18 SYMMETRY OF POLYTOPES AND POLYHEDRA

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INTRODUCTION

Symmetry of geometric figures is among the most frequently recurring themes in science. The present chapter discusses symmetry of discrete geometric structures, namely of polytopes, polyhedra, and related polytope-like figures. These structures have an outstanding history of study unmatched by almost any other geometric object. The most prominent symmetric figures, the regular solids, occur from very early times and are attributed to Plato (427-347 B.C.E.). Since then, many changes in point of view have occurred about these figures and their symmetry. With the arrival of group theory in the 19th century, many of the early approaches were consolidated and the foundations were laid for a more rigorous development of the theory. In this vein, Schläfli (1814-1895) extended the concept of regular polytopes and tessellations to higher dimensional spaces and explored their symmetry groups as reflection groups.

Today we owe much of our present understanding of symmetry in geometric figures (in a broad sense) to the influential work of Coxeter, which provided a unified approach to regularity of figures based on a powerful interplay of geometry and algebra [Cox73]. Coxeter's work also greatly influenced modern developments in this area, which received a further impetus from work by Grünbaum and Danzer [Grü77a, DS82]. In the past 20 years, the study of regular figures has been extended in several directions that are all centered around an abstract combinatorial polytope theory and a combinatorial notion of regularity [McS02].

History teaches us that the subject has shown an enormous potential for revival. One explanation for this is the appearance of polyhedral structures in many contexts that have little apparent relation to regularity, such as the occurrence of many of them in nature as crystals [Fej64, Se95, We77].

18.1 REGULAR CONVEX POLYTOPES AND REGULAR TESSELLATIONS IN $\mathbb{E}^d$

Perhaps the most important (but certainly the most investigated) symmetric polytopes are the regular convex polytopes in Euclidean spaces. See [Grü67] and [Zi95] for general properties of convex polytopes, or Chapter 15 in this Handbook. The most comprehensive text on regular convex polytopes and regular tessellations is [Cox73]; many combinatorial aspects are also discussed in [McS02].

GLOSSARY

Convex d-polytope: The intersection P of finitely many closed halfspaces in a

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Euclidean space, which is bounded and d-dimensional.

Face: The empty set and P itself are ***improper faces*** of dimension -1 and d, respectively. A ***proper face*** F of P is the (nonempty) intersection of P with a supporting hyperplane of P. (Recall that a hyperplane H ***supports*** P at F if $P \cap H = F$ and P lies in one of the closed halfspaces bounded by H.)

Vertex, edge, i-face, facet: Face of P of dimension 0, 1, i, or $d-1$, respectively.

Vertex figure: A vertex figure of P at a vertex x is the intersection of P with a hyperplane H that strictly separates x from the other vertices of P. (If P is regular, one can take H to be the hyperplane passing through the midpoints of the edges that contain x.)

Face lattice of a polytope: The set $\mathcal{F}(P)$ of all faces of P, ordered by inclusion. As a partially ordered set, this is a ranked lattice. Also, $\mathcal{F}(P) \setminus \{P\}$ is called the ***boundary complex*** of P.

Flag: A maximal totally ordered subset of $\mathcal{F}(P)$.

Isomorphism of polytopes: A bijection $\varphi : \mathcal{F}(P) \mapsto \mathcal{F}(Q)$ between the face lattices of two polytopes P and Q such that φ preserves incidence in both directions; that is, $F \subseteq G$ in $\mathcal{F}(P)$ if and only if $F\varphi \subseteq G\varphi$ in $\mathcal{F}(Q)$. If such an isomorphism exists, P and Q are ***isomorphic***.

Dual of a polytope: A convex d-polytope Q is the dual of P if there is a ***duality*** $\varphi : \mathcal{F}(P) \mapsto \mathcal{F}(Q)$; that is, a bijection reversing incidences in both directions, meaning that $F \subseteq G$ in $\mathcal{F}(P)$ if and only if $F\varphi \supseteq G\varphi$ in $\mathcal{F}(Q)$. A polytope has many duals but any two are isomorphic, justifying speaking of "the dual". (If P is regular, one can take Q to be the convex hull of the facet centers of P, or a rescaled copy of this.)

Self-dual polytope: A polytope that is isomorphic to its dual.

Symmetry: A Euclidean isometry of the ambient space (affine hull of P) that maps P to itself.

Symmetry group of a polytope: The group $G(P)$ of all symmetries of P.

Regular polytope: A polytope whose symmetry group $G(P)$ is transitive on the flags.

Schläfli symbol: A symbol $\{p_1, \ldots, p_{d-1}\}$ that encodes the local structure of a regular polytope. For each $i = 1, \ldots, d-1$, if F is any $(i{+}1)$-face of P, then p_i is the number of i-faces of F that contain a given $(i{-}2)$-face of F.

Tessellation: A family T of convex d-polytopes in Euclidean d-space $\mathbb{E}^d$, called the ***tiles*** of T, such that the union of all tiles of T is $\mathbb{E}^d$, and any two distinct tiles do not have interior points in common. All tessellations are assumed to be ***locally finite***, meaning that each point of $\mathbb{E}^d$ has a neighborhood meeting only finitely many tiles, and ***face-to-face***, meaning that the intersection of any two tiles is a face of each (possibly the empty face); see Chapter 4. The concept of a tessellation extends to other spaces including spherical space (Euclidean unit sphere) and hyperbolic space.

Face lattice of a tessellation: A ***proper face*** of T is a nonempty face of a tile of T. ***Improper faces*** of T are the empty set and the whole space $\mathbb{E}^d$. The set $\mathcal{F}(T)$ of all (proper and improper) faces is a ranked lattice called the face lattice of T. Concepts like isomorphism and duality carry over from polytopes.

Symmetry group of a tessellation: The group $G(T)$ of all symmetries of T; that is, of all isometries of the ambient (spherical, Euclidean, or hyperbolic) space that preserve T. Concepts like regularity and Schläfli symbol carry over from polytopes.

Apeirogon: A tessellation of the real line with closed intervals of the same length. This can also be regarded as an infinite polygon whose edges are given by the intervals.

ENUMERATION AND CONSTRUCTION

The convex regular polytopes P in $\mathbb{E}^d$ are known for each d. If $d = 1$, P is a line segment and $|G(P)| = 2$. In all other cases, up to similarity, P can be uniquely described by its Schläfli symbol $\{p_1, \ldots, p_{d-1}\}$. For convenience one writes $P = \{p_1, \ldots, p_{d-1}\}$. If $d = 2$, P is a convex regular p-gon for some $p \geq 3$, and $P = \{p\}$; also, $G(P) = D_p$, the dihedral group of order $2p$.

The regular polytopes P with $d \geq 3$ are summarized in Table 18.1.1, which also includes the numbers f_0 and f_{d-1} of vertices and facets, the order of $G(P)$, and the diagram notation (Section 18.6) for the group (following [Hum90]). Here and below, p^n will be used to denote a string of n consecutive p's. For $d = 3$ the list consists of the five Platonic solids (Figure 18.1.1). The regular d-simplex, d-cube, and d-cross-polytope occur in each dimension d. (These are line segments if $d = 1$, and triangles or squares if $d = 2$.) The dimensions 3 and 4 are exceptional in that there are 2 respectively 3 more regular polytopes. If $d \geq 3$, the facets and vertex figures of $\{p_1, \ldots, p_{d-1}\}$ are the regular (d–1)-polytopes $\{p_1, \ldots, p_{d-2}\}$ and $\{p_2, \ldots, p_{d-1}\}$, respectively, whose Schläfli symbols, when superposed, give the original. The dual of $\{p_1, \ldots, p_{d-1}\}$ is $\{p_{d-1}, \ldots, p_1\}$. Self-duality occurs only for $\{3^{d-1}\}$, $\{p\}$, and $\{3, 4, 3\}$. Except for $\{3^{d-1}\}$ and $\{p\}$ with p odd, all regular polytopes are centrally symmetric.

TABLE 18.1.1 The convex regular polytopes in $\mathbb{E}^d$ ($d \geq 3$).

DIMENSION	NAME	SCHLÄFLI SYMBOL	f_0	f_{d-1}	$\|G(P)\|$	DIAGRAM
$d \geq 3$	d-simplex	$\{3^{d-1}\}$	d+1	d+1	(d+1)!	A_d
	d-cross-polytope	$\{3^{d-2}, 4\}$	$2d$	2^d	$2^d d!$	B_d (or C_d)
	d-cube	$\{4, 3^{d-2}\}$	2^d	$2d$	$2^d d!$	B_d (or C_d)
$d = 3$	icosahedron	$\{3, 5\}$	12	20	120	H_3
	dodecahedron	$\{5, 3\}$	20	12	120	H_3
$d = 4$	24-cell	$\{3, 4, 3\}$	24	24	1152	F_4
	600-cell	$\{3, 3, 5\}$	120	600	14400	H_4
	120-cell	$\{5, 3, 3\}$	600	120	14400	H_4

The regular tessellations T in $\mathbb{E}^d$ are also known. If $d = 1$, T is an apeirogon and $G(T)$ is the infinite dihedral group. For $d \geq 2$ see the list in Table 18.1.2. The first $d - 1$ entries in $\{p_1, \ldots, p_d\}$ give the Schläfli symbol for the (regular) tiles of T, the last $d - 1$ that for the (regular) vertex figures. (A vertex figure at a vertex x is the convex hull of the midpoints of the edges emanating from x.) The cubical

FIGURE 18.1.1
The five Platonic solids.

tessellation occurs for each d, while for $d = 2$ and $d = 4$ there is a dual pair of exceptional tessellations.

TABLE 18.1.2 The regular tessellations in $\mathbb{E}^d$ $(d \geq 2)$.

DIMENSION	SCHLÄFLI SYMBOL	TILES	VERTEX-FIGURES
$d \geq 2$	$\{4, 3^{d-2}, 4\}$	d-cubes	d-cross-polytopes
$d = 2$	$\{3, 6\}$ $\{6, 3\}$	triangles hexagons	hexagons triangles
$d = 4$	$\{3, 3, 4, 3\}$ $\{3, 4, 3, 3\}$	4-cross-polytopes 24-cells	24-cells 4-cross-polytopes

As vertices of the plane polygon $\{p\}$ we can take the points corresponding to the p-th roots of unity. The d-simplex can be defined as the convex hull of the $d+1$ points in $\mathbb{E}^{d+1}$ corresponding to the permutations of $(1, 0, \ldots, 0)$. As vertices of the d-cross-polytope in $\mathbb{E}^d$ choose the $2d$ permutations of $(\pm 1, 0, \ldots, 0)$, and for the d-cube take the 2^d points $(\pm 1, \ldots, \pm 1)$. The midpoints of the edges of a 4-cross-polytope are the 24 vertices of a regular 24-cell. The coordinates for the remaining regular polytopes are more complicated [Cox73, pp. 52,157].

For the cubical tessellation $\{4, 3^{d-2}, 4\}$ take the vertex set to be $\mathbb{Z}^d$ (giving the square tessellation if $d = 2$). For the triangle tessellation $\{3, 6\}$ choose as vertices the integral linear combinations of two unit vectors inclined at $\pi/3$. Locating the face centers gives the vertices of the hexagonal tessellation $\{6, 3\}$. For $\{3, 3, 4, 3\}$ in $\mathbb{E}^4$ take the alternating vertices of the cubical tessellation; that is, the integral points with an even coordinate sum. Its dual $\{3, 4, 3, 3\}$ (with 24-cells as tiles) has the vertices at the centers of the tiles of $\{3, 3, 4, 3\}$.

The regular polytopes and tessellations have been with us since before recorded history, and a strong strain of mathematics since classical times has centered on them. The classical theory intersects with diverse mathematical areas such as Lie algebras and Lie groups, Tits buildings [Ti74], finite and combinatorial group theory [Bu95, Mag74], geometric and algebraic combinatorics, graphs and combinatorial designs [BCN89], singularity theory, and Riemann surfaces.

SYMMETRY GROUPS

For a convex regular d-polytope P in $\mathbb{E}^d$, pick a fixed (***base***) flag Φ, and consider the maximal simplex C (***chamber***) in the barycentric subdivision (***chamber complex***) of P whose vertices are the centers of the nonempty faces in Φ. Then

C is a fundamental region for $G(P)$ in P and $G(P)$ is generated by the reflections $R_0, \ldots, R_{d-1}$ in the walls of C that contain the center of P, where R_i is the reflection in the wall opposite to the vertex of C corresponding to the i-face in Φ. If $P = \{p_1, \ldots, p_{d-1}\}$, then

$$\begin{cases} R_i^2 = (R_j R_k)^2 = 1 & (0 \le i, j, k \le d-1,\ |j-k| \ge 2) \\ (R_{i-1} R_i)^{p_i} = 1 & (1 \le i \le d-1) \end{cases}$$

is a presentation for $G(P)$ in terms of these generators. In particular, $G(P)$ is a finite (spherical) Coxeter group with string diagram

$$\bullet \overset{}{\underset{p_1}{\text{———}}} \bullet \underset{p_2}{\text{———}} \bullet \cdots\cdots \bullet \underset{p_{d-2}}{\text{———}} \bullet \underset{p_{d-1}}{\text{———}} \bullet$$

(see Section 18.6).

If T is a regular tessellation of $\mathbb{E}^d$, pick Φ and C as before. Now $G(T)$ is generated by the $d+1$ reflections in all walls of C giving $R_0, \ldots, R_d$ (as above). The presentation for $G(T)$ carries over, but now $G(T)$ is an infinite (Euclidean) Coxeter group.

18.2 REGULAR STAR-POLYTOPES

The regular star-polyhedra and star-polytopes are obtained by allowing the faces or vertex figures to be *starry* (star-like). This leads to very beautiful figures that are closely related to the regular convex polytopes. See Coxeter [Cox73] for a comprehensive account; see also McMullen and Schulte [McS02]. In defining star-polytopes, we shall combine the approach of [Cox73] and McMullen [McM68] and introduce them via the associated starry polytope-configuration.

GLOSSARY

d-polytope-configuration: A finite family Π of affine subspaces, called ***elements***, of Euclidean d-space $\mathbb{E}^d$, ordered by inclusion, such that the following conditions are satisfied. Π contains the empty set $\emptyset$ and $\mathbb{E}^d$ as (*improper*) elements. The dimensions of the other (*proper*) elements can take the values $0, 1, \ldots, d-1$, and the affine hull of their union is $\mathbb{E}^d$. As a partially ordered set, Π is a ranked lattice. For $F, G \in \Pi$ with $F \subseteq G$ call $G/F := \{H \in \Pi | F \subseteq H \subseteq G\}$ the ***subconfiguration*** of Π defined by F and G; this has itself the structure of a $(dim(G) - dim(F) - 1)$-polytope-configuration. As further conditions, each G/F contains at least 2 proper elements if $dim(G) - dim(F) = 2$, and as a partially ordered set, each G/F (including Π itself) is connected if $dim(G) - dim(F) \ge 3$. (See the definition of an abstract polytope in Section 18.8.) It can be proved that in $\mathbb{E}^d$ every Π satisfies the stronger condition that each G/F contains exactly 2 proper elements if $dim(G) - dim(F) = 2$.

Regular polytope-configuration: A polytope-configuration Π whose symmetry group $G(\Pi)$ is flag-transitive. (A flag is a maximal totally ordered subset of Π.)

Regular star-polygon: For positive integers n and k with $(n, k) = 1$ and $1 < k < \frac{n}{2}$, up to similarity the regular star-polygon $\{\frac{n}{k}\}$ is the connected plane polygon whose consecutive vertices are $(\cos(\frac{2\pi kj}{n}), \sin(\frac{2\pi kj}{n}))$ for $j = 0, 1, \ldots, n-1$. If $k = 1$, the same plane polygon bounds a (nonstarry) convex n-gon with Schläfli symbol $\{n\}$ $(= \{\frac{n}{1}\})$. With each regular (convex or star-) polygon $\{\frac{n}{k}\}$ is associated a regular 2-polytope-configuration obtained by replacing each edge by its affine hull.

Star-polytope-configuration: A d-polytope-configuration Π is ***nonstarry*** if it is the family of affine hulls of the faces of a convex d-polytope. It is ***starry***, or a ***star-polytope-configuration***, if it is not nonstarry. For instance, among the 2-polytope-configurations that are associated with a regular (convex or star-) polygon $\{\frac{n}{k}\}$ for a given n, the one with $k = 1$ is nonstarry and those for $k > 1$ are starry. In the first case the corresponding n-gon is convex, and in the second case it is genuinely star-like. In general, the starry polytope configurations are those that belong to genuinely star-like polytopes (that is, star-polytopes).

Regular star-polytope: If $d = 2$, a regular star-polytope is a regular star-polygon. Defined inductively, if $d \geq 3$, a regular d-star-polytope P is a finite family of regular convex $(d-1)$-polytopes or regular $(d-1)$-star-polytopes such that the family consisting of their affine hulls as well as the affine hulls of their "faces" is a regular d-star-polytope-configuration $\Pi = \Pi(P)$. Here, the faces of the polytopes can be defined in such a way that they correspond to the elements in the associated polytope-configuration. The symmetry groups of P and Π are the same.

ENUMERATION AND CONSTRUCTION

Regular star-polytopes P can only exist for $d = 2$, 3, or 4. As regular convex polytopes, they are also uniquely determined by the Schläfli symbol $\{p_1, \ldots, p_{d-1}\}$, but now at least one entry is not integral. Again the symbols for the facets and vertex figures, when superposed, give the original. If $d = 2$, $P = \{\frac{n}{k}\}$ for some k with $(n, k) = 1$ and $1 < k < \frac{n}{2}$, and $G(P) = D_n$. For $d = 3$ and 4 the star-polytopes are listed in Table 18.2.1 together with the numbers f_0 and f_{d-1} of vertices and facets, respectively.

FIGURE 18.2.1
The four Kepler-Poinsot polyhedra.

Every regular d-star-polytope has the same vertices and symmetry group as a regular convex d-polytope. The four regular star-polyhedra (3-star-polytopes) are also known as the ***Kepler-Poinsot polyhedra*** (Figure 18.2.1). They can be constructed from the icosahedron $\{3, 5\}$ or dodecahedon $\{5, 3\}$ by two kinds of operations, ***stellation*** or ***faceting*** [Cox73]. Loosely speaking, in the former operation one extends the faces of a polyhedron symmetrically until they again form a polyhedron, while in the latter operation the vertices of a polyhedron are redis-

TABLE 18.2.1 The regular star-polytopes in $\mathbb{E}^d$ $(d \geq 3)$.

DIMENSION	SCHLÄFLI SYMBOL	f_0	f_{d-1}
$d = 3$	$\{3, \frac{5}{2}\}$	12	20
	$\{\frac{5}{2}, 3\}$	20	12
	$\{5, \frac{5}{2}\}$	12	12
	$\{\frac{5}{2}, 5\}$	12	12
$d = 4$	$\{3, 3, \frac{5}{2}\}$	120	600
	$\{\frac{5}{2}, 3, 3\}$	600	120
	$\{3, 5, \frac{5}{2}\}$	120	120
	$\{\frac{5}{2}, 5, 3\}$	120	120
	$\{3, \frac{5}{2}, 5\}$	120	120
	$\{5, \frac{5}{2}, 3\}$	120	120
	$\{5, 3, \frac{5}{2}\}$	120	120
	$\{\frac{5}{2}, 3, 5\}$	120	120
	$\{5, \frac{5}{2}, 5\}$	120	120
	$\{\frac{5}{2}, 5, \frac{5}{2}\}$	120	120

tributed in classes that are then the vertex sets for the faces of a new polyhedron. Regarded as regular maps on surfaces (Section 18.3), the polyhedra $\{3, \frac{5}{2}\}$ (***great icosahedron***) and $\{\frac{5}{2}, 3\}$ (***great stellated dodecahedron***) are of genus 0, while $\{5, \frac{5}{2}\}$ (***great dodecahedron***) and $\{\frac{5}{2}, 5\}$ (***small stellated dodecahedron***) are of genus 4.

The ten regular star-polytopes in $\mathbb{E}^4$ all have the same vertices and symmetry groups as the 600-cell $\{3, 3, 5\}$ or 120-cell $\{5, 3, 3\}$ and can be derived from these by 4-dimensional stellation or faceting operations [Cox73, McM68]. See also [Cox93] for their names, which describe the various relationships among the polytopes. For presentations of their symmetry groups which reflect the finer combinatorial structure of the star-polytopes, see also [McS02].

The dual of $\{p_1, \ldots, p_{d-1}\}$ (which is obtained by dualizing the associated star-polytope-configuration using reciprocation with respect to a sphere) is $\{p_{d-1}, \ldots, p_1\}$. Regarded as abstract polytopes (Section 18.8), the star-polytopes $\{p_1, \ldots, p_{d-1}\}$ and $\{q_1, \ldots, q_{d-1}\}$ are isomorphic if and only if the symbol $\{q_1, \ldots, q_{d-1}\}$ is obtained from $\{p_1, \ldots, p_{d-1}\}$ by replacing each entry 5 by $\frac{5}{2}$ and each $\frac{5}{2}$ by 5.

18.3 REGULAR SKEW POLYHEDRA

Regular skew polyhedra are finite or infinite polyhedra whose vertex figures are skew (antiprismatic) polygons. The standard reference is Coxeter [Cox68]. Topologically, these polyhedra are regular maps on surfaces. For general properties of regular maps see Coxeter and Moser [CM80], McMullen and Schulte [McS02], or Chapter 20 of this Handbook.

GLOSSARY

(Right) prism, antiprism (with regular bases): A convex 3-polytope whose vertices are contained in two parallel planes and whose set of 2-faces consists of the two ***bases*** (contained in the parallel planes) and the 2-faces in the ***mantle*** that connects the bases. The bases are congruent regular polygons. For a (right) prism, each base is a translate of the other by a vector perpendicular to its affine hull, and the mantle 2-faces are rectangles. For a (right) antiprism, each base is a translate of a reciprocal (dual) of the other by a vector perpendicular to its affine hull, and the mantle 2-faces are isosceles triangles. (The prism or antiprism is ***semi-regular*** if its mantle 2-faces are squares or equilateral triangles, respectively; see Section 18.5.)

Map on a surface: A decomposition (tessellation) P of a closed surface S into nonoverlapping simply connected regions, the 2-***faces*** of P, by arcs, the ***edges*** of P, joining pairs of points, the ***vertices*** of P, such that two conditions are satisfied. First, each edge belongs to exactly two 2-faces. Second, if two distinct edges intersect, they meet in one vertex or in two vertices.

Regular map: A map P on S whose combinatorial automorphism group $\Gamma(P)$ is transitive on the flags (incident triples consisting of a vertex, an edge, and a 2-face).

Polyhedron: A map P on a closed surface S embedded (without self-intersections) into a Euclidean space, such that two conditions are satisfied. Each 2-face of P is a convex plane polygon, and any two adjacent 2-faces do not lie in the same plane. See also the more general definition in the next section.

Skew polyhedron: A polyhedron P such that for at least one vertex x, the vertex figure of P at x is not a plane polygon; the ***vertex figure*** at x is the polygon whose vertices are the vertices of P adjacent to x and whose edges join consecutive vertices as one goes around x.

Regular polyhedron: A polyhedron P whose symmetry group $G(P)$ is flag-transitive. (For a regular skew polyhedron P in $\mathbb{E}^3$ or $\mathbb{E}^4$, each vertex figure must be a 3-dimensional antiprismatic polygon, meaning that it contains all edges of an antiprism that are not edges of a base. See also Section 18.4.)

ENUMERATION

In $\mathbb{E}^3$ all, and in $\mathbb{E}^4$ all finite, regular skew polyhedra are known [Cox68]. In these cases the (orientable) polyhedron P is completely determined by the extended Schläfli symbol $\{p, q|r\}$, where the 2-faces of P are convex p-gons such that q meet at each vertex, and r is the number of edges in each edge path of P that leaves, at each vertex, exactly two 2-faces of P on the right. The group $G(P)$ is isomorphic to $\Gamma(P)$ and has the presentation

$$\rho_0{}^2 = \rho_1{}^2 = \rho_2{}^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_0\rho_2)^2 = (\rho_0\rho_1\rho_2\rho_1)^r = 1$$

(but the generators ρ_i are not all hyperplane reflections). The polyhedra $\{p, q|r\}$ and $\{q, p|r\}$ are duals, and the vertices of one can be obtained as the centers of the 2-faces of the other.

In $\mathbb{E}^3$ there are just three regular skew polyhedra: $\{4, 6|4\}$, $\{6, 4|4\}$, and $\{6, 6|3\}$. These are the (infinite) *Petrie-Coxeter polyhedra*. For example, $\{4, 6|4\}$ consists of half the square faces of the cubical tessellation $\{4, 3, 4\}$ in $\mathbb{E}^3$.

TABLE 18.3.1 The finite regular skew polyhedra in $\mathbb{E}^4$.

SCHLÄFLI SYMBOL	f_0	f_2	GROUP ORDER	GENUS
$\{4,4\|r\}$	r^2	r^2	$8r^2$	1
$\{4,6\|3\}$	20	30	240	6
$\{6,4\|3\}$	30	20	240	6
$\{4,8\|3\}$	144	288	2304	73
$\{8,4\|3\}$	288	144	2304	73

The finite regular skew polyhedra in $\mathbb{E}^4$ (or equivalently, in spherical 3-space) are listed in Table 18.3.1. There is an infinite sequence of toroidal polyhedra as well as two pairs of duals related to the (self-dual) 4-simplex $\{3,3,3\}$ and 24-cell $\{3,4,3\}$. For drawings of projections of these polyhedra into 3-space see [BoW88, SWi91]; Figure 18.3.1 represents $\{4,8|3\}$.

FIGURE 18.3.1
A projection of $\{4,8|3\}$ *into* $\mathbb{R}^3$.

These projections are examples of *combinatorially regular polyhedra* in ordinary 3-space; see [BrW93] and Chapter 20 in this Handbook. For regular polyhedra in $\mathbb{E}^4$ with planar, but not necessarily convex, 2-faces, see also [ABM00, Bra00]. For regular skew polyhedra in hyperbolic 3-space, see [Gar67].

18.4 THE GRÜNBAUM-DRESS POLYHEDRA

A new impetus to the study of regular figures came from Grünbaum [Grü77b], who generalized the regular skew polyhedra by allowing skew polygons as faces as well as vertex figures. This restored the symmetry in the definition of polyhedra. For the classification of these "new" regular polyhedra in $\mathbb{E}^3$, see [Grü77b], [Dre85], and [McS02]. The proper setting for this subject is, strictly speaking, in the context of realizations of abstract regular polytopes (see Section 18.8).

GLOSSARY

Polygon: A figure P in Euclidean space $\mathbb{E}^d$ consisting of a (finite or infinite) sequence of distinct points, called the ***vertices*** of P, joined in successive pairs, and closed cyclicly if finite, by line segments, called the ***edges*** of P, such that each compact set in $\mathbb{E}^d$ meets only finitely many edges.

Zigzag polygon: A (zigzag-shaped) infinite plane polygon P whose vertices alternately lie on two parallel lines and whose edges are all of the same length.

Antiprismatic polygon: A closed polygon P in 3-space whose vertices are alternately vertices of each of the two (regular convex) bases of a (right) antiprism Q (Section 18.3), such that the orthogonal projection of P onto the plane of a base gives a regular star-polygon (Section 18.2). This star-polygon (and thus P) has twice as many vertices as each base, and is a convex polygon if and only if the edges of P are just those edges of Q that are not edges of a base.

Prismatic polygon: A closed polygon P in 3-space whose vertices are alternately vertices of each of the two (regular convex) bases of a (right) prism Q (Section 18.3), such that the orthogonal projection of P onto the plane of a base traverses twice a regular star-polygon in that plane (Section 18.2). Each base of Q (and thus the star-polygon) is assumed to have an odd number of vertices. The star-polygon is a convex polygon if and only if each edge of P is a diagonal in a rectangular 2-face in the mantle of Q.

Helical polygon: An infinite polygon in 3-space whose vertices lie on a helix given parametrically by $(a\cos\beta t, a\sin\beta t, bt)$, where $a, b \neq 0$ and $0 < \beta < \pi$, and are obtained as t ranges over the integers. Successive integers correspond to successive vertices.

Polyhedron: A (finite or infinite) family P of polygons in $\mathbb{E}^d$, called the 2-***faces*** of P, such that three conditions are satisfied. First, each edge of one of the 2-faces is an edge of exactly one other 2-face. Second, for any two edges F and F' of (2-faces of) P there exist chains $F = G_0, G_1, \dots, G_n = F'$ of edges and $H_1, \dots, H_n$ of 2-faces such that each H_i is incident with G_{i-1} and G_i. Third, each compact set in $\mathbb{E}^d$ meets only finitely many 2-faces.

Regular: A polygon or polyhedron P is regular if its symmetry group $G(P)$ is transitive on the flags.

Petrie polygon of a polyhedron: A polygonal path along the edges of a regular polyhedron P such that any two successive edges, but no three, are edges of a 2-face of P.

Petrie dual: The family of all Petrie polygons of a regular polyhedron P. This is itself a regular polyhedron, and its Petrie dual is P itself.

ENUMERATION

For a systematic discussion of regular polygons in arbitrary Euclidean spaces see [Cox93]. In light of the geometric classification scheme for the new regular polyhedra in $\mathbb{E}^3$ proposed in [Grü77b], it is useful to classify the regular polygons in $\mathbb{E}^3$

into seven groups: convex polygons, plane star-polygons (Section 18.2), apeirogons (Section 18.1), zigzag polygons, antiprismatic polygons, prismatic polygons, and helical polygons. These correspond to the four kinds of isometries in $\mathbb{E}^3$: rotation, rotatory reflection (a reflection followed by a rotation in the reflection plane), glide reflection, and twist.

The 2-faces and vertex figures of a regular polyhedron P in $\mathbb{E}^3$ are regular polygons of the above kind. (The vertex figure at a vertex x is the polygon whose vertices are the vertices of P adjacent to x and whose edges join two such vertices y and z if and only if $\{y, x\}$ and $\{x, z\}$ are edges of a 2-face in P. For a regular P, this is a single polygon.) It is convenient to group the regular polyhedra in $\mathbb{E}^3$ into 8 classes. The first four are the traditional regular polyhedra: the five Platonic solids; the three planar tessellations; the four regular star-polyhedra (Kepler-Poinsot polyhedra); and the three infinite regular skew polyhedra (Petrie-Coxeter polyhedra). The four other classes and their polyhedra can be described as follows: the class of nine finite polyhedra with finite skew (antiprismatic) polygons as faces; the class of infinite polyhedra with finite skew (prismatic or antiprismatic) polygons as faces, which includes three infinite families as well as three individual polyhedra; the class of polyhedra with zigzag polygons as faces, which contains six infinite families; and the class of polyhedra with helical polygons as faces, which has three infinite families and six individual polyhedra.

Alternatively, these forty-eight polyhedra can be described as follows [McS02]. There are eighteen finite regular polyhedra, namely the nine classical finite regular polyhedra (Platonic solids and Kepler-Poinsot polyhedra), and their Petrie-duals. The regular tessellations of the plane, and their Petrie duals (with zigzag 2-faces), are the six planar polyhedra in the list. From those, twelve further polyhedra are obtained as blends (in the sense of Section 18.8) with a line segment or an apeirogon (Section 18.1). The six blends with a line segment have finite skew, or (infinite planar) zigzag, 2-faces with alternate vertices on a pair of parallel planes; the six blends with an apeirogon have helical polygons or zigzag polygons as 2-faces. Finally, there are twelve further polyhedra which are not blends; they fall into a single family and are related to the cubical tessellation of $\mathbb{E}^3$. Each polyhedron can be described by a generalized Schläfli symbol, which encodes the geometric structure of the polygonal faces and vertex figures, tells whether or not the polyhedron is a blend, and indicates a presentation of the symmetry group. For more details see [McS02] (or [Grü77b, Dre85, Joh]).

18.5 SEMI-REGULAR AND UNIFORM CONVEX POLYTOPES

The very stringent requirements in the definition of regularity of polytopes can be relaxed in many different ways, yielding a great variety of weaker regularity notions. We shall only consider polytopes and polyhedra that are convex. See Johnson [Joh] for a detailed discussion, or Martini [Mar93] for a survey.

GLOSSARY

Semi-regular: A convex d-polytope P is semi-regular if its facets are regular and its symmetry group $G(P)$ is transitive on the vertices of P.

Uniform: A convex polygon is uniform if it is regular. Recursively, if $d \geq 3$, a convex d-polytope P is uniform if its facets are uniform and its symmetry group $G(P)$ is transitive on the vertices of P.

Regular-faced: P is regular-faced if all its facets (and lower-dimensional faces) are regular.

ENUMERATION

Each regular polytope is semi-regular, and each semi-regular polytope is uniform. Also, by definition each uniform 3-polytope is semi-regular. For $d = 3$ the family of semi-regular (uniform) convex polyhedra consists of the Platonic solids, two infinite classes of prisms and antiprisms, as well as the thirteen polyhedra known as Archimedean solids [Fej64]. The seven semi-regular polyhedra whose symmetry group is edge-transitive are also called the ***quasi-regular*** polyhedra.

Besides the regular polytopes, there are only seven semi-regular polytopes in higher dimensions: three for $d = 4$, and one for each of $d = 5, 6, 7, 8$ (for a short proof, see [BB91]). However, there are many more uniform polytopes but a complete list is known only for $d = 4$ [Joh]. Except for the regular 4-polytopes and the prisms over uniform 3-polytopes, there are exactly 40 uniform 4-polytopes.

For $d = 3$ all, for $d = 4$ all save one, and for $d \geq 5$ many, uniform polytopes can be obtained by a method called ***Wythoff's construction***. This method proceeds from a finite Euclidean reflection group W in $\mathbb{E}^d$, or the even (rotation) subgroup W^+ of W, and constructs the polytopes as the convex hull of the orbit under W or W^+ of a point, the initial vertex, in the fundamental region of the group, which is a d-simplex (chamber) or the union of two adjacent d-simplices in the corresponding chamber complex of W, respectively; see Sections 18.1 and 18.6.

The regular-faced polytopes have also been described for each dimension. In general, such a polytope can have different kinds of facets (and vertex figures). For $d = 3$ the complete list contains exactly 92 regular-faced convex polyhedra and includes all semi-regular polyhedra. For each $d \geq 5$, there are only two regular-faced d-polytopes that are not semi-regular. Except for $d = 4$, each regular-faced d-polytope has a nontrivial symmetry group.

There are many further generalizations of the notion of regularity [Mar93]. However, in most cases complete lists of the corresponding polytopes are either not known or available only for $d = 3$. The variants that have been considered include: ***isogonal*** polytopes (requiring vertex-transitivity of $G(P)$), or ***isohedral*** polytopes, the reciprocals of the isogonal polytopes, with a facet-transitive group $G(P)$; more generally, ***k-face-transitive*** polytopes (requiring transitivity of $G(P)$ on the k-faces), for a single value or several values of k; ***congruent-faceted***, or ***monohedral***, polytopes (requiring congruence of the facets); and ***equifaceted*** polytopes (requiring combinatorial isomorphism of the facets). Similar problems have also been considered for nonconvex polytopes or polyhedra, and for tilings [GS87].

18.6 REFLECTION GROUPS

Symmetry properties of geometric figures are closely tied to the algebraic structure of their symmetry groups, which are often subgroups of finite or infinite reflection groups. A classical reference for reflection groups is Coxeter [Cox73]. A more recent text is Humphreys [Hum90].

GLOSSARY

Reflection group: A group generated by (hyperplane) reflections in a finite-dimensional space V. The space can be a real or complex vector space (or affine space). A ***reflection*** is a linear (or affine) transformation whose eigenvalues, save one, are all equal to 1, while the remaining eigenvalue is a primitive k-th root of unity for some $k \geq 2$; in the real case, it is -1. If the space is equipped with further structure, the reflections are assumed to preserve it. For example, if V is real Euclidean, the reflections are Euclidean reflections.

Coxeter group: A group W, finite or infinite, that is generated by finitely many generators $\sigma_1, \ldots, \sigma_n$ and has a presentation of the form $(\sigma_i\sigma_j)^{m_{ij}} = 1$ $(i, j = 1, \ldots, n)$, where the m_{ij} are positive integers or ∞ such that $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$ $(i \neq j)$. The matrix $(m_{ij})_{ij}$ is the ***Coxeter matrix*** of W.

Coxeter diagram: A labeled graph $\mathcal{D}$ that represents a Coxeter group W as follows. The nodes of $\mathcal{D}$ represent the generators σ_i of W. The i-th and j-th node are joined by a (single) branch if and only if $m_{ij} > 2$. In this case, the branch is labeled m_{ij} if $m_{ij} \neq 3$ (and remains unlabeled if $m_{ij} = 3$).

Irreducible Coxeter group: A Coxeter group W whose Coxeter diagram is connected. (Each Coxeter group W is the direct product of irreducible Coxeter groups, with each factor corresponding to a connected component of the diagram of W.)

Root system: A finite set $\mathcal{R}$ of non-zero vectors, the ***roots***, in $\mathbb{E}^d$ satisfying the following conditions. $\mathcal{R}$ spans $\mathbb{E}^d$, and $\mathcal{R} \cap \mathbb{R}e = \{\pm e\}$ for each $e \in \mathcal{R}$. For each $e \in \mathcal{R}$, the Euclidean reflection S_e in the linear hyperplane orthogonal to e maps $\mathcal{R}$ onto itself. Moreover, the numbers $2(e, e')/(e', e')$, with $e, e' \in \mathcal{R}$, are integers (*Cartan integers*); here (,) denotes the standard inner product on $\mathbb{E}^d$. (These conditions define ***crystallographic*** root systems. Sometimes the integrality condition is omitted to give a more general notion of root system.) The group W generated by the reflections S_e $(e \in \mathcal{R})$ is a finite Coxeter group, called the ***Weyl group*** of $\mathcal{R}$.

GENERAL PROPERTIES

Every Coxeter group $W = \langle \sigma_1, \ldots, \sigma_n \rangle$ admits a faithful representation as a reflection group in the real vector space R^n. This is obtained as follows. If W has Coxeter matrix $M = (m_{ij})_{ij}$ and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$, define the symmetric bilinear form $\langle\ ,\ \rangle_M$ by

$$\langle e_i, e_j \rangle_M := -\cos(\pi/m_{ij}) \quad (i, j = 1, \ldots, n),$$

with appropriate interpretation if $m_{ij} = \infty$. For $i = 1, \ldots, n$ the linear transformation $S_i : \mathbb{R}^n \mapsto \mathbb{R}^n$ given by

$$xS_i := x - 2\langle e_i, x\rangle_M \, e_i \qquad (x \in \mathbb{R}^n)$$

is the orthogonal reflection in the hyperplane orthogonal to e_i. Let $O(M)$ denote the orthogonal group corresponding to $\langle\ ,\ \rangle_M$. Then $\sigma_i \mapsto S_i$ $(i = 1, \ldots, n)$ defines a faithful representation $\rho : W \mapsto GL(\mathbb{R}^n)$, called the ***canonical representation***, such that $W\rho \subseteq O(M)$.

The group W is finite if and only if the associated form $\langle\ ,\ \rangle_M$ is positive definite; in this case, $\langle\ ,\ \rangle_M$ determines a Euclidean geometry on $\mathbb{R}^n$. In other words, each finite Coxeter group is a finite Euclidean reflection group. Conversely, every finite Euclidean reflection group is a Coxeter group. The finite Coxeter groups have been completely classified by Coxeter and are usually listed in terms of their Coxeter diagrams.

The finite irreducible Coxeter groups with string diagrams are precisely the symmetry groups of the convex regular polytopes, with a pair of dual polytopes corresponding to a pair of groups that are related by reversing the order of the generators. See Section 18.1 for an explanation about how the generators act on the polytopes. Table 18.1.1 also lists the names for the corresponding Coxeter diagrams.

For $p_1, \ldots, p_{n-1} \geq 2$ write $[p_1, \ldots, p_{n-1}]$ for the Coxeter group with string diagram $\bullet\underset{p_1}{\frac{\qquad}{}}\bullet\underset{p_2}{\frac{\qquad}{}}\bullet \cdots\cdots \bullet\underset{p_{n-2}}{\frac{\qquad}{}}\bullet\underset{p_{n-1}}{\frac{\qquad}{}}\bullet$. Then $[p_1, \ldots, p_{n-1}]$ is the automorphism group of the universal abstract regular n-polytope $\{p_1, \ldots, p_{n-1}\}$; see Section 18.8. The regular honeycombs $\{p_1, \ldots, p_{n-1}\}$ on the sphere (convex regular polytopes) or in Euclidean or hyperbolic space are examples of such universal polytopes. The spherical honeycombs are exactly the finite universal regular polytopes (with $p_i > 2$ for all i). The Euclidean honeycombs arise exactly when $p_i > 2$ for all i and the bilinear form $\langle\ ,\ \rangle_M$ for $[p_1, \ldots, p_{n-1}]$ is positive semi-definite (but not positive definite). Similarly, the hyperbolic honeycombs correspond exactly to the groups $[p_1, \ldots, p_{n-1}]$ that are Coxeter groups of "hyperbolic type" [McS02].

There are exactly two sources of finite Coxeter groups, to some extent overlapping: the symmetry groups of convex regular polytopes, and the Weyl groups of (crystallographic) root systems, which are important in Lie Theory. Every root system $\mathcal{R}$ has a set of ***simple roots***; this is a subset $\mathcal{S}$ of $\mathcal{R}$, which is a basis of $\mathbb{E}^d$ such that every $e \in \mathcal{R}$ is a linear combination of vectors in $\mathcal{S}$ with integer coefficients which are all non-negative or all non-positive. The distinguished generators of the Weyl group W are given by the reflections S_e in the linear hyperplane orthogonal to e $(e \in \mathcal{S})$, for some set $\mathcal{S}$ of simple roots of $\mathcal{R}$. The irreducible Weyl groups in $\mathbb{E}^2$ are the symmetry groups of the triangle, square or hexagon. The diagrams A_d, B_d, C_d and F_4 of Table 18.1.1 all correspond to irreducible Weyl groups and root systems (with B_d and C_d corresponding to a pair of dual root systems), but H_3 and H_4 do not (they correspond to a non-crystallographic root system [CMP98]). There is one additional series of irreducible Weyl groups in $\mathbb{E}^d$ with $d \geq 4$ (a certain subgoup of index 2 in B_d), whose diagram is denoted by D_d. The remaining irreducible Weyl groups occur in dimensions 6, 7 and 8, with diagrams E_6, E_7 and E_8, respectively.

Each Weyl group W stabilizes the lattice spanned by a set $\mathcal{S}$ of simple roots, the ***root lattice*** of $\mathcal{R}$. These lattices have many interesting geometric properties and occur also in the context of sphere packings (see Conway and Sloane [CS88] and Chapter 60). The irreducible Coxeter groups W of euclidean type, or, equivalently, the infinite discrete irreducible euclidean reflection groups, are intimately related to Weyl groups; they are also called ***affine Weyl groups***.

The complexifications of the reflection hyperplanes for a finite Coxeter group give an example of a complex ***hyperplane arrangement*** (see [Bj93], [OT92] and Chapter 7). The topology of the set-theoretic complement of these ***Coxeter arrangements*** in complex space has been extensively studied.

For hyperbolic reflection groups, see Vinberg [Vi85]. In hyperbolic space, a discrete irreducible reflection group need not have a fundamental region which is a simplex.

18.7 COMPLEX REGULAR POLYTOPES

Complex regular polytopes are subspace configurations in unitary complex space that share many properties with regular polytopes in real spaces. For a detailed account see Coxeter [Cox93]. The subject originated from Shephard [Sh52].

GLOSSARY

Complex d-polytope: A d-polytope-configuration as defined in Section 18.2, but now the elements, or ***faces***, are subspaces in unitary complex d-space $\mathbb{C}^d$. However, unlike in real space, the subconfigurations G/F with $dim(G)-dim(F)=2$ can contain more than 2 proper elements. A ***complex polygon*** is a complex 2-polytope.

Regular complex polytope: A complex polytope P whose (unitary) symmetry group $G(P)$ is transitive on the flags (the maximal sets of mutually incident faces).

ENUMERATION AND GROUPS

The regular complex d-polytopes P are completely known for each d. Every d-polytope can be uniquely described by a ***generalized Schläfli symbol***

$$p_0\{q_1\}p_1\{q_2\}p_2\dots p_{d-2}\{q_{d-1}\}p_{d-1},$$

which we explain below. For $d=1$, the regular polytopes are precisely the point sets on the complex line, which in corresponding real 2-space are the vertex sets of regular convex polygons; the Schläfli symbol is simply p if the real polygon is a p-gon. In general, the entry p_i is the Schläfli symbol for the complex 1-polytope that occurs as the 1-dimensional subconfiguration G/F of P, where F is an $(i-1)$-face and G an $(i+1)$-face of P such that $F \subseteq G$. As is further explained below, the p_i i-faces in this subconfiguration are cyclicly permuted by a hyperplane reflection that leaves the whole polytope invariant. Note that, unlike in real Euclidean space, a hyperplane reflection in unitary complex space need not have period 2 but can have any finite period greater than 1. The meaning of the entries q_i is also given below.

The regular complex polytopes P with $d \geq 2$ are summarized in Table 18.7.1, which includes the numbers f_0 and f_{d-1} of vertices and facets ((d–1)-faces) and the group order. Listed are only the nonreal polytopes as well as only one polytope from each pair of duals. A complex polytope is ***real*** if, up to an affine transformation of $\mathbb{C}^d$, all its faces are subspaces that can be described by linear equations over the reals. In particular, $p_0\{q_1\}p_1 \ldots p_{d-2}\{q_{d-1}\}p_{d-1}$ is real if and only if $p_i = 2$ for each i; in this case, $\{q_1, \ldots, q_{d-1}\}$ is the Schläfli symbol for the related regular polytope in real space. As in real space, each polytope $p_0\{q_1\}p_1 \ldots p_{d-2}\{q_{d-1}\}p_{d-1}$ has a dual (reciprocal) and its Schläfli symbol is $p_{d-1}\{q_{d-1}\}p_{d-2} \ldots p_1\{q_1\}p_0$; the symmetry groups are the same and the numbers of vertices and facets are interchanged. The polytope $p\{4\}2\{3\}2 \ldots 2\{3\}2$ is the ***generalized complex d-cube***, and its dual $2\{3\}2 \ldots 2\{3\}2\{4\}p$ the ***generalized complex d-cross-polytope***; if $p = 2$, these are the real d-cubes and d-cross-polytopes, respectively.

TABLE 18.7.1 The nonreal complex regular polytopes (up to duality).

DIMENSION	POLYTOPE	f_0	f_{d-1}	$\lvert G(P)\rvert$
$d \geq 1$	$p\{4\}2\{3\}2 \ldots 2\{3\}2$	p^d	pd	$p^d d!$
$d = 2$	$3\{3\}3$	8	8	24
	$3\{6\}2$	24	16	48
	$3\{4\}3$	24	24	72
	$4\{3\}4$	24	24	96
	$3\{8\}2$	72	48	144
	$4\{6\}2$	96	48	192
	$4\{4\}3$	96	72	288
	$3\{5\}3$	120	120	360
	$5\{3\}5$	120	120	600
	$3\{10\}2$	360	240	720
	$5\{6\}2$	600	240	1200
	$5\{4\}3$	600	360	1800
$d = 3$	$3\{3\}3\{3\}3$	27	27	648
	$3\{3\}3\{4\}2$	72	54	1296
$d = 4$	$3\{3\}3\{3\}3\{3\}3$	240	240	155 520

The symmetry group $G(P)$ of a complex regular d-polytope P is a finite unitary reflection group in $\mathbb{C}^d$; if $P = p_0\{q_1\}p_1 \ldots p_{d-2}\{q_{d-1}\}p_{d-1}$, then the notation for the group $G(P)$ is $p_0[q_1]p_1 \ldots p_{d-2}[q_{d-1}]p_{d-1}$. If $\Phi = \{\emptyset = F_{-1}, F_0, \ldots, F_{d-1}, F_d = \mathbb{C}^d\}$ is a flag of P, then for each $i = 0, 1, \ldots, d-1$ there is a unitary reflection R_i that fixes F_j for $j \neq i$ and cyclicly permutes the p_i i-faces in the subconfiguration F_{i+1}/F_{i-1} of P. These generators R_i can be chosen in such a way that in terms of $R_0, \ldots, R_{d-1}$, the group $G(P)$ has a presentation of the form

$$\left\{ \begin{array}{ll} R_i^{p_i} = 1 & (0 \leq i \leq d-1), \\ R_i R_j = R_j R_i & (0 \leq i < j-1 \leq d-2), \\ \multicolumn{2}{l}{R_i R_{i+1} R_i R_{i+1} R_i \ldots = R_{i+1} R_i R_{i+1} R_i R_{i+1} \ldots} \\ \multicolumn{2}{l}{\text{with } q_{i+1} \text{ generators on each side } (0 \leq i \leq d-2).} \end{array} \right.$$

This explains the entries q_i in the Schläfli symbol. Conversely, any d unitary reflections that satisfy the first two sets of relations, and generate a finite group, can be used to determine a regular complex polytope by a complex analogue of Wythoff's construction (see Section 18.5). If P is real, then $G(P)$ is conjugate, in the general linear group of $\mathbb{C}^d$, to a finite (real) Coxeter group (see Section 18.6). Complex regular polytopes are only one source for finite unitary reflection groups; there are also others [Cox93, ShT54].

See Cuypers [Cuy95] for the classification of quaternionic regular polytopes (polytope-configurations in quaternionic space).

18.8 ABSTRACT REGULAR POLYTOPES

Abstract regular polytopes are combinatorial structures that generalize the familiar regular polytopes. The terminology adopted is patterned after the classical theory. Many symmetric figures discussed in earlier sections could be treated (and their structure clarified) in this more general framework. Much of the research in this area is quite recent. For a comprehensive account see McMullen and Schulte [McS02].

GLOSSARY

Abstract d-polytope: A partially ordered set P, with elements called ***faces***, that satisfies the following conditions. P is equipped with a ***rank function*** with range $\{-1, 0, \ldots, d\}$, which associates with a face F its ***rank*** rank F; if rank $F = j$, F is a ***j-face***, or a ***vertex***, an ***edge***, or a ***facet*** if $j = 0, 1$, or $d - 1$, respectively. P has a unique minimal element F_{-1} of rank -1 and a unique maximal element F_d of rank d. These two elements are the ***improper*** faces; the others are ***proper***. The ***flags*** (maximal totally ordered subsets) of P all contain exactly $d + 2$ faces (including F_{-1} and F_d). If $F < G$ in P, then $G/F := \{H \in P | F \leq H \leq G\}$ is said to be a ***section*** of P. All sections of P (including P itself) are ***connected***, meaning that, given two proper faces H, H' of a section G/F, there is a sequence $H = H_0, H_1, \ldots, H_k = H'$ of proper faces of G/F (for some k) such that H_{i-1} and H_i are incident for each $i = 1, \ldots, k$. (That is, P is ***strongly connected***.) Finally, if $F < G$ with $0 \leq \text{rank } F + 1 = j = \text{rank } G - 1 \leq d - 1$, there are exactly two j-faces H such that $F < H < G$. (Note that this last condition basically says that P is topologically real. The condition is violated for nonreal complex polytopes.)

Faces and co-faces: We can safely identify a face F of P with the section $F/F_{-1} = \{H \in P | H \leq F\}$. The section $F_d/F = \{H \in P | F \leq H\}$ is the ***co-face*** of P, or the ***vertex figure*** if F is a vertex.

Regular polytope: An abstract polytope P whose ***automorphism group*** $\Gamma(P)$ (the group of order-preserving permutations of $\mathcal{P}$) is transitive on the flags. (Then $\Gamma(P)$ must be simply flag-transitive.)

C-group: A group Γ generated by involutions $\sigma_1, \ldots, \sigma_m$ (that is, a quotient of a Coxeter group) such that the ***intersection property*** holds:

$$\langle \sigma_i | i \in I \rangle \cap \langle \sigma_i | i \in J \rangle = \langle \sigma_i | i \in I \cap J \rangle \quad \text{for all } I, J \subset \{1, \ldots, m\}.$$

The letter "C" stands for "Coxeter". (Coxeter groups are C-groups, but not vice versa.)

String C-group: A C-group $\Gamma = \langle \sigma_1, \ldots, \sigma_m \rangle$ such that $(\sigma_i \sigma_j)^2 = 1$ if $1 \leq i < j - 1 \leq m - 1$. (Then Γ is a quotient of a Coxeter group with a string Coxeter diagram.)

Realization: For a regular (abstract) d-polytope P with vertex-set $\mathcal{F}_0$, a surjection $\beta : \mathcal{F}_0 \mapsto V$ onto a set V of points in a Euclidean space, such that each automorphism of P induces an isometric permutation of V. Then V is the ***vertex set*** of the realization β.

Chiral polytope: An abstract polytope P whose automorphism group $\Gamma(P)$ has exactly two orbits on the flags, with adjacent flags in different orbits. (Two flags are adjacent if they differ in exactly one face.) Chiral polytopes are an important class of nearly regular polytopes.

GENERAL PROPERTIES

Abstract 2-polytopes are isomorphic to ordinary n-gons or apeirogons (Section 18.2). Except for some degenerate cases, the abstract 3-polytopes with finite faces and vertex figures are in one-to-one correspondence with the maps on surfaces (Section 18.3). Accordingly, a finite (abstract) 4-polytope P has facets and vertex figures that are isomorphic to maps on surfaces.

The group $\Gamma(P)$ of every regular d-polytope P is a string C-group. Fix a flag $\Phi := \{F_{-1}, F_0, \ldots, F_d\}$, the ***base flag*** of P. Then $\Gamma(P)$ is generated by ***distinguished generators*** $\rho_0, \ldots, \rho_{d-1}$ (*with respect to* Φ), where ρ_i is the unique automorphism that keeps all but the i-face of Φ fixed. These generators satisfy relations

$$(\rho_i \rho_j)^{p_{ij}} = 1 \quad (i, j = 0, \ldots, d-1),$$

with $p_{ii} = 1$, $p_{ij} = p_{ji} \geq 2$ $(i \neq j)$, and $p_{ij} = 2$ if $|i - j| \geq 2$; in particular, $\Gamma(P)$ is a string C-group with generators $\rho_0, \ldots, \rho_{d-1}$. The numbers $p_i := p_{i-1,i}$ determine the (*Schläfli*) ***type*** $\{p_1, \ldots, p_{d-1}\}$ of P. The group $\Gamma(P)$ is a quotient of the Coxeter group $[p_1, \ldots, p_{d-1}]$ (Section 18.6), but in general the quotient is proper.

Conversely, if Γ is a string C-group with generators $\rho_0, \ldots, \rho_{d-1}$, then it is the group of a regular d-polytope P, and $\rho_0, \ldots, \rho_{d-1}$ are the distinguished generators with respect to some base flag of P. The i-faces of P are the right cosets of the subgroup $\Gamma_i := \langle \rho_k | k \neq i \rangle$ of Γ, and in P, $\Gamma_i \varphi \leq \Gamma_j \psi$ if and only if $i \leq j$ and $\Gamma_i \varphi \cap \Gamma_j \psi \neq \emptyset$. For any $p_1, \ldots, p_{d-1} \geq 2$, $[p_1, \ldots, p_{d-1}]$ is a string C-group and the corresponding d-polytope is the ***universal*** regular d-polytope $\{p_1, \ldots, p_{d-1}\}$; every other regular d-polytope of the same type $\{p_1, \ldots, p_{d-1}\}$ is derived from it by making identifications. Examples are the regular spherical, Euclidean, and hyperbolic honeycombs. The one-to-one correspondence between string C-groups and the groups of regular polytopes sets up a powerful dialogue between groups on one hand and polytopes on the other.

There is also a similar such dialogue for chiral polytopes (see Schulte and Weiss [SWe94]). If P is chiral and $\Phi := \{F_{-1}, F_0, \ldots, F_d\}$ is its base flag, then $\Gamma(P)$ is generated by automorphisms $\sigma_1, \ldots, \sigma_{d-1}$, where σ_i fixes all the faces in $\Phi \setminus \{F_{i-1}, F_i\}$ and cyclically permutes consecutive i-faces of P in the (polygonal) section F_{i+1}/F_{i-2} of rank 2. The orientation of each σ_i can be chosen in such a way that the resulting ***distinguished generators*** $\sigma_1, \ldots, \sigma_{d-1}$ of $\Gamma(P)$ satisfy relations

$$\sigma_i^{p_i} = (\sigma_j \sigma_{j+1} \ldots \sigma_k)^2 = 1 \quad (i, j, k = 1, \ldots, d-1 \text{ and } j < k),$$

with p_i determined by the type $\{p_1, \ldots, p_{d-1}\}$ of P. Moreover, a certain intersection property (resembling that for C-groups) holds for $\Gamma(P)$. Conversely, if Γ is a group generated by $\sigma_1, \ldots, \sigma_{d-1}$, and if these generators satisfy the above relations and the intersection property, then Γ is the group of a chiral polytope, or the rotation subgroup of index 2 in the group of a regular polytope. Each isomorphism type of chiral polytope occurs combinatorially in two ***enantiomorphic*** (mirror image) forms; these correspond to two sets of generators σ_i of the group determined by a pair of adjacent base flags.

Abstract polytopes are closely related to buildings and diagram geometries [Bu95, Ti74]. They are essentially the "thin diagram geometries with a string diagram". The universal regular polytopes $\{p_1, \ldots, p_{d-1}\}$ correspond to "thin buildings".

CLASSIFICATION BY TOPOLOGICAL TYPE

Abstract polytopes are not a priori embedded into an ambient space. Therefore for abstract polytopes, the traditional enumeration of regular polytopes is replaced by the classification by global or local topological type. On the group level, this translates into the enumeration of finite string C-groups with certain kinds of presentations.

Every ***locally spherical*** abstract regular polytope P of rank $d+1$ is a quotient of a regular tessellation $\{p_1, \ldots, p_d\}$ in spherical, Euclidean or hyperbolic d-space; in other words, P is a regular tessellation on the corresponding spherical, Euclidean or hyperbolic space-form. In this context, the classical regular convex polytopes are precisely the abstract regular polytopes that are locally spherical and globally spherical. The ***projective regular polytopes*** are the regular tessellations in real projective d-space, and are obtained as quotients of the centrally symmetric regular convex polytopes under the central inversion.

Much work has also been done in the toroidal and locally toroidal case [McS02]. A ***regular toroid*** of rank $d+1$ is the quotient of a regular tessellation $\{p_1, \ldots, p_d\}$ in Euclidean d-space by a lattice that is invariant under all symmetries of the vertex figure of $\{p_1, \ldots, p_d\}$; in other words, a regular toroid is a regular tessellation on the d-torus. If $d = 2$, these are the reflexible regular torus maps of [CM80]. For $d \geq 3$ there are three infinite sequences of ***cubical toroids*** of type $\{4, 3^{d-2}, 4\}$, and for $d = 4$ there are two infinite sequences of ***exceptional toroids*** for each of the types $\{3, 3, 4, 3\}$ and $\{3, 4, 3, 3\}$. Their groups are known in terms of generators and relations.

For $d \geq 2$, the d-torus is the only d-dimensional compact Euclidean space-form which can admit a regular or chiral tessellation. Further, chirality can only occur if $d = 2$ (yielding the irreflexible torus maps of [CM80]). Little is known about regular tessellations on hyperbolic space forms (again, see [CM80] and [McS02]).

For regular d-polytopes P_1 and P_2, let $\langle P_1, P_2 \rangle$ denote the class of all regular $(d+1)$-polytopes with facets isomorphic to P_1 and vertex figures isomorphic to P_2. Each nonempty class $\langle P_1, P_2 \rangle$ contains a ***universal polytope*** denoted by $\{P_1, P_2\}$, which "covers" all other polytopes in its class. Classification by local topological type means enumeration of all finite universal polytopes $\{P_1, P_2\}$ where P_1 and P_2 are of the prescribed (global) topological type. There are variants of this definition. A polytope Q in $\langle P_1, P_2 \rangle$ is ***locally toroidal*** if P_1 and P_2 are regular convex polytopes (spheres) or regular toroids, with at least one of the latter kind.

Locally toroidal regular polytopes can only exist in ranks 4, 5, and 6 [McS02]. The enumeration is complete for rank 5, and nearly complete for rank 4. In rank 6, a list of finite polytopes is known that is conjectured to be complete. The enumeration in rank 4 involves analysis of the Schläfli types $\{4, 4, r\}$ with $r = 3, 4$, $\{6, 3, r\}$ with $r = 3, 4, 5, 6$, and $\{3, 6, 3\}$, and their duals. Here, complete lists of finite universal regular polytopes are known for each type except $\{4, 4, 4\}$ and $\{3, 6, 3\}$; the type $\{4, 4, 4\}$ is almost settled, and for $\{3, 6, 3\}$ partial results were known. In rank 5, only the types $\{3, 4, 3, 4\}$ and its dual occur. Finally, in rank 6, there are $\{3, 3, 3, 4, 3\}$, $\{3, 3, 4, 3, 3\}$ and $\{3, 4, 3, 3, 4\}$, and their duals. On the group level, the classification of toroidal and locally toroidal polytopes amounts to the classification of certain C-groups which are defined in terms of generators and relations. These groups are quotients of Euclidean or hyperbolic Coxeter groups and are obtained from those by either one or two extra defining relations. Very little is known about the corresponding classification for chiral polytopes.

REALIZATIONS

A good number of the geometric figures discussed in the earlier sections could be described in the general context of realizations of abstract regular polytopes. For an account of realizations see [McS02] or McMullen [McM94].

Let $\beta : \mathcal{F}_0 \mapsto V$ be a realization of a regular d-polytope P, and let $\mathcal{F}_j$ denote the set of j-faces of P ($j = -1, 0, \ldots, d$). With $\beta_0 := \beta$, $V_0 := V$, then for $j = 1, \ldots, d$, β recursively induces a surjection $\beta_j : \mathcal{F}_j \mapsto V_j$, with $V_j \subset 2^{V_{j-1}}$, given by

$$F\beta_j := \{G\beta_{j-1} | G \in \mathcal{F}_{j-1}, G \leq F\}$$

for each $F \in \mathcal{F}_j$. It is convenient to identify β and $\left\{\beta_j\right\}_{j=0}^{d}$ and also call the latter a realization of $\mathcal{P}$. The realization is ***faithful*** if each β_j is a bijection; otherwise, it is ***degenerate***. Its ***dimension*** is the dimension of the affine hull of V. Each realization corresponds to a (not necessarily faithful) representation of the automorphism group $\Gamma(P)$ as a group of Euclidean isometries.

The traditional approach in the study of regular figures starts from a Euclidean (or other) space and describes all figures of a specified kind that are regular according to some geometric definition of regularity. For example, the Grünbaum-Dress polyhedra of Section 18.4 are the realizations in $\mathbb{E}^3$ of abstract regular 3-polytopes P, which are both discrete and faithful; their symmetry group is flag-transitive and is isomorphic to the automorphism group $\Gamma(P)$.

A rather new approach proceeds from a given abstract regular polytope P and describes all the realizations of P. For a finite P, each realization β is uniquely determined by its ***diagonal vector*** Δ, whose components are the squared lengths of the diagonals (pairs of vertices) in the diagonal classes of P modulo $\Gamma(P)$. Each orthogonal representation of $\Gamma(P)$ yields one or more (possibly degenerate) realizations of P. Then taking the sum of two representations of $\Gamma(P)$ is equivalent to an operation for the related realizations called a ***blend***, which in turn amounts to adding the corresponding diagonal vectors. If we identify the realizations with their diagonal vectors, then the space of all realizations of P becomes a closed convex cone $C(P)$, the ***realization cone of*** P, whose finer structure is given by the irreducible representations of $\Gamma(P)$. The extreme rays of $C(P)$ correspond to the ***pure*** (unblended) realizations, which are given by the irreducible representations of $\Gamma(P)$. Each realization of P is a blend of pure realizations.

For instance, a regular n-gon P has $\lfloor\frac{1}{2}n\rfloor$ diagonal classes, and for each $k = 1, \ldots, \lfloor\frac{1}{2}n\rfloor$, there is a planar regular star-polygon $\{\frac{n}{k}\}$ if $(n,k) = 1$ (Section 18.2), or a "degenerate star-polygon $\{\frac{n}{k}\}$" if $(n,k) > 1$; the latter is a degenerate realization of P, which reduces to a line segment if $n = 2k$. For the regular icosahedron P there are 3 pure realizations. Apart from the usual icosahedron $\{3,5\}$ itself, there is another 3-dimensional pure realization, namely the great icosahedron $\{3, \frac{5}{2}\}$ (Section 18.2). The final pure realization is induced by its covering of $\{3,5\}/2$, the ***hemi-icosahedron*** (obtained from P by identifying antipodal vertices), all of whose diagonals are edges; thus its vertices must be those of a 5-simplex. The regular d-simplex has (up to similarity) a unique realization. The regular d-cross-polytope and d-cube have 2 and d pure realizations, respectively. For other polytopes see [BS00, McS02, MW99, MW00].

18.9 SOURCES AND RELATED MATERIAL

SURVEYS

[Ba95]: A popular book on the geometry and visualization of polyhedral and non-polyhedral figures with symmetries in higher dimensions.

[Bj93]: A monograph on oriented matroids and their applications.

[BrW93]: A survey on polyhedral manifolds and their embeddings in real space.

[BCN89]: A monograph on distance-regular graphs and their symmetry properties.

[Bu95]: A Handbook of Incidence Geometry, with articles on buildings and diagram geometries.

[CS88]: A monograph on sphere packings and related topics.

[Cox70]: A short text on certain chiral tessellations of 3-dimensional manifolds.

[Cox73]: A monograph on the traditional regular polytopes, regular tessellations, and reflection groups.

[Cox93]: A monograph on complex regular polytopes and complex reflection groups.

[CM80]: A monograph on discrete groups and their presentations.

[DGS81]: A collection of papers on various aspects of symmetry, contributed in honor of H.S.M. Coxeter's 80-th birthday.

[DuV64]: A monograph on geometric aspects of the quaternions with applications to symmetry.

[Fej64]: A monograph on regular figures, mainly in 3 dimensions.

[Grü67]: A monograph on convex polytopes.

[GS87]: A monograph on plane tilings and patterns.

[Hum90]: A monograph on Coxeter groups and reflection groups.

[Joh]: A monograph on uniform polytopes and semi-regular figures.

[Mag74]: A book on discrete groups of Möbius transformations and non-euclidean tessellations.

[Mar93]: A survey on symmetric convex polytopes and a hierarchical classification by symmetry.

[Mo87]: A book on the topology of the three-manifolds of classical plane tessellations.

[McM94]: A survey on abstract regular polytopes with emphasis on geometric realizations.

[McS02]: A monograph on abstract regular polytopes and their groups.

[OT92]: A monograph on hyperplane arrangements.

[Ro84]: A text about symmetry classes of convex polytopes.

[Se95]: An introduction to the geometry of mathematical quasicrystals and related tilings.

[SF88]: A text on interdisciplinary aspects of polyhedra and their symmetries.

[ShM95]: A collection of twenty-six papers by H.S.M. Coxeter.

[Ti74]: A text on buildings and their classification.

[We77]: A monograph on three-dimensional polyhedral geometry and its applications in crystallography.

[Zi95]: A graduate textbook on convex polytopes.

RELATED CHAPTERS

Chapter 4: Tilings
Chapter 7: Oriented matroids
Chapter 15: Basic properties of convex polytopes
Chapter 20: Polyhedral maps
Chapter 60: Sphere packing and coding theory
Chapter 61: Crystals and quasicrystals

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