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MATHEMATICS

ON THE NUMBER OF POSITIVE INTEGERS $\leqslant x$ AND FREE OF PRIME FACTORS $>y$, II

BY

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1. *Introduction.*

If $x>0$, $y\geqslant 2$, we denote by $\Psi(x, y)$ the number of positive integers not exceeding x which contain no prime factors greater than y. An upper bound for $\Psi(x, y)$ was given by RANKIN [5]:

$$\Psi(x, y)<x \exp \left\{-\frac{\log_3 y}{\log y} \log x+\log_2 y+O\left(\frac{\log_2 y}{\log_3 y}\right)\right\} \tag{1.1}$$

($\log_2 y$ stands for log log y, $\log_3 y$ for log log log y). For the region

$$(\log x)^2<y<x^{1/3} \tag{1.2}$$

the following improvement was derived in [2] by a different method:

$$\Psi(x, y)<x(\log y)^2 \exp (-u \log u-u \log_2 u+O(u)) \tag{1.3}$$

where $u=(\log x)/(\log y)$. Moreover, [2] contains a quite accurate asymptotic formula for $\Psi(x, y)$, certainly valid if $\log y>(\log x)^{2/3}$:

$$\Psi(x, y) \sim x\varrho(u) \tag{1.4}$$

where $u=(\log x)/\log y$, and $\varrho(u)$ is the solution of the differential-difference equation $u\, \varrho'(u)=-\varrho(u-1)(u>1)$ with initial conditions $\varrho(u)=1$ $(0\leqslant u\leqslant 1)$, $\varrho(u)$ continuous at $u=1$.

Formula (1.4) was established previously in the smaller region $\log y>c \log x$ (for any positive $c<1$) by K. DICKMAN [3], S. D. CHOWLA, T. VIJAYARAGHAVAN, V. RAMASWAMI, and A. A. BUCHSTAB (for references see [2]).

With this function ϱ, which is asymptotically (for $u\to\infty$)

$$\varrho(u)=\exp \left[-u\left\{\log u+\log_2 u-1+\frac{\log_2 u}{\log u}-\frac{1}{\log u}+O\left(\frac{(\log_2 u)^2}{(\log u)^2}\right)\right\}\right] \tag{1.5}$$

(see [1]), we can rewrite (1.3) as

$$\Psi(x, y)<x\,(\log y)^2\, \varrho(u)\, e^{O(u)} \qquad ((\log x)^2<y<x^{1/3}). \tag{1.6}$$

If y is less than about $\log x$, the behaviour of Ψ is entirely different. A special result was obtained by P. Erdös [2], who showed that

$$\log \Psi(x, \log x) \sim (\log 4)(\log x)/(\log_2 x), \tag{1.7}$$

if $x \to \infty$.

In the present paper we have two main results, both derived (as far as upper estimates are concerned) by the method Rankin used for (1.1). The first one (theorem 1) is an asymptotic formula for $\log \Psi(x, y)$, effective if $y \to \infty$, $u \to \infty$. It shows clearly how the asymptotic behaviour of $\log \Psi$ changes around $y = \log x$. The second one (theorem 2) is an improvement of (1.3), but it is the method rather than the improvement that makes it worth-while to insert it in this paper.

Theorem 1. With the abbreviation

$$Z = \left\{\log\left(1+\frac{y}{\log x}\right)\right\}\cdot\frac{\log x}{\log y} + \left\{\log\left(1+\frac{\log x}{y}\right)\right\}\cdot\frac{y}{\log y} \tag{1.8}$$

we have, uniformly for $2 < y \leqslant x$,

$$\log \Psi(x, y) = Z\{1 + O((\log y)^{-1}) + O((\log_2 x)^{-1}) + O((u+1)^{-1})\}, \tag{1.9}$$

where $u = (\log x)/(\log y)$.

Theorem 2. If c is a constant, $c > 1$, we have for

$$(\log x)^c \leqslant y \leqslant x, \quad x \geqslant 2$$

uniformly

$$\left\{\begin{aligned} \log \Psi(x, y) \leqslant \log(x\varrho(u)) + \tfrac{1}{2}\log(1+u) + \\ + O(\log_2 y) + O((\log x)^2/y) + O(R), \end{aligned}\right. \tag{1.10}$$

where

$$R = \int_1^{\log y} \exp(s\eta/(\log y))\, V(e^s)\, ds,$$

$$u = (\log x)/(\log y), \quad \eta = \log u + \log_2(u+1),$$

and V is a function connected with the error term in the prime number theorem (see (4.1)).

The term $O(R)$ on the right-hand side of (1.10) is not easily simplified without losing something in some region or other. It is certainly small compared to the main term $\log(x\varrho(u))$. A rough estimate can be obtained by taking $V(e^s) = \exp(-s^{\frac{1}{2}})$, and splitting the interval $1 \leqslant s \leqslant \log y$ into $1 \leqslant s \leqslant \beta \log y$, $\beta \log y \leqslant s \leqslant \log y$, where β is a constant between 0 and 1. A quite strong result can be obtained if something in the direction of the Riemann hypothesis is true. It can be shown that if δ and θ are constants, $0 < \theta < 1$, $\delta > (1-\theta)^{-1}$, and if the Riemann zeta function has no zeros with real part $> \theta$, then $R = O(1)$ if $y > (\log x)^\delta$.

If c and d are constants, $1<c<d$, and $(\log x)^c \leqslant y \leqslant (\log x)^d$, the estimate (1.10) is almost contained in (1.9), although the error term in (1.10) is smaller. We have

$$x\varrho(u)=\exp\,(Z+O(u))$$

in such a region. If, however, $y=(\log x)^v$ and $v \to \infty$, then (1.10) is better than the upper estimate contained in (1.9).

2. *A lower bound.*

There is an almost trivial lower bound for $\Psi(x, y)$ which fits remarkably well to our upper bounds. This lower bound is $W(x, y)$, defined by

$$W(x, y)=\binom{N+K}{K} \tag{2.1}$$

where $N=\pi(y)$ (i.e. the number of primes $\leqslant y$), $K=[(\log x)/(\log y)]$. It is easily seen that $W(x, y)$ represents the number of solutions of

$$\sum_{p\leqslant y} \xi_p \leqslant (\log x)/(\log y)$$

in non-negative integers ξ_p. This number of solutions does not exceed the number of solutions of

$$\sum_{p\leqslant y} \xi_p \log p \leqslant \log x,$$

whence

$$W(x, y) \leqslant \Psi(x, y) \qquad (2<y\leqslant x).$$

We infer, by Stirling's formula

$$\log \Psi(x, y) \geqslant (N+K)\log(N+K)-N\log N-K\log K+O(1), \tag{2.2}$$

uniformly for $2 \leqslant y \leqslant x$.

The main terms on the right-hand side of (2.2) can be written as

$$\int_0^N \log((K+t)/t)\,dt \quad \text{or} \quad \int_0^K \log((N+t)/t)\,dt,$$

and these formulas show us the effect of relatively small perturbations in N and K. Using $N=y(\log y)^{-1}+O(y(\log y)^{-2})$, $K=u+O(1)$, we easily verify that (2.2) implies

$$\log \Psi(x, y) \geqslant Z\,\{1+O((\log y)^{-1})+O((u+1)^{-1})\}. \tag{2.3}$$

3. *Rankin's method.*

Rankin's method is easily explained. Let Y represent the set of all integers which are entirely composed of prime factors $\leqslant y$. Then we have, for every $\eta>0$

$$\Psi(x, y)=\sum_{d\leqslant x, d\in Y} 1 \leqslant \sum_{d\leqslant x, d\in Y} (x/d)^\eta \leqslant \sum_{d\in Y} (x/d)^\eta = x^\eta \prod_{p\leqslant y} (1-p^{-\eta})^{-1}.$$

Therefore

$$\Psi(x, y) \leqslant x^{\eta} \prod_{p \leqslant y} (1-p^{-\eta})^{-1} \tag{3.1}$$

The difficulty lies in selecting η such that the right-hand side of (3.1) is small. Rankin used $\eta = 1-(\log_3 y)/(\log y)$. We shall use instead $\eta = \sigma$, where σ is defined by

$$\sigma = \left\{\log\left(1+\frac{y}{\log x}\right)\right\} \Big/ \log y, \tag{3.2}$$

which is particularly successful at least in the region $y \leqslant (\log x)^c$, where c is any constant >1. For larger values of y we shall obtain better results with $\eta = \tau$ (see 3.7).

We owe to the reader some motivation for the choice of σ or τ. It will turn out that the expression

$$\eta \log x + \int_e^y \log\{(1-t^{-\eta})^{-1}\} (\log t)^{-1}\, dt, \tag{3.3}$$

is a good approximation to the logarithm of the right-hand side of (3.1). (The error is D, studied in sec. 4). Therefore, it seems to be very reasonable to take η such that (3.3) is minimal. This means that η has to satisfy

$$\log x = \int_e^y (t^{\eta}-1)^{-1}\, dt. \tag{3.4}$$

In a region $y \leqslant (\log x)^c$ a reasonable approximation to the solution of (3.4) is obtained if we replace $t^{\eta}-1$ by $y^{\eta}-1$. This leads to the choice $\eta = \sigma$ (see (3.2)).

On the other hand, if, for example

$$(\log x)^c \leqslant y \leqslant x \tag{3.5}$$

then we can "streamline" (3.4) by writing

$$\log x = \int_1^{\log y} e^{(1-\eta)v}\, dv. \tag{3.6}$$

In sec. 6 we shall use $\eta = \tau$, where τ satisfies (3.6). That is

$$\tau = 1 - \xi/(\log y), \tag{3.7}$$

where $e^{\xi} - 1 = \xi(\log x)/(\log y)$, $\xi > 0$.

4. *Application of the prime number theorem.*

We use the following notation: li y is the logarithmic integral

$$\mathrm{li}\, y = \int_e^y (\log t)^{-1}\, dt$$

(instead of the slightly more cumbersome usual definition as the principal value of $\int_0^y (\log t)^{-1}\, dt$). Furthermore, let V be a continuous positive function whose integral can be used as error term in the prime number theorem:

$$\pi(y)-\operatorname{li} y=O\left(\int_e^y V(t)\,dt\right) \qquad (y\to\infty). \tag{4.1}$$

We can take $V(t)=\exp(-c(\log t)^{\frac{1}{2}})$ (de la Vallée-Poussin), even $V(t)=\exp(-(\log t)^{4/7-\varepsilon})$ (Tchudakoff), and $V(t)=t^{-\frac{1}{2}}\log t$ if the Riemann hypothesis is true.

We put

$$D=\sum_{p\leqslant y}\log\{(1-p^{-\eta})^{-1}\}-\int_e^y\{\log(1-t^{-\eta})^{-1}\}\,d\operatorname{li} t. \tag{4.2}$$

We shall estimate D if, for the time being, nothing is assumed about η apart from $\eta>0$. Let f be any positive monotonically decreasing function with a continuous derivative for $e\leqslant t<\infty$. Then we have, if $y>e$, integrating by parts,

$$\int_e^y f(t)\,d(\pi(t)-\operatorname{li} t)=f(y)\,(\pi(y)-\operatorname{li} y)-f(e)\,(\pi(e)-\operatorname{li} e)-\int_e^y f'(t)\,(\pi(t)-\operatorname{li} t)\,dt.$$

Since f' does not change sign we have, by (4.1),

$$\int_e^y f'(t)\,(\pi(t)-\operatorname{li} t)\,dt=O\left(\int_e^y(-f'(t))\,dt\int_e^t V(s)\,ds\right).$$

Integrating by parts, we obtain for the right-hand side

$$O\left(\int_e^y (f(t)-f(y))\,V(t)\,dt\right)=O\left(\int_e^y f(t)\,V(t)\,dt\right).$$

Hence

$$\int_e^y f(t)\,d(\pi(t)-\operatorname{li} t)=O\left(\int_e^y f(t)\,V(t)\,dt\right).$$

Applying this to (4.2), taking $f(t)=\log\{(1-t^{-\eta})^{-1}\}$, we obtain

$$D=O(U_1)+h(\eta), \tag{4.3}$$

where $h(\eta)=O(\log\eta^{-1})$ if $0<\eta<\frac{1}{2}$, $h(\eta)=O(2^{-\eta})$ if $\eta\geqslant\frac{1}{2}$, and U_1 is defined by

$$U_1=\int_e^y\log\{(1-t^{-\eta})^{-1}\}\,V(t)\,dt. \tag{4.4}$$

5. *Proof of theorem* 1.

In this section, we shall replace η by σ (see (3.2)). We first estimate W_1-W_2, where

$$W_1=\int_e^y\log\{(1-t^{-\sigma})^{-1}\}\,(\log t)^{-1}\,dt, \tag{5.1}$$

$$W_2=\log\{(1-y^{-\sigma})^{-1}\}\int_e^y(\log t)^{-1}\,dt. \tag{5.2}$$

We have

$$W_1 - W_2 = \sigma \int_e^y (s^\sigma - 1)^{-1} s^{-1}\, ds \int_e^s (\log t)^{-1}\, dt,$$

whence $W_1 - W_2 = O(S)$, where

(5.3) $$S = \sigma \int_e^y (s^\sigma - 1)^{-1} (\log s)^{-1}\, ds = \sigma \int_1^{\log y} (e^{\sigma v} - 1)^{-1} e^v v^{-1}\, dv.$$

We can deal in a similar way with (4.4). This leads (with $\eta = \sigma$) to

(5.4) $$U_1 = O(\log \{(1 - y^{-\sigma})^{-1}\} \int_e^y V(t)\, dt) + O(\sigma \int_e^y (s^\sigma - 1)^{-1} V(s)\, ds).$$

The second term is $O(S)$, as $V(s) = (\log s)^{-1}$ is a trivial possibility for V.

Next we remark that $\log \{(1 - y^{-\sigma})^{-1}\} = \log (1 + y^{-1} \log x)$, whence

(5.5) $$\sigma \log x + W_2 = \log\left(1 + \frac{y}{\log x}\right) \frac{\log x}{\log y} + (\operatorname{li} y) \log\left(1 + \frac{\log x}{y}\right).$$

Finally, combining (3.1), (4.2), (4.3) with the above formulas, we obtain

(5.6) $$\begin{cases} \log \Psi(x, y) \leqslant \log\left(1 + \dfrac{y}{\log x}\right) \dfrac{\log x}{\log y} + \\ \qquad + \log\left(1 + \dfrac{\log x}{y}\right) (\operatorname{li} y + O(\int_e^y V(t)\, dt)) \\ \qquad + O(h(\sigma)) + O(S). \end{cases}$$

We shall study S, given by (5.3), in three different regions:

(i) $0 < \sigma \leqslant (\log 2)/(\log y)$, corresponding to the interval $y \leqslant \log x$,

(ii) $(\log 2)/\log y < \sigma \leqslant \frac{1}{2}$, corresponding to values of y from $\log x$ to roughly $(\log x)^2$,

(iii) $\frac{1}{2} < \sigma < 1 - (\log \log x + O(1))/(\log x)$, corresponding to values of y from about $(\log x)^2$ to x.

In case (i) we have

$$S = O(\sigma \int_1^{\log y} (\sigma v)^{-1} e^v v^{-1}\, dv) = O(y/(\log y)^2).$$

In case (ii) we have

$$\begin{aligned} S &= \sigma \int_1^{(\log 2)/\sigma} + \sigma \int_{(\log 2)/\sigma}^{\log y} = \\ &= O(\int_1^{(\log 2)/\sigma} v^{-2} e^v\, dv) + O(\sigma \int_{(\log 2)/\sigma}^{\log y} v^{-1} e^{(1-\sigma) v}\, dv) = \\ &= O(\sigma^2 e^{(\log 2)/\sigma}) + O(\sigma e^{(1-\sigma) \log y} (\log y)^{-1}) = \\ &= O(\sigma^2 e^{(\log 2)/\sigma}) + O(\sigma (\log x) (\log y)^{-1}). \end{aligned}$$

We shall show that in the latter expression the first term is not of larger order than the second one. We put $y=Q\log x$, whence $1\leqslant Q\leqslant\log x$. Then

$$\sigma^2\exp((\log 2)/\sigma)=\sigma\cdot\frac{\log(1+Q)}{\log y}\cdot\exp\left(\frac{\log 2\cdot\log y}{\log(1+Q)}\right)=$$
$$=O\left\{\sigma\cdot\frac{\log x}{\log y}\cdot\frac{\log(1+Q)}{\log x}\cdot\exp\left(\frac{\log 2\cdot\log\log x}{\log(1+Q)}\right)\right\}.$$

We have to show that

$$\exp\left(\frac{\log 2\cdot\log\log x}{\log(1+Q)}\right)=O\left(\frac{\log x}{\log(1+Q)}\right) \tag{5.7}$$

if $x\to\infty$, $1\leqslant Q\leqslant\log x$.

It is easily seen that (5.7) is true if $1\leqslant Q\leqslant 3$. We next assume $3\leqslant Q\leqslant\log x$. Then the left-hand side of (5.7) is $O((\log x)^{\frac{1}{2}})$, and this is $O(\log x/\log_2 x)$. This proves that (5.7) holds, and that in case (ii)

$$S=O(\sigma(\log x)(\log y)^{-1}).$$

In case (iii) we have $(1-\sigma)\log y=\log_2 x+O(1)$, whence

$$S=O(\int_1^{\log y}v^{-1}e^{(1-\sigma)v}\,dv)=$$
$$=O(\int_{1-\sigma}^{(1-\sigma)\log y}t^{-1}e^t\,dt)=O(\int_{1-\sigma}^{1})+O(\int_1^{(1-\sigma)\log y})=$$
$$=O(\log(1-\sigma)^{-1})+O(e^{(1-\sigma)\log y}(1-\sigma)^{-1}(\log y)^{-1})=$$
$$=O(\log_2 x)+O((\log x)/(\log_2 x))=O((\log x)/(\log_2 x)).$$

In case (i) and in case (ii) we derived

$$S=O(y(\log y)^{-2}),\qquad S=O(\sigma(\log x)(\log y)^{-1}),$$

respectively. It easily follows that $S=O(Z/\log y)$ in both cases. In case (iii) we have $S=O(\log x/\log_2 x)$, and $Z>\sigma\log x>\frac{1}{2}\log x$, whence $S=O(Z/\log_2 x)$.

We have to devote some attention to the error term $O(h(\sigma))$ of (5.6) (cf. (4.3)). If $y<\log x$ we have $\sigma^{-1}=O((\log y)(\log x)/y)$, whence

$$h(\sigma)=O(\log\sigma^{-1})=O(\log_2 y)+O(\log((\log x)/y))=$$
$$=O(\log_2 y)+O(\log(1+(\log x)/y))=O(Z\,y^{-1}(\log y)(\log_2 y)).$$

If $\log x\leqslant y<x$ we have $\sigma^{-1}=O(\log y)$ whence

$$h(\sigma)=O(\log_2 y)=O(Z\,y^{-1}(\log y)(\log_2 y)).$$

So finally, (5.6) leads to

$$\log\Psi(x,y)\leqslant Z(1+O((\log y)^{-1})+O((\log_2 x)^{-1})) \tag{5.8}$$

uniformly for $2<y\leqslant x$.

From (2.3) and (5.8) theorem 1 follows.

6. *Proof of theorem* 2.

We take a constant $c>1$, and we restrict x and y by $x>e^c$ and

(6.1) $$\exp\{(c\log_2 x-c\log c)/(1-(\log x)^{-1}c)\}\leqslant y<x.$$

The lower bound is roughly $c^{-c}(\log x)^c$.

We shall apply Rankin's method with $\eta=\tau$, $\tau=1-\xi/(\log y), \xi>0$, $e^\xi-1=\xi u$, $u=(\log x)/(\log y)$ (see (3.7)). If x is fixed, and y increases through the interval (6.1), then ξ decreases through the interval

(6.2) $$\log_2 x-\log c\geqslant\xi>0,$$

And, since $\tau=1-(e^\xi-1)/(\log x)$, τ increases through the interval

(6.3) $$1-c^{-1}+(\log x)^{-1}\leqslant\tau<1.$$

We apply (3.1), (4.2), and (4.3) with $\eta=\tau$:

(6.4) $$\log\Psi(x,y)\leqslant\tau\log x+\int_e^y\log\{(1-t^{-\tau})^{-1}\}\{(\log t)^{-1}+O(V(t))\}\,dt+O(1).$$

We have $\log\{(1-t^{-\tau})^{-1}\}=t^{-\tau}+O(t^{-2\tau})$, $V(t)=O((\log t)^{-1})$, whence

(6.5) $$\left\{\begin{aligned}&\log\Psi(x,y)\leqslant\log x-u\xi+\int_e^y t^{-\tau}(\log t)^{-1}\,dt+\\&\qquad+O\Big(\int_e^y t^{-\tau}V(t)\,dt\Big)+O\Big(\int_e^y t^{-2\tau}(\log t)^{-1}\,dt\Big).\end{aligned}\right.$$

Substituting $t=y^{s/\xi}$ we obtain

(6.6) $$\int_e^y t^{-\tau}(\log t)^{-1}\,dt=\int_{1-\tau}^{\xi}e^s s^{-1}\,ds.$$

It is not difficult to find a few terms of the asymptotic behaviour of the latter integral in terms of u, but accidentally a very close approximation can be obtained from the asymptotic formula for $\varrho(u)$, which is, according to [1],

(6.7) $$\varrho(u)\sim\frac{e^\gamma}{(2\pi u)^{\frac12}}\exp\left\{-\int_0^\xi\frac{se^s-e^s+1}{s}\,ds\right\}\qquad(u\to\infty),$$

also with $e^\xi-1=\xi u$; γ is Euler's constant. It follows that (6.6) equals

(6.8) $$u\,\xi+\log\varrho(u)+\log_2 y+\tfrac12\log(1+u)+O(1).$$

It results that for the region (6.1) we have

(6.9) $$\left\{\begin{aligned}&\log\Psi(x,y)\leqslant\log\{x\,\varrho(u)\,(\log y)\,(1+u)^{\frac12}\}+O(1)+\\&\qquad+O\Big(\int_e^y t^{-2\tau}(\log t)^{-1}\,dt\Big)+O\Big(\int_e^y t^{-\tau}\,V(t)\,dt\Big).\end{aligned}\right.$$

The first error term is

$$(6.10)\qquad O\Big(\int_1^{\log y} e^{\varrho s}\, s^{-1}\, ds\Big), \text{ with } \varrho = -1 + 2\xi/(\log y).$$

By (6.2) we have $\xi < \log_2 x$. It follows that (6.10) is

$$(6.11)\qquad O\Big(\int_1^{\log y} e^{\omega s}\, s^{-1}\, ds\Big), \text{ with } \omega = -1 + 2(\log_2 x)/(\log y).$$

If $-\infty < \omega < (\log y)^{-1}$ we have $e^{\omega s} = O(1)$, and (6.11) is $O(\log_2 y)$ (It can even be reduced to $O(1)$ if $y > (\log x)^{2+\varepsilon}$). If $\omega > (\log y)^{-1}$ we estimate

$$\int_1^{\log y} e^{\omega s}\, s^{-1}\, ds = \int_1^{1/\omega} + \int_{1/\omega}^{\log y} = O(\log_2 y) + O\Big(\int_1^{\omega \log y} e^z\, z^{-1}\, dz\Big) =$$
$$= O(\log_2 y) + O(e^{\omega \log y}/(\omega \log y)).$$

Since ω is given by (6.11) it follows that

$$(6.12)\qquad \int_e^y t^{-2\tau} (\log t)^{-1}\, dt = O(\log_2 y) + O((\log x)^2/y).$$

Next we turn to the second error term in (6.9). If we put $t = e^s$ it becomes

$$O\Big(\int_1^{\log y} \exp\, (s\, \xi/(\log y))\; V(e^s)\, ds\Big).$$

Since $\xi = \eta + O(1)$, where $\eta = \log u + \log_2 (u+1)$, it does not do any harm to replace ξ by η. Therefore theorem 2 follows from (6.9), at least if $x > c_1$, where c_1 is such that $x > c_1$ implies that the left-hand side of (6.1) is less than $(\log x)^c$.

Finally, (1.10) is trivial for $2 \leqslant x \leqslant c_1$, $(\log x)^c \leqslant y \leqslant x$, since the term $(\log x)^2/y$ has a positive lower bound and $\log \Psi(x, y)$ has a positive upper bound under these circumstances.

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