}$$

where

$$C_{t,j}(q) = \sum_{n\geq 0} a_{t,j}(n)q^n,$$

$a_{t,j}(n)$ denotes the number of t-cores of n with BG-rank $= j$ and

$$E(q) = \prod_{j=1}^{\infty}(1-q^j),$$

$$F(t,q) = \frac{E^{t-4}(q^{2t})E^2(q^t)E^3(q^2)}{E^2(q)}.$$

We observe that (1.3) suggests that $C_{t,j}(q)$ is an even (odd) function of q if j is even (odd).

It is instructive to compare (1.10, 1.11) with the well-known identity [5] for unrestricted t-cores

$$\sum_{n\geq 0} a_t(n)q^n = \frac{E^t(q^t)}{E(q)}. \tag{1.12}$$

Here $a_t(n)$ denotes the number of t-cores of n.

The rest of this paper is organised as follows.

In Section 2 we discuss the Littlewood decomposition of π in terms of t-core and t-quotient of π. We describe the Garvan, Kim, Stanton bijection for t-cores and use a constant term technique to provide a simple proof of the Klyachko identity [8]

$$\sum_{\substack{\vec{n}\in\mathbb{Z}^t \\ \vec{n}\cdot\vec{1}_t=0}} q^{\frac{t}{2}\vec{n}\cdot\vec{n}+\vec{b}_t\cdot\vec{n}} = \frac{E^t(q^t)}{E(q)}. \tag{1.13}$$

Here $\vec{1}_t = (1,1,\ldots,1) \in \mathbb{Z}^t$, $\vec{b}_t = (0,1,2,\ldots,t-1)$.

In Section 3 we establish a fundamental identity connecting BG-rank and the Littlewood decomposition.

In Section 4 we discuss a combinatorial proof of (1.8).

Section 5 is devoted to the proof of the identities (1.10, 1.11).

Section 6 deals with 5-cores with prescribed BG-rank. There we derive the explicit formulas for the coefficients $a_{5,j}(n)$, $j = 0, \pm 1$.

In Section 7 we give a generalization of the BG-rank and state a number of results.

2. Two Bijections

In this section we will follow closely the discussion in [4], [5] to recall some basic facts about t-cores and t-quotients. A region r in the extended t-residue diagram of π is the set of cells (i,j) satisfying $t(r-1) \leq j-i < tr$. A cell of π is called exposed if it is at the end of a row. One can construct t bi-infinite words $W_0, W_1, \dots, W_{t-1}$ of two letters N, E as

$$\text{The } r\text{-th letter of } W_i = \begin{cases} E, & \text{if there is an exposed cell labelled } i \text{ in the region } r \\ N, & \text{otherwise.} \end{cases}$$

It is easy to see that the word set $\{W_0, W_1, \dots, W_{t-1}\}$ fixes π uniquely.

Let P be the set of all partitions and $P_{t\text{-core}}$ be the set of all t-cores. There is a well-known bijection

$$\phi_1 : P \to P_{t\text{-core}} \times P \times P \times P \ldots \times P$$

which goes back to Littlewood [9]

$$\phi_1(\pi) = (\pi_{t\text{-core}}, \hat{\pi}_0, \hat{\pi}_1, \dots, \hat{\pi}_{t-1})$$

such that

$$|\pi| = |\pi_{t\text{-core}}| + t \sum_{i=0}^{t-1} |\hat{\pi}_i|.$$

Multipartition $(\hat{\pi}_0, \hat{\pi}_1, \dots, \hat{\pi}_{t-1})$ is called the t-quotient of π. We remark that (1.12) is the immediate corollary of the Littlewood bijection. We describe ϕ_1 in full detail a bit later.

The second bijection

$$\phi_2 : P_{t\text{-core}} \to \{\vec{n} : \vec{n} \in \mathbb{Z}^t, \vec{n} \cdot \vec{1}_t = 0\}$$

was introduced in [5]. It is for t-cores only

$$\phi_2(\pi_{t\text{-core}}) = \vec{n} = (n_0, n_1, \dots, n_{t-1})$$

where for $0 \le i \le t-2$

$$n_i = r_i(\pi_{t\text{-core}}, t) - r_{i+1}(\pi_{t\text{-core}}, t) \tag{2.1}$$

and

$$n_{t-1} = r_{t-1}(\pi_{t\text{-core}}, t) - r_0(\pi_{t\text{-core}}, t). \tag{2.2}$$

Clearly,

$$\sum_{i=0}^{t-1} n_i = \vec{n} \cdot \vec{1}_t = 0.$$

Moreover,

$$|\pi_{t\text{-core}}| = \frac{t}{2} \vec{n} \cdot \vec{n} + \vec{b}_t \cdot \vec{n}, \tag{2.3}$$

as shown in [5]. And so

$$\sum_{n \ge 0} a_t(n) q^n = \sum_{\substack{\vec{n} \in \mathbb{Z}^t \\ \vec{n} \cdot \vec{1}_t = 0}} q^{\frac{t}{2} \vec{n} \cdot \vec{n} + \vec{b}_t \cdot \vec{n}}. \tag{2.4}$$

Note that (1.12), (2.4) imply the Klyachko identity (1.13). The reader may wonder if (2.1, 2.2) can be used to define $\phi_2(\pi) = \vec{n}$ for any partition π. This, of course, can be done. However, in general ϕ_2 is not a $1-1$ function and so ϕ_2^{-1} can't be defined. Indeed, if $\pi_1 \ne \pi_2$, but $\pi_{t\text{-core}}$ is a t-core of both π_1 and π_2 then

$$\phi_2(\pi_1) = \phi_2(\pi_2) = \phi_2(\pi_{t\text{-core}}).$$

When a partition is a t-core, ϕ_2 can be inverted. To do this we recall that the partition is a t-core iff for $0 \le i \le t-1$

$$\begin{array}{lclccccc} \text{Region} & : & \cdots\cdots\cdots & n_i-1 & n_i & n_i+1 & n_i+2 & \cdots\cdots\cdots \\ W_i & : & \cdots\cdots\cdots & E & E & N & N & \cdots\cdots\cdots \end{array}$$

as explained in [5]. For example, the word image of $\phi_2^{-1}((2,-1,-1))$ is

$$\begin{array}{rcccccccc} \text{Region}: & \cdots\cdots & -1 & 0 & 1 & 2 & 3 & \cdots\cdots \\ W_0: & \cdots\cdots & E & E & E & E & N & \cdots\cdots \\ W_1: & \cdots\cdots & E & N & N & N & N & \cdots\cdots \\ W_2: & \cdots\cdots & E & N & N & N & N & \cdots\cdots. \end{array}$$

This means that

$$\phi_2^{-1}((2,-1,-1)) = (4,2). \tag{2.5}$$

More generally, if

$$\phi_1(\pi) = (\pi_{t\text{-core}}, \hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{t-1})$$

with

$$\hat{\pi}_i = (\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_{m_i}^{(i)}), \quad 0 \le i \le t-1,$$

then cells colored i are not exposed only in the regions

$$n_i + j - \lambda_j^{(i)}, \quad 1 \le j \le m_i$$

and

$$n_i + m_i + k, \quad k \ge 1.$$

For example, if $\hat{\pi}_i = (\lambda_1, \lambda_2, \lambda_3)$ then

$$\begin{array}{rccccccccccccccccc} \text{Region}: & \cdots\cdots & & n_i+1-\lambda_1 & & \cdots\cdots & & n_i+2-\lambda_2 & & \cdots\cdots & & n_i+3-\lambda_3 & & \cdots\cdots & n_i+4 & & \cdots\cdots \\ W_i: & \cdots\cdots & E & N & E & \cdots\cdots & E & N & E & \cdots\cdots & E & N & E & \cdots\cdots & E & N & \cdots\cdots \end{array}$$

Clearly, one can easily determine $\vec{n}$ and $(\hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{t-1})$ from the word set $\{W_0, W_1, \ldots, W_{t-1}\}$. And so

$$\phi_1(\pi) = (\phi_2^{-1}(\vec{n}), \hat{\pi}_0, \ldots, \hat{\pi}_{t-1}).$$

We illustrate the above with the following example. If $t = 3$ and $\pi = (7,5,4,3,2)$ then

$$\begin{array}{rcccccccccc} \text{Region}: & \cdots\cdots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots\cdots \\ W_0: & \cdots\cdots & E & E & E & N & E & E & N & N & \cdots\cdots \\ W_1: & \cdots\cdots & E & N & N & E & N & N & N & N & \cdots\cdots \\ W_2: & \cdots\cdots & E & N & E & N & N & N & N & N & \cdots\cdots. \end{array}$$

We have

$$\begin{aligned} n_0 &= 2, & \widehat{\pi}_0 &= (2), \\ n_1 &= -1, & \widehat{\pi}_1 &= (1,1), \\ n_2 &= -1, & \widehat{\pi}_2 &= (1). \end{aligned}$$

Using (2.5), we obtain

$$\phi_1((7,5,4,3,2)) = ((4,2),(2),(1,1),(1)).$$

To proceed further we recall some standard q-hypergeometric notations [6]:

$$(a_1, a_2, a_3, \ldots; q)_N = (a_1; q)_N (a_2; q)_N (a_3; q)_N \ldots$$

where

$$(a;q)_N = (a)_N = \begin{cases} \prod_{j=0}^{N-1}(1-aq^j), & N>0 \\ 1, & N=0 \\ \prod_{j=1}^{-N}(1-aq^{-j})^{-1}, & N<0. \end{cases}$$

We shall also require the Jacobi triple product identity [6, (II.28)]

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2, -zq, -\frac{q}{z}; q^2)_\infty. \tag{2.6}$$

We are now ready to prove the Klyachko identity (1.13). We will employ a so-called constant term technique. To this end we rewrite the left hand side of (1.13) as

$$\text{LHS (1.13)} \ = [z^0] \sum_{\vec{n}\in\mathbb{Z}^t} q^{\frac{t}{2}\vec{n}\cdot\vec{n}+\vec{b}_t\cdot\vec{n}} z^{\vec{n}\cdot\vec{1}_t} = [z^0] \prod_{i=0}^{t-1} \sum_{n_i=-\infty}^{\infty} q^{\frac{t}{2}n_i^2+in_i} z^{n_i}$$

where $[z^i]f(z)$ is the coefficient of z^i in the expansion of $f(z)$ in powers of z. With the aid of (2.6) we derive

$$\begin{aligned} \text{LHS (1.13)} \ &= [z^0] \prod_{i=0}^{t-1} \left(q^t, -q^{i+\frac{t}{2}} z, -\frac{q^{\frac{t}{2}}}{q^i z}; q^t \right)_\infty \\ &= [z^0] \frac{E^t(q^t)}{E(q)} \left(q, -q^{\frac{t}{2}} z, -\frac{q}{q^{\frac{t}{2}} z}; q \right)_\infty \\ &= [z^0] \Bigg(\frac{E^t(q^t)}{E(q)} \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}+\frac{t-1}{2}n} z^n \Bigg) \\ &= \frac{E^t(q^t)}{E(q)}, \end{aligned}$$

as desired. The above proof is just a warm-up excercise to prepare the reader for a more sophisticated proof of (1.10) discussed in Section 5.

3. The Littlewood decomposition and BG-rank

The main goal of this section is to establish the following identities for BG-rank. If t is even and $(n_0, \ldots, n_{t-1}) = \phi_2(\pi)$, then

$$\text{BG-rank}(\pi) = \sum_{i=0}^{\frac{t-2}{2}} n_{2i}. \tag{3.1}$$

If t is odd then

$$\text{BG-rank}(\pi_{t\text{-core}}) = bg(\vec{n}), \tag{3.2}$$

where $\vec{n} = \phi_2(\pi_{t\text{-core}})$ and

$$bg(\vec{n}) := \frac{1-\sum_{j=0}^{t-1}(-1)^{j+n_j}}{4}. \tag{3.3}$$

Moreover, if t is odd and $\phi_1(\pi) = (\pi_{t\text{-core}}, \widehat{\pi}_0, \ldots, \widehat{\pi}_{t-1})$ then

$$\text{BG-rank}(\pi) = \text{BG-rank}(\pi_{t\text{-core}}) + \sum_{j=0}^{t-1}(-1)^{j+n_j}\text{BG-rank}(\widehat{\pi}_j). \tag{3.4}$$

The proof of (3.1) is straightforward. It is sufficient to observe that if some cell is colored i in the t-residue diagram of π, then it is colored $\frac{1-(-1)^i}{2}$ in the 2-residue diagram of π. And so we obtain with the aid of (1.2)

$$\begin{aligned}\text{BG-rank}(\pi) &= (r_0 + r_2 + r_4 + \cdots + r_{t-2}) - (r_1 + r_3 + r_5 + \cdots + r_{t-1}) \\ &= (r_0 - r_1) + (r_2 - r_3) + \cdots + (r_{t-2} - r_{t-1}) \\ &= n_0 + n_2 + \cdots + n_{t-2},\end{aligned}$$

as desired. Next, let $D(\pi) = D$ denote the size of the Durfee square of π. To prove (3.2) we begin by rewriting (1.1) as

$$\text{BG-rank}(\pi) = \frac{1}{2}\left(\text{par}(\nu) + \sum_{j=1}^{\nu}(-1)^{\lambda_j - j}\right). \tag{3.5}$$

Here $\pi = (\lambda_1, \lambda_2, \ldots, \lambda_\nu)$ and par(x) is defined as

$$\text{par}(x) := \frac{1-(-1)^x}{2}.$$

Next, let π_1, π_2 denote the partitions constructed from the first $D = D(\pi_{t\text{-core}})$ rows, columns of $\pi_{t\text{-core}}$, respectively. Let π_3 denote a partition whose diagram is the Durfee square of $\pi_{t\text{-core}}$. It is plain that

$$\begin{aligned}\text{BG-rank}(\pi_{t\text{-core}}) &= \text{BG-rank}(\pi_1) + \text{BG-rank}(\pi_2) - \text{BG-rank}(\pi_3) \\ &= \text{BG-rank}(\pi_1) + \text{BG-rank}(\pi_2) - \text{par}(D).\end{aligned} \tag{3.6}$$

We shall also require the following sets

$$\begin{aligned}P_+ &:= \{i \in \mathbb{Z} : \quad 0 \le i \le t-1, \quad n_i > 0\}, \\ P_- &:= \{i \in \mathbb{Z} : \quad 0 \le i \le t-1, \quad n_i < 0\}.\end{aligned}$$

Here n_i's are the components of $\phi_2(\pi_{t\text{-core}})$. Note that if $i \in P_+$, then i is exposed in all positive regions $\le n_i$ of π_1. This observation together with (3.5) implies that

$$\begin{aligned}\text{BG-rank}(\pi_1) &= \frac{1}{2}\left(\text{par}(D) + \sum_{i\in P_+}\sum_{k=1}^{n_i}(-1)^{t(k-1)+i}\right) \\ &= \frac{1}{2}\left(\text{par}(D) + \sum_{i\in P_+}(-1)^i\text{par}(n_i)\right)\end{aligned} \tag{3.7}$$

In [5], the authors showed that under conjugation $\phi_2(\pi_{t\text{-core}})$ transforms as

$$(n_0, n_1, n_2, \ldots, n_{t-1}) \to (-n_{t-1}, -n_{t-2}, -n_{t-3}, \ldots, -n_0).$$

Also it is easy to see that

$$\text{BG-rank}(\pi_2) = \text{BG-rank}(\pi_2').$$

It follows that

$$\text{BG-rank}(\pi_2) = \frac{1}{2}\left(\text{par}(D) + \sum_{i\in P_-}(-1)^i\text{par}(n_i)\right). \tag{3.8}$$

Combining (3.6, 3.7, 3.8) and taking into account that $\mathrm{par}(0) = 0$ we get

$$\begin{aligned}\text{BG-rank}(\pi_t\text{-core}) &= \frac{1}{2}\sum_{i\in P_-\cup P_+}(-1)^i \mathrm{par}(n_i)\\ &= \frac{1}{2}\sum_{i=0}^{t-1}(-1)^i\mathrm{par}(n_i) = \frac{1-\sum_{i=0}^{t-1}(-1)^{i+n_i}}{4},\end{aligned}$$

as desired. Note that formula (3.2) implies that BG-rank of odd t-core is bounded, as stated in (1.9). Next, let $\widetilde{\pi}_{0,i}, \widetilde{\pi}_{2,i}, \widetilde{\pi}_{3,i}, \ldots$ denote the partitions constructed from $\phi_1(\pi) = (\pi_t\text{-core}, \widehat{\pi}_0, \widehat{\pi}_1, \ldots, \widehat{\pi}_{t-1})$, for odd t as follows

$$\begin{aligned}\widetilde{\pi}_{0,i} &= \phi_1^{-1}(\pi_t\text{-core}, \widehat{\pi}_0, \widehat{\pi}_1, \ldots, \widehat{\pi}_{i-1}, (0), \widehat{\pi}_{i+1}, \ldots, \widehat{\pi}_{t-1}),\\ \widetilde{\pi}_{1,i} &= \phi_1^{-1}(\pi_t\text{-core}, \widehat{\pi}_0, \widehat{\pi}_1, \ldots, \widehat{\pi}_{i-1}, (\lambda_1), \widehat{\pi}_{i+1}, \ldots, \widehat{\pi}_{t-1}),\\ \widetilde{\pi}_{2,i} &= \phi_1^{-1}(\pi_t\text{-core}, \widehat{\pi}_0, \widehat{\pi}_1, \ldots, \widehat{\pi}_{i-1}, (\lambda_1,\lambda_2), \widehat{\pi}_{i+1}, \ldots, \widehat{\pi}_{t-1}),\\ &\cdots\cdots\cdots.\end{aligned}$$

Here $\widehat{\pi}_i = (\lambda_1, \lambda_2, \ldots, \lambda_\nu)$. Note that the W_i word of $\widetilde{\pi}_{0,i}$ is

$$\begin{array}{ccccccc}\text{Region} & : & \cdots\cdots\cdots & n_i & n_i+1 & \cdots\cdots\cdots\\ W_i & : & \cdots\cdots\cdots & E & N & \cdots\cdots\cdots.\end{array}$$

To convert $\widetilde{\pi}_{0,i}$ into $\widetilde{\pi}_{1,i}$ we attach a rim hook of length $t\lambda_1$ to $\widetilde{\pi}_{0,i}$ so that W_i becomes

$$\begin{array}{cccccccccccc}\text{Region} & : & \cdots\cdots\cdots & & n_i+1-\lambda_1 & & \cdots\cdots\cdots & & n_i+2, & & \cdots\cdots\cdots\\ W_i & : & \cdots\cdots\cdots & E & N & E & \cdots\cdots\cdots & E & N & N & \cdots\cdots\cdots.\end{array}$$

It is not hard to verify that the color of the head (north-eastern) cell of the added rim-hook in the 2-residue diagram of $\widetilde{\pi}_{1,i}$ is given by $\mathrm{par}(tn_i + i) = \mathrm{par}(n_i + i)$. Observe that zeros and ones alternate along the added hook rim. This means that BG-rank does not change if λ_1 is even. If λ_1 is odd then the change is determined by the color of the added head cell, i.e.

$$\begin{aligned}\text{BG-rank}(\widetilde{\pi}_{1,i}) &= \text{BG-rank}(\widetilde{\pi}_{0,i}) + \mathrm{par}(\lambda_1)(1 - 2\mathrm{par}(n_i + i))\\ &= \text{BG-rank}(\widetilde{\pi}_{0,i}) + \mathrm{par}(\lambda_1)(-1)^{n_i+i},\end{aligned}$$

Next, we convert $\widetilde{\pi}_{1,i}$ into $\widetilde{\pi}_{2,i}$ by adding the new hook rim of length $t\lambda_2$ to $\widetilde{\pi}_{1,i}$ so that W_i becomes

$$\begin{array}{cccccccccccccc}\text{Region} & : & \cdots\cdots & & n_i+1-\lambda_1 & & \cdots\cdots & & n_i+2-\lambda_2 & & \cdots\cdots & n_i+3 & & \cdots\cdots\\ W_i & : & \cdots\cdots & E & N & E & \cdots\cdots & E & N & E & \cdots\cdots & E & N & \cdots\cdots.\end{array}$$

The color of the new head cell is given by

$$\mathrm{par}(t(n_i + 1) + i) = \mathrm{par}(n_i + 1 + i),$$

and so

$$\begin{aligned}\text{BG-rank}(\widetilde{\pi}_{2,i}) &= \text{BG-rank}(\widetilde{\pi}_{1,i}) + \mathrm{par}(\lambda_2)(1 - 2\mathrm{par}(n_i + 1 + i))\\ &= \text{BG-rank}(\widetilde{\pi}_{0,i}) + (-1)^{n_i+i}(\mathrm{par}(\lambda_1) - \mathrm{par}(\lambda_2)).\end{aligned}$$

Proceeding as above we arrive at

$$\begin{aligned}\text{BG-rank}(\pi) &= \text{BG-rank}(\widetilde{\pi}_{0,i}) + (-1)^{n_i+i}\sum_{j=1}^{\nu}(-1)^{j+1}\text{par}(\lambda_j)\\ &= \text{BG-rank}(\widetilde{\pi}_{0,i}) + (-1)^{n_i+i}\text{BG-rank}(\widehat{\pi}_i). \qquad (3.9)\end{aligned}$$

Formula (3.4) follows easily from (3.9). Let us now define $\vec{B}_t, \tilde{\vec{B}}_t \in \mathbb{Z}^t$ as

$$\vec{B}_t = \begin{cases} \sum_{i=0}^{\frac{t-1}{2}} \vec{e}_{2i}, & \text{if } t \equiv 1 \pmod 4 \\ \sum_{i=0}^{\frac{t-3}{2}} \vec{e}_{1+2i}, & \text{if } t \equiv -1 \pmod 4 \end{cases}$$

and

$$\tilde{\vec{B}}_t = \vec{B}_t + \sum_{i=0}^{t-1} \vec{e}_i = \begin{cases} \sum_{i=0}^{\frac{t-3}{2}} \vec{e}_{1+2i}, & \text{if } t \equiv 1 \pmod 4 \\ \sum_{i=0}^{\frac{t-1}{2}} \vec{e}_{2i}, & \text{if } t \equiv -1 \pmod 4 \end{cases}$$

Here $\vec{e}_i$'s are standard unit vectors in $\mathbb{Z}^t$ defined as $e_0 = (1, 0, \ldots, 0), \ldots, \vec{e}_{t-1} = (0, \ldots, 0, 1)$.

We conclude this section with the following important observation. If odd $t > 1$, $k = 0, 1, \ldots, \frac{t-1}{2}$ and $\vec{n} \in \mathbb{Z}^t$, $\vec{n} \cdot \vec{1}_t = 0$, then

$$bg(\vec{n}) = (-1)^{\frac{t-1}{2}} \left(\left\lfloor \frac{t}{4} \right\rfloor - k \right) \qquad (3.10)$$

iff $\vec{n} \equiv \vec{B}_t + \vec{e}_{i_0} + \vec{e}_{i_1} + \cdots + \vec{e}_{i_{2k}} \pmod 2$ for some $0 \le i_0 < i_1 < i_2 < \cdots < i_{2k} \le t-1$. In particular, if $\vec{n} \in \mathbb{Z}^t$, $\vec{n} \cdot \vec{1}_t = 0$, then

$$bg(\vec{n}) = (-1)^{\frac{t+1}{2}} \left\lfloor \frac{t+1}{4} \right\rfloor \qquad (3.11)$$

iff $\vec{n} \equiv \tilde{\vec{B}}_t \pmod 2$. We leave the proof as an exercise for the interested reader.

4. Combinatorial proof of $p_j(5n+4) \equiv 0 \pmod 5$

Throughout this section we assume that

$$|\pi| \equiv 4 \pmod 5$$

and

$$|\pi_{5\text{-core}}| \equiv 4 \pmod 5.$$

To prove (1.8) we shall require a few definitions. Following [5], we define the 5-core crank as

$$c_5(\pi) := 2(r_0(\pi, 5) - r_4(\pi, 5)) + (r_1(\pi, 5) - r_3(\pi, 5)) + 1 \pmod 5. \qquad (4.1)$$

Note that if $|\pi_{5\text{-core}}| \equiv 4 \pmod 5$, then obviously

$$n_0 + n_1 + n_2 + n_3 + n_4 = 0, \qquad (4.2)$$

$$n_1 + 2n_2 + 3n_3 + 4n_4 \equiv 4 \pmod 5. \qquad (4.3)$$

Here, $\vec{n} = (n_0, n_1, n_2, n_3, n_4) = \phi_2(\pi_5\text{-core})$. Let's introduce a new vector $\vec{\alpha}(\vec{n}) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$, defined as

$$\alpha_0 = \frac{n_0 - 3n_1 - 2n_2 - n_3 + 1}{5}, \tag{4.4}$$

$$\alpha_1 = \frac{-3n_0 - n_1 - 4n_2 - 2n_3 + 2}{5}, \tag{4.5}$$

$$\alpha_2 = \frac{-3n_0 - n_1 + n_2 - 2n_3 + 2}{5}, \tag{4.6}$$

$$\alpha_3 = \frac{n_0 + 2n_1 + 3n_2 + 4n_3 + 1}{5}, \tag{4.7}$$

$$\alpha_4 = \frac{4n_0 + 3n_1 + 2n_2 + n_3 - 1}{5}. \tag{4.8}$$

Using (4.2, 4.3) it is easy to verify that $\vec{\alpha}(\vec{n}) \in \mathbb{Z}^5$ and that

$$(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 1. \tag{4.9}$$

Inverting (4.4–4.8) we find that

$$n_0 = \alpha_0 + \alpha_4, \tag{4.10}$$

$$n_1 = -\alpha_0 + \alpha_1 + \alpha_4, \tag{4.11}$$

$$n_2 = -\alpha_1 + \alpha_2, \tag{4.12}$$

$$n_3 = -\alpha_2 + \alpha_3 - \alpha_4, \tag{4.13}$$

$$n_4 = -\alpha_3 - \alpha_4, \tag{4.14}$$

Note that in terms of these new variables we have

$$c_5(\pi) \equiv \sum_{i=0}^{4} i\alpha_i \pmod 5, \tag{4.15}$$

$$|\pi| = 5Q(\vec{\alpha}) - 1 + 5\sum_{i=0}^{4} |\widehat{\pi}_i|, \tag{4.16}$$

and

$$\begin{aligned} \text{BG-rank}(\pi) = {} & \frac{1 - (-1)^{\alpha_0+\alpha_1} - (-1)^{\alpha_1+\alpha_2} - \cdots - (-1)^{\alpha_4+\alpha_0}}{4} \\ & + (-1)^{\alpha_0+\alpha_4}\text{BG-rank}(\widehat{\pi}_0) \\ & + (-1)^{\alpha_2+\alpha_3}\text{BG-rank}(\widehat{\pi}_1) \\ & + (-1)^{\alpha_1+\alpha_2}\text{BG-rank}(\widehat{\pi}_2) \\ & + (-1)^{\alpha_0+\alpha_1}\text{BG-rank}(\widehat{\pi}_3) \\ & + (-1)^{\alpha_3+\alpha_4}\text{BG-rank}(\widehat{\pi}_4). \end{aligned} \tag{4.17}$$

Here $\phi_1(\pi) = (\pi_5\text{-core}, \widehat{\pi}_0, \ldots, \widehat{\pi}_4)$ and $Q(\vec{\alpha}) := \vec{\alpha} \cdot \vec{\alpha} - (\alpha_0\alpha_1 + \alpha_1\alpha_2 + \cdots + \alpha_4\alpha_0)$. It is convenient to combine $\phi_1, \phi_2, \vec{\alpha}$ into a new invertible function Φ, defined as

$$\Phi(\pi) = (\vec{\alpha}(\phi_2(\pi_5\text{-core})), \vec{\widehat{\pi}}),$$

where $\vec{\widehat{\pi}} := (\widehat{\pi}_0, \ldots, \widehat{\pi}_4)$. Following [2] we define

$$\widehat{C}_1(\vec{\alpha}) = (\alpha_4, \alpha_0, \alpha_1, \alpha_2, \alpha_3),$$
$$\widehat{C}_2(\vec{\widehat{\pi}}) = (\widehat{\pi}_4, \widehat{\pi}_2, \widehat{\pi}_3, \widehat{\pi}_0, \widehat{\pi}_1),$$
$$\widehat{O}(\pi) = \Phi^{-1}(\widehat{C}_1(\vec{\alpha}), \widehat{C}_2(\vec{\widehat{\pi}})).$$

We observe that operator $\widehat{O}$ has the following properties

$$|\widehat{O}(\pi)| = |\pi|,$$
$$\widehat{O}^5(\pi) = \pi,$$
$$\text{BG-rank}\left(\widehat{O}(\pi)\right) = \text{BG-rank}(\pi),$$
$$c_5\left(\widehat{O}(\pi)\right) \equiv 1 + c_5(\pi) \pmod 5. \tag{4.18}$$

Clearly, $\widehat{O}$ preserves the norm and the BG-rank of the partition. And so we can assemble all partitions of $5n+4$ with BG-rank $= j$ into disjoint orbits:

$$\pi, \quad \widehat{O}(\pi), \quad \widehat{O}^2(\pi), \quad \widehat{O}^3(\pi), \quad \widehat{O}^4(\pi).$$

Here, π is some partition of $5n+4$ with BG-rank $= j$. Formula (4.18) suggests that all five members of the same orbit are distinct. Clearly,

$$p_j(5n+4) = 5 \cdot (\text{number of orbits}).$$

Hence, $p_j(5n+4) \equiv 0 \pmod 5$, as desired. In fact, we have the following

Theorem 4.1. *Let j be any fixed integer. The residue of the 5-core crank mod 5 divides the partitions enumerated by $p_j(5n+4)$ into five equal classes.*

We note that this theorem generalizes Theorem 4.1 [2, p.717].

5. Identities for odd t-cores with extreme BG-rank values

The main object of this section is to provide a proof of formulas (1.10) and (1.11). Thoughout this section t is presumed to be a positive odd integer. We will prove (1.11) first. To this end we employ the observation (3.10) together with (2.3) to rewrite it as

$$\sum_{\substack{\vec{n}\in\mathbb{Z}^t, \vec{n}\cdot\vec{1}_t=0 \\ \vec{n}\equiv\vec{\tilde{B}}_t \pmod 2}} q^{\tilde{Q}(\vec{n})} = q^{\frac{t^2-1}{8}} \frac{E^t(q^{4t})}{E(q^4)}, \tag{5.1}$$

where

$$\tilde{Q}(\vec{n}) := \frac{t}{2}\vec{n}\cdot\vec{n} + \vec{b}_t \cdot \vec{n}. \tag{5.2}$$

Next we introduce new summation variables $\vec{\tilde{n}} = (\tilde{n}_0, \ldots, \tilde{n}_{t-1}) \in \mathbb{Z}^t$ as follows

$$\vec{n} = 2\vec{\tilde{n}} + \sum_{i=0}^{\lfloor \frac{t-3}{4} \rfloor} \left(\vec{e}_{\frac{t-3}{2}-2i} - \vec{e}_{\frac{t+1}{2}+2i}\right). \tag{5.3}$$

Obviously, $\vec{\tilde{n}}$ is subject to the constraint

$$\vec{\tilde{n}} \cdot \vec{1}_t = 0. \tag{5.4}$$

Note that in terms of new variables we have

$$\tilde{Q}(\vec{n}) = \tilde{Q}(\vec{n}) + (t-1)\vec{1}_t \cdot \tilde{\vec{n}} = \frac{t^2-1}{8} + 4\Big\{\frac{t}{2}\tilde{\vec{n}}\cdot\tilde{\vec{n}} + \sigma_1 + \sigma_2 + \sigma_3\Big\}, \tag{5.5}$$

where

$$\sigma_1 = \sum_{i=0}^{\lfloor \frac{t-3}{4} \rfloor} (t-1-i)\tilde{n}_{\frac{t-3}{2}-2i},$$

$$\sigma_2 = \sum_{i=0}^{\lfloor \frac{t-3}{4} \rfloor} i\tilde{n}_{2i+\frac{t+1}{2}},$$

$$\sigma_3 = \sum_{i=-\lfloor \frac{t-1}{4} \rfloor}^{\lfloor \frac{t-1}{4} \rfloor} (\frac{t-1}{2}+i)\tilde{n}_{\frac{t-1}{2}+2i}.$$

At this point it is natural to perform further changes:

$$\tilde{n}_{\frac{t-3}{2}-2i} \to \tilde{n}_{t-1-i}, \qquad 0 \le i \le \left\lfloor \frac{t-3}{4} \right\rfloor$$

$$\tilde{n}_{\frac{t+1}{2}+2i} \to \tilde{n}_{i}, \qquad 0 \le i \le \left\lfloor \frac{t-3}{4} \right\rfloor$$

$$\tilde{n}_{\frac{t-1}{2}+2i} \to \tilde{n}_{\frac{t-1}{2}+i}, \qquad -\left\lfloor \frac{t-1}{4} \right\rfloor \le i \le \left\lfloor \frac{t-1}{4} \right\rfloor.$$

This way we obtain

$$\tilde{Q}(\vec{n}) = \frac{t^2-1}{8} + 4\tilde{Q}(\tilde{\vec{n}}),$$
$$\tilde{\vec{n}} \in \mathbb{Z}^t, \quad \tilde{\vec{n}}\cdot\vec{1}_t = 0.$$

And so with the aid of the Klyachko identity (1.13) we find that

$$C_{t,(-1)^{\frac{t+1}{4}}\lfloor\frac{t+1}{4}\rfloor}(q) = \sum_{\substack{\tilde{\vec{n}}\in\mathbb{Z}^t \\ \tilde{\vec{n}}\cdot\vec{1}_t=0}} q^{\frac{t^2-1}{8}+4\tilde{Q}(\tilde{\vec{n}})} = q^{\frac{t^2-1}{8}}\frac{E^t(q^{4t})}{E(q^4)}, \tag{5.6}$$

as desired. To prove (1.10) we shall require the following lemma.

Lemma 5.1. *For a positive odd t*

$$\psi^2(q^2) = q^{\frac{t-1}{2}}\psi^2(q^{2t}) + \frac{E^3(q^{4t})}{f(-q^t,-q^{3t})}\sum_{i=0}^{\frac{t-3}{2}} q^i \frac{f(q^{t-1-2i},-q^{1+2i})}{f(-q^{4i+2},-q^{4t-2-4i})} \tag{5.7}$$

holds.

In the above we employed the Ramanujan notations

$$\psi(q) := \frac{E^2(q^2)}{E(q)} = \sum_{n\ge 0} q^{\binom{n+1}{2}}, \tag{5.8}$$

$$f(a,b) := (ab,-a,-b;ab)_\infty. \tag{5.9}$$

Using (2.6) we can easily show that

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}. \tag{5.10}$$

Setting $a = q^{t-1-2i}, b = -q^{1+2i}, 0 \le i \le \frac{t-3}{2}$ in (5.10) and dissecting we obtain

$$\begin{aligned} f(q^{t-1-2i}, -q^{1+2i}) &= f(-q^{2+t+4i}, -q^{3t-2-4i}) \\ &+ q^{t-1-2i} f(-q^{2-t+4i}, -q^{5t-2-4i}). \end{aligned} \tag{5.11}$$

To prove the above lemma we start with the Ramanujan ${}_1\psi_1$-summation formula [6, II.29]

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az, \frac{q}{az}, q, \frac{b}{a}; q)_\infty}{(z, \frac{b}{az}, b, \frac{q}{a}; q)_\infty}, \qquad |\frac{b}{a}| < |z| < 1. \tag{5.12}$$

We set $b = aq$ to obtain

$$\sum_{n=-\infty}^{\infty} \frac{z^n}{1 - aq^n} = \frac{(az, \frac{q}{az}, q, q; q)_\infty}{(z, \frac{q}{z}, a, \frac{q}{a}; q)_\infty} = \frac{E^3(q) f(-az, -\frac{q}{az})}{f(-z, -\frac{q}{z}) f(-a, -\frac{q}{a})}, \quad |q| < |z| < 1. \tag{5.13}$$

If we replace $q \to q^4, z = q, a = q^2$ in (5.13) we find that

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^{2+4n}} = \psi^2(q^2). \tag{5.14}$$

Next we split the sum on the left of (5.14) as

$$\psi^2(q^2) = \sum_{\substack{i=0 \\ i \ne \frac{t-1}{2}}}^{t-1} \sum_{m_i=-\infty}^{\infty} q^i \frac{q^{tm_i}}{1 - q^{2+4i} q^{4tm_i}} + \sum_{m=-\infty}^{\infty} q^{\frac{t-1}{2}} \frac{q^{tm}}{1 - q^{2t} q^{4tm}}. \tag{5.15}$$

Using (5.14) with $q \to q^t$ it is easy to recognize the last sum in (5.15) as $q^{\frac{t-1}{2}} \psi^2(q^{2t})$. And so we have

$$\psi^2(q^2) = q^{\frac{t-1}{2}} \psi^2(q^{2t}) + \frac{E^3(q^{4t})}{f(-q^t, -q^{3t})} \sum_{\substack{i=0 \\ i \ne \frac{t-1}{2}}}^{t-1} q^i \frac{f(-q^{2+4i+t}, -q^{3t-2-4i})}{f(-q^{2+4i}, -q^{4t-2-4i})}, \tag{5.16}$$

where we have made a multiple use of (5.13). Finally, folding the last sum in half and using (5.11) we arrive at

$$\begin{aligned} \psi^2(q^2) &= q^{\frac{t-1}{2}} \psi^2(q^{2t}) + \sum_{i=0}^{\frac{t-3}{2}} \frac{E^3(q^{4t}) q^i}{f(-q^t, -q^{3t}) f(-q^{2+4i}, -q^{4t-2-2i})} \\ &\quad \times \left\{ f\left(-q^{2+4i+t}, -q^{3t-2-4i}\right) + q^{t-1-2i} f\left(-q^{5t-2-4i}, -q^{2-t+4i}\right) \right\} \\ &= q^{\frac{t-1}{2}} \psi^2(q^{2t}) + \frac{E^3(q^{4t})}{f(-q^t, -q^{3t})} \sum_{i=0}^{\frac{t-3}{2}} q^i \frac{f(q^{t-1-2i}, -q^{1+2i})}{f(-q^{2+4i}, -q^{4t-2-4i})}. \end{aligned} \tag{5.17}$$

This concludes the proof of Lemma 5.1.

We now move on to prove (1.10). Again, using the observation (3.10), we can rewrite it as

$$\sum_{j=0}^{t-1} \sum_{\substack{\vec{n} \in \mathbb{Z}^t, \vec{n} \cdot \vec{1}_t = 0 \\ \vec{n} \equiv \vec{B}_t + \vec{e}_j \pmod 2}} q^{\tilde{Q}(\vec{n})} = q^{\frac{(t-1)(t-3)}{8}} F(t, q^2). \tag{5.18}$$

Remarkably, (5.18) is just the constant term in z of the following more general identity

$$\sum_{j=0}^{t-1} \sum_{\substack{\vec{n}\in\mathbb{Z}^t \\ \vec{n}\equiv\vec{B}_t+\vec{e}_j \pmod 2}} q^{\tilde{Q}(\vec{n})} z^{\frac{\vec{n}\cdot\vec{1}_t}{2}}$$

$$= q^{\frac{(t-1)(t-3)}{8}} F(t,q^2) \sum_{n=-\infty}^{\infty} q^{2n^2+(t-1)n} z^n. \tag{5.19}$$

To prove (5.19) we observe that its right hand side satisfies the *first* order functional equation

$$\widehat{D}_{t,q}(f(z)) = f(z), \tag{5.20}$$

where

$$\widehat{D}_{t,q}(f(z)) := zq^{t+1} f(zq^4).$$

After a bit of labor one can verify that for $0 \le i \le t-1$

$$\widehat{D}_{t,q}\left(\sum_{\substack{\vec{n}\in\mathbb{Z}^t \\ \vec{n}\equiv\vec{B}_t+\vec{e}_i \pmod 2}} q^{\tilde{Q}(\vec{n})} z^{\frac{\vec{n}\cdot\vec{1}_t}{2}} \right) = \sum_{\substack{\vec{n}\in\mathbb{Z}^t \\ \vec{n}\equiv\vec{B}_t+\vec{e}_{i+2} \pmod 2}} q^{\tilde{Q}(\vec{n})} z^{\frac{\vec{n}\cdot\vec{1}_t}{2}}, \tag{5.21}$$

where $\vec{e}_t := \vec{e}_0$ and $\vec{e}_{t+1} := \vec{e}_1$. Clearly, (5.21) implies that the left hand side of (5.19) satisfies (5.20), as well. It remains to verify (5.19) at one nontrivial point. To this end we set

$$z = \begin{cases} 1, & \text{if } t \equiv -1 \pmod 4, \\ q^2, & \text{if } t \equiv 1 \pmod 4) \end{cases}$$

in (5.19), and then replace $q^2 \to q$ to get with the help of (2.6)

$$q^{\frac{t-1}{2}} \psi(q^{2t}) \prod_{j=0}^{\frac{t-3}{2}} f^2\left(q^{1+2j}, q^{2t-1-2j}\right)$$

$$\times \left\{ 1 + \sum_{i=1}^{\frac{t-1}{2}} q^{-i} \frac{f(q^t,q^t) f(q^{2i},q^{2t-2i})}{\psi(q^{2t}) f(q^{t+2i},q^{t-2i})} \right\} = \psi(q^2) F(t,q). \tag{5.22}$$

To proceed further we need to verify two product identities

$$\psi(q^2) \prod_{j=0}^{\frac{t-3}{2}} f^2\left(q^{1+2j}, q^{2t-1-2j}\right) = \psi(q^{2t}) F(t,q)$$

and

$$\psi(q^{2t}) \frac{f(q^t,q^t) f(q^{2i},q^{2t-2i})}{f(q^{t+2i},q^{t-2i})} = E^3(q^{4t}) \frac{f(q^{2i},-q^{t-2i})}{f(-q^t,-q^{3t}) f(-q^{2t+4i},-q^{2t-4i})}, \quad i \in \mathbb{N}.$$

Next, we multiply both sides of (5.22) by $\frac{\psi(q^2)}{F(t,q)}$ and simplify to arrive at

$$q^{\frac{t-1}{2}} \psi^2(q^{2t}) + \frac{E^3(q^{4t})}{f(-q^t,-q^{3t})} \sum_{i=1}^{\frac{t-1}{2}} q^{\frac{t-1}{2}-i} \frac{f(q^{2i},-q^{t-2i})}{f(-q^{2t+4i},-q^{2t-4i})} = \psi^2(q^2), \tag{5.23}$$

which is essentially the identity in Lemma 5.1. This concludes our proof of (5.19). It follows that (5.18), (1.10) hold true.

6. 5-cores with prescribed BG-rank

Formula (1.9) suggests that BG-rank($\pi_{5\text{-core}}$) can assume just three values: $0, \pm 1$. This means that

$$a_5(n) = a_{5,-1}(n) + a_{5,0}(n) + a_{5,1}(n). \tag{6.1}$$

The generating function of version (6.1) is

$$\frac{E^5(q^5)}{E(q)} = C_{5,-1}(q) + C_{5,0}(q) + C_{5,1}(q). \tag{6.2}$$

In the last section we proved (1.10), (1.11). These identities with $t = 5$ state that

$$C_{5,-1}(q) = q^3 \frac{E^5(q^{20})}{E(q^4)}, \tag{6.3}$$

$$C_{5,1}(q) = qF(5, q^2). \tag{6.4}$$

By (1.3) we observe that $C_{t,j}(q)$ is either an odd or an even functions of q with parity determined by the parity of j. Therefore, $C_{5,0}(q)$ is an even function of q, and $C_{5,\pm1}(q)$ are odd functions of q. Consequently, we see that

$$\mathrm{ep}\left(\frac{E^5(q^5)}{E(q)}\right) = C_{5,0}(q) \tag{6.5}$$

where

$$\mathrm{ep}(f(x)) := \frac{f(x) + f(-x)}{2}.$$

In this section we will show that $C_{5,0}(q)$ can be expressed as a sum of two infinite products

$$C_{5,0}(q) = R(q^2), \tag{6.6}$$

where

$$R(q) := \frac{E^4(q^{10})E(q^5)E^2(q^4)}{E^2(q^{20})E(q)} + q\frac{E^2(q^{20})E^3(q^5)E^6(q^2)}{E^2(q^{10})E^2(q^4)E^3(q)}. \tag{6.7}$$

It is easy to rewrite (6.7) in a manifestly positive way as

$$R(q) = f(q, q^4)f(q^2, q^3)\{\varphi(q^5)\psi(q^2) + q\varphi(q)\psi(q^{10})\},$$

where

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{E^5(q^2)}{E^2(q^4)E^2(q)},$$

and $\psi(q)$ is defined in (5.8). Formula (6.6) enabled us to discover and prove the new Lambert series identity

$$R(q) = \sum_{i=0}^{1} \sum_{n=-\infty}^{\infty} (-1)^i q^{5n+i} \frac{1 + q^{1+2i+10n}}{(1 - q^{1+2i+10n})^2}. \tag{6.8}$$

In what follows we will require three identities:

$$\left[ux, \frac{u}{x}, vy, \frac{v}{y}; q\right]_\infty = \left[uy, \frac{u}{y}, vx, \frac{v}{x}; q\right]_\infty + \frac{v}{x}\left[xy, \frac{x}{y}, uv, \frac{u}{v}; q\right]_\infty, \tag{6.9}$$

([6, ex. 5.21])

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af\left(\frac{b}{c},\frac{c}{b}(abcd)\right)f\left(\frac{b}{d},\frac{d}{b}(abcd)\right), \tag{6.10}$$

provided $ab = cd$ ([1]) and

$$\frac{E^5(q^5)}{E(q)} = \sum_{i=1}^{2}\sum_{n=-\infty}^{\infty}(-1)^{i+1}\frac{q^{5n+i-1}}{(1-q^{5n+i})^2}, \tag{6.11}$$

([6, ex. 5.7], [5, p.8]). Here

$$[a;q]_\infty = \left(a,\frac{q}{a};q\right)_\infty,$$

$$[a_1,a_2,\ldots,a_n;q]_\infty = \prod_{i=1}^{n}[a_i;q]_\infty.$$

Next, we wish to establish the validity of

$$F(5,q) = \frac{E(q^{10})E^2(q^5)E^3(q^2)}{E^2(q)} = \frac{E^5(q^5)}{E(q)} + q\frac{E^5(q^{10})}{E(q^2)}. \tag{6.12}$$

To this end we multiply both sides of (6.12) by

$$\frac{\left[q,q^3;q^{10}\right]_\infty^2\left[q^2,q^4;q^{10}\right]_\infty}{E^4(q^{10})}$$

to obtain after simplification that

$$[q^2,q^2,q^4,q^6;q^{10}]_\infty = [q,q^3,q^5,q^5;q^{10}]_\infty + q[q,q,q^3,q^3;q^{10}]_\infty. \tag{6.13}$$

But the last equation is nothing else but (6.9) with q replaced by q^{10} and $u = q^2, v = q^5, x = 1, y = q$. We now combine

$$\text{ep}\left(q\frac{E^5(q^5)}{E(q)}\right) = qC_{5,-1}(q) + qC_{5,1}(q),$$

with (6.3), (6.5), and (6.12) to obtain

$$\text{ep}\left(q\frac{E^5(q^5)}{E(q)}\right) = 2q^4\frac{E^5(q^{20})}{E(q^4)} + q^2\frac{E^5(q^{10})}{E(q^2)}. \tag{6.14}$$

This can be stated as the following eigenvalue problem

$$T_2\left(q\frac{E^5(q^5)}{E(q)}\right) = q\frac{E^5(q^5)}{E(q)}, \tag{6.15}$$

where for prime p the Hecke operator T_p is defined by its action as

$$T_p\left(\sum_{n\geq 0}a_nq^n\right) = \sum_{n\geq 0}a_{pn}q^n + p\left(\frac{p}{5}\right)\sum_{n\geq 0}a_nq^{pn},$$

with $\left(\frac{a}{b}\right)$ being the Legendre symbol. We remark that (6.15) is the $p = 2$ case of the more general formula

$$T_p\left(q\frac{E^5(q^5)}{E(q)}\right) = \left(p + \left(\frac{p}{5}\right)\right)\left(q\frac{E^5(q^5)}{E(q)}\right), \tag{6.16}$$

which can be deduced from (6.11). We shall not supply the details. Instead, we note that (6.16) together with (6.3, 6.4, 6.5) implies that

$$T_{\tilde{p}}(qC_{5,j}(q)) = \left(\tilde{p} + \left(\frac{\tilde{p}}{5}\right)\right)(qC_{5,j}(q)), \quad j = 0 \pm 1. \tag{6.17}$$

Here, $\tilde{p}$ is an odd prime.

To prove (6.6) we use (6.12) to deduce that

$$\mathrm{ep}\left(\frac{E^5(q^5)}{E(q)}\right) = \mathrm{ep}(F(5,q)) = E(q^{10})E^3(q^2) \cdot \mathrm{ep}\left(\frac{E^2(q^5)}{E^2(q)}\right). \tag{6.18}$$

To proceed further we employ (6.10) with $a = q, b = q^9, c = q^3, d = q^7$ to get

$$\begin{aligned}\frac{E(q^5)}{E(q)} &= \frac{E(q^4)}{E(q^{20})E^2(q^2)} f(q,q^9)f(q^3,q^7) \\ &= \frac{E(q^4)}{E(q^{20})E^2(q^2)}\{f(q^4,q^{16})f(q^8,q^{12}) + qf(q^6,q^{14})f(q^2,q^{18})\} \\ &= \frac{E^2(q^{20})E(q^8)}{E(q^{40})E^2(q^2)} + q\frac{E(q^{40})E(q^{10})E^3(q^4)}{E(q^{20})E(q^8)E^3(q^2)}.\end{aligned} \tag{6.19}$$

It is clear that

$$\mathrm{ep}\left(\frac{E^2(q^5)}{E^2(q)}\right) = \frac{E^4(q^{20})E^2(q^8)}{E^2(q^{40})E^4(q^2)} + q^2\frac{E^2(q^{40})E^2(q^{10})E^6(q^4)}{E^2(q^{20})E^2(q^8)E^6(q^2)}. \tag{6.20}$$

Combining (6.18) and (6.20) we find that

$$\mathrm{ep}\left(\frac{E^5(q^5)}{E(q)}\right) = R(q^2). \tag{6.21}$$

The last formula together with (6.5) implies (6.6). Next, we rewrite (6.11) as

$$\frac{E^5(q^5)}{E(q)} = \sum_{i=1}^{2}\sum_{n=-\infty}^{\infty}(-1)^{i+1}\frac{q^{5n+i-1}(1+2q^{5n+i}+q^{10n+2i})}{(1-q^{10n+2i})^2}.$$

Clearly,

$$\begin{aligned}\mathrm{ep}\left(\frac{E^5(q^5)}{E(q)}\right) &= \sum_{i=1}^{2}\sum_{\substack{n=-\infty \\ n\equiv i-1 \pmod 2}}^{\infty}(-1)^{i+1}\frac{q^{5n+i-1}(1+q^{10n+2i})}{(1-q^{10n+2i})^2} \\ &= \sum_{i=0}^{1}\sum_{n=-\infty}^{\infty}(-1)^{i}\frac{q^{10n+i}(1+q^{20n+4i+2})}{(1-q^{20n+4i+2})^2}.\end{aligned} \tag{6.22}$$

Formula (6.8) with $q \to q^2$ follows easily from (6.21) and (6.22). Before we move on we wish to summarize some of the above observations in the formula below

$$\begin{aligned}\frac{E^5(q^5)}{E(q)} &= \Big\{\frac{E^4(q^{20})E(q^{10})E^2(q^8)}{E^2(q^{40})E(q^2)} + q^2\frac{E^2(q^{40})E^3(q^{10})E^6(q^4)}{E^2(q^{20})E^2(q^8)E^3(q^2)}\Big\} \\ &\quad + q\Big\{\frac{E^5(q^{10})}{E(q^2)} + 2q^2\frac{E^5(q^{20})}{E(q^4)}\Big\}.\end{aligned} \tag{6.23}$$

In [5], the authors used (6.11) to find explicit formulas for the coefficients

$$a_5(n) = \frac{2^{d+1}+(-1)^d}{3}\cdot 5^c \cdot \prod_{i=1}^{s}\frac{p_i^{a_i+1}-1}{p_i-1}\prod_{j=1}^{t}\frac{q_j^{b_j+1}+(-1)^{b_j}}{q_j+1}. \tag{6.24}$$

Here

$$n+1 = 2^d 5^c \prod_{i=1}^{s} p_i^{a_i} \prod_{j=1}^{t} q_j^{b_j} \tag{6.25}$$

is the prime factorization of $n+1$ and $p_i \equiv \pm 1 \pmod 5, 1 \le i \le s$ and $q_j \equiv \pm 2 \pmod 5, 1 \le j \le t$ are odd primes. Formulas (6.3), (6.4), (6.5) and (6.12) suggest the following relations. For $n \in \mathbb{N}$ and $r = 0, 1, 2, 3$ one has

$$a_{5,0}(n) = \begin{cases} a_5(n), & \text{if } n \equiv 0 \pmod 2, \\ 0, & \text{otherwise,} \end{cases} \tag{6.26}$$

$$a_{5,-1}(4n+r) = \begin{cases} a_5(n), & \text{if } r = 3, \\ 0, & \text{otherwise,} \end{cases} \tag{6.27}$$

$$a_{5,1}(4n+r) = \begin{cases} a_5(2n), & \text{if } r = 1, \\ a_5(n) + a_5(2n+1), & \text{if } r = 3, \\ 0, & \text{if } r = 0, 2. \end{cases} \tag{6.28}$$

These relations together with (6.24) enabled us to derive explicit formulas for $a_{5,j}(n)$ with $-1 \le j \le 1$. In particular, if the prime factorization of $n+1$ is given by (6.25), then

$$a_{5,1}(4n+3) = 2^{d+1} 5^c \prod_{i=1}^{s} \frac{p_i^{a_i+1} - 1}{p_i - 1} \prod_{j=1}^{t} \frac{q_j^{b_j+1} + (-1)^{b_j}}{q_j + 1}. \tag{6.29}$$

We would like to conclude this section with the following discussion. It is easy to check that (6.17) implies that

$$a_{5,j}(pn+p-1) + p\left(\frac{p}{5}\right) a_{5,j}\left(\frac{n+1}{p} - 1\right) = \left(p + \left(\frac{p}{5}\right)\right) a_{5,j}(n), \quad j = 0, \pm 1, \tag{6.30}$$

where p is odd prime, $n \in \mathbb{N}$ and $a_{5,j}(x) = 0$ if $x \notin \mathbb{Z}$. Setting $p = 5$ we find that

$$a_{5,j}(5n+4) = 5 a_{5,j}(n), \quad j = 0, \pm 1. \tag{6.31}$$

This is a refinement of the well-known result

$$a_5(5n+4) = 5 a_5(n), \tag{6.32}$$

proven in [5]. We can prove (6.31) by adapting the combinatorial proof in [5].

Let's define

$$\vec{n} = (n_0, n_1, n_2, n_3, n_4) = \phi_2(\pi_5\text{-core})$$

for some π_5-core with BG-rank(π_5-core) $= j$ and $|\pi_5\text{-core}| = n$. Consider map $\vec{n} \to \vec{\tilde{n}} = (\tilde{n}_0, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4)$ with

$$\begin{aligned} \tilde{n}_0 &= n_1 + 2n_2 + 2n_4 + 1, \\ \tilde{n}_1 &= -n_1 - n_2 + n_3 + n_4 + 1, \\ \tilde{n}_2 &= 2n_1 + n_2 + 2n_3, \\ \tilde{n}_3 &= -2n_2 - 2n_3 - n_4 - 1, \\ \tilde{n}_4 &= -2n_1 - n_3 - 2n_4 - 1. \end{aligned}$$

Obviously $\vec{\tilde{n}} \in \mathbb{Z}^5$ and $\vec{\tilde{n}} \cdot \vec{1}_5 = 0$ and so we can define $\tilde{\pi}_{5\text{-core}} = \phi_2^{-1}(\vec{\tilde{n}})$. It is easy to check that

$$|\tilde{\pi}_{5\text{-core}}| = 5n + 4,$$

and that

$$\text{BG-rank}(\tilde{\pi}_{5\text{-core}}) = j,$$

and

$$c_5(\tilde{\pi}_{5\text{-core}}) \equiv 4 \pmod 5.$$

Recall that the orbit $\{\tilde{\pi}_{5\text{-core}}, \widehat{O}(\tilde{\pi}_{5\text{-core}}), \ldots, \widehat{O}^4(\tilde{\pi}_{5\text{-core}})\}$ contains just one member with $c_5 \equiv 4 \pmod 5$. And so each 5-core of n with BG-rank j is in $1-1$ correspondence with an appropriate 5-member orbit of t-cores of $5n+4$ with BG-rank j. This observation yields a combinatorial proof of (6.31).

7. Outlook

Given our combinatorial proof of

$$p_j(5n+4) \equiv 0 \pmod 5, \quad j \in \mathbb{Z}$$

one may wonder about a combinatorial proof of the other mod 5 congruences (1.4-1.7). We strongly suspect that such proof will be dramatically different from the one discussed in Section 4. In addition, one would like to have combinatorial insights into (6.30) for $p \neq 5$.

In this paper we found "positive" *eta*-quotient representations for $C_{5,j}(q), -1 \leq j \leq 1$. In the general case (odd t, $-\lfloor \frac{t-1}{4} \rfloor \leq j \leq \lfloor \frac{t+1}{4} \rfloor$), we established such representation only for $C_{t,\pm\lfloor \frac{t\pm1}{4} \rfloor}(q)$. Clearly, one wants to find "positive" *eta*-quotient representations for other admissible values of BG-rank. (See [3] for a fascinating discussion of the $t = 7$ case).

Finally, we observe that (1.2) is the $s = 2$ case of the following more general definition

$$\text{gbg-rank}(\pi, s) = \sum_{j=0}^{s-1} r_j(\pi, s)\omega_s^j$$

with

$$\omega_s = e^{i\frac{2\pi}{s}}.$$

Many identities, proven here, can be generalized further. For example, we can prove that if $(s,t) = 1$ then

$$\text{gbg-rank}(\pi_{t\text{-core}}, s) = \frac{\sum_{i=0}^{t-1} \omega_s^{i+1}(\omega_s^{tn_i} - 1)}{(1-\omega_s^t)(1-\omega_s)} \tag{7.1}$$

and for $1 \leq i \leq s-1$ that

$$\sum_{\text{gbg-rank}(\pi_{t\text{-core}}, s) = g(i)} q^{|\pi_{t\text{-core}}|} = q^{a(i)} F_i(q^s). \tag{7.2}$$

Here,

$$(n_0, n_1, \ldots, n_{t-1}) = \phi_2(\pi_{t\text{-core}}),$$

$$a(i) = \frac{(t^2-1)(s^2-1)}{24} - \frac{(t-1)(s-i)i}{2},$$

$$g(i) = \frac{1}{(1-\omega_s)(1-\frac{1}{\omega_s})} - \omega_s^{\frac{t-1}{2}} \frac{1 + \frac{t-1}{\omega_s^i}}{(1-\omega_s^t)(1-\frac{1}{\omega_s})},$$

$$F_i(q) = E(q^s)E(q^{st})^{t-2}\frac{[q^{it};q^{st}]_\infty}{[q^i;q^s]_\infty}.$$

Setting $s = 2$ in (7.1), (7.2) we obtain (3.2), (1.10), respectively.

In addition we can show that

$$\sum_{\text{gbg-rank}(\pi_t\text{-core},s)=g(0)} q^{|\pi_t\text{-core}|} = q^{a(0)}\frac{E(q^{s^2t})^t}{E(q^{s^2})}. \tag{7.3}$$

Setting $s = 2$ in (7.3) we get (1.11).

In [10] Olsson and Stanton defined so-called (s,t)-good partitions. Surprisingly, t-cores with gbg-rank $= g(0)$ coincide with (t,s)-good partitions.

Let $\nu(t,s)$ denote a number of distinct values that gbg-rank$(\pi_t\text{-core}, s)$ may assume. Then it can be shown that

$$\nu(s,t) \le \frac{\binom{t+s}{t}}{t+s},$$

provided that $(s,t) = 1$. Morever, if s is prime or if s is a composite number and $t < 2p$ then

$$\nu(s,t) = \frac{\binom{t+s}{t}}{t+s}.$$

Here, p is a smallest prime divisor of s and $(s,t) = 1$.

Details of these and related results will be left to a later paper.

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