Chapter 3

Shallow Water Equations and the Ocean

Over most of the globe the ocean has a rather distinctive vertical structure, with an upper layer ranging from 20 m to 200 m in thickness, consisting of water heated by solar radiation, which is significantly warmer than the underlying deep water. The water in this upper layer is therefore less dense than deep ocean water. This *density stratification* is weak or absent only in certain high-latitude regions of the North Atlantic and in a few other locations. In these regions surface water has become sufficiently cold and salty (via evaporation) to sink into the deep ocean. Such regions therefore convert surface water to deep ocean water. Deep water returns to the surface layer over broad regions of the tropics and subtropics in a manner that is not completely understood. However, it is clear that this transformation must take place by the entrainment of deep ocean water into the surface layer via turbulent mixing processes. The complete circulation, consisting of sinking in selected polar regions, deep transport to the subtropics, entrainment into the surface layer, and surface transport back to the polar regions, is called the *thermohaline circulation*.

In addition to the thermohaline circulation, there are important horizontal circulations confined within the surface layer. These *gyres* are driven by wind stress acting on the ocean surface. Rather special conditions apply where the surface gyres impinge on the east coasts of continents, which result in poleward-flowing *western boundary currents*. The Gulf Stream off of the east coast of the United States is an example of a western boundary current.

Figure 3.1 shows an idealized model of the ocean structure discussed above. It is unrealistic in a number of respects: In the real ocean the density gradient with depth is continuous and the ocean bottom is not a flat surface. However, these simplifications make the most important characteristic behaviors of the ocean approachable mathematically.

3.1 Derivation of shallow water equations

Before tackling the dynamics of the two-layer ocean illustrated in figure 3.1, it is useful to "warm up" on a simpler problem, that of a single shallow layer of flowing water. We first derive the *shallow water equations* and then examine the linearized solutions about a state of rest.

Though it is possible to obtain the equations for this situation from the general fluid equations, it is actually easier to derive the shallow water equations directly from first principles.

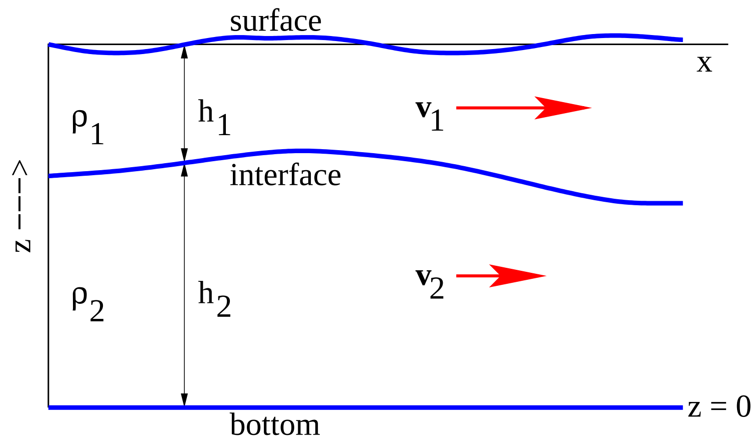

Figure 3.1: Idealized model for the vertical structure of the ocean. The density of the upper layer ρ_1 is less than that of the lower layer ρ_2.

In so doing, we make the following assumptions:

1. The water is of uniform density ρ and the layer of water has thickness $h(x, y, t)$.

2. The water flows over a horizontal, flat surface, so that h is also the elevation of the water surface.

3. The slope of the water surface is small compared to unity and the horizontal scale of flow features is large compared to the depth of the water.

4. The flow velocity is independent of depth, so that $\mathbf{v} = \mathbf{v}(x, y, t)$. This velocity is assumed to be almost horizontal.

5. Friction with the underlying surface is neglected.

6. The water within the layer is in *hydrostatic balance*. The pressure at the upper surface is zero. (This is trivially extendable to the case of constant pressure at the surface.)

As noted previously, the hydrostatic equation in geometrical vertical coordinates is

$$\frac{dp}{dz} = -g\rho. \tag{3.1}$$

Integrating downward from the surface yields pressure as a function of depth equal to

$$p = g\rho(h - z). \tag{3.2}$$

Throughout the treatment of layered models we define the gradient operator as being two-dimensional: $\nabla = (\partial/\partial x, \partial/\partial y)$. We need the horizontal gradient of the pressure for the momentum equation. From equation (3.2) we find that $-\rho^{-1}\nabla p = -g\nabla h$ is the horizontal pressure gradient force per unit mass.

The Coriolis force per unit mass is $-2\Omega \times \mathbf{v}$. In deriving the relationship between the vertical profiles of pressure and density, we have neglected the vertical component of this

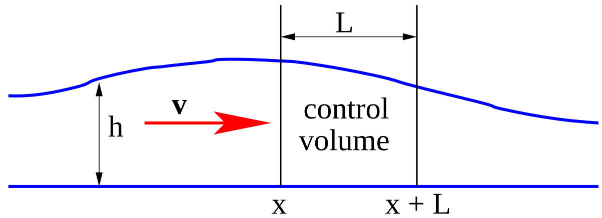

Figure 3.2: Sketch for derivation of the shallow water equations.

force because it is so small compared to the gravitational and pressure gradient forces. If we resolve $\Omega = \Omega_h + \Omega_z \hat{\mathbf{z}}$ into vertical and horizontal components, the Coriolis force becomes $-(2\Omega_h \times \mathbf{v} + 2\Omega_z \hat{\mathbf{z}} \times \mathbf{v})$. Since $\mathbf{v}$ is essentially horizontal, the first term represents mainly the vertical component of the Coriolis force, and we drop it for the sake of consistency. The quantity $f = 2\Omega_z = 2|\Omega| \sin\phi$ where ϕ is the latitude, is called the *Coriolis parameter*. The retained part of the Coriolis force is thus $-f\hat{\mathbf{z}} \times \mathbf{v}$.

Using the equations for the Coriolis force and the pressure gradient force on a unit mass of fluid, the horizontal momentum equation may be obtained directly from Newton's second law,

$$\frac{d\mathbf{v}}{dt} + g\nabla h + f\hat{\mathbf{z}} \times \mathbf{v} = 0, \tag{3.3}$$

where $\mathbf{v}$ is the (horizontal) velocity of a fluid parcel. Since the total time derivative follows the evolution of a parcel, it can be expanded using the chain rule as follows:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} \tag{3.4}$$

where we have set $dx/dt = v_x$ and $dy/dt = v_y$ according to the above ideas. This may be written more compactly as

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}. \tag{3.5}$$

The components of equation (3.3) represent two of the equations needed to determine the three variables v_x, v_y, and the thickness h of the fluid layer. The third equation needed to close the problem comes from the conservation of mass.

Referring to figure 3.2, the time rate of change of mass in the control volume, taken as a square column of side L topped by the water surface, is equal to the net flow of mass in from the sides:

$$\frac{dM_{CV}}{dt} = \rho L \left[h(x)v_x(x) - h(x+L)v_x(x+L) + h(y)v_y(y) - h(y+L)v_y(y+L) \right]. \tag{3.6}$$

The mass in the control volume may be written $M_{CV} = \rho L^2 h$ and the differences on the right side of this equation may be approximated as follows: $h(x)v_x(x) - h(x+L)v_x(x+L) = -L(\partial h v_x / \partial x)$, leading to the following governing equation for h:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{v}) = 0. \tag{3.7}$$

Note that by making a product rule expansion of the divergence term, this equation may also be written as

$$\frac{dh}{dt} + h\nabla \cdot \mathbf{v} = 0 \tag{3.8}$$

where the total time derivative has the same meaning as in equation (3.5).

3.2 Linear solutions

Let us assume a basic state of rest in which $\mathbf{v} = \mathbf{v}_0 = 0$ and $h = h_0 =$ constant. These values trivially satisfy the mass continuity and momentum equations. We now examine the behavior of small perturbations about this base state by assuming that $h = h_0(1 + \eta)$ where $|\eta| \ll 1$. We also assume that $\mathbf{v}$ is small enough that quadratic terms in $\mathbf{v}$ and η can be ignored. The mass continuity and momentum equations become

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \mathbf{v} = 0 \tag{3.9}$$

and

$$\frac{\partial \mathbf{v}}{\partial t} + gh_0 \nabla \eta + f\hat{\mathbf{z}} \times \mathbf{v} = 0. \tag{3.10}$$

We now assume that η and $\mathbf{v}$ are proportional to $\exp[i(k_x x + k_y y - \omega t)]$, resulting in three linear, homogeneous algebraic equations which may be written in matrix form:

$$\begin{pmatrix} -\omega & k_x & k_y \\ k_x g h_0 & -\omega & if \\ k_y g h_0 & -if & -\omega \end{pmatrix} \begin{pmatrix} \eta \\ v_x \\ v_y \end{pmatrix} = 0. \tag{3.11}$$

This equation only has non-trivial solutions when the determinant of the matrix of coefficients is zero, which yields a polynomial equation for ω:

$$\omega^3 - \omega(f^2 + k^2 g h_0) = 0, \tag{3.12}$$

where $k^2 = k_x^2 + k_y^2$. This has solutions

$$\omega = 0, \quad \omega = \pm(f^2 + k^2 g h_0)^{1/2}. \tag{3.13}$$

3.2.1 Geostrophic balance

The solution with $\omega = 0$ appears trivial, but is not. Substitution of this frequency value back into equation (3.11) results in the following conditions in the special case in which $k_y = 0$:

$$v_x = 0, \quad v_y = (ik_x g h_0/f)\eta. \tag{3.14}$$

This is a special case of *geostrophic balance*. The flow pattern is illustrated in figure 3.3. The sinusoidally varying pressure gradient force in the x direction induced by the corresponding fluctuations in layer thickness is countered by the Coriolis force, which is produced by the sinusoidally varying flow velocity.

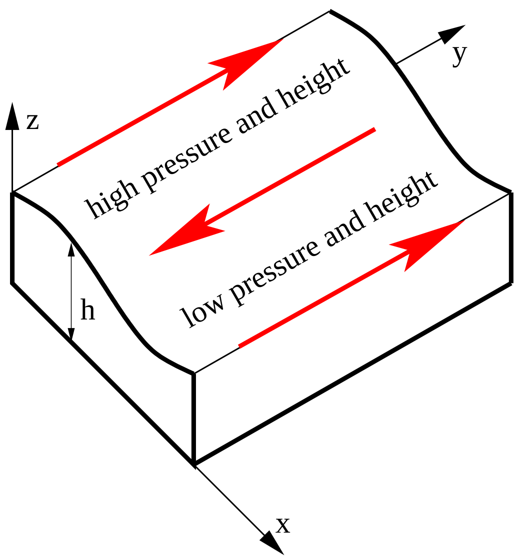

Figure 3.3: Relationship between flow and layer thickness in northern hemisphere geostrophic flow.

More generally, geostrophic balance occurs with any flow in which parcels do not accelerate, i. e., when $d\mathbf{v}/dt = 0$. In this case equation (3.3) reduces to

$$g\nabla h + f\hat{\mathbf{z}} \times \mathbf{v} = 0, \tag{3.15}$$

which shows that flow is parallel to lines of constant thickness, with lower thicknesses to the left in the northern hemisphere and to the right in the southern hemisphere. The stronger the thickness gradient, the stronger the flow.

In many circumstances where the parcel acceleration is not zero, but occurs over time scales long compared with the rotational period of the earth, i. e., when $|d\mathbf{v}/dt| \ll |f\hat{\mathbf{z}} \times \mathbf{v}|$, the actual flow is approximately geostrophic. The small deviations from geostrophic flow are much more important than their relative magnitude would suggest, and constitute the focus of much of large-scale geophysical fluid dynamics. We return to this subject later.

3.2.2 Gravity-inertia waves

The other solutions to the linearized shallow water equations are wave-like. Let us first look at the special case in which $f = 0$, i. e., for waves on the equator. In this case $\omega^2 = k^2 g h_0$, which means that the wave has phase speed $c = \omega/k = (gh_0)^{1/2}$. Equation (3.10) with f set to zero shows us that the velocity vector is in the direction of wave propagation, i. e., the wave is longitudinal. Furthermore the magnitude of the longitudinal velocity is related to the fractional thickness perturbation by

$$v_l = c\eta. \tag{3.16}$$

Thus thicker regions have fluid velocities in the direction of wave propagation, while thinner regions have fluid velocities pointing in the opposite direction.

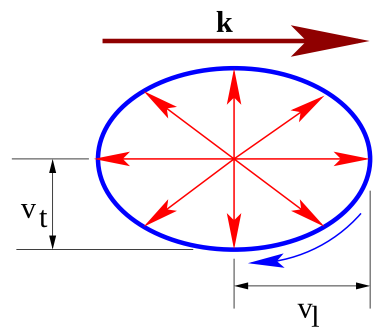

Figure 3.4: Rotation (looking down) at the rotation of the velocity vector in a gravity-inertia wave in the northern hemisphere.

If $f \neq 0$, the Coriolis force acts to deflect fluid parcels which are otherwise oscillating back and forth in the direction of wave motion. Interestingly, the frequency remains finite at zero wavenumber in this case, which means that the phase speed tends to infinity for long wavelengths:

$$u_{phase} = (f^2/k^2 + c^2)^{1/2}, \tag{3.17}$$

where $c^2 \equiv gh_0$ as before. The velocity vector rotates through 360° in an elliptical trajectory with one cycle per wave period, as shown in figure 3.4. The rotation is clockwise looking down in the northern hemisphere, and counterclockwise in the southern. The ratio of semi-minor to semi-major axes of the elliptical trajectory is given by

$$\frac{v_t}{v_l} = \frac{f}{(f^2 + k^2c^2)^{1/2}}. \tag{3.18}$$

Equation (3.16) generalizes to

$$v_l = u_p \eta \tag{3.19}$$

when $f \neq 0$.

A basin of water open at the top supports waves of all horizontal wavelengths. The shallow water solutions we have discussed here are valid for horizontal wavelengths much greater than the depth of the basin, as expected from the initial assumptions. A more exact analysis reveals the form of the waves at shorter wavelengths, but we do not pursue this analysis here. Numerous alternative sources (e. g., Kundu, 1990; Faber, 1995) present this solution.

3.3 Two-layer ocean

We now approach the problem of disturbances in the real ocean by analyzing the two-layer case illustrated in figure 3.1. For the most part, each layer can be treated separately, in the manner used to deal with the single layer case. The factor which couples the layers together is the horizontal pressure gradient term.

From the hydrostatic equation the pressure in the first or upper layer is

$$p_1 = g\rho_1(h_1 + h_2 - z), \tag{3.20}$$

where we have assumed that the pressure is zero above the water surface. From this we infer that the pressure at the interface between the two layers is

$$p_I = g\rho_1 h_1. \tag{3.21}$$

Finally, the pressure in the second layer is

$$p_2 = p_I + g\rho_2(h_2 - z) = g\rho_1 h_1 + g\rho_2(h_2 - z). \tag{3.22}$$

The mass continuity equations for each layer are identical to that for a single layer:

$$\frac{\partial h_{1,2}}{\partial t} + \nabla \cdot (h_{1,2}\mathbf{v}_{1,2}) = 0. \tag{3.23}$$

Using the fact that $\nabla p_1 = g\nabla[\rho_1(h_1 + h_2)]$ and $\nabla p_2 = g\nabla[\rho_1 h_1 + \rho_2 h_2]$, the momentum equations for the two layers are

$$\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 + g\nabla(h_1 + h_2) + f\hat{\mathbf{z}} \times \mathbf{v}_1 = 0 \tag{3.24}$$

and

$$\frac{\partial \mathbf{v}_2}{\partial t} + \mathbf{v}_2 \cdot \nabla \mathbf{v}_2 + g\nabla[(\rho_1/\rho_2)h_1 + h_2] + f\hat{\mathbf{z}} \times \mathbf{v}_2 = 0. \tag{3.25}$$

To understand how this system of equations works, it is sufficient to examine linearized solutions about a state of rest at the earth's equator, where $f = 0$. Letting $h_1 = h_{01}(1 + \eta_1)$ and $h_2 = h_{02}(1 + \eta_2)$ as before, and assuming a plane wave moving in the x direction with form $\exp[i(kx - \omega t)]$, the mass continuity equations and the x components of the momentum equations reduce to a set of four linear, homogeneous equations. In matrix form these equations are

$$\begin{pmatrix} -\omega & k & 0 & 0 \\ kgh_{01} & -\omega & kgh_{02} & 0 \\ 0 & 0 & -\omega & k \\ kg(\rho_1/\rho_2)h_{01} & 0 & kgh_{02} & -\omega \end{pmatrix} \begin{pmatrix} \eta_1 \\ v_{x1} \\ \eta_2 \\ v_{x2} \end{pmatrix} = 0. \tag{3.26}$$

Taking the determinant of the matrix of coefficients yields

$$\omega^4 - gk^2(h_{01} + h_{02})\omega^2 + g^2k^4(1 - \rho_1/\rho_2)h_{01}h_{02} = 0, \tag{3.27}$$

which has the solutions

$$\omega^2 = \frac{gk^2(h_{01} + h_{02})}{2}\left\{1 \pm \left[1 - 4\left(1 - \frac{\rho_1}{\rho_2}\right)\frac{h_{01}h_{02}}{(h_{01} + h_{02})^2}\right]^{1/2}\right\}. \tag{3.28}$$

In the ocean the fractional difference between ρ_1 and ρ_2 is tiny, which means that $|1 - \rho_1/\rho_2| \ll 1$. Using $(1 + \epsilon)^{1/2} \approx 1 + \epsilon/2$, which is valid when $|\epsilon| \ll 1$, equation (3.28) simplifies to

$$\omega^2 = \frac{gk^2(h_{01} + h_{02})}{2}\left\{1 \pm \left[1 - 2\left(1 - \frac{\rho_1}{\rho_2}\right)\frac{h_{01}h_{02}}{(h_{01} + h_{02})^2}\right]\right\}. \tag{3.29}$$

The two solutions are now explored.

3.3.1 External mode

The wave mode associated with the plus sign is called the *external mode*. The term involving the density ratio can be ignored due to its small size, resulting in

$$\omega^2 = gk^2(h_{01} + h_{02}). \tag{3.30}$$

This mode is just like that which occurs for a single shallow water layer of undisturbed depth $h_{01} + h_{02}$. The fact that this layer is made up of two sub-layers of slightly different density is of no signficance here. In the ocean this could be called the “tsunami mode”, representing the rapidly moving waves generated by sub-surface earthquakes and landslides. The phase speed of these waves is $c = [g(h_{01} + h_{02})]^{1/2}$, which is very fast for deep ocean basins. This mode is of little interest in climate applications but of great interest for tsunami predictions.

3.3.2 Internal mode

The minus sign in equation (3.29) results in

$$\omega^2 = gk^2 \left(1 - \frac{\rho_1}{\rho_2}\right) \frac{h_{01}h_{02}}{h_{01} + h_{02}} = gk^2 h_e, \tag{3.31}$$

where

$$h_e = \left(1 - \frac{\rho_1}{\rho_2}\right) \frac{h_{01}h_{02}}{h_{01} + h_{02}} \tag{3.32}$$

is called the *equivalent depth*. The significance of h_e is that a single layer of water of undisturbed depth h_e would support gravity waves of the same phase speed as this mode of the two-layer system. Note that if $h_{02} \gg h_{01}$ (the usual case), then $h_{01}h_{02}/(h_{01} + h_{02}) \approx h_{01}$.

A significant part of ocean dynamics, especially that part pertaining to climate, is contained in the dynamics of the internal mode. From equation (3.26) we infer that

$$(h_{01} - h_e)\eta_1 + h_{02}\eta_2 = 0. \tag{3.33}$$

This follows from the realization that $\omega^2/k^2 = gh_e$. The displacement of the surface of the ocean is $h'_S = h_{01}\eta_1 + h_{02}\eta_2$ and the displacement of the interface between the two layers is $h'_I = h_{02}\eta_2$. Since $h_e \ll h_{01}$, we have $\eta_1/\eta_2 \approx -h_{02}/h_{01}$ from equation (3.33). Appealing again to equation (3.33), we find that

$$h'_S = h_e\eta_1 = -h_e\eta_2 h_{02}/h_{01} = -(h_e/h_{01})h'_I. \tag{3.34}$$

In other words, the surface displacement is a small fraction h_e/h_{01} of the displacement of the interface between the two layers, and is of opposite sign.

This fact turns out to be very useful for remote sensing of the state of the ocean. Satellite-based radars can measure the vertical displacement of the ocean surface to a high degree of accuracy. This in combination with equation (3.34) allows the vertical displacement of the interface between surface and deep water to be inferred.

Equation (3.26) also yields information about the relative flow velocities in the shallow and deep layers. In particular, we find that

$$v_{x1}h_{01} = -v_{x2}h_{02}. \tag{3.35}$$

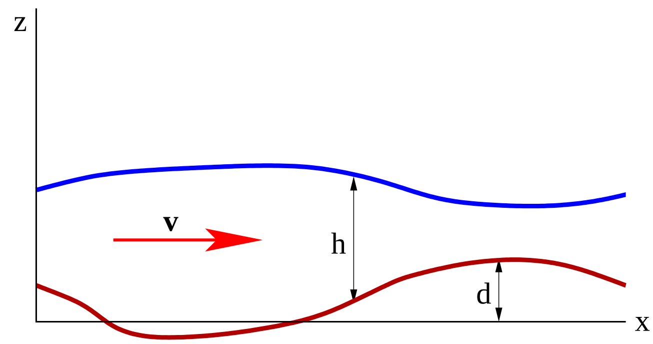

Figure 3.5: Sketch for deriving shallow water equations with terrain.

Thus, the flows in the shallow and deep layers are opposite in direction. The net horizontal mass flux is $\rho_1 v_{x1} h_{01} + \rho_2 v_{x2} h_{02}$. Since $\rho_1 \approx \rho_2$, we see that the net horizontal mass flux due to internal mode transports is nearly zero – transport in the shallow layer in a particular direction is compensated by transport in the opposite direction in the deep layer. In the usual case in which $h_{02} \gg h_{01}$, we also see that $|v_{x2}| \ll |v_{x1}|$, i. e., velocities in the surface layer are much stronger than velocities in the deep layer.

This analysis was carried out for the restricted case of linearized gravity waves on the equator. However, to the extent that (a) the density in the surface layer is only slightly less than the density of deep water, and (b) the thickness of the deep layer is much greater than the thickness of the surface layer, the result is much more general. In particular, the dynamics of the surface layer are essentially the dynamics of a shallow water flow with depth equal to the equivalent depth calculated above. The deep ocean layer responds passively to this mode with only minimal flow velocity, as governed by the condition of zero net mass flux integrated over the depth of the ocean. Under these circumstances, the assumption of a flat ocean bottom, which is highly unrealistic, has no significant effect on the results as long as the thickness of the deep layer greatly exceeds the thickness of the surface layer.

3.4 Effects of topography

We now derive the equations for shallow water flow for the case in which the underlying surface is not flat, but varies in height, with an elevation above some reference level of $d(x, y)$, as illustrated in figure 3.5. The only change from the flat bottom case is that the elevation of the upper surface of the water is now $h + d$ rather than h. The pressure as a function of height is thus $p = g\rho(h + d - z)$, with the result that the momentum equation becomes

$$\frac{d\mathbf{v}}{dt} + g\nabla(h + d) + f\hat{\mathbf{z}} \times \mathbf{v} = 0. \tag{3.36}$$

The mass continuity equation (3.7) is unchanged.

3.4.1 Steady, two-dimensional, non-rotating flow over topography

Let us imagine the special case of time-independent flow in the x direction over two-dimensional topography, $d = d(x)$, and where the environment is not rotating, i. e., $f = 0$.

In this case the governing equations (3.7) and (3.36) reduce to

$$\frac{\partial h v_x}{\partial x} = 0 \tag{3.37}$$

and

$$v_x \frac{\partial v_x}{\partial x} + g \frac{\partial (h+d)}{\partial x} = 0. \tag{3.38}$$

These easily integrate to

$$h v_x = M \tag{3.39}$$

and

$$v_x^2/2 + g(h+d) = H, \tag{3.40}$$

where M and H are constants. Note that these equations are purely algebraic, and can in principle be solved for v_x and h in terms of $d(x)$, M, and H. Unfortunately, the system is cubic, so analytic solutions, are cumbersome.

3.5 References

Faber, T. E., 1995. *Fluid dynamics for physicists.* Cambridge University Press, 440 pp.

Kundu, P. K., 1990: *Fluid mechanics.* Academic Press, 638 pp.

3.6 Laboratory

1. Given your previous density profile inferred from TAO mooring data, approximate this profile by a two-layer profile, each layer with constant density. Then, compute the speed of internal gravity waves for this profile. Assume that the thickness of the bottom layer is much greater than the thickness of the upper layer.

3.7 Problems

1. Moving base state:

   (a) Solve the steady state shallow water equations for the thickness field $h_0(y)$ in geostrophic balance with uniform flow in the x direction, $\mathbf{v} = \mathbf{v}_0 = (v_{x0}, 0)$, assuming that $h_0(0) = h_R > 0$.

   (b) Linearize the shallow water equations about this state, assuming that $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}'$ and $h = h_0 + h'$ where $\mathbf{v}'$ and h' are small. (Note that these equations no longer have constant coefficients!)

2. In the steady case on the equator where $f = 0$ and $\partial \mathbf{v}/\partial t = 0$, show that $\mathbf{v} \cdot \nabla \mathbf{v}$ can be represented as the gradient of something when $\partial v_y/\partial x = \partial v_x/\partial y$, so that the momentum equation can be written in the form $\nabla(\text{stuff}) = 0$, from which you can infer that stuff = constant. This constant is called the *Bernoulli constant*.

3. Verify equations (3.33), (3.34), and (3.35).

4. Assume an ocean 5 km deep with a surface layer 100 m deep. The density of the deep ocean is $\rho_2 = 1028$ kg m^{-3} while the density of the surface layer is $\rho_1 = 1023$ kg m^{-3}. (Assume that ocean water is incompressible.)

   (a) What is the equivalent depth of the internal mode?

   (b) Compute the speed of external and internal wave modes on the equator.

   (c) If the interface deflects downward 20 m in the internal wave, how much does the surface deflect upward?

   (d) What is the magnitude of the surface layer parcel velocity in the internal mode in this case?

   (e) What is the corresponding magnitude of the deep layer parcel velocity?

5. Consider a steady, non-rotating, two-dimensional flow over topography in which $v_x = v_0$ and $h = h_0$ when $d = 0$, where v_0 and h_0 are constant.

   (a) Compute M and H in equations (3.39) and (3.40) in terms of v_0 and h_0.

   (b) Eliminate v_x between these two equations, resulting in a cubic equation for h.

   (c) Write h in the form $h = h_0(1 + \eta)$ and assume that d is small in magnitude so that the above cubic equation for h can be linearized in η. Solve for η.

   (d) Using the above approximate solution, find the height of the surface of the shallow water layer, $h + d$ as a function of d. Consider how this height varies as a function of d, taking the cases $v_0^2 < gh_0$ and $v_0^2 > gh_0$ separately. How does the flow velocity v_x vary with d in the above two cases?