THE ANDRÉ-OORT CONJECTURE.

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ABSTRACT. In this paper we prove, assuming the Generalized Riemann Hypothesis, the André-Oort conjecture on the Zariski closure of sets of special points in a Shimura variety. In the case of sets of special points satisfying an additional assumption, we prove the conjecture without assuming the GRH.

CONTENTS

1. Introduction

1.1. **The André-Oort conjecture.** The purpose of this paper is to prove, under certain assumptions, the André-Oort conjecture on special subvarieties of Shimura varieties.

Before stating the André-Oort conjecture we provide some motivation from algebraic geometry. Let Z be a smooth complex algebraic variety and let $\mathcal{F} \longrightarrow Z$ be a variation of polarizable $\mathbb{Q}$-Hodge structures on Z (for example $\mathcal{F} = R^i f_* \mathbb{Q}$ for a smooth proper morphism $f : Y \longrightarrow Z$). To every $z \in Z$ one associates a reductive algebraic $\mathbb{Q}$-group $\mathbf{MT}(z)$, called the Mumford-Tate group of the Hodge structure $\mathcal{F}_z$. This group is the stabilizer of the Hodge classes in the rational Hodge structures tensorially generated by $\mathcal{F}_z$ and its dual. A point $z \in Z$ is said to be Hodge generic if $\mathbf{MT}(z)$ is maximal. If Z is irreducible, two Hodge generic points of Z have the same Mumford-Tate group, called the generic Mumford-Tate group $\mathbf{MT}_Z$. The complement of the Hodge generic locus is a countable union of closed irreducible algebraic subvarieties of Z, each not contained in the union of the others. This is proved in [7]. Furthermore, it is shown in [38] that when Z is defined over $\overline{\mathbb{Q}}$ (and under certain simple assumptions) these components are also defined over $\overline{\mathbb{Q}}$. The irreducible components of the intersections of these subvarieties are called *special subvarieties* (or subvarieties of Hodge type) of Z relative to $\mathcal{F}$. Special subvarieties of dimension zero are called *special points*.

Example : Let Z be the modular curve $Y(N)$ (with $N \geq 4$) and let $\mathcal{F}$ be the variation of polarizable $\mathbb{Q}$-Hodge structures $R^1 f_* \mathbb{Q}$ of weight one on Z associated to the universal elliptic curve $f : E \longrightarrow Z$. Special points on Z parametrize elliptic curves with complex multiplication. The generic Mumford-Tate group on Z is $\mathbf{GL}_{2,\mathbb{Q}}$. The Mumford-Tate group of a special point corresponding to an elliptic curve with complex multiplication by a quadratic imaginary field K is the torus $\mathrm{Res}_{K/\mathbb{Q}} \mathbf{G}_{\mathbf{m},K}$ obtained by restriction of scalars from K to $\mathbb{Q}$ of the multiplicative group $\mathbf{G}_{\mathbf{m},K}$ over K.

The general Noether-Lefschetz problem consists in describing the geometry of these special subvarieties, in particular the distribution of special points. Griffiths transversality condition prevents, in general, the existence of moduli spaces for variations of polarizable $\mathbb{Q}$-Hodge structures. Shimura varieties naturally appear as solutions to such moduli problems with additional data (c.f. [11], [12], [21]). Recall that a $\mathbb{Q}$-Hodge structure on a $\mathbb{Q}$-vector space V is a structure of $\mathbf{S}$-module on $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$, where $\mathbf{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_{\mathbf{m},\mathbb{C}}$. In other words it is a morphism of real algebraic groups

$$h \colon \mathbf{S} \longrightarrow \mathbf{GL}(V_{\mathbb{R}}) \ .$$

The Mumford-Tate group $\mathbf{MT}(h)$ is the smallest algebraic $\mathbb{Q}$-subgroup $\mathbf{H}$ of $\mathbf{GL}(V)$ such that h factors through $\mathbf{H}_{\mathbb{R}}$. A *Shimura datum* is a pair $(\mathbf{G}, X)$, with $\mathbf{G}$ a linear connected reductive group over $\mathbb{Q}$ and X a $\mathbf{G}(\mathbb{R})$-conjugacy class in the set of morphisms of real algebraic groups $\mathrm{Hom}(\mathbf{S}, \mathbf{G}_{\mathbb{R}})$, satisfying the "Deligne's conditions" [12, 1.1.13]. These conditions imply, in particular, that the connected components of X are Hermitian symmetric domains and that $\mathbb{Q}$-representations of $\mathbf{G}$ induce polarizable variations of $\mathbb{Q}$-Hodge structures on X. A morphism of Shimura data from $(\mathbf{G}_1, X_1)$ to $(\mathbf{G}_2, X_2)$ is a $\mathbb{Q}$-morphism $f : \mathbf{G}_1 \longrightarrow \mathbf{G}_2$ that maps X_1 to X_2.

Given a compact open subgroup K of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ (where $\mathbf{A}_{\mathrm{f}}$ denotes the ring of finite adèles of $\mathbb{Q}$) the set $\mathbf{G}(\mathbb{Q}) \backslash (X \times \mathbf{G}(\mathbf{A}_{\mathrm{f}})/K)$ is naturally the set of $\mathbb{C}$-points of a quasi-projective variety (a *Shimura variety*) over $\mathbb{C}$, denoted $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. The projective limit $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}} = \varprojlim_K \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is a $\mathbb{C}$-scheme on which $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ acts continuously by multiplication on the right (c.f. section 4.1.1). The multiplication by $g \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ on $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ induces an algebraic correspondence T_g on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, called a *Hecke correspondence*. One easily shows that a subvariety $V \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is *special* (with respect to some variation of Hodge structure associated to a faithful $\mathbb{Q}$-representation of $\mathbf{G}$) if and only if there is a Shimura datum $(\mathbf{H}, X_{\mathbf{H}})$, a morphism of Shimura data $f : (\mathbf{H}, X_{\mathbf{H}}) \longrightarrow (\mathbf{G}, X)$ and an element $g \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ such that V is an irreducible component of the image of the morphism :

$$\text{Sh}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}} \xrightarrow{\text{Sh}(f)} \text{Sh}(\mathbf{G}, X)_{\mathbb{C}} \xrightarrow{\cdot g} \text{Sh}(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \text{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \ .$$

It can also be shown that the Shimura datum $(\mathbf{H}, X_{\mathbf{H}})$ can be chosen in such a way that $\mathbf{H}$ is the generic Mumford-Tate group on $X_{\mathbf{H}}$ (see Lemma 2.1 of [37]). A *special* point is a special subvariety of dimension zero. One sees that a point $\overline{(x, g)} \in \text{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$ (where $x \in X$ and $g \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$) is *special* if and only the group $\mathbf{MT}(x)$ is commutative (in which case $\mathbf{MT}(x)$ is a torus).

Given a special subvariety V of $\text{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, the set of special points of $\text{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$ contained in V is dense in V for the strong (and in particular for the Zariski) topology. Indeed, one shows that V contains a special point, say s. Let $\mathbf{H}$ be a reductive group defining V and let $\mathbf{H}(\mathbb{R})^+$ denotes the connected component of the identity in the real Lie group $\mathbf{H}(\mathbb{R})$. The fact that $\mathbf{H}(\mathbb{Q}) \cap \mathbf{H}(\mathbb{R})^+$ is dense in $\mathbf{H}(\mathbb{R})^+$ implies that the "$\mathbf{H}(\mathbb{Q}) \cap \mathbf{H}(\mathbb{R})^+$-orbit" of s, which is contained in V, is dense in V. This "orbit" (sometimes referred to as the Hecke orbit of s) consists of special points. The André-Oort conjecture is the converse statement.

Definition 1.1.1. *Given a set Σ of subvarieties of $\text{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ we denote by $\boldsymbol{\Sigma}$ the subset $\cup_{V \in \Sigma} V$ of $\text{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$.*

Conjecture 1.1.2 (André-Oort)**.** *Let $(\mathbf{G}, X)$ be a Shimura datum, K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and let Σ a set of special points in $\text{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$. Then every irreducible component of the Zariski closure of $\boldsymbol{\Sigma}$ in $\text{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is a special subvariety.*

One may notice an analogy between this conjecture and the so-called Manin-Mumford conjecture (first proved by Raynaud) which asserts that irreducible components of the Zariski closure of a set of *torsion* points in an Abelian variety are translates of Abelian subvarieties by torsion points. There is a large (and constantly growing) number of proofs of the Manin-Mumford conjecture. A proof of the Manin-Mumford conjecture using *exactly* the same strategy as the one used in this paper was recently given by Ullmo and Ratazzi (see [36]).

1.2. **The results.** Our main result is the following :

Theorem 1.2.1. *Let $(\mathbf{G}, X)$ be a Shimura datum, K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and let Σ be a set of special points in $\text{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$. We make **one** of the following assumptions :*

   (1) Assume the Generalized Riemann Hypothesis (GRH) for CM fields.

(2) *Assume that there exists a faithful representation $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ such that with respect to this representation, the Mumford-Tate groups $\mathbf{MT}_s$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class as s ranges through Σ.*

Then every irreducible component of the Zariski closure of $\boldsymbol{\Sigma}$ in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is a special subvariety.

In fact we prove the following

Theorem 1.2.2. *Let $(\mathbf{G}, X)$ be a Shimura datum, K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and let Σ be a set of special subvarieties in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. We make **one** of the following assumptions :*

(1) *Assume the Generalized Riemann Hypothesis (GRH) for CM fields.*
(2) *Assume that there exists a faithful representation $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ such that with respect to this representation, the generic Mumford-Tate groups $\mathbf{MT}_V$ of V lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class as V ranges through Σ.*

Then every irreducible component of the Zariski closure of $\boldsymbol{\Sigma}$ in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is a special subvariety.

The case of theorem 1.2.2 where Σ is a set of special points is theorem 1.2.1.

1.3. **The history of the André-Oort conjecture.** For history and results obtained before 2002, we refer to the introduction of [16]. We just mention that conjecture 1.1.2 was stated by André in 1989 in the case of an irreducible curve in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing a Zariski dense set of special points, and in 1995 by Oort for irreducible subvarieties of moduli spaces of polarised Abelian varieties containing a Zariski-dense set of special points.

Let us mention some results we will use in the course of our proof.

In [9] (further generalized in [34] and [37]), the conclusion of the theorem 1.2.2 is proved for sets Σ of *strongly special* subvarieties in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ without assuming (1) or (2) (c.f. section 2). The statement is proved using ergodic theoretic techniques.

Using Galois-theoretic techniques and geometric properties of Hecke correspondences, Edixhoven and the second author (see [17]) proved the conjecture for curves in Shimura varieties containing infinite sets of special points satisfying our assumption (2). Subsequently, the second author (in [41]) proved the André-Oort conjecture for curves in Shimura varieties assuming the GRH. The main new ingredient in [41] is a theorem on lower bounds for Galois orbits of special points. In the work [15], Edixhoven proves, assuming the GRH, the André-Oort conjecture for products of modular curves. In [40], the second author proves the André-Oort conjecture for sets of special points satisfying an additional condition.

The authors started working together on this conjecture in 2003 trying to generalize the Edixhoven-Yafaev strategy to the general case of the André-Oort conjecture. In the process two main difficulties occur. One is the question of irreducibility of transforms of subvarieties under Hecke correspondences. This problem is dealt with in sections 6 and 7. The other difficulty consists in dealing with higher dimensional special subvarieties. Our strategy is to proceed by induction on the generic dimension of elements of Σ. The main ingredient for controlling the induction was the discovery by Ullmo and the second author in [37] of a possible combination of Galois theoretic and ergodic techniques. It took form while the second author was visiting the University of Paris-Sud in January-February 2005.

1.4. **Conventions.** Let F be a number field or $\mathbb{C}$. An F-algebraic variety is a reduced separated scheme over F, not necessarily irreducible. It is of finite type over F unless mentioned. A subvariety is always assumed to be a closed subvariety. In sections 2 and 3 we freely use notations recalled in section 4.

2. Equidistribution and Galois orbits.

In this section we recall a crucial ingredient in the proof of the theorem 1.2.2 : the Galois/ ergodic alternative from [37].

2.1. **Some definitions.**

2.1.1. *Shimura subdata defining special subvarieties.* Let $(\mathbf{G}, X)$ be a Shimura datum and K be a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. We fix X^+ a connected component of X and denote by $S_K(\mathbf{G}, X)_{\mathbb{C}}$ the connected component of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ image of $X^+ \times \{1\}$ in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Thus $S_K(\mathbf{G}, X)_{\mathbb{C}} = \Gamma \backslash X^+$, where $\Gamma = \mathbf{G}(\mathbb{Q})_+ \cap K$ is a neat arithmetic subgroup of $\mathbf{G}(\mathbb{Q})_+$.

Definition 2.1.1. *Let V be a special subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$. A Shimura subdatum $(\mathbf{H}_V, X_V)$ of $(\mathbf{G}, X)$ is said to define V if :*

- *the algebraic group $\mathbf{H}_V$ is the generic Mumford-Tate group on X_V.*
- *V is the image in $S_K(\mathbf{G}, X)_{\mathbb{C}}$ of a connected component of $\mathrm{Sh}_{K \cap \mathbf{H}_V(\mathbf{A}_{\mathrm{f}})}(\mathbf{H}_V, X_V)_{\mathbb{C}}$ under the natural morphism*

$$f : \mathrm{Sh}_{K \cap \mathbf{H}_V(\mathbf{A}_{\mathrm{f}})}(\mathbf{H}_V, X_V)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$$

(we emphasize here that no Hecke correspondence is involved).

Remark 2.1.2. Let V be a special subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$. If there exists a Shimura subdatum $(i : \mathbf{H}_V \hookrightarrow \mathbf{G}, X_V \subset X)$ of $(\mathbf{G}, X)$ defining V then any other Shimura subdatum

defining V is of the form $(\mathrm{Int}\gamma \circ i : \mathbf{H}_V \hookrightarrow \mathbf{G}, \gamma X_V \subset X)$ for some element $\gamma \in \Gamma_K :=$ $\mathbf{G}(\mathbb{Q})_+ \cap K$. In other words : the algebraic group $\mathbf{H}_V$ is uniquely defined by V but its embedding into $\mathbf{G}$ is uniquely defined only up to conjugation by Γ_K.

Lemma 2.1.3. *Let* $(\mathbf{G}, X)$ *be a Shimura datum with* $\mathbf{G}$ *semisimple of adjoint type. Let* K *be a neat compact open subgroup of* $\mathbf{G}(\mathbf{A}_\mathrm{f})$.

Let $(\mathbf{H}, X_{\mathbf{H}})$ *be a Shimura subdatum of* $(\mathbf{G}, X)$ *with* $\mathbf{H}$ *the generic Mumford-Tate group on* $X_{\mathbf{H}}$ *and denote by* $K_{\mathbf{H}}$ *the compact open subgroup* $K \cap \mathbf{H}(\mathbf{A}_\mathrm{f})$ *of* $\mathbf{H}(\mathbf{A}_\mathrm{f})$. *Let* V *be a special subvariety of* $S_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}$. *Then there exists a Shimura subdatum* $(\mathbf{H}_V, X_V)$ *of* $(\mathbf{H}, X_{\mathbf{H}})$ *defining* V.

Proof. The case $(\mathbf{H}, X_{\mathbf{H}}) = (\mathbf{G}, X)$ is proven in [37, lemma 2.1]. In general let $p(V)$ be the special subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ image of V under the natural morphism

$$\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}} \xrightarrow{p} \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \ .$$

As $\mathbf{G}$ is semi-simple of adjoint type there exists a Shimura subdatum $(\mathbf{H}_V, X_V)$ of $(\mathbf{G}, X)$ defining $p(V)$. One checks immediately that $(\mathbf{H}_V, X_V)$ is a Shimura subdatum of $(\mathbf{H}, X_{\mathbf{H}})$ defining V. $\qquad\qquad\square$

2.1.2. *The measure* μ_V. Let $(\mathbf{G}, X)$ be a Shimura datum with $\mathbf{G}$ semisimple of adjoint type. Let $(\mathbf{H}_V, X_V)$ be a Shimura subdatum of (G, X) defining a special subvariety V of $\S_K(\mathbf{G}, X)_{\mathbb{C}}$. Thus there exists a connected component X_V^+ of X_V, a neat arithmetic group Γ_V of the stabiliser $\mathbf{H}_V(\mathbb{Q})_+$ of X_V^+ in $\mathbf{H}_V(\mathbb{Q})^+$ and a morphism

$$f : \Gamma_V \backslash X_V^+ \longrightarrow S_K(\mathbf{G}, X)_{\mathbb{C}}$$

whose image is V.

Remark 2.1.4. The morphism f is generically injective by lemma 2.2 of [37].

Definition 2.1.5. *We define* μ_V *to be the probability measure on* $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$ *supported on* V, *push-forward by* f *of the standard probability measure on the Hermitian locally symmetric space* $\Gamma_V \backslash X_V^+$ *induced by the Haar measure on* $\mathbf{H}_V(\mathbb{R})_+$.

Remark 2.1.6. One immediately checks that the measure μ_V depends only on V and not on the choice of the embedding $\mathbf{H}_V \hookrightarrow \mathbf{G}$.

2.1.3. $\mathbf{T}$-*special subvarieties*.

Definition 2.1.7. *Let* $(\mathbf{G}, X)$ *be a Shimura datum and let* $\lambda : \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ *be the canonical morphism. Fix a (possibly trivial) subtorus* $\mathbf{T}$ *of* $\mathbf{G}^{\mathrm{ad}}$ *such that* $\mathbf{T}(\mathbb{R})$ *is compact. A* $\mathbf{T}$-*special subdatum* $(\mathbf{H}, X_{\mathbf{H}})$ *of* $(\mathbf{G}, X)$ *is a Shimura subdatum such that* $\mathbf{H}$ *is the generic Mumford-Tate group of* $X_{\mathbf{H}}$ *and* $\mathbf{T}$ *is the connected centre of* $\lambda(\mathbf{H})$.

A special subvariety V of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ is $\mathbf{T}$-special if there exists a $\mathbf{T}$-special subdatum $(\mathbf{H}_V, X_V)$ of $(\mathbf{G}, X)$ defining V.

In the case where $\mathbf{T}$ is trivial, we call V strongly special.

Remark 2.1.8. The definition of *strongly special* given in [9] requires that $\mathbf{H}_V$ is not contained in a proper parabolic subgroup of $\mathbf{G}$ but as explained in [34, rem. 3.9] this last condition is automatically satisfied.

2.2. **The rough alternative.** With these definitions, the alternative from [37] can roughly be stated as follows.

Let $(\mathbf{G}, X)$ be a Shimura datum with $\mathbf{G}$ semisimple of adjoint type, X^+ a connected component of X, K be a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and F a number field over which $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ admits a canonical model (c.f. section 4.1.2). Let $(V_n)_{n \in \mathbb{N}}$ be a sequence of special subvarieties of $S_K(\mathbf{G}, X)_{\mathbb{C}}$.

- If there exists a finite collection $\{\mathbf{T}_1, \cdots, \mathbf{T}_r\}$ of $\mathbb{R}$-anisotropic $\mathbb{Q}$-subtori of $\mathbf{G}$ such that each V_n, $n \in \mathbb{N}$, is $\mathbf{T}_i$-special for some $i \in \{1, \cdots, r\}$, then the sequence (V_n) is equidistributed in the following sense : after possibly passing to a subsequence the sequence of probability measures μ_{V_n} weakly converges to the probability measure μ_V of some special subvariety V and for n large, V_n is contained in V.

  This implies that all irreducible components of the Zariski closure of $\bigcup_{n \in \mathbb{N}} V_n$ in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ are special. Indeed, let n_k be a subsequence such that $\mu_{V_{n_k}} \longrightarrow \mu_V$ and $V_{n_k} \subset V$ for all k large enough. Let Y be the closure of the union of V_{n_k} for k such that $V_{n_k} \subset V$. Then $Y \subset V$ and by convergence of measures, the support of μ_V whose closure is V is contained in Y hence $Y = V$ and is special. Because the sequence formed by the remaining elements of the original sequence V_n is again equidistributed, we reiterate the process. Thus we show that the components of $\bigcup_{n \in \mathbb{N}} V_n$ are special.

- otherwise the function $\deg_{L_K}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V_n)$ is an unbounded function of n and we can use Galois-theoretic methods to study the Zariski closure of $\bigcup_{n \in \mathbb{N}} V_n$ in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ (c.f. definition 4.2.4 for the definition of the degree $\deg_{L_K}$) .

We now explain this alternative in more details.

2.3. **Equidistribution results.** Ratner's classification of probability measures on homogeneous spaces of the form $\Gamma \backslash \mathbf{G}(\mathbb{R})^+$ (where Γ denotes a lattice in $\mathbf{G}(\mathbb{R})^+$), ergodic under some unipotent flows [29], and Dani-Margulis recurrence lemma [10] enable Clozel and Ullmo [9] to prove the following equidistribution result in the strongly special case, generalized by Ullmo and Yafaev [37, theorem 3.8] to the $\mathbf{T}$-special case :

Theorem 2.3.1 (Clozel-Ullmo, Ullmo-Yafaev). *Let* $(\mathbf{G}, X)$ *be a Shimura datum with* $\mathbf{G}$ *semisimple of adjoint type,* K *a compact open subgroup of* $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. *Let* $\mathbf{T}$ *be an* $\mathbb{R}$-*anisotropic* $\mathbb{Q}$-*subtorus of* $\mathbf{G}$. *Let* $(V_n)_{n \in \mathbb{N}}$ *be a sequence of* $\mathbf{T}$-*special subvarieties of* $S_K(\mathbf{G}, X)_{\mathbb{C}}$. *Let* μ_{V_n} *be the canonical probability measure on* $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ *supported by* V_n. *There exists a* $\mathbf{T}$-*special subvariety* V *of* $S_K(\mathbf{G}, X)_{\mathbb{C}}$ *and a subsequence* $(\mu_{n_k})_{k \in \mathbb{N}}$ *weakly converging to* μ_V. *Furthermore* V *contains* V_{n_k} *for all* k *sufficiently large. In particular, the irreducible components of the Zariski closure of a set of* $\mathbf{T}$-*special subvarieties of* $S_K(\mathbf{G}, X)_{\mathbb{C}}$ *are special.*

Remarks 2.3.2. (1) Note that a special point of $S_K(\mathbf{G}, X)_{\mathbb{C}}$, whose Mumford-Tate group is a non-central torus, is not strongly special. Moreover, given an $\mathbb{R}$-anisotropic $\mathbb{Q}$-subtorus $\mathbf{T}$ of $\mathbf{G}$, the connected Shimura variety $S_K(\mathbf{G}, X)_{\mathbb{C}}$ contains only a finite number of $\mathbf{T}$-special points (c.f. [37, lemma 3.7]). Thus theorem 2.3.1 says nothing *directly* on the André-Oort conjecture.

(2) In fact the conclusion of the theorem 2.3.1 is simply not true for special points : they are dense for the Archimedian topology in $S_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$, so just consider a sequence of special points converging to a non-special point in $S_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$ (or diverging to a cusp if $S_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$ is non-compact). In this case the corresponding sequence of Dirac delta measures will converge to the Dirac delta measure of the non-special point (respectively escape to infinity).

(3) There is a so-called equidistribution conjecture which implies André-Oort and much more. A sequence (x_n) of points of $\mathbf{S}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$ is called *strict* if any for any proper special subvariety V of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$, the set

$$\{n : x_n \in V\}$$

is finite. Let E be a field of definition of canonical model of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$. To any special point x, one associates a probability measure Δ_x on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$ as follows :

$$\Delta_x = \frac{1}{|\mathrm{Gal}(\overline{E}/E) \cdot x|} \sum_{y \in \mathrm{Gal}(\overline{E}/E) \cdot x} \delta_y$$

where δ_y is the Dirac measure at the point y and $|\mathrm{Gal}(\overline{E}/E) \cdot x|$ denotes the cardinality of the Galois orbit $\mathrm{Gal}(\overline{E}/E) \cdot x$. The equidistribution conjecture predicts that if (x_n) is a strict sequence of special points, then the sequence of measures Δ_{x_n} weakly converges to the canonical probability measure attached to $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$. This statement implies the André-Oort conjecture. The equidistribution conjecture is known for modular curves and is open in general. There

are some recent conditional results for Hilbert modular varieties due to Zhang (see [42]). For more on this, we refer to the survey [35].

2.4. **Galois orbits of non-strongly special subvarieties.** In this paragraph, we recall the lower bound obtained in [37] for the degree of the Galois orbit of a non-strongly special subvariety in a Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, $\mathbf{G}$ semisimple of adjoint type.

Definition 2.4.1. *Let $(\mathbf{G}, X)$ be a Shimura datum. Let $K = \prod_{p \, \mathrm{prime}} K_p$ be a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let $(\mathbf{H}_V, X_V)$ be a Shimura subdatum of $(\mathbf{G}, X)$ defining a special subvariety V of $S_K(\mathbf{G}, X)_{\mathbb{C}}$.*

We denote by :

- *$E_{\mathbf{H}_V}$ the reflex field of $(\mathbf{H}_V, X_{\mathbf{H}_V})$*
- *$\mathbf{T}_V$ the connected centre of $\mathbf{H}_V$. It is a (possibly trivial) torus.*
- *$K_{\mathbf{T}_V}^{\mathrm{m}}$ the maximal compact open subgroup of $\mathbf{T}_V(\mathbf{A}_{\mathrm{f}})$.*
- *$K_{\mathbf{T}_V}$ the compact open subgroup $\mathbf{T}_V(\mathbf{A}_{\mathrm{f}}) \cap K \subset K_{\mathbf{T}_V}^{\mathrm{m}}$.*
- *$i(\mathbf{T}_V)$ the number of primes p such that $K_{\mathbf{T}_{V,p}}^{m} \neq K_{\mathbf{T}_{V,p}}$.*
- *$\mathbf{C}_V$ the torus $\mathbf{H}_V/\mathbf{H}_V^{\mathrm{der}}$ isogenous to $\mathbf{T}_V$.*
- *$d_{\mathbf{T}_V}$ the absolute value of the discriminant of the splitting field L_V of $\mathbf{C}_V$, and n_V the absolute degree of L_V.*
- *$\beta_V := \log(d_{\mathbf{T}_V})$.*

Remark 2.4.2. Notice that the group $K_{\mathbf{T}_V}$ depends on the particular embedding $\mathbf{H}_V \hookrightarrow \mathbf{G}$ (which is determined by V up to conjugation by Γ) while the other quantities defined above (and also the index $|K_{\mathbf{T}_V}^{m}/K_{\mathbf{T}_V}|$) only depend on V.

One of the main ingredients of our proof is the following lower bounds for the degree of Galois orbits of non-strongly special subvarieties obtained in [37, theorem 2.13] :

Theorem 2.4.3 (Ullmo-Yafaev). *Let $(\mathbf{G}, X)$ be a Shimura datum with $\mathbf{G}$ semisimple of adjoint type. Let $K = \prod_{p \, \mathrm{prime}} K_p$ be a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let E be a number field over which $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ admits a canonical model.*

Assume the GRH for CM fields. There exists a real number $B > 0$ and, for each positive integer N, a real number $C(N) > 0$ such that the following holds.

Let $(\mathbf{H}, X_{\mathbf{H}})$ be a Shimura subdatum of $(\mathbf{G}, X)$ with $\mathbf{H}$ the generic Mumford-Tate group of $X_{\mathbf{H}}$ and let $K_{\mathbf{H}} = K \cap \mathbf{H}(\mathbf{A}_{\mathrm{f}})$. Let V be a special subvariety of $S_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}$. Then the following inequality holds :

$$(2.1) \qquad \deg_{L_{K_{\mathbf{H}}}}(\mathrm{Gal}(\overline{\mathbb{Q}}/E) \cdot V) > C(N) \cdot \max(1, B^{i(\mathbf{T}_V)} \cdot |K_{\mathbf{T}_V}^{m}/K_{\mathbf{T}_V}| \cdot \beta_V^N) \ .$$

Furthermore, if one considers only the subvarieties V such that the associated tori $\mathbf{T}_V$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class, then the assumption of the GRH can be dropped.

2.5. **The precise alternative.** Throughout the paper we will be using the following notations.

Definition 2.5.1. *Let $(\mathbf{G}, X)$ be a Shimura datum with $\mathbf{G}$ semisimple of adjoint type. Let $K = \prod_{p \text{ prime}} K_p$ be a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$.*

Let $(\mathbf{H}, X_{\mathbf{H}})$ be a Shimura subdatum of $(\mathbf{G}, X)$ with $\mathbf{H}$ being the generic Mumford-Tate group of $X_{\mathbf{H}}$ and let $K_{\mathbf{H}} = K \cap \mathbf{H}(\mathbf{A}_{\mathrm{f}})$. Let V be a special subvariety of $S_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}$.

With the notations of definition 2.4.1 and theorem 2.4.3 we define $\alpha_V = 0$ if $\mathbf{T}_V$ is trivial, otherwise :

$$\alpha_V := B^{i(\mathbf{T}_V)} \cdot |K_{\mathbf{T}_V}^m / K_{\mathbf{T}_V}| \ .$$

The alternative roughly explained in the introduction to section 2 can now be formulated in the following theorem (easy adaptation of [37, theor. 3.9]) :

Theorem 2.5.2. *Let $(\mathbf{G}, X)$ be a Shimura datum with $\mathbf{G}$ semisimple of adjoint type and let X^+ be a connected component of X. Let $K = \prod_{p \text{ prime}} K_p$ be a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$.*

Assume the GRH for CM fields.

Let Σ be a set of special subvarieties V of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ such that $\alpha_V \beta_V$ is bounded as V ranges through Σ. There exists a finite set $\{\mathbf{T}_1, \cdots, \mathbf{T}_r\}$ of $\mathbb{Q}$-subtori of $\mathbf{G}$ such that any V in Σ is $\mathbf{T}_i$-special for some $i \in \{1, \cdots, r\}$.

Furthermore, if one considers only the subvarieties V such that the associated tori $\mathbf{T}_V$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class, then the assumption of the GRH can be dropped.

Proof. Choose a number field F such that $S_K(\mathbf{G}, X)$ admits a canonical model over F. Then the assumption that $\alpha_V \beta_V$ is bounded as V ranges through Σ implies that the degrees of $\mathrm{Gal}(\overline{F}/F)$ orbits of varieties in Σ are unbounded. The conclusion now follows from [37, theorem 3.9]. $\square$

3. Reduction and strategy.

From now on we will use the following convenient terminology :

Definition 3.0.3. *Let $(\mathbf{G}, X)$ be a Shimura datum and K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let Σ be a set of special subvarieties of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. A subset Λ of Σ is called a modification of Σ if Λ and Σ have the same Zariski closure in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Given a subtorus $\mathbf{T}$ of $\mathbf{G}^{\mathrm{ad}}$ we say that Σ is $\mathbf{T}$-special if any element in Σ is a $\mathbf{T}$-special subvariety.*

3.1. **First reduction.** We first have the following reduction of the proof of theorem 1.2.2 :

Theorem 3.1.1. *Let $(\mathbf{G}, X)$ be a Shimura datum and K a compact open subgroup of $\mathbf{G}(\mathbf{A}_f)$. Let Z be an irreducible subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Suppose that Z contains a Zariski dense set Σ, which is a union of special subvarieties V, $V \in \Sigma$, all of the same dimension $n(\Sigma) < \dim Z$.*

We make **one** *of the following assumptions :*

(1) *Assume the Generalized Riemann Hypothesis (GRH) for CM fields.*

(2) *Assume that there is a faithful representation $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ such that with respect to this representation, the connected centres $\mathbf{T}_V$ of the generic Mumford-Tate groups $\mathbf{H}_V$ of V lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class as V ranges through Σ.*

Then

(a) *The variety Z contains a Zariski dense set Σ' of special subvarieties of constant dimension $n(\Sigma') > n(\Sigma)$.*

(b) *Furthermore, if Σ satisfies the condition (2), one can choose Σ' also satisfying (2).*

Proposition 3.1.2. *Theorem 3.1.1 implies the main theorem 1.2.2.*

Proof. Let Σ as in the main theorem 1.2.2. Without loss of generality one can assume that the Zariski closure Z of Σ is irreducible. Moreover by Noetherianity one can assume that all the $V \in \Sigma$ have the same dimension $n(\Sigma)$.

Notice that the assumption (2) of the theorem 1.2.2 implies the assumption (2) of the theorem 3.1.1. We then apply theorem 3.1.1,(a) to Σ : the subvariety Z contains a Zariski-dense set Σ' of special subvarieties V', $V' \in \Sigma'$, of constant dimension $n(\Sigma') > n(\Sigma)$.

By theorem 3.1.1,(b) one can replace Σ by Σ'. Applying this process recursively and as $n(\Sigma') \leq \dim(Z)$, we conclude that Z is special. $\qquad\square$

3.2. **Second reduction.** Part (b) of theorem 3.1.1 is easy, we deal with it in section 5. Part (a) of theorem 3.1.1 can itself be reduced to the following main theorem (we refer to section 4 for relevant facts about reflex fields and to definition 6.0.4 for the (usual) definition of an F-irreducible F-variety) :

Theorem 3.2.1. *Let $(\mathbf{G}, X)$ be a Shimura datum with $\mathbf{G}$ semisimple of adjoint type and let X^+ be a connected component of X. Let K be a compact open subgroup of $\mathbf{G}(\mathbf{A}_f)$. Let F be a number field containing the reflex field $E(\mathbf{G}, X)$.*

Let Z be a Hodge generic F-irreducible F-subvariety of the connected component $S_K(\mathbf{G}, X)_{\mathbb{C}}$ of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Suppose that Z contains a Zariski dense set Σ, which is a union of special subvarieties V, $V \in \Sigma$, all of the same dimension $n(\Sigma)$ and such that for any modification Σ' of Σ the set $\{\alpha_V \beta_V, V \in \Sigma'\}$ is unbounded.

We make **one** *of the following assumptions :*

(1) *Assume the Generalized Riemann Hypothesis (GRH) for CM fields.*
(2) *Assume that there is a faithful representation* $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ *such that with respect to this representation, the connected centres* $\mathbf{T}_V$ *of the generic Mumford-Tate groups* $\mathbf{H}_V$ *of* V *lie in one* $\mathbf{GL}_n(\mathbb{Q})$-*conjugacy class as* V *ranges through* Σ.

After possibly replacing Σ *by a modification, for every* V *in* Σ *there exists a special subvariety* V' *such that* $V \subsetneq V' \subset Z$.

Proposition 3.2.2. *Theorem 3.2.1 implies theorem 3.1.1 (a).*

Proof. Let $(\mathbf{G}, X)$, K, Z and Σ be as in theorem 3.1.1.

Notice that the image of a special subvariety by a morphism of Shimura varieties deduced from a morphism of Shimura data is a special subvariety. Conversely any irreducible component of the preimage of a special subvariety by such a morphism is special.

This first implies that if $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ is a compact open subgroup and if $K' \subset K$ is a finite index subgroup then theorem 3.1.1(a) is true at level K if and only if it is true at level K'.

Let $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ be the Shimura datum adjoint to $(\mathbf{G}, X)$ and $\lambda : (\mathbf{G}, X) \longrightarrow (\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ the natural morphism of Shimura datas. For $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ sufficiently small let K^{ad} be a neat compact open subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbf{A}_{\mathrm{f}})$ containing $\lambda(K)$. Consider the (finite) morphism of Shimura varieties

$$f : \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}} \ .$$

By construction 2.3 in [17] if $Z \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ satisfies assumption (2) then its image $f(Z)$ in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ also satisfies (2). Applying our first remark to the morphism f we obtain that theorem 3.1.1 (a) for $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ implies theorem 3.1.1 (a) for $(\mathbf{G}, X)$. Thus we reduced the proof of theorem 3.1.1 (a) to the case where $\mathbf{G}$ is semisimple of adjoint type.

We can assume that the variety Z in theorem 3.1.1 is Hodge generic. To fulfill this condition, replace $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ by the smallest special subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing Z (c.f. [17, prop.2.1]). This comes down to replacing $\mathbf{G}$ with the generic Mumford-Tate group on Z.

We can also assume that Z is contained in $S_K(\mathbf{G}, X)_{\mathbb{C}}$ as proving theorem 3.1.1 for Z is equivalent to proving theorem 3.1.1 for any irreducible component of its image under some Hecke correspondence.

As Z contains a Zariski-dense set of special points, and any special point is $\overline{\mathbb{Q}}$-valued, the variety Z is defined over some number field $F \subset \mathbb{C}$ containing the reflex field $E(\mathbf{G}, X)$: $Z = Z_F \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \mathbb{C}$. As Z is irreducible Z_F is geometrically irreducible.

If for some modification Σ' of Σ the set $\{\alpha_V \beta_V,\ V \in \Sigma'\}$ is bounded, by theorem 2.5.2 and by Noetherianity there exists a $\mathbb{Q}$-subtorus $\mathbf{T}$ of $\mathbf{G}$ and a $\mathbf{T}$-special modification of Σ. Applying theorem 2.3.1 one obtains that Z is special.

Thus we can assume that Z satisfies the hypothesis of theorem 3.2.1 : we have reduced the proof of theorem 3.1.1 to the case where Z satisfies moreover the assumptions of theorem 3.2.1.

Let Σ' be the set of the special subvarieties V' obtained from theorem 3.2.1 applied to Z. Thus Z contains the Zariski-dense set $\mathbf{\Sigma'} = \cup_{V' \in \Sigma'} V'$. After possibly replacing Σ' by a modification, we can assume by Noetherianity of Z that the subvarieties in Σ' have the same dimension $n(\Sigma')$. Of course $n(\Sigma') > n(\Sigma)$. This proves the theorem 3.1.1 (a) assuming theorem 3.2.1. $\qquad\square$

3.3. **Sketch of the proof of the André-Oort conjecture in the case where Z is a curve.** The strategy for proving theorem 3.2.1 is fairly complicated. We first recall the strategy developed in [17] in the case where Z is a curve. In the next section we explain why this strategy *cannot* be directly generalized to higher dimensional cases.

As already noticed in the proof of theorem 1.2.1 one can assume without loss of generality that the group $\mathbf{G}$ is semisimple of adjoint type, Z is Hodge generic (i.e. its generic Mumford-Tate group is equal to $\mathbf{G}$), and Z is contained in the connected component $S_K(\mathbf{G}, X)_{\mathbb{C}}$ of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. The proof of the theorem 1.2.1 in the case where Z is a curve then relies on three ingredients.

3.3.1. The first one is a geometric *criterion* for a Hodge generic subvariety Z to be special in terms of Hecke correspondences. Given a Hecke correspondence T_m, $m \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ (c.f. section 4.1.1) we denote by T_m^0 the correspondence it induces on $S_K(\mathbf{G}, X)_{\mathbb{C}}$. This correspondence decomposes as $T_m^0 = \sum_i T_{q_i}$, where the q_i's are elements of $\mathbf{G}(\mathbb{Q})_+ \cap KmK$ defined by

$$\mathbf{G}(\mathbb{Q})_+ \cap KmK = \coprod \Gamma_K q_i^{-1} \Gamma_K \ .$$

Theorem 3.3.1. [17, theorem 7.1] *Let $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ be a Shimura variety, with $\mathbf{G}$ semisimple of adjoint type. Let $Z \subset S_K(\mathbf{G}, X)_{\mathbb{C}}$ be a Hodge generic subvariety of the connected component $S_K(\mathbf{G}, X)_{\mathbb{C}}$ of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Suppose there exist a prime l and an element $m \in \mathbf{G}(\mathbb{Q}_l)$ such that the neutral component $T_m^0 = \sum_{i=1}^n T_{q_i}$ of the Hecke correspondence T_m associated with m has the following properties :*

(1) $Z \subset T_m^0 Z$.
(2) For any $i \in \{1, \cdots n\}$, the variety $T_{q_i} Z$ is irreducible.
(3) For any $i \in \{1, \cdots n\}$ the $T_{q_i} + T_{q_i^{-1}}$-orbit is dense in $S_K(\mathbf{G}, X)$.

Then $Z = S_K(\mathbf{G}, X)$, in particular Z is special.

From (1) and (2) one deduces the existence of one index i such that $Z = T_{q_i} Z$. It follows that Z contains an $T_{q_i} + T_{q_i^{-1}}$-orbit. The equality $Z = S_K$ follows from (3).

In the case where Z is a curve one proves the existence of a prime l and of an element $m \in \mathbf{G}(\mathbb{Q}_l)$ satisfying these properties as follows. The property (3) is satisfied for essentially any m. The property (2), which is crucial for this strategy, is obtained by showing that for any prime l outside a finite set of primes $\mathcal{P}_Z$ and any $q \in \mathbf{G}(\mathbb{Q})^+ \cap (\mathbf{G}(\mathbb{Q}_l) \times \prod_{p \neq l} K_p)$, the variety $T_q Z$ is irreducible. This is an easy corollary of a result due independently to Weisfeiler and Nori (c.f. theorem 4.3.3) applied to the Zariski closure of the image of the monodromy representation. This result implies that for all l except those in a finite set $\mathcal{P}_Z$, the closure in $\mathbf{G}(\mathbb{Q}_l)$ of the image of the monodromy representation for the $\mathbb{Z}$-variation of Hodge structure on the smooth locus Z^{sm} of Z coincides with the closure of $K \cap \mathbf{G}(\mathbb{Q})^+$ in $\mathbf{G}(\mathbb{Q}_l)$ of the open compact subgroup $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$. To prove the property (1) one uses Galois orbits of special points contained in Z and the fact that Hecke correspondences commute with the Galois action. First one notices that Z is defined over a number field F, finite extension of the reflex field $E(\mathbf{G}, X)$ (c.f. section 4.1.2). If $s \in Z$ is a special point, r_s the associated reciprocity morphism and $m \in \mathbf{G}(\mathbb{Q}_l)$ belongs to $r_s((\mathbb{Q}_l \otimes F)^*) \subset \mathbf{MT}(s)(\mathbb{Q}_l)$ then the Galois orbit $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot s$ is contained in the intersection $Z \cap T_m Z$. If this intersection is proper its cardinality $Z \cap T_m Z$ is bounded above by a uniform constant times the degree $[K_l : K_l \cap m K_l m^{-1}]$ of the correspondence T_m. To find l and m such that $Z \subset T_m Z$ it is then enough to exhibit $m \in r_s((\mathbb{Q}_l \otimes F)^*)$ such that the cardinality $|\mathrm{Gal}(\overline{\mathbb{Q}}/F).s|$ is larger than $[K_l : K_l \cap m K_l m^{-1}]$. This is dealt with by the next two ingredients :

3.3.2. The second ingredient claims the existence of "unbounded" Hecke correspondences of controlled degree defined by elements in $r_s((\mathbb{Q}_l \otimes F)^*)$:

Theorem 3.3.2. [17, corollary 7.4.4] *There exists an integer k such that for all $s \in \Sigma$ and for any prime l such that $\mathbf{MT}(s)_{\mathbb{F}_l}$ is a split torus, there exists $m \in r_s((\mathbb{Q}_l \otimes F)^*) \subset \mathbf{MT}(s)(\mathbb{Q}_l)$ such that*

(1) for any factor $\mathbf{G}_i$ of $\mathbf{G}$ the image of m in $\mathbf{G}_i(\mathbb{Q}_l)$ is not in a compact subgroup.
(2) $[K_l : K_l \cap m K_l m^{-1}] \ll l^k$

3.3.3. The third ingredient is a lower bound for $|\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot s|$ due to Edixhoven, and improved in theorem 2.4.3.

3.3.4. Finally using this lower bound for $|\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot s|$ and the effective Cebotarev theorem consequence of GRH one proves the existence for any special point $s \in \Sigma$ with a sufficiently big Galois orbit of a prime l outside $\mathcal{P}_Z$, splitting $\mathbf{MT}(s)$, such that $\mathbf{MT}(s)_{\mathbb{F}_l}$ is a torus and such that $|\mathrm{Gal}(\overline{\mathbb{Q}}/F).s| \gg l^k$. Effective Cebotarev is not needed under the assumption that the $\mathbf{MT}(s)$, $s \in \Sigma$, are isomorphic. The reason being that in this case, the splitting field of the $\mathbf{MT}(s)$ is constant and the classical Chebotarev theorem provides us with a suitable l.

We then choose an m satisfying the conditions of the theorem 3.3.2. As $|\mathrm{Gal}(\overline{\mathbb{Q}}/F).s| \gg [K_l : K_l \cap mK_lm^{-1}]$ one obtains $Z \subset T_mZ$ and by the criterion 3.3.1 the subvariety Z is special.

3.4. Strategy for proving the theorem 1.2.2 : the general case.

Our strategy for dealing with the general case of the theorem 3.2.1 is as follows :

Let $(\mathbf{G}, X)$ be a Shimura datum with $\mathbf{G}$ semisimple of adjoint type and let X^+ be a connected component of X. Let K be a compact open subgroup of $\mathbf{G}(\mathbf{A}_\mathrm{f})$. Let Z be a subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$. Suppose that Z contains a Zariski dense set $\boldsymbol{\Sigma}$, which is a union of special subvarieties V, $V \in \Sigma$, all of the same dimension $n(\Sigma)$ and such that the set $\{\alpha_V\beta_V,\ V \in \Sigma\}$ is unbounded.

Notice that the idea of the proof of [17] generalizes to the case where $\dim Z = n(\Sigma) + 1$ (c.f. section 8.5.1). In the general case, for a V in Σ with $\alpha_V\beta_V$ sufficiently large we want to exhibit V' special subvariety in Z containing V properly.

Our first step (section 6) is geometric : we give a criterion (theorem 6.1) similar to criterion 3.3.1 saying that an inclusion $Z \subset T_mZ$, for a prime l and an element $m \in \mathbf{H}_V(\mathbb{Q}_l)$ satisfying certain conditions, implies that V is properly contained in a special subvariety V' of Z.

The criterion we need has to be much more subtle than the one in [17]. In the characterization of [17], in order to obtain the irreducibility of T_mZ the prime l must be outside some finite set $\mathcal{P}_Z$ of primes. It seems impossible to make the set of bad primes $\mathcal{P}_Z$ explicit in terms of numerical invariants of Z, except in a few cases where the Chow ring of the Baily-Borel compactification of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is easy to describe (like the case considered by Edixhoven, where $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is a product $\prod_{i=1}^{n} X_i$ of modular curves, and where he shows that for a k-dimensional subvariety Z dominant on all factors X_i, $1 \leq i \leq n$, the bad primes $p \in \mathcal{P}_Z$ are smaller than the supremum of the degree of the projections of Z on the k-factors $X_{i_1} \times \cdots \times X_{i_k}$ of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$). In particular that characterization is not suitable for our induction.

Our criterion 6.1 for an irreducible subvariety Z containing a non-strongly special subvariety V and satisfying $Z \subset T_m Z$ for some $m \in \mathbf{T}_V(\mathbb{Q}_l)$ to contain a special subvariety V' containing V properly does no longer require the irreducibility of $T_m Z$. In particular it is valid for *any* prime l, outside $\mathcal{P}_Z$ or not. Instead we notice that the inclusion $Z \subset T_m Z$ implies that Z contains the image Z' in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ of the $\langle K_l', (k_1 m k_2)^n \rangle$-orbit of (one irreducible component of) the preimage of V in the pro-l-covering of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Here k_1 and k_2 are some elements of K_l, n some positive integer and K_l' the l-adic closure of the image of the monodromy of Z. If the group $\langle K_l', (k_1 m k_2)^n \rangle$ is not compact, then the irreducible component of Z' containing V contains a special subvariety V' of Z containing V properly.

The main problem with this criterion is that the group $\langle K_l', k_1 m k_2 \rangle$ can be compact, containing K_l' with very small index. This is the case in Edixhoven's counter-example [14, Remark 7.2]. In this case $\mathbf{G} = \mathrm{PGL}_2 \times \mathrm{PGL}_2$, $K_l' := \Gamma_0(l) \times \Gamma_0(l)$ and $k_1 m k_2$ is $w_l \times w_l$, the product of two Atkin-Lehner involutions. The index $[\langle K_l', k_1 m k_2 \rangle : K_l']$ is four.

Our second step (section 7) consists in getting rid of this problem and is purely group-theoretic. We notice that if K_l is *not a maximal* compact open subgroup but is contained in an *Iwahori* subgroup of $\mathbf{G}(\mathbb{Q}_l)$, then for "many" m in $\mathbf{T}_V(\mathbb{Q}_l)$ the element $k_1 m k_2$ is not contained in a compact subgroup for any k_1 and k_2 in K_l. This is our theorem 7.1 about the existence of adequate Hecke correspondences. The proof relies on simple properties of the Bruhat-Tits decomposition of $\mathbf{G}(\mathbb{Q}_l)$.

Our third step (section 8) is Galois-theoretic and geometric. We use theorem 2.4.3, theorem 6.1, theorem 7.1 to show (under one of the assumptions of theorem 3.1.1) that the existence of a prime number l satisfying certain conditions forces a subvariety Z of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing a non-strongly special subvariety V to contain a special subvariety V' containing V properly. The proof is a nice geometric induction on $r = \dim Z - \dim V$.

Our last step (section 9) is number-theoretic : we complete the proof of the theorem 3.2.1 and hence of theorem 1.2.2 by exhibiting, using effective Chebotarev under GRH (or usual Chebotarev under the second assumption of theorem 1.2.2), a prime l satisfying our desiderata. For this step it is crucial that both the index of an Iwahori subgroup in a maximal compact subgroup of $\mathbf{G}(\mathbb{Q}_l)$ and the degree of the correspondence T_m are bounded by a uniform power of l.

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4. Preliminaries.

4.1. Notations. In this section we define some notations and recall some standard facts about Shimura varieties that we will use in this paper. We refer to [11], [12], [21] for details.

As far as groups are concerned, reductive algebraic groups are assumed to be connected. The exponent ^{0} denotes the algebraic neutral component and the exponent $^+$ the topological neutral component. Thus if $\mathbf{G}$ is a $\mathbb{Q}$-algebraic group $\mathbf{G}(\mathbb{R})^+$ denotes the topological neutral component of the real Lie group of $\mathbb{R}$-points $\mathbf{G}(\mathbb{R})$. We also denote by $\mathbf{G}(\mathbb{Q})^+$ the intersection $\mathbf{G}(\mathbb{R})^+ \cap \mathbf{G}(\mathbb{Q})$.

When $\mathbf{G}$ is reductive we denote by $\mathbf{G}^{\mathrm{ad}}$ the adjoint group of $\mathbf{G}$ (the quotient of $\mathbf{G}$ by its center) and by $\mathbf{G}(\mathbb{R})_+$ the preimage in $\mathbf{G}(\mathbb{R})$ of $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$. The notation $\mathbf{G}(\mathbb{Q})_+$ denotes the intersection $\mathbf{G}(\mathbb{R})_+ \cap \mathbf{G}(\mathbb{Q})$. In particular when $\mathbf{G}$ is adjoint then $\mathbf{G}(\mathbb{Q})^+ = \mathbf{G}(\mathbb{Q})_+$.

For any topological space Z, we denote by $\pi_0(Z)$ the set of connected components of Z.

4.1.1. Shimura varieties. Let $(\mathbf{G}, X)$ be a Shimura datum. We fix X^+ a connected component of X. Given K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ one obtains the homeomorphic decomposition

$$(4.1) \qquad \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} = \mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbf{A}_{\mathrm{f}})/K \simeq \coprod_{g \in \mathcal{C}} \Gamma_g \backslash X^+ \ ,$$

where $\mathcal{C}$ denotes a set of representatives for the (finite) double coset space $\mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbf{A}_{\mathrm{f}})/K$, and Γ_g denotes the arithmetic subgroup $gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$ of $\mathbf{G}(\mathbb{Q})_+$. We denote by Γ_K the group Γ_e corresponding to the identity element $e \in \mathcal{C}$ and by $S_K(\mathbf{G}, X)_{\mathbb{C}} = \Gamma_K \backslash X^+$ the corresponding connected component of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$.

The Shimura variety $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ is the $\mathbb{C}$-scheme projective limit of the $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, K ranging through compact open subgroups of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. The group $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ acts continuously on the right on $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$. The set of $\mathbb{C}$-points of $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ is

$$\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C}) = \frac{\mathbf{G}(\mathbb{Q})}{\mathbf{Z}(\mathbb{Q})} \backslash (X \times \mathbf{G}(\mathbf{A}_{\mathrm{f}}) / \overline{\mathbf{Z}(\mathbb{Q})}) \ ,$$

where $\mathbf{Z}$ denotes the centre of $\mathbf{G}$ and $\overline{\mathbf{Z}(\mathbb{Q})}$ denotes the closure of $\mathbf{Z}(\mathbb{Q})$ in $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ [12, prop.2.1.10]. The action of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ on the right is given by : $\overline{(x, h)} \xrightarrow{\cdot g} \overline{(x, h \cdot g)}$. For $m \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$, we denote by T_m the Hecke correspondence

$$\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \longleftarrow \mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}} \xrightarrow{\cdot m} \mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \ .$$

4.1.2. *Reciprocity morphisms and canonical models.* Given $(\mathbf{G}, X)$ a Shimura datum, where X is the $\mathbf{G}(\mathbb{R})$-conjugacy class of some $h : \mathbf{S} \longrightarrow \mathbf{G}_{\mathbb{R}}$, we denote by $\mu_h : \mathbf{G}_{m, \mathbb{C}} \longrightarrow \mathbf{G}_{\mathbb{C}}$ the $\mathbb{C}$-morphism of $\mathbb{Q}$-groups obtained by composing the embedding of tori

$$\begin{array}{ccc} \mathbf{G}_{m, \mathbb{C}} & \longrightarrow & \mathbf{S}_{\mathbb{C}} \\ z & \longrightarrow & (z, 1) \end{array}$$

with $h_{\mathbb{C}}$. Let $E(\mathbf{G}, X)$ be the field of definition of the $\mathbf{G}(\mathbb{C})$-conjugacy class of μ_h, it is called the reflex field of $(\mathbf{G}, X)$. In the case where $\mathbf{G}$ is a torus $\mathbf{T}$ and $X = \{h\}$ we denote by

$$r_{(\mathbf{T}, \{h\})} : \mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}} \longrightarrow \mathbf{T}(\mathbf{A}_{\mathrm{f}}) / \overline{\mathbf{T}(\mathbb{Q})}$$

the reciprocity morphism defined in [12, 2.2.3] for any field $E \subset \mathbb{C}$ containing $E(\mathbf{T}, \{h\})$. Let $x = \overline{(h, g)}$ be a special point in $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ image of the pair $(h : \mathbf{S} \longrightarrow \mathbf{T} \subset \mathbf{G}, g) \in X \times \mathbf{G}(\mathbf{A}_{\mathrm{f}})$. The field $E(h) = E(\mathbf{T}, \{h\})$ depends only on h and is an extension of $E(\mathbf{G}, X)$ [12, 2.2.1]. The Shimura variety $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ admits a unique model $\mathrm{Sh}(\mathbf{G}, X)$ over $E(\mathbf{G}, X)$ such that the $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$-action on the right is defined over $E(\mathbf{G}, X)$, the special points are algebraic and if $x = \overline{(h, g)}$ is a special point of $\mathrm{Sh}(\mathbf{G}, X)(\mathbb{C})$ then an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E(h)) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/E(\mathbf{G}, X))$ acts on x by $\sigma(x) = \overline{(h, \tilde{r}(\sigma)g)}$, where $\tilde{r}(\sigma) \in \mathbf{T}(\mathbf{A}_{\mathrm{f}})$ is any lift of $r_{(\mathbf{T}, \{h\})}(x) \in \mathbf{T}(\mathbf{A}_{\mathrm{f}}) / \overline{\mathbf{T}(\mathbb{Q})}$, c.f. [12, 2.2.5]. This is called the canonical model of $\mathrm{Sh}(\mathbf{G}, X)$. For any compact open subgroup K of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$, one obtains the canonical model for $\mathrm{Sh}_K(\mathbf{G}, X)$ over $E(\mathbf{G}, X)$. For details on this definition, sketches of proofs of the existence and uniqueness and all the relevant references we refer the reader to Chapters 12-14 of [21].

For $m \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ the Hecke correspondence T_m is defined over $E(\mathbf{G}, X)$. We will denote by $\pi_K : \mathrm{Sh}(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)$ the natural projection.

4.1.3. *Galois action on the set of connected components of a special subvariety.* In this subsection we recall the explicit description of the Galois action on the set of connected components of special subvarieties of Shimura varieties. All of this is taken from [12], sections 2.4-2.6.

Let $(\mathbf{G}, X)$ be a Shimura datum and let $(\mathbf{H}, X_{\mathbf{H}})$ be a Shimura sub datum of $(\mathbf{G}, X)$. Let $\mathbf{T}$ be the connected centre of $\mathbf{H}$. Note that we do not rule out the case when $\mathbf{T}$ is trivial. The group $\mathbf{H}$ is an almost direct product $\mathbf{T}\mathbf{H}^{\mathrm{der}}$.

Let $K_{\mathbf{H}}$ be the compact open subgroup $\mathbf{H}(\mathbf{A}_{\mathrm{f}}) \cap K$ of $\mathbf{H}(\mathbf{A}_{\mathrm{f}})$. We describe the Galois action on the set of components of $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})$. Let $\pi_0(\mathbf{H}, K_{\mathbf{H}})$ be the set of geometric components of $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})$. Recall ([12] 2.1.3.1) that

$$\pi_0(\mathbf{H}, K_{\mathbf{H}}) = \mathbf{H}(\mathbb{Q})_+ \backslash \mathbf{H}(\mathbf{A}_{\mathrm{f}}) / K_{\mathbf{H}} = \mathbf{H}(\mathbf{A}_{\mathrm{f}}) / \mathbf{H}(\mathbb{Q})_+ K_{\mathbf{H}}$$

where $\mathbf{H}(\mathbb{Q})_+$ is the stabiliser of a connected component of $X_{\mathbf{H}}$ in $\mathbf{H}(\mathbb{Q})$. Let $E = E_{\mathbf{H}}$ be the reflex field of $(\mathbf{H}, X_{\mathbf{H}})$ and $\mathbf{T}_E := \mathrm{Res}_{E_{\mathbf{H}}/\mathbb{Q}} \mathbf{G}_{\mathbf{m}E_{\mathbf{H}}}$.

Following Deligne ([12] 2.0.15.1) we define for any reductive $\mathbb{Q}$-group $\mathbf{N}$

$$\pi(\mathbf{N}) := \mathbf{N}(\mathbf{A})/\mathbf{N}(\mathbb{Q})\rho(\widetilde{\mathbf{N}}(\mathbf{A})).$$

Here $\rho \colon \widetilde{\mathbf{N}} \longrightarrow \mathbf{N}^{\mathrm{der}}$ denotes the universal covering of $\mathbf{N}^{\mathrm{der}}$. We set

$$\overline{\pi_0}(\pi(\mathbf{N})) := \pi_0(\pi(\mathbf{N}))/\pi_0(\mathbf{N}(\mathbb{R})_+),$$

then by ([12] 2.1.3.2) we have

$$\pi_0(\mathbf{H}, K_{\mathbf{H}}) = \overline{\pi_0}(\pi(\mathbf{H}))/K_{\mathbf{H}}.$$

The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E_{\mathbf{H}})$ on $\pi_0(\mathbf{H}, K_{\mathbf{H}})$ is given by the reciprocity morphism ([12] 2.6.1.1)

$$r_{(\mathbf{H}, X_{\mathbf{H}})} \colon \mathrm{Gal}(\overline{\mathbb{Q}}/E_{\mathbf{H}}) \longrightarrow \overline{\pi_0}(\pi(\mathbf{H})) \ .$$

The morphism $r_{(\mathbf{H}, X_{\mathbf{H}})}$ factors through $\mathrm{Gal}(\overline{\mathbb{Q}}/E_{\mathbf{H}})^{\mathrm{ab}}$ which is identified with $\pi_0(\pi(\mathbf{T}_E))$ by the global class field theory.

Let $\mathbf{C}$ be $\mathbf{H}/\mathbf{H}^{\mathrm{der}}$. To $(\mathbf{H}, X_{\mathbf{H}})$ one associates two Shimura data $(\mathbf{C}, \{x\})$ and $(\mathbf{H}^{\mathrm{ad}}, X_{\mathbf{H}^{\mathrm{ad}}})$. The field $E_{\mathbf{H}}$ is the composite of $E(\mathbf{C}, \{x\})$ and $E(\mathbf{H}^{\mathrm{ad}}, X_{\mathbf{H}^{\mathrm{ad}}})$. There are morphisms of Shimura data

$$\theta^{\mathrm{ab}} \colon (\mathbf{H}, X_{\mathbf{H}}) \longrightarrow (\mathbf{C}, \{x\}) \text{ and } \theta^{\mathrm{ad}} \colon (\mathbf{H}, X_{\mathbf{H}}) \longrightarrow (\mathbf{H}^{\mathrm{ad}}, X_{\mathbf{H}^{\mathrm{ad}}}).$$

Note that $(\mathbf{C}, \{x\})$ is a special Shimura datum. Let $r_{(\mathbf{C}, \{x\})}$ be the reciprocity morphism associated with $(\mathbf{C}, \{x\})$. The morphism θ^{ab} induces a morphism $\overline{\pi_0}(\pi(\mathbf{H})) \to \overline{\pi_0}(\pi(\mathbf{C}))$. This morphism preceded by $r_{(\mathbf{H}, X_{\mathbf{H}})}$ is $r_{(\mathbf{C}, \{x\})}$. We let L be the Galois closure of E_H. Note that the degree of L over $\mathbb{Q}$ is bounded uniformly on $(\mathbf{H}, X_{\mathbf{H}})$. We will keep the notations and assumptions introduced above throughout this section.

4.1.4. *The tower of Shimura varieties at a prime l.* Let l be a prime. Suppose $K^l \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}}^l)$ is a compact open subgroup, where $\mathbf{A}_{\mathrm{f}}^l$ denotes the ring of finite adèles outside l.

Definition 4.1.1. *We denote by $\mathrm{Sh}_{K^l}(\mathbf{G}, X)$ the $E(\mathbf{G}, X)$-scheme $\varprojlim \mathrm{Sh}_{K^l \cdot U_l}(\mathbf{G}, X)$ for U_l compact open subgroup of $\mathbf{G}(\mathbb{Q}_l)$.*

The scheme $\mathrm{Sh}_{K^l}(\mathbf{G}, X)$ is the quotient $\mathrm{Sh}(\mathbf{G}, X)/K^l$. It admits a continuous $\mathbf{G}(\mathbb{Q}_l)$-action on the right. Given a compact open subgroup $U_l \subset \mathbf{G}(\mathbb{Q}_l)$ we denote by $\pi_{U_l} : \mathrm{Sh}_{K^l}(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_{K^l U_l}(\mathbf{G}, X)$ the canonical projection.

4.1.5. *Neatness.* Let $\mathbf{G}$ be a linear algebraic group over $\mathbb{Q}$. We recall the definition of *neatness* for subgroups of $\mathbf{G}(\mathbb{Q})$ and its generalization to subgroups of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. We refer to [3] and [26, 0.6] for more details.

Given an element $g \in \mathbf{G}(\mathbb{Q})$ let $\mathrm{Eig}(g)$ be the subgroup of $\overline{\mathbb{Q}}^*$ generated by the eigenvalues of g. We say that $g \in \mathbf{G}(\mathbb{Q})$ is *neat* if the subgroup $\mathrm{Eig}(g)$ is torsion-free. A subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is neat if any element of Γ is neat. In particular such a group is torsion-free.

Given an element $g_p \in \mathbf{G}(\mathbb{Q}_p)$ let $\mathrm{Eig}_p(g_p)$ be the subgroup of $\overline{\mathbb{Q}_p}^*$ generated by all eigenvalues of g_p. Let $\overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}_p}$ be some embedding and consider the torsion part $(\overline{\mathbb{Q}}^* \cap \mathrm{Eig}_p(g_p))_{\mathrm{tors}}$. Since every subgroup of $\overline{\mathbb{Q}}^*$ consisting of roots of unity is normalized by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, this group does not depend on the choice of the embedding $\overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}_p}^*$. We say that g_p is *neat* if

$$(\overline{\mathbb{Q}}^* \cap \mathrm{Eig}_p(g))_{\mathrm{tors}} = \{1\} \ .$$

We say that $g = (g_p)_p \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ is neat if

$$\bigcap_p (\overline{\mathbb{Q}}^* \cap \mathrm{Eig}_p(g_p))_{\mathrm{tors}} = \{1\} \ .$$

A subgroup $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ is neat if any element of K is neat. Of course if the projection K_p of K in $\mathbf{G}(\mathbb{Q}_p)$ is neat then K is neat. Notice that if K is a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ then all of the Γ_g in the decomposition (4.1) are.

Neatness is preserved by conjugacy and intersection with an arbitrary subgroup. Moreover if $\rho : \mathbf{G} \longrightarrow \mathbf{H}$ is a $\mathbb{Q}$-morphism of linear algebraic $\mathbb{Q}$-groups and $g \in \mathbf{G}(\mathbb{Q})$ (resp. $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$) is neat then its image $\rho(g)$ is also neat.

We recall the following well-known lemma :

Lemma 4.1.2. *Let $K = \prod_p K_p$ be a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and let $l \geq 3$ be a prime number. There exists an open subgroup K_l' of K_l such that the subgroup $K' := K_l' \times \prod_{p \neq l} K_p$ of K is neat.*

Proof. As noticed above if K_l' is neat then $K' := K_l' \times \prod_{p \neq l} K_p$ is neat. As a subgroup of a neat group is neat, it is enough to show that a special maximal compact open subgroup

$K_l \subset \mathbf{G}(\mathbb{Q}_l)$ contains a neat subgroup K_l' with finite index. By [26, p.5] one can take, $K_l' = K_l^{(1)}$ the first congruence kernel. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. **Baily-Borel compactification and degrees of subvarieties.** In this section we recall the results we will need on projective geometry of Shimura varieties. We will also prove a proposition (proposition 4.2.11) which compares the degrees of a subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)$ with respect to two different line bundles.

4.2.1. *Degrees.* We will need only basics on numerical intersection theory as recalled in [20, chap.1, p.15-17]. Let X be a complete irreducible complex variety and L a line bundle on X with topological first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$. Given $V \subset X$ an irreducible subvariety we define the degree of V with respect to L by

$$\deg_L V = c_1(L)^{\dim V} \cap [V] \in H_0(X, \mathbb{Z}) = \mathbb{Z}\ ,$$

where $[V] \in H_{2\dim V}(X, \mathbb{Z})$ denotes the fundamental class of V and $\cap$ denotes the cap product between $H^{2\dim V}(X, \mathbb{Z})$ and $H_{2\dim V}(X, \mathbb{Z})$. We also write $\deg_L V = \int_V c_1(L)^{\dim V}$. It satisfies the projection formula : given $f : Y \longrightarrow X$ a generically finite surjective proper map one has

$$\deg_{f^*L} Y = (\deg f) \deg_L X\ .$$

All this naturally extends to the case where X or V are possibly non-reduced complete complex schemes provided V is of pure dimension.

4.2.2. *Nefness.* Recall (c.f. [20, def. 1.4.1]) that a line bundle L on a complete scheme X is said to be *nef* if $\deg_L C \geq 0$ for every irreducible curve $C \subset X$. We will need the following basic result (c.f. [20, theor.1.4.9]) :

Theorem 4.2.1 (Kleiman). *Let L be a line bundle on a complete complex scheme X. Then L is nef if and only if for every irreducible subvariety $V \subset X$ one has $\deg_L V \geq 0$.*

4.2.3. *Baily-Borel compactification.* Let $(\mathbf{G}, X)$ be a Shimura datum. Given $K \subset \mathbf{G}(\mathbf{A}_f)$ a neat compact open subgroup, let $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ the corresponding complex Shimura variety.

Definition 4.2.2. *We denote by $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ the Baily-Borel compactification of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, c.f. [2].*

The Baily-Borel compactification $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ is a normal projective variety. Its boundary $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}} \setminus \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ has complex codimension > 1 if and only if $\mathbf{G}$ has no split $\mathbb{Q}$-simple factors of dimension 3. The following proposition summarizes basic properties of $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ that we will use.

Proposition 4.2.3. *(1) The line bundle of holomorphic forms of maximal degree on X descends to $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ and extends uniquely to an ample line bundle L_K on $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ such that, at the generic points of the boundary components of codimension one, it is given by forms with logarithmic poles. Let K_1 and K_2 be neat compact open subgroups of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and g in $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ such that $K_2 \subset gK_1g^{-1}$. Then the morphism from $\mathrm{Sh}_{K_2}(\mathbf{G}, X)_{\mathbb{C}}$ to $\mathrm{Sh}_{K_1}(\mathbf{G}, X)_{\mathbb{C}}$ induced by g extends to a morphism $f : \overline{\mathrm{Sh}_{K_2}(\mathbf{G}, X)_{\mathbb{C}}} \longrightarrow \overline{\mathrm{Sh}_{K_1}(\mathbf{G}, X)_{\mathbb{C}}}$, and the line bundle $f^*L_{K_1}$ is canonically isomorphic to L_{K_2} .*

(2) The canonical model $\mathrm{Sh}_K(\mathbf{G}, X)$ of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ over the reflex field $E(\mathbf{G}, X)$ admits a unique extension to a model $\overline{\mathrm{Sh}_K(\mathbf{G}, X)}$ of $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ over $E(\mathbf{G}, X)$. The line bundle L_K is naturally defined over $E(\mathbf{G}, X)$.

(3) Let $\varphi : (\mathbf{H}, Y) \longrightarrow (\mathbf{G}, X)$ be a morphism of Shimura data and $K_{\mathbf{H}} \subset \mathbf{H}(\mathbf{A}_{\mathrm{f}})$, $K_{\mathbf{G}} \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ neat compact open subgroups with $\varphi(K_{\mathbf{H}}) \subset K_{\mathbf{G}}$. Then the canonical map $\phi : \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y) \longrightarrow \mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G}, X)$ induced by φ extends to a morphism still denoted by $\phi : \overline{\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)} \longrightarrow \overline{\mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G}, X)}$.

Proof. The first statement is [2, lemma 10.8] and [26, prop.8.1, sections 8.2, 8.3]. The second one is [26, theor.12.3.a]. The third statement is [30, theorem p.231] (over $\mathbb{C}$) and [26, theor. 12.3.b] (over $E(\mathbf{G}, X)$). $\qquad\square$

Definition 4.2.4. *Given a complex subvariety $Z \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ we will denote by $\deg_{L_K} Z$ the degree of the compactification $\overline{Z} \subset \overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ with respect to the line bundle L_K. We will write $\deg Z$ when it is clear to which line bundle we are referring to.*

Remark 4.2.5. More generally given a connected semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$ of Hermitian type (and of non-compact type) with associated Hermitian domain X and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ a neat arithmetic lattice, the Baily-Borel compactification $\overline{\Gamma \backslash X}$ of the quasi-projective complex variety $\Gamma \backslash X$ and the bundle L_Γ on $\overline{\Gamma \backslash X}$ are well-defined.

4.2.4. *Comparison of degrees for sub-Shimura data.*

Proposition 4.2.6. *Let $\phi : \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_{K'}(\mathbf{G}', X')_{\mathbb{C}}$ be a morphism of Shimura varieties associated to a Shimura sub-datum $\varphi : (\mathbf{G}, X) \longrightarrow (\mathbf{G}', X')$, a neat compact open subgroup K of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and a neat compact open subgroup K' of $\mathbf{G}'(\mathbf{A}_{\mathrm{f}})$ containing $\varphi(K)$. Then the line bundle*

$$\Lambda_{K,K'} := \phi^*L_{K'} \otimes L_K^{-1}$$

on $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ is nef.

This proposition is a corollary of the following

Proposition 4.2.7. *Let $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$ be a $\mathbb{Q}$-morphism of connected semisimple algebraic $\mathbb{Q}$-groups of Hermitian type (and of non-compact type) inducing a holomorphic totally geodesic embedding of the associated Hermitian domains $\phi : X^+ \longrightarrow X'^+$. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a neat arithmetic lattice and $\Gamma' \subset \mathbf{G}'(\mathbb{Q})$ a neat arithmetic lattice containing $\varphi(\Gamma)$. Then the line bundle*

$$\Lambda_{\Gamma,\Gamma'} := \phi^* L_{\Gamma'} \otimes L_\Gamma^{-1}$$

on $\overline{\Gamma \backslash X}$ is nef.

Proposition 4.2.7 implies the proposition 4.2.6. Let $C \subset \overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ be an irreducible curve. To prove that $\deg_{\Lambda_{K,K'}} C \geq 0$ one can assume without loss of generality that C is contained in the connected component $\overline{S_K} = \overline{\Gamma_K \backslash X^+}$ and that $\phi : \overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}} \longrightarrow \overline{\mathrm{Sh}_{K'}(\mathbf{G}', X')_{\mathbb{C}}}$ maps $\overline{S_K}$ to $\overline{S_{K'}} = \overline{\Gamma_{K'} \backslash X'^+}$. The morphism of reductive $\mathbb{Q}$-groups $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$ induces a $\mathbb{Q}$-morphism $\overline{\varphi} : \mathbf{G}^{\mathrm{der}} \longrightarrow \mathbf{G}'^{\mathrm{ad}}$ of semisimple $\mathbb{Q}$-groups. Let Γ denote the neat lattice $\mathbf{G}^{\mathrm{der}}(\mathbb{Q}) \cap K \subset \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ and Γ' the neat lattice of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q})$ image of $\Gamma_{K'}$. Notice that $\Gamma' \backslash X'^+ = \Gamma_{K'} \backslash X'^+$. Consider the diagram

(4.2)
$$
\begin{array}{c}
\overline{\Gamma \backslash X^+} \\
 \\
\end{array}
$$

$$
\begin{array}{ccc}
 & \overline{\Gamma \backslash X^+} & \\
 \pi \downarrow & \searrow {\scriptstyle \phi \circ \pi} & \\
 \overline{\Gamma_K \backslash X^+} & \xrightarrow{\phi} & \overline{\Gamma' \backslash X'^+}
\end{array}
$$

with π the natural finite étale map. The proposition 4.2.3 (1) easily extends to this setting :

$$\pi^*(L_{\Gamma_K}) = L_\Gamma \ .$$

Thus

$$\pi^* \Lambda_{K,K'} = \Lambda_{\Gamma,\Gamma'} \ .$$

Let d denote the degree of π. By the projection formula one obtains :

$$\deg_{\Lambda_{K,K'}} C = \frac{1}{d} \deg_{\Lambda_{K,K'}} \pi^{-1}(C) \ .$$

Now $\deg_{\Lambda_{K,K'}} \pi^{-1}(C) \geq 0$ by proposition 4.2.7.

<div style="text-align: right;">□</div>

Proof of the proposition 4.2.7. Let $C \subset \overline{\Gamma \backslash X}$ be an irreducible curve. We want to show that $\deg_{\Lambda_{\Gamma,\Gamma'}} C \geq 0$. First notice that by the projection formula and by proposition 4.2.3 (1), we can assume that the group $\mathbf{G}$ is simply connected and the group $\mathbf{G}'$ is adjoint.

Let $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$ be the decomposition of $\mathbf{G}$ into $\mathbb{Q}$-simple factors. Let $\varphi_i : \mathbf{G}_i \longrightarrow \mathbf{G}'$, $1 \leq i \leq r$ denote the components of $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$. If $\Gamma_1 \subset \Gamma$ is a finite

index subgroup and $p : \overline{\Gamma_1 \backslash X^+} \longrightarrow \overline{\Gamma \backslash X^+}$ is the corresponding finite étale morphism, by proposition 4.2 the line bundle $\Lambda_{\Gamma_1, \Gamma'}$ corresponding to $\phi \circ p$ is isomorphic to $p^* \Lambda_{\Gamma, \Gamma'}$. The fact that $\deg_{\Lambda_{\Gamma, \Gamma'}} C \geq 0$ is once more implied by $\deg_{\Lambda_{\Gamma_1, \Gamma'}} p^{-1}(C) \geq 0$. Thus we can assume that $\Gamma = \Gamma_1 \times \cdots \times \Gamma_r$, with Γ_i a neat arithmetic subgroup of $\mathbf{G}_i(\mathbb{Q})$. The variety $\overline{\Gamma \backslash X^+}$ decomposes into a product

$$\overline{\Gamma \backslash X^+} = \overline{\Gamma_1 \backslash X_1^+} \times \cdots \times \overline{\Gamma_r \backslash X_r^+}$$

and the line bundle $\Lambda_{\Gamma, \Gamma'}$ on $\overline{\Gamma \backslash X^+}$ decomposes as

$$\Lambda_{\Gamma, \Gamma'} = \Lambda_{\Gamma_1, \Gamma'} \boxtimes \cdots \boxtimes \Lambda_{\Gamma_r, \Gamma'} \ ,$$

with $\Lambda_{\Gamma_i, \Gamma'} = \phi_i^* L_{\Gamma'} \otimes L_{\Gamma_i}^{-1}$ the corresponding line bundle on $\overline{\Gamma_i \backslash X_i^+}$. Let $p_i : \overline{\Gamma \backslash X^+} \longrightarrow \overline{\Gamma_i \backslash X_i^+}$ be the natural projection. As

$$\deg_{\Lambda_{\Gamma, \Gamma'}} C = \sum_{i=1}^{r} \deg_{p_i^* \Lambda_{\Gamma_i, \Gamma'}} C \ ,$$

we have reduced the proof of the proposition to the case where $\mathbf{G}$ is $\mathbb{Q}$-simple. It then follows from the more precise following proposition 4.2.8. $\qquad \square$

Proposition 4.2.8. *Assume that $\mathbf{G}$ is $\mathbb{Q}$-simple.*

(1) *If $\mathbf{G}$ is $\mathbb{Q}$-anisotropic then the line bundle $\Lambda_{\Gamma, \Gamma'}$ on the smooth complex projective variety $\Gamma \backslash X^+$ admits a metric of non negative curvature.*

(2) *If $\mathbf{G}$ is $\mathbb{Q}$-isotropic then either the line bundle $\Lambda_{\Gamma, \Gamma'}$ on $\overline{\mathrm{Sh}_K(\mathbf{G}, X)}$ is trivial or it is ample.*

Proof. Let $\mathbf{G}' = \mathbf{G}_1' \times \cdots \times \mathbf{G}_{r'}'$ be the decomposition of $\mathbf{G}'$ into $\mathbb{Q}$-simple factor and $\varphi_j : \mathbf{G} \longrightarrow \mathbf{G}_j'$, $1 \leq j \leq r'$, the components of $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$. By naturality of L_Γ and $L_{\Gamma'}$ (c.f. proposition 4.2) one can assume that $\Gamma' = \Gamma_1' \times \cdots \Gamma_{r'}'$. Accordingly one has

$$\Gamma' \backslash X'^+ = \Gamma_1' \backslash X_1'^+ \times \cdots \times \Gamma_{r'}' \backslash X_{r'}'^+ \ .$$

As $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$ is injective and $\mathbf{G}$ is $\mathbb{Q}$-simple we can without loss of generality assume that $\varphi_1 : \mathbf{G} \longrightarrow \mathbf{G}_1'$ is injective. As

$$\Lambda = (\phi_1^* L_{\Gamma_1'} \otimes L_\Gamma^{-1}) \otimes \phi_2^* L_{\Gamma_2'} \otimes \cdots \phi_{r'}^* L_{\Gamma_r'} \ ,$$

and the $L_{\Gamma_j'}$, $j \geq 2$, are ample on $\overline{\Gamma_j' \backslash X_j'^+}$ it is enough to prove the statement replacing $\Lambda_{\Gamma, \Gamma'}$ by $\phi_1^* L_{\Gamma_1'} \otimes L_\Gamma^{-1}$. Thus we can assume $\mathbf{G}'$ is $\mathbb{Q}$-simple.

By the adjunction formula the line bundle $\Lambda_{\Gamma, \Gamma' | \Gamma \backslash X^+}$ restriction of $\Lambda_{\Gamma, \Gamma'}$ coincides with $\Lambda^{\max} N^*$, where N denotes the automorphic bundle on $\Gamma \backslash X^+$ associated to the normal bundle of X in X' and N^* denotes its dual. As X is totally geodesic in X' the curvature

form on N is the restriction to N of the curvature form on TX'. As X' is non-positively curved, the automorphic bundle N^* and thus also the automorphic line bundle $\Lambda_{\Gamma, \Gamma'|\Gamma \backslash X^+}$ admits a Hermitian metric of non-negative curvature. This concludes the proof of the proposition in the case $\mathbf{G}$ is $\mathbb{Q}$-anisotropic.

Suppose now $\mathbf{G}$ is $\mathbb{Q}$-isotropic. For simplicity we denote $\Lambda_{\Gamma, \Gamma'}$ by Λ from now on. We have to prove that the boundary components of $\overline{\Gamma \backslash X^+}$ do not essentially modify the positivity of $\Lambda_{|\Gamma \backslash X^+}$. We use the notation and the results of Dynkin [13], Ihara [19] and Satake [31]. Let $X = X_1 \times \cdots \times X_r$ (resp. $X' = X'_1 \times \cdots \times X'_{r'}$) be the decomposition of X (resp. X') into irreducible factors. Each X_i (resp. X'_j) is the Hermitian symmetric domain associated to an $\mathbb{R}$-isotropic $\mathbb{R}$-simple factor $\mathbf{G}_i$ (resp. $\mathbf{G}'_j$) of $\mathbf{G}_{\mathbb{R}}$ (resp. $\mathbf{G}'_{\mathbb{R}}$). The group $\mathbf{G}_{\mathbb{R}}$ (resp. $\mathbf{G}'_{\mathbb{R}}$) decomposes as $\mathbf{G}_0 \times \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$ (resp. $\mathbf{G}'_0 \times \mathbf{G}'_1 \times \cdots \times \mathbf{G}'_{r'}$) with $\mathbf{G}_0$ (resp. $\mathbf{G}'_0$) an $\mathbb{R}$-anisotropic group. Let $\mathbf{m}$ (resp. $\mathbf{m}'$) be the r-tuple (resp. r'-tuple) of non-negative integers defining the automorphic line bundle L_K (resp. $L_{K'}$) (c.f. [31, lemma 2]) and M_ϕ be the $r' \times r$-matrix with integral coefficients associated to $\varphi : \mathbf{G} \hookrightarrow \mathbf{G}'$ (c.f. [31, section 2.1]). The automorphic line bundle $\Lambda_{|\Gamma \backslash X^+}$ on $\Gamma \backslash X^+$ is associated to the r-tuple of integers $\boldsymbol{\lambda} = \mathbf{m}' M_\varphi - \mathbf{m}$ (where $\mathbf{m}$ and $\mathbf{m}'$ are seen as row vectors). It admits a locally homogeneous Hermitian metric of non-negative curvature if and only if $\lambda_i \geq 0$, $1 \leq i \leq r$ (in which case we say that $\boldsymbol{\lambda}$ is non-negative).

Lemma 4.2.9. *The row vector $\boldsymbol{\lambda}$ is non-negative.*

Proof. As $\mathbf{G}$ and $\mathbf{G}'$ are defined over $\mathbb{Q}$, both $\mathbf{m}$ and $\mathbf{m}'$ are of rational type by [31, p.301]. So $m_i = m$ for all i, $m'_j = m'$ for all j. The equality $\boldsymbol{\lambda} = \mathbf{m}' M_\varphi - \mathbf{m}$ can be written in coordinates

$$(4.3) \qquad \forall i \in \{1, \cdots, r\}, \ \lambda_i = \sum_{1 \leq j \leq r'} m_{j,i} \, m' - m \ ,$$

with $M_\varphi = (m_{j,i})$. Fix i in $\{1, \cdots r\}$ and let prove that $\lambda_i \geq 0$. As the $m_{i,j}$'s and m' are non-negative, it is enough to exhibit one j, $1 \leq j \leq r'$, with $m_{j,i} \, m' - m \geq 0$. Choose j such that the component $\varphi_{i,j} : X_i \longrightarrow X'_j$ of the map $\varphi : X_1 \times \cdots \times X_r \longrightarrow X'_1 \times \cdots \times X'_{r'}$ induced by $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$ is an embedding. Recall that with the notation of [31, p.290] one has

$$m_i = < H_{1,i}, H_{1,i} >_i \ ,$$

where $\mathfrak{h}_i$ denotes the chosen Cartan subalgebra of $\mathfrak{g}_i(\mathbb{R})$ and $<, >_i$ denotes the canonical scalar product on $\sqrt{-1}\mathfrak{h}_i$. This gives the equality :

$$(4.4) \qquad m_{j,i} \, m'_j - m_i = < \phi_j(H_{1,i}), \phi_j(H_{1,i}) >_j - < H_{1,i}, H_{1,i} >_i \ .$$

As $\mathbf{G}_i$ is $\mathbb{R}$-simple, any two invariant non-degenerate forms on $\sqrt{-1}\mathfrak{h}_i$ are proportional : there exists a positive real constant $c_{i,j}$ (called by Dynkin [13, p.130] the index of $\varphi_{i,j}$: $\mathbf{G}_i \longrightarrow \mathbf{G}_j$) such that

$$\forall X, Y \in \sqrt{-1}\mathfrak{h}_i, \ < \phi_j(X), \phi_j(Y) >_j = c_{i,j} < X, Y >_i \ .$$

Equation (4.4) thus gives :

(4.5)
$$m_{j,i}\, m'_j - m_i = (c_{i,j} - 1) < H_{1,i}, H_{1,i} >_i \ .$$

By [13, theorem 2.2. p.131] the constant $c_{i,j}$ is a positive integer. Thus $m_{j,i}\, m'_j - m_i$ is non-negative and this finishes the proof that $\boldsymbol{\lambda}$ is non-negative. $\qquad\square$

By [31, cor.2 p.298] the sum $M = \sum_{1 \leq j \leq r'} m_{j,i}$ is independent of i ($1 \leq i \leq r$). This implies that $\boldsymbol{\lambda}$ is of rational type : one of the λ_i is non-zero if and only if all are. In this case $\boldsymbol{\lambda}$ is positive of rational type and Λ is ample on $\overline{\Gamma\backslash X^+}$ by [31, theor.1].

If $\boldsymbol{\lambda} = 0$, the line bundle $\Lambda_{|\Gamma\backslash X^+}$ is trivial. As $\mathbf{G}$ is $\mathbb{Q}$-simple, if $\mathbf{G}$ is not locally isomorphic to $\mathbf{SL}_2$ the line bundle Λ on $\overline{\Gamma\backslash X^+}$ is trivial.

The last case is treated in the following lemma :

Lemma 4.2.10. *If $\boldsymbol{\lambda} = 0$ and $\mathbf{G}$ is locally isomorphic to $\mathbf{SL}_2$, then $\phi : \mathbf{G} \longrightarrow \mathbf{G}'$ is a local isomorphism and the line bundle Λ on $\overline{\Gamma\backslash X^+}$ is trivial.*

Proof. It follows from the equation (4.3) that there exists a unique integer j such that the morphism $\varphi_j : \mathbf{G}_{\mathbb{R}} \longrightarrow \mathbf{G}_j$ is non trivial. In particular $\mathbf{G}'$ is $\mathbb{R}$-simple. Moreover the equation (4.5) implies that index c of $\phi : \mathbf{G} \longrightarrow \mathbf{G}'$ is equal to 1. Thus by [13, theorem 6.2 p.152] the Lie algebra $\mathfrak{g}$ is a regular subalgebra of $\mathfrak{g}'$. If $\mathbf{G}'_{\mathbb{R}}$ is classical, the equality [13, (2.36) p.136] shows that necessarily $\phi : \mathbf{G} \longrightarrow \mathbf{G}'$ is a local isomorphism. In particular the line bundle Λ on $\overline{\Gamma\backslash X^+}$ is trivial. If the group $\mathbf{G}'_{\mathbb{R}}$ is an exceptional simple Lie group of Hermitian type (thus E_6 or E_7), Dynkin shows in [13, Tables 16, 17 p.178-179] that there is a unique realization of $\mathfrak{g}$ as a regular subalgebra of $\mathfrak{g}'$ of index 1. However this realization is not of Hermitian type : the coefficient $\alpha'_1(\varphi(H_1))$ is zero. Thus this case is impossible. $\qquad\square$

This finishes the proof of proposition 4.2.8.

$\qquad\square$

From the nefness of $\Lambda_{K,K'}$ we now deduces the following crucial corollary :

Corollary 4.2.11. *Let $\phi : \mathrm{Sh}_K(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_{K'}(\mathbf{G}', X')$ be a morphism of Shimura varieties associated to a Shimura sub-datum $\varphi : (\mathbf{G}, X) \longrightarrow (\mathbf{G}', X')$, K' a compact open subgroup of $\mathbf{G}(\mathbf{A}_f)$ and $K = K' \cap \mathbf{G}(\mathbf{A}_f)$. Then for any irreducible subvariety Z of $\mathrm{Sh}_K(\mathbf{G}, X)$ whose irreducible components are Hodge generic one has $\deg_{L_K} Z \leq \deg_{L_{K'}} \phi(Z)$.*

Proof. As the irreducible components of Z are Hodge generic in $\mathrm{Sh}_K(\mathbf{G}, X)$ we know by lemma 2.2 in [37] (and its proof) that $\phi_{|Z} : Z \longrightarrow Z'$ is generically injective. In particular by the projection formula one has

$$\deg_{L_{K'}} Z' = \deg_{\phi^* L_{K'}} Z \ .$$

So the inequality $\deg_{L_K} Z \leq \deg_{L_{K'}} Z'$ is equivalent to the inequality $\deg_{L_{K'}} Z \geq \deg_{L_K} Z$.

As $\phi^* L_{K'} = L_K \otimes \Lambda_{K,K'}$ one has

$$\deg_{\phi^* L_{K'}} Z = \sum_{i=0}^{\dim Z} \binom{i}{\dim Z} \int_Z c_1(L_K)^i \wedge c_1(\Lambda_{K,K'})^{\dim Z - i} \ .$$

The inequality $\deg_{\phi^* L'_K} Z \geq \deg_{L_K} Z$ thus follows if we show :

$$\forall i \ , 1 \leq i \leq \dim Z, \quad \int_Z c_1(L_K)^i \wedge c_1(\Lambda_{K,K'})^{\dim Z - i} \geq 0 \ .$$

As L_K is ample it follows from the nefness of $\Lambda_{K,K'}$ and Kleiman's theorem 4.2.1. $\qquad\square$

4.3. p-adic closure of Zariski-dense groups. We recall the following well-known result :

Proposition 4.3.1. *Let H be a subgroup of $\mathbf{GL}_n(\mathbb{Z})$ and let $\mathbf{H}$ be the Zariski closure of H in $\mathbf{GL}_{n,\mathbb{Z}}$. Suppose that $\mathbf{H}^0_{\mathbb{Q}}$ is semisimple. Then for any prime number p the closure of H in $\mathbf{H}(\mathbb{Z}_p)$ is open.*

Proof. The case when H is finite is obvious. Suppose that H is infinite. Since $\mathbf{H}(\mathbb{Z}_p)$ is compact and H is infinite, the closure H_p of H in $\mathbf{H}(\mathbf{Z}_p)$ is not discrete. Then it is a p-adic analytic group and it has a Lie algebra L which is a Lie subalgebra of the Lie algebra Lie $\mathbf{H}$ of $\mathbf{H}$ and projects non-trivially on any factor of Lie $\mathbf{H}$. By construction L is invariant under the adjoint action of H, thus also under the adjoint action of the Zariski closure $\mathbf{H}$ of H. As $\mathbf{H}^0_{\mathbb{Q}}$ is semisimple one deduces $L_{\mathbb{Q}} = $ Lie $\mathbf{H}_{\mathbb{Q}}$, which implies that H_p is open in $\mathbf{H}(\mathbb{Z}_p)$. $\qquad\square$

Remark 4.3.2. The easy proposition 4.3.1 can be strengthened to the following remarkable theorem, due independently to Weisfeiler and Nori, which was used in [17] but which we will not need :

Theorem 4.3.3 ([39], [25])**.** *Let H be a finitely generated subgroup of $\mathbf{GL}_n(\mathbb{Z})$ and let $\mathbf{H}$ be the Zariski closure of H in $\mathbf{GL}_{n,\mathbb{Z}}$. Suppose that $\mathbf{H}(\mathbb{C})$ has finite fundamental group. Then the closure of H in $\mathbf{GL}_n(\mathbf{A}_\mathrm{f})$ is open in the closure of $\mathbf{H}(\mathbb{Z})$ in $\mathbf{GL}_n(\mathbf{A}_\mathrm{f})$.*

5. Inclusion of Shimura subdatum.

In this section we prove the following proposition which implies part (b) of the theorem 3.1.1.

Proposition 5.1. *Suppose that the set Σ in the theorem 3.2.1 is such that with respect to a faithful representation $\rho\colon \mathbf{G} \longrightarrow \mathbf{GL}_n$ the centres $\mathbf{T}_V$ of the generic Mumford-Tate groups $\mathbf{H}_V$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-orbit as V ranges through Σ. Then the set Σ' obtained in the proposition 3.2.2 admits a modification Σ'' such that the centres $\mathbf{T}_{V'}$ of the generic Mumford-Tate groups $\mathbf{H}_{V'}$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-orbit as V' ranges through Σ''.*

We first prove the following general fact about Shimura data which will also be used at another point in this paper.

Lemma 5.2. *Let $(\mathbf{G}, X)$ be a Shimura datum such that $\mathbf{G}$ is the generic Mumford-Tate group on X and $(\mathbf{H}, X_{\mathbf{H}})$ be a Shimura subdatum of $(\mathbf{G}, X)$. Let $\mathbf{T}$ (resp. $\mathbf{Z}$) be the connected centre of $\mathbf{G}$ (resp. $\mathbf{H}$). Then*

$$\mathbf{T} \subset \mathbf{Z} \ .$$

Proof. The proof uses in a crucial way the fact that $\mathbf{G}$ is the generic Mumford-Tate group on X. We write

$$\mathbf{G} = \mathbf{T}\mathbf{G}^{\mathrm{der}} \ .$$

As $\mathbf{T} \cap \mathbf{H}$ is contained in the centre $\mathbf{Z}$ of $\mathbf{H}$, we can write

$$\mathbf{H} = (\mathbf{T} \cap \mathbf{H})\mathbf{H}'$$

for some subgroup $\mathbf{H}'$ of $\mathbf{G}^{\mathrm{der}}$.

Fix α an element of X that factors through $\mathbf{H}_{\mathbb{R}} = (\mathbf{T} \cap \mathbf{H})_{\mathbb{R}}\mathbf{H}'_{\mathbb{R}}$. As X is the $\mathbf{G}(\mathbb{R})$ conjugacy class of α any element $x \in X$ is of the form $g\alpha g^{-1} = \alpha^g$ for some g of $\mathbf{G}(\mathbb{R})$. Thus x factors through

$$((\mathbf{T} \cap \mathbf{H})_{\mathbb{R}})^g\, (\mathbf{G}^{\mathrm{der}}_{\mathbb{R}})^g = (\mathbf{T} \cap \mathbf{H})_{\mathbb{R}}\, \mathbf{G}^{\mathrm{der}}_{\mathbb{R}} \ .$$

It follows that the Mumford-Tate group of x is contained in $(\mathbf{T} \cap \mathbf{H})\mathbf{G}^{\mathrm{der}}$. For x Hodge generic, we obtain

$$(\mathbf{T} \cap \mathbf{H})\mathbf{G}^{\mathrm{der}} = \mathbf{G} \ ,$$

hence $\mathbf{T} \cap \mathbf{H} = \mathbf{T}$, therefore $\mathbf{T} \subset \mathbf{Z}$. $\qquad\qquad\square$

To prove the proposition, first note that an inclusion of special subvarieties $V \subset V'$ corresponds to an inclusion of Shimura data $(\mathbf{H}, X_{\mathbf{H}}) \subset (\mathbf{H}', X_{\mathbf{H}'})$. The lemma above implies that the centres $\mathbf{T}'$ of the groups $\mathbf{H}'$ are contained in a $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class of a fixed torus $\mathbf{T}$. It follows that the tori $\mathbf{T}'$ are split by the same field L. As there are only

finitely many subfields of L, a modification of Σ' satisfies the condition that the splitting field of the tori $\mathbf{T}'$ is constant, say L. As in the discussion before the lemma 2.4 of [37], we identify $X^*(\mathbf{T}')$ with a submodule of $X^*(\mathrm{Res}_{L/\mathbb{Q}}\mathbf{G}_{\mathbf{m}L})$ which has a canonical basis. By the lemma 2.4 of [37], the coordinates of the characters (with respect to this basis) occurring in the representation $\mathbf{T}' \subset \mathbf{GL}_n$ are uniformly bounded. It follows that the tori $\mathbf{T}'$ lie in finitely many $\mathbf{GL}_n(\mathbb{Q})$-conjugacy classes. The result follows.

6. THE GEOMETRIC CRITERION.

In this section we show that for certain elements $m \in \mathbf{G}(\mathbb{Q}_l)$ and under certain assumptions on a subvariety Z, the inclusion $Z \subset T_m Z$ implies that Z contains a special subvariety V' containing V properly.

Definition 6.0.4. *Let $(\mathbf{G}, X)$ be a Shimura datum and $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ a compact open subgroup. Let $F \subset \mathbb{C}$ be a number field containing the reflex field $E(\mathbf{G}, X)$. We use the following common abuse of notation : a subvariety $Z \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is called an F-irreducible F-subvariety if $Z = Z_F \times_{\mathrm{Spec}\,F} \mathrm{Spec}\,\mathbb{C}$, where $Z_F \subset \mathrm{Sh}_K(\mathbf{G}, X)_F$ is an irreducible closed subscheme.*

Our main theorem in this section is the following :

Theorem 6.1. *Let $(\mathbf{G}, X)$ be a Shimura datum, X^+ a connected component of X and $K = \prod_{p\,\mathrm{prime}} K_p \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ an open compact subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. We assume that there exists a prime p_0 such that the compact open subgroup $K_{p_0} \subset \mathbf{G}(\mathbb{Q}_{p_0})$ is neat. Let F be a number field containing the field of definition of $S_K(\mathbf{G}, X)_{\mathbb{C}}$.*

Let V be a non-strongly special subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ contained in a Hodge generic F-irreducible F-subvariety Z of $S_K(\mathbf{G}, X)_{\mathbb{C}}$.

Let $l \neq p_0$ be a prime number splitting $\mathbf{T}_V$ and m an element of $\mathbf{T}_V(\mathbb{Q}_l)$.

Suppose that Z satisfies the conditions

(1) $Z \subset T_m Z$.

(2) *for every k_1 and k_2 in K_l the image of $k_1 m k_2$ in $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$ generates an unbounded (for the l-adic topology) subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$.*

Then Z contains a special subvariety V' containing V properly.

Proof.

Lemma 6.2. *If the conclusion of the theorem 6.1 holds for all Shimura data $(\mathbf{G}, X)$ with $\mathbf{G}$ semisimple of adjoint type then the conclusion of the theorem 6.1 holds for all Shimura data.*

Proof. Let $\mathbf{G}$, X, K, V, Z, F, l and m be as in the statement of theorem 6.1. Let $\lambda \colon \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ be the natural morphism. Let $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ be the adjoint Shimura datum attached to $(\mathbf{G}, X)$ and let $K^{\mathrm{ad}} = \prod_{p \text{ prime}} K_p^{\mathrm{ad}}$ be the compact open subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbf{A}_{\mathrm{f}})$ defined as follows :

(1) $K_{p_0}^{\mathrm{ad}} \subset \mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_{p_0})$ is the compact open subgroup image of K_{p_0} by λ.
(2) $K_l^{\mathrm{ad}} \subset \mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$ is the compact open subgroup image of K_l by λ.
(3) If $p \neq p_0, l$, K_p^{ad} is a maximal compact open subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_p)$ containing the image of K_p by λ.

The group K^{ad} is neat because K_{p_0}, and therefore $K_{p_0}^{\mathrm{ad}}$, is. As the reflex field $E(\mathbf{G}, X)$ contains the reflex field $E(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$, there is a finite morphism of Shimura varieties

$$f : \mathrm{Sh}_K(\mathbf{G}, X)_F \longrightarrow \mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_F$$

Let V^{ad} be the image $f_{\mathbb{C}}(V)$. As V is non-strongly special, V^{ad} is a non-strongly special subvariety of $S_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$. Thus $\mathbf{T}_{V^{\mathrm{ad}}} = \lambda(\mathbf{T}_V)$ is a non-trivial torus.

We define the F-irreducible subvariety Z_F^{ad} of $\mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_F$ to be the image of Z_F in $\mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_F$ by this morphism. Of course $Z^{\mathrm{ad}} := Z_F \times_F \mathbb{C}$ is contained in $S_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$. Let m^{ad} be the image of m in $\mathbf{T}_V^{\mathrm{ad}}(\mathbb{Q}_l)$. The inclusion $Z \subset T_m Z$ implies that $Z^{\mathrm{ad}} \subset T_{m^{\mathrm{ad}}} Z^{\mathrm{ad}}$.

As $\mathbf{G}^{\mathrm{ad}}$ is of adjoint type, we can apply the theorem 6.1 to $\mathbf{G}^{\mathrm{ad}}$, X^{ad}, K^{ad}, V^{ad}, Z^{ad}, F, l and m^{ad}. So Z^{ad} contains a special subvariety V'^{ad} containing V^{ad} properly. As irreducible components of the preimage by a finite Shimura morphism of a special subvariety are special, Z contains a special subvariety V' containing V properly. $\square$

For the rest of this section, we are assuming the group $\mathbf{G}$ to be semisimple of adjoint type. Moreover for simplicity of notations we replace in this proof the field $E(\mathbf{G}, X)$ by the field F. Thus $\mathrm{Sh}(\mathbf{G}, X)$ denotes the canonical model of $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ over F, $S_{K^l}(\mathbf{G}, X)$ is the connected component, of $\mathrm{Sh}_{K^l}(\mathbf{G}, X) = \mathrm{Sh}_{K^l}(\mathbf{G}, X)_F$ (image of $X^+ \times \{1\}$), etc. Moreover we will drop the label $(\mathbf{G}, X)$ when it is obvious which Shimura datum we are referring to.

Lemma 6.3. *Let $Z = Z_1 \cup \cdots \cup Z_n$ be the decomposition of Z into geometrically irreducible components. Each irreducible component Z_i, $1 \leq i \leq n$, is Hodge generic.*

Proof. As Z is Hodge generic, at least one irreducible component, say Z_1, is Hodge generic. As Z_F is irreducible, any irreducible component Z_j, $1 \leq j \leq n$, is of the form Z_1^{σ} for some element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. As the conjugate under any element of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ of a special subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is still special, one gets the result. This is a consequence of

a theorem of Kazhdan's. See [22] for a comprehensive exposition of the proof in full generality and all the relevant references. □

We fix a $\mathbb{Z}$-structure on $\mathbf{G}$ and its subgroups by choosing a finitely generated free $\mathbb{Z}$-module W, a faithful representation $\xi \colon \mathbf{G} \hookrightarrow \mathbf{GL}(W_{\mathbb{Q}})$ and taking the Zariski closures in the $\mathbb{Z}$-group-scheme $\mathbf{GL}(W)$. We choose the representation ξ in such a way that K is contained in $\mathbf{GL}(\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} W)$ (i.e. K stabilizes $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} W$). This induces canonically a $\mathbb{Z}$-variation of Hodge structure on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$: c.f. [17, section 3.2].

Let z be a Hodge generic point of the smooth locus Z_1^{sm} of Z_1. Let $\pi_1(Z_1^{\mathrm{sm}}, z)$ be the topological fundamental group of Z_1^{sm} at the point z. The representation $\xi \colon \mathbf{G} \longrightarrow \mathbf{GL}(W_{\mathbb{Q}})$ induces a polarizable variation of $\mathbb{Z}$-Hodge structure $\mathcal{F}$ on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, in particular on its irreducible component $S_K(\mathbf{G}, X)_{\mathbb{C}}$. We choose a point $\widetilde{z}$ of X lying above z. This choice canonically identifies the fibre at z of the locally constant sheaf underlying $\mathcal{F}$ with the $\mathbb{Z}$-module W. The action of $\pi_1(Z_1^{\mathrm{sm}}, z)$ on this fibre is described by the monodromy representation

$$\rho \colon \pi_1(Z_1^{\mathrm{sm}}, z) \longrightarrow \Gamma = \pi_1(S_K(\mathbf{G}, X)_{\mathbb{C}}, z) = \mathbf{G}(\mathbb{Q}) \cap K \xrightarrow{\xi} \mathbf{GL}(W) \ \ .$$

As Γ is Zariski-dense in $\mathbf{G}$ the algebraic monodromy group is $\mathbf{G}$. As Z is Hodge generic the group $\rho(\pi_1(Z_1^{\mathrm{sm}}, z))$ is Zariski-dense in $\mathbf{G}$ by [1, theor. 1.4].

Let l be a prime as in the statement. The proposition 4.3.1 implies that the l-adic closure of $\rho(\pi_1(Z_1^{\mathrm{sm}}, z))$ in $\mathbf{G}(\mathbb{Q}_l)$ is a compact open subgroup $K'_l \subset K_l$.

Write $K = K^l K_l$ with $K^l = \prod_{p \neq l} K_p$. Let $\pi_{K_l} \colon \mathrm{Sh}_{K^l} \longrightarrow \mathrm{Sh}_K$ be the Galois pro-étale cover with group K_l as defined in section 4.1.1. Let $\widetilde{Z_1}$ be an irreducible component of the preimage of Z_1^{sm} in Sh_{K^l} and let $\widetilde{V}$ be an irreducible component of the preimage of V in $\widetilde{Z_1}$. As $\pi_{K_l} \colon \mathrm{Sh}_{K^l} \longrightarrow \mathrm{Sh}_K$ is pro-étale, the smooth locus $\widetilde{Z_1}^{\mathrm{sm}}$ of $\widetilde{Z_1}$ naturally identifies with an irreducible component $\widetilde{Z_1^{\mathrm{sm}}}$ of $\pi_{K_l}^{-1}(Z_1^{\mathrm{sm}})$.

The idea of the proof is to show that the inclusion $Z \subset T_m Z$ implies that $\widetilde{Z_1}$ is stabilized by a "big" group and then consider the orbit of $\widetilde{V}$ under the action of this group.

Lemma 6.4. *The variety $\widetilde{Z_1}$ is stabilized by the group K'_l. The set of irreducible components of $\pi_{K_l}^{-1}(Z_1)$ naturally identifies with the finite set K_l/K'_l.*

Proof. Let $\widetilde{z}$ be a geometric point of $\widetilde{Z_1^{\mathrm{sm}}}$ lying over z. Let $\varpi(Z_1^{\mathrm{sm}}, z)$ denote the algebraic fundamental group of Z_1^{sm} at z. The set of irreducible components of $\pi_{K_l}^{-1}(Z_1)$ naturally identifies with the quotient $K_l/\rho_{\mathrm{alg}}(\varpi(Z_1^{\mathrm{sm}}, z))$, where $\rho_{\mathrm{alg}} \colon \varpi(Z_1^{\mathrm{sm}}, z) \longrightarrow K_l \subset \mathbf{G}(\mathbb{Q}_l)$ denotes the (continuous) monodromy representation of the K_l-pro-étale cover $\pi_{K_l} \colon \pi_{K_l}^{-1}(Z_1^{\mathrm{sm}}) \longrightarrow Z_1^{\mathrm{sm}}$. The group $\varpi(Z_1, z)$ naturally identifies with the profinite completion

of $\pi_1(Z_1^{\mathrm{sm}}, z)$. One has the commutative diagram

$$(6.1) \qquad \begin{array}{ccc} \pi_1(Z_1^{\mathrm{sm}}, z) & \xrightarrow{\ \rho\ } & \mathbf{G}(\mathbb{Q}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ \varpi_1(Z_1^{\mathrm{sm}}, z) & \xrightarrow[\rho_{\mathrm{alg}}]{} & \mathbf{G}(\mathbb{Q}_l) \end{array}$$

where $i : \pi_1(Z_1^{\mathrm{sm}}, z) \longrightarrow \varpi_1(Z_1^{\mathrm{sm}}, z)$ and $j : \mathbf{G}(\mathbb{Q}) \longrightarrow \mathbf{G}(\mathbb{Q}_l)$ denote the natural homomorphisms. As $i(\pi_1(Z_1^{\mathrm{sm}}, z))$ is dense in $\varpi_1(Z_1^{\mathrm{sm}}, z)$ and ρ_{alg} is continuous one deduces that $\rho_{\mathrm{alg}}(\varpi_1(Z_1^{\mathrm{sm}}, z)) = K_l'$. Thus the set of irreducible components of $\pi_{K_l}^{-1}(Z_1^{\mathrm{sm}})$ identifies with K_l/K_l' and $\widetilde{Z_1^{\mathrm{sm}}}$ is K_l'-stable. $\qquad\square$

Lemma 6.5. *There exist elements k_1, k_2 of K_l and an integer $n \geq 1$ such that*

$$\widetilde{Z_1} = \widetilde{Z_1} \cdot (k_1 m k_2)^n$$

Proof. The inclusion $Z \subset T_m Z$ implies that for every geometrically irreducible component Z_i, $1 \leq i \leq n$, of Z, there is a geometric irreducible component $\widetilde{Z_i}$ of $\pi_{K_l}^{-1}(Z_i)$ which is also a geometric irreducible component of the preimage of $T_m Z$ by $\pi_{K_l} : \mathrm{Sh}_{K^l} \longrightarrow \mathrm{Sh}_K$. As the geometric irreducible components of $\pi_{K_l}^{-1}(T_m Z)$ are of the form $\widetilde{Z_i} \cdot (k_1 m k_2)$, $k_1, k_2 \in K_l$, there exists an index i, $1 \leq i \leq n$, and two elements k_1, k_2 in K_l such that

$$\widetilde{Z_1} = \widetilde{Z_i} \cdot k_1 m k_2 \ .$$

As Z is F-irreducible there exists σ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that $Z_i = \sigma(Z_1)$. As the morphism $\pi_{K_l} : \mathrm{Sh}_{K^l} \longrightarrow \mathrm{Sh}_K$ is defined over F, the subvariety $\sigma(\widetilde{Z_1})$ of Sh_{K^l} satisfies $\pi_{K_l}(\sigma(\widetilde{Z_1})) = Z_i$. Thus the subvarieties $\sigma(\widetilde{Z_1})$ and $\widetilde{Z_i}$ of Sh_{K^l} are both irreducible components of $\pi_{K_l}^{-1}(Z_i)$. Thus there exists an element k of K_l such that

$$\widetilde{Z_i} = \sigma(\widetilde{Z_1}) \cdot k \ .$$

By replacing k_1 with $k k_1$, we obtain k_1, k_2 in K_l such that

$$(6.2) \qquad \widetilde{Z_1} = \sigma(\widetilde{Z_1}) \cdot (k_1 m k_2) \ .$$

As the $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$-action is defined over F, the previous equation implies :

$$(6.3) \qquad \forall\, i \in \mathbb{N}, \quad \widetilde{Z_1} = \sigma^i(\widetilde{Z_1}) \cdot (k_1 m k_2)^i \ .$$

As the set of irreducible components of Z is finite, there exists a positive integer m such that $\sigma^m(Z_1) = Z_1$. Thus the Abelian group $(\sigma^m)^{\mathbf{Z}}$ acts on the set of irreducible components of $\pi_{K_l}^{-1}(Z_1)$. By the previous lemma this set is finite. So there exists a positive integer n (multiple of m) such that $\sigma^n(\widetilde{Z_1}) = \widetilde{Z_1}$. The equality (6.3) applied to $i = n$ concludes the proof of the lemma. $\qquad\square$

From the lemmas 6.4 and 6.5 one obtains the

Corollary 6.6. *Let U_l be the group $\langle K'_l, (k_1 m k_2)^n \rangle$. The variety $\widetilde{Z_1}$ is stabilized by U_l.*

We now conclude the proof of theorem 6.1. Let $\mathbf{G} = \prod_{i=1}^{s} \mathbf{G}_i$ be the decomposition of $\mathbf{G}$ into $\mathbb{Q}$-simple factors. Without loss of generality we can assume that $K = K_1 \times \cdots \times K_s$, where K_i, $1 \leq i \leq s$, is a compact open subgroup of $\mathbf{G}_i(\mathbf{A}_f)$. Let $(\mathbf{G}_{>1}, X_{>1})$ be the product of Shimura data $(\prod_{i=2}^{s} \mathbf{G}_i, \prod_{i=2}^{s} X_i)$, and $K_{>1}$ be the compact open subgroup $\prod_{i=2}^{s} K_i$ of $\mathbf{G}_{>1}(\mathbf{A}_f)$. The connected component $S_K(\mathbf{G}, X)_{\mathbb{C}}$ of the Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ decomposes as a product

$$S_K(\mathbf{G}, X)_{\mathbb{C}} = S_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}} \times S_{K_{>1}}(\mathbf{G}_{>1}, X_{>1})_{\mathbb{C}}$$

with $S_{K_{>1}}(\mathbf{G}_{>1}, X_{>1})_{\mathbb{C}} = \prod_{i=2}^{s} S_{K_i}(\mathbf{G}_i, X_i)_{\mathbb{C}}$.

Let $p_i : \mathbf{G} \longrightarrow \mathbf{G}_i$ denote the natural projections. By the assumption made on m, the group U_l is unbounded in $\mathbf{G}(\mathbb{Q}_l)$. After possibly renumbering the factors, we can assume that $p_1(U_l)$ is unbounded in $\mathbf{G}_1(\mathbb{Q}_l)$. In particular the torus $p_1(\mathbf{T})$ is non-trivial. Indeed if it was trivial, then the group $p_1(U_l)$ would be contained in $p_1(K_l)$ which is compact and therefore bounded.

Similarly let $\mathbf{G}_{1, \mathbb{Q}_l} = \prod_{j=1}^{r} \mathbf{H}_i$ be the decomposition of $\mathbf{G}_{1, \mathbb{Q}_l}$ into simple $\mathbb{Q}_l$-factors. Again, up to renumbering we can assume that the image of U_l under the projection $h_1 : \mathbf{G}_{\mathbb{Q}_l} \longrightarrow \mathbf{H}_1$ is unbounded in $\mathbf{H}_1(\mathbb{Q}_l)$. Let $\mathbf{H}_{>1} = \prod_{j=2}^{r} \mathbf{H}_j$. Let $\tau : \widetilde{\mathbf{G}_{\mathbb{Q}_l}} \longrightarrow \mathbf{G}_{\mathbb{Q}_l}$ (resp. $\tau_1 : \widetilde{\mathbf{H}_1} \longrightarrow \mathbf{H}_1$) be the universal cover of $\mathbf{G}_{\mathbb{Q}_l}$ (resp. $\mathbf{H}_1$).

Sublemma 6.7. *The group $U_l \cap \mathbf{H}_1(\mathbb{Q}_l)$ contains the group $\tau_1(\widetilde{\mathbf{H}_1}(\mathbb{Q}_l))$ with finite index.*

Proof. Let $\widetilde{h_1} : \widetilde{\mathbf{G}_{\mathbb{Q}_l}} \longrightarrow \widetilde{\mathbf{H}_1}$ be the canonical projection. Let $\widetilde{U_l} = \tau^{-1}(U_l) \subset \widetilde{\mathbf{G}_{\mathbb{Q}_l}}(\mathbb{Q}_l)$. As U_l is an open non-compact subgroup of $\mathbf{G}_{\mathbb{Q}_l}(\mathbb{Q}_l)$, the group $\widetilde{U_l}$ is open non-compact in $\widetilde{\mathbf{G}_{\mathbb{Q}_l}}(\mathbb{Q}_l)$. As $h_1(U_l)$ is non-compact in $\mathbf{H}_1(\mathbb{Q}_l)$ the projection $\widetilde{h_1}(\widetilde{U_l})$ is open non-compact in the group $\widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$. As the group $\widetilde{\mathbf{H}_1}$ is simple and simply connected, we obtain by the theorem (T) of [28] the equality $\widetilde{h_1}(\widetilde{U_l}) = \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$. This implies that the group $\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$ is normal in $\widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$: given $h \in \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$, let $g \in \widetilde{U_l}$ satisfying $\widetilde{h_1}(g) = h$. Then

$$(\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l))^h = (\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l))^g = (\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)) \ .$$

As the group $\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$ is an open normal subgroup of $\widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$ and the group $\widetilde{\mathbf{H}_1}$ is simply-connected, we obtain the equality $\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l) = \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$. As τ_1 is an isogeny of algebraic groups, we get that $U_l \cap \mathbf{H}_1(\mathbb{Q}_l)$ contains $\tau_1(\widetilde{\mathbf{H}_1}(\mathbb{Q}_l))$ with finite index. $\qquad \square$

As a corollary, there exists compact open subgroups $U_{l,1}$ in $K_1 \cap \mathbf{H}_{>1}(\mathbb{Q}_l)$ and $U_{l,>1}$ in $K_{>1}$ such that U_l contains the unbounded open subgroup $\tau_1(\widetilde{\mathbf{H}_1}(\mathbb{Q}_l)) \cdot U_{l,1} \cdot U_{l,>1}$.

Definition 6.8. *We replace U_l by its subgroup $\tau_1(\widetilde{\mathbf{H}}_1(\mathbb{Q}_l)) \cdot U_{l,1} \cdot U_{l,>1}$. We denote by V' the Zariski closure $\overline{\pi_{K_l}(\widetilde{V} \cdot U_l)}^{\mathrm{Zar}}$.*

As $\widetilde{Z_1}$ is stabilized by U_l the variety V' is a subvariety of Z.

Lemma 6.9. *The subvariety V' of Z is special.*

Proof. Let $V_{>1}$ denote the special subvariety of $S_{K_{>1}}(\mathbf{G}_{>1}, X_{>1})_{\mathbb{C}}$ projection of V. By definition of U_l the inclusion

$$V' \subset S_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}} \times V_{>1}$$

holds. On the other hand as $U_l \cap \mathbf{G}_1(\mathbb{Q}_l)$ is open unbounded in $\mathbf{G}_1(\mathbb{Q}_l)$ one can choose q in $U_l \cap \mathbf{G}_1(\mathbb{Q})^+$ and not contained in any bounded subgroup of U_l. Let Γ_q be the subgroup of $\mathbf{G}_1(\mathbb{Q})^+$ generated by $\Gamma := K_1 \cap \mathbf{G}_1(\mathbb{Q})^+$ and q. As q is not contained in a compact subgroup of U_l the group Γ_q contains Γ with infinite index. Let $x = (x_1, x_{>1})$ be any point of V (the coordinates correspond to the above decomposition of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ as a direct product). By definition of Γ_q the closure of $(\Gamma_q x_1, x_{>1})$ is contained in V'. By the proof of [17, th.6.1 p.637] this closure is $S_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}} \times \{x_{>1}\}$. Thus V' contains $S_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}} \times V_{>1}$. Finally

$$V' = S_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}} \times V_{>1} \ .$$

In particular V' is special. $\qquad\square$

Lemma 6.10. *The subvariety V' of Z contains V properly.*

Proof. As the Mumford-Tate group $\mathbf{H}$ of V centralizes the torus $\mathbf{T}$, the projection $\mathbf{H}_1$ of $\mathbf{H}$ on $\mathbf{G}_1$ centralizes the non-trivial torus $\mathbf{T}_1$ projection of $\mathbf{T}$ on $\mathbf{G}_1$. In particular $\mathbf{H}_1$ is a proper algebraic subgroup of $\mathbf{G}_1$. But as

$$V' = S_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}} \times V_{>1} \ ,$$

the group $\mathbf{G}_1$ is a direct factor of the Mumford-Tate group of V'. $\qquad\square$

$\qquad\square$

7. Existence of suitable Hecke correspondences.

In this section we prove, under some assumptions on the compact open subgroup K_l, the existence of Hecke correspondences of small degree candidates for applying theorem 6.1 assuming the Galois orbit of V is sufficiently big.

Definition 7.0.5. *Let* $\mathbf{G}$ *be a reductive* $\mathbb{Q}$-*group and* $\mathbf{T} \subset \mathbf{G}$ *a split torus. Let* l *be a prime number. A compact open subgroup* U_l *of* $\mathbf{G}(\mathbb{Q}_l)$ *is said to be* in good position *with respect to* $\mathbf{T}$ *if* $U_l \cap \mathbf{T}(\mathbb{Q}_l)$ *is the maximal compact open subgroup of* $\mathbf{T}(\mathbb{Q}_l)$.

Our main result in this section is the following :

Theorem 7.1. *Let* $(\mathbf{G}, X)$ *be a Shimura datum,* X^+ *a connected component of* X, $K \subset$ $\mathbf{G}(\mathbf{A}_f)$ *a neat open compact subgroup of* $\mathbf{G}(\mathbf{A}_f)$ *and* F *a number field containing a field of definition of* $S_K(\mathbf{G}, X)_{\mathbb{C}}$. *There is a positive integer* k *such that the following holds.*

Let V *be a special, not strongly special subvariety contained in a Hodge generic* F-*irreducible* F-*subvariety* Z *of* $S_K(\mathbf{G}, X)_{\mathbb{C}}$.

Let l *be a prime number splitting* $\mathbf{T}_V$ *and* m *an element of* $\mathbf{T}_V(\mathbb{Q}_l)$. *We assume that the compact open subgroup* K *is of the form* $K = K^l \cdot K_l$, *where* K^l *is a compact open subgroup of* $\mathbf{G}(\mathbf{A}_f^l)$ *and* K_l *is a compact open subgroup of* $\mathbf{G}(\mathbb{Q}_l)$ *contained in an Iwahori subgroup* I_l *of* $\mathbf{G}(\mathbb{Q}_l)$ *(c.f. next paragraph) in good position with respect to* $\mathbf{T}_V$.

Then there exists an element $m \in \mathbf{T}_V(\mathbb{Q}_l)$ *satisfying the following conditions :*

(1) $\mathrm{Gal}(\overline{F}/F) \cdot V \subset Z \cap T_m Z$.
(2) *For every* $k_1, k_2 \in K_l$ *the image of* $k_1 m k_2$ *in* $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$ *generates an unbounded subgroup of* $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$.
(3) $[K_l : K_l \cap m K_l m^{-1}] < l^k$.

Remark 7.0.6. As noticed in the introduction, the restriction $K_l \subset I_l$ is a necessary condition. One easily constructs a counter-example to the conclusion of theorem 7.1 if $K_l \subset \mathbf{G}(\mathbb{Q}_l)$ is a special maximal open compact subgroup. For explicit counterexamples see remark 7.2 of [14].

7.1. **Some properties of Iwahori subgroups.** We refer to [5], [6] and [18] for basic facts about buildings, Iwahori subgroups and Iwahori-Hecke algebras.

We first recall the definition of an Iwahori subgroup. Let l be a prime number. Let $\mathbf{G}$ be a reductive algebraic isotropic $\mathbb{Q}_l$-group and $\mathbf{A} \subset \mathbf{G}$ a maximal split torus of $\mathbf{G}$. We denote by $\mathbf{M} \subset \mathbf{G}$ the centraliser of $\mathbf{A}$ in $\mathbf{G}$. We choose $\mathbf{P} = \mathbf{M} \cdot \mathbf{N}$ a minimal parabolic subgroup of $\mathbf{G}$, where $\mathbf{N}$ denotes the unipotent radical of $\mathbf{P}$. Let $\mathcal{X}$ be the (extended) Bruhat-Tits building of $\mathbf{G}$, $\mathcal{A} \subset \mathcal{X}$ the apartment of $\mathcal{X}$ associated to $\mathbf{A}$. Let $K_l^{\mathrm{m}} \subset \mathbf{G}(\mathbb{Q}_l)$ be a special maximal subgroup (c.f [5, (I), def. 1.3.7 p.22, def. 4.4.1 p.79]) of $\mathbf{G}(\mathbb{Q}_l)$ such that $K_{l,\mathbf{A}}^{\mathrm{m}} = K_l^{\mathrm{m}} \cap \mathbf{A}(\mathbb{Q}_l)$ is the maximal compact open subgroup of $\mathbf{A}(\mathbb{Q}_l)$. We denote by $x_0 \in \mathcal{A}$ the unique K_l^{m}-fixed vertex in $\mathcal{X}$, by $\mathcal{C} \subset \mathcal{A}$ the unique Weyl chamber with apex at x_0 whose stabilizer at infinity is $\mathbf{P}(\mathbb{Q}_l)$, by C the unique chamber (or alcove) of $\mathcal{C}$ having x_0 for one of its vertices and by $I_l \subset K_l^{\mathrm{m}}$ the Iwahori subgroup fixing C pointwise.

Remark 7.1.1. Strictly speaking (i.e. with the notations of Bruhat-Tits [5]) the group I_l as defined above is an Iwahori subgroup only in the case where the group $\mathbf{G}$ is simply-connected. Our terminology is a well-established abuse of notations.

Definition 7.1.2. *We denote by* $\mathrm{ord}_{\mathbf{M}} : \mathbf{M}(\mathbb{Q}_l) \longrightarrow X_*(\mathbf{M})$ *the homomorphism characterized by*

$$< \mathrm{ord}_{\mathbf{M}}(m), \alpha >= \mathrm{ord}_{\mathbb{Q}_l}(\alpha(m)) \ ,$$

where $\mathrm{ord}_{\mathbb{Q}_l}$ *denotes the normalized (additive) valuation on* $\mathbb{Q}_l^*$ *and* $X_*(\mathbf{M})$ *denotes the group of cocharacters of* $\mathbf{M}$. *We denote by* $\Lambda \subset X_*(\mathbf{M})$ *the free* $\mathbf{Z}$-*module* $\mathrm{ord}_{\mathbf{M}}(\mathbf{M}(\mathbb{Q}_l))$.

The group $\mathbf{M}(\mathbb{Q}_l)$ (in particular the group $\mathbf{A}(\mathbb{Q}_l)$) acts on $\mathcal{A}$ via Λ-translations.

Definition 7.1.3. *Let* $\Lambda^+ \subset \Lambda$ *be the positive cone associated to the Weyl chamber* $\mathcal{C}$.

Elements of Λ^+ acting on $\mathcal{A}$ map $\mathcal{C}$ to $\mathcal{C}$.

Proposition 7.1.4. *Let m be an element of* $\mathbf{A}(\mathbb{Q}_l)$ *with non-trivial image* $\mathrm{ord}_M(m) \in \Lambda^+$. *Then for any elements $i_1, i_2 \in I_l$, the element $i_1 m i_2 \in \mathbf{G}(\mathbb{Q}_l)$ is not contained in a compact subgroup of* $\mathbf{G}(\mathbb{Q}_l)$.

Proof. Let W_0 be the finite Weyl group of $\mathbf{G}$, let W be the modified affine Weyl group associated to $\mathcal{A}$ and Ω the finite subgroup of W taking the chamber C to itself. Let $\Delta = \{\alpha_1, \cdots, \alpha_m\}$ be the set of affine roots on $\mathcal{A}$ which are positive on C and whose null set H_α is a wall of C. For $\alpha \in \Delta$ we denote by S_α the reflexion of $\mathcal{A}$ along the wall H_α. The group W is generated by Ω and the S_α's, $\alpha \in \Delta$. It identifies with the semi-direct product $W_0 \ltimes \Lambda$ (c.f. [6, p.140]).

Recall the Bruhat-Tits decomposition :

(7.1) $$\mathbf{G}(\mathbb{Q}_l) = I_l \cdot W \cdot I_l \quad,$$

where by abuse of notations we still write W for a set of representatives of W in $\mathbf{G}(\mathbb{Q}_l)$ Let $r : \mathbf{G}(\mathbb{Q}_l) \longrightarrow W$ be the map sending $g \in \mathbf{G}(\mathbb{Q}_l)$ to the unique $r(g) \in W$ such that $r(g) \in I_l g I_l$. Geometrically speaking the map r essentially coincides with the retraction $\rho_{\mathcal{A},C}$ of the Bruhat-Tits building $\mathcal{X}$ with centre the chamber C onto the apartment $\mathcal{A}$ ([5, I, theor.2.3.4]).

Let $\mathcal{H}(\mathbf{G}, I_l)$ be the Hecke algebra (for the convolution product) of bi-I_l-invariant compactly supported continuous complex functions on $\mathbf{G}(\mathbb{Q}_l)$. By the equation (7.1) this is an associative algebra with a vector space basis $T_w = 1_{I_l w I_l}$, $w \in W$, where $1_{I_l w I_l}$ denotes the characteristic function of the double coset $I_l w I_l$. A presentation of the algebra $\mathcal{H}(\mathbf{G}, I_l)$ with generators T_ω, $\omega \in \Omega$, and T_α, $\alpha \in \Delta$, is given in [6, theorem 3.6 p.142] (or [4, p.242-243]). Given $w \in W$ let $l(w) \in \mathbb{N}$ be the number of hyperplanes H_α separating the

two chambers C and wC. One obtains in particular (c.f. [6, theorem 3.6 (b)] or [3, section 3.2, 1) and 6)]) :

(7.2) $$\forall w, w' \in W, \quad T_w \cdot T_{w'} = T_{ww'} \text{ if } l(ww') = l(w) + l(w') \ .$$

Let $\delta \in X^*(\mathbf{M})$ be the determinant of the adjoint action of $\mathbf{M}$ on the Lie algebra of $\mathbf{N}$. For $\lambda \in \Lambda^+ \subset W$ one easily shows the equality (c.f. [18, (1.11)]) :

(7.3) $$l(\lambda) = \langle \delta, \lambda \rangle \ .$$

In particular any two elements λ, μ in $\Lambda^+ \subset W$ satisfy $l(\lambda \cdot \mu) = l(\lambda) + l(\mu)$. Thus the equation (7.2) implies the relation :

(7.4) $$T_\lambda T_\mu = T_{\lambda + \mu} \ .$$

Remark 7.1.5. Equality (7.4) is stated in [18, (1.15)]) for the Iwahori-Hecke algebra of a split adjoint group, but generalizes easily.

Let m, i_1, i_2 as in the statement of the proposition and denote by g the element $i_1 m i_2 \in \mathbf{G}(\mathbb{Q}_l)$. By equation (7.4) one has the equality :

$$r(g^n) = n \cdot r(g) = n \cdot \mathrm{ord}_{\mathbf{M}}(m) \ .$$

This implies that the chamber $\rho_{\mathcal{A},C}(g^n C) = n \cdot \mathrm{ord}_{\mathbf{M}}(m) + C$ leaves any compact of $\mathcal{A}$ as n tends to infinity. As a corollary the chamber $g^n C$ of $\mathcal{X}$ also leaves any compact of $\mathcal{X}$ when n tends to infinity. This proves that the group $g^{\mathbb{Z}}$ is not contained in a compact subgroup of $\mathbf{G}(\mathbb{Q}_l)$. $\qquad\square$

7.2. Some uniformity results.
In this section we prove some uniformity results concerning Shimura data and reciprocity morphisms. The first is this simple observation.

Lemma 7.2.1. *Let $(\mathbf{G}, X)$ be a Shimura datum. There is constant R such that for any Shimura sub-datum $(\mathbf{H}, X_{\mathbf{H}})$, the degree of the reflex field $E(\mathbf{H}, X_{\mathbf{H}})$ over $E(\mathbf{G}, X)$ is bounded by R.*

Proof. This is a direct consequence of the definition of the reflex field. $\qquad\square$

Proposition 7.2.2. *Let $(\mathbf{G}, X)$ be a Shimura datum, $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ a neat open compact subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. There is a positive integer h such that the following holds. Let V be a special subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ which is not strongly special and l be a prime splitting $\mathbf{T}_V$. For any m in $\mathbf{T}_V(\mathbb{Q}_l)$, m^h satisfies the condition that for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$*

$$\sigma(V) \subset T_{m^h}(V).$$

Proof. Let V be as above. For simplicity of notations we write $\mathbf{T}$ for $\mathbf{T}_V$, $\mathbf{H}$ for $\mathbf{H}_V$ and $\mathbf{C}$ for $\mathbf{C}_V$. By definition a constant is called uniform if it is independent of V.

To show the existence of an element h as in the statement, we will prove several lemmas. The first one is the following.

Lemma 7.2.3. *Replace F by a compositum of it with the reflex field $E(\mathbf{C}, \{x\})$. There is a uniform integer n_1 such that for any $m \in \mathbf{T}(\mathbb{Q}_l)$, the power m^{n_1} is in the preimage of $r_{(\mathbf{C}, \{x\})}((\mathbb{Q}_l \otimes F)^*)$ in $\mathbf{T}(\mathbb{Q}_l)$ by the natural map $\mathbf{T}(\mathbb{Q}_l) \longrightarrow \mathbf{C}(\mathbb{Q}_l)$.*

Proof. Let $\mathbf{L}$ be the torus $\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}_{\mathbf{m}F}$. The element x gives a cocharacter $\mu_{\mathbf{C}} \colon \mathbf{G}_{\mathbf{m}\mathbb{C}} \longrightarrow \mathbf{C}_{\mathbb{C}}$ defined by $\mu_{\mathbf{C}}(z) = x_{\mathbb{C}}(z, 1)$. The morphism $r_{(\mathbf{C}, \{x\})} \colon \mathbf{L} \longrightarrow \mathbf{C}$ corresponds to the morphism on cocharacter groups $X_*(\mathbf{L}) \longrightarrow X_*(\mathbf{C})$ which sends the cocharacter $\mu_\sigma \in X_*(\mathbf{L})$ (induced by $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$) to $\sigma(\mu_{\mathbf{C}})$. The lemma 2.4 of [37] says that there is a basis (χ_i) of characters of $\mathbf{C}$ such that the $< \chi_i, \sigma(\mu_{\mathbf{C}}) >$ are uniformly bounded. It follows that the index of $r_{(\mathbf{C}, \{x\})}((\mathbb{Q}_l \otimes F)^*)$ in $\mathbf{C}(\mathbb{Q}_l)$ is finite (this is the consequence of the fact that $r_{(\mathbf{C}, \{x\})}$ is surjective as a morphism of tori) and uniformly bounded. Let n_1 be a uniform bound on this index. It follows that for any $m \in \mathbf{T}(\mathbb{Q}_l)$, the power m^{n_1} is in the preimage of $r_{(\mathbf{C}, \{x\})}((\mathbb{Q}_l \otimes F)^*)$ in $\mathbf{T}(\mathbb{Q}_l)$. $\qquad\square$

Recall that we have assumed that F contains the reflex field of $(\mathbf{C}, \{x\})$ and that we have the following sequence of morphisms

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F) \overset{r_{(\mathbf{H}, X_{\mathbf{H}})}}{\longrightarrow} \overline{\pi_0}(\pi(\mathbf{H})) \overset{p}{\longrightarrow} \overline{\pi_0}(\pi(\mathbf{C})) \ .$$

For an element t of $\mathbf{H}(\mathbf{A})$ (resp. $\mathbf{C}(\mathbf{A})$), we write $\overline{m}$ for its image in $\pi(\mathbf{H})$ (resp. $\pi(\mathbf{C})$).

We know that $p(\overline{m}^{n_1})$ is in $p \circ r_{(\mathbf{H}, X_{\mathbf{H}})}(\mathrm{Gal}(\overline{\mathbb{Q}}/F)) = r_{(\mathbf{C}, \{x\})}(\mathrm{Gal}(\overline{\mathbb{Q}}/F))$. Hence, there is an element σ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that

$$p(\overline{m}^{n_1}) = (p \circ r_{(\mathbf{H}, X_{\mathbf{H}})})(\sigma) \ .$$

It follows that there exists an element y in the kernel of p such that

$$\overline{m}^{n_1} = y r_{(\mathbf{H}, X_{\mathbf{H}})}(\sigma) \ ,$$

Our next aim is to show that a uniform power of $\overline{m}$ is actually in $r_{(\mathbf{H}, X_{\mathbf{H}})}(\mathrm{Gal}(\overline{\mathbb{Q}}/F))$. This follows directly from the following lemma.

Lemma 7.2.4. *There exists a uniform integer n such that any element of the kernel of p is killed by n.*

Proof. We start by noticing that to prove the lemma, it suffices to prove that the natural morphism

$$\pi(\mathbf{H}) \longrightarrow \pi(\mathbf{C})$$

has a kernel killed by a uniform integer. Indeed, suppose this to be the case. Then, passing to the group of connected components, we see that the kernel of

$$\pi_0(\pi(\mathbf{H})) \longrightarrow \pi_0(\pi(\mathbf{C}))$$

satisfies the same property. The lemma now follows from the fact that the groups $\pi_0(\mathbf{H}(\mathbb{R})_+)$ and $\pi_0(\mathbf{C}(\mathbb{R})_+)$ are finite of uniformly bounded order.

Let us now turn to showing that the kernel of $\pi(\mathbf{H}) \longrightarrow \pi(\mathbf{C})$ is killed by a uniformly bounded integer. To simplify notations, let us still denote this morphism by p. Recall that $\pi(\mathbf{H}) = \mathbf{H}(\mathbf{A})/\mathbf{H}(\mathbb{Q})\rho(\widetilde{\mathbf{H}}(\mathbb{Q}))$. Let y be an element of $\mathbf{H}(\mathbf{A})$ such that its image $\overline{y}$ in $\pi(\mathbf{H})$ belongs to the kernel of p.

Using that $\mathbf{H} = \mathbf{T}\mathbf{H}^{\mathrm{der}}$ and that $\mathbf{T} \cap \mathbf{H}^{\mathrm{der}}$ is finite of uniformly bounded order we see that there is a uniform integer n_2, an element t in $\mathbf{T}(\mathbf{A})$ and α in $\mathbf{H}^{\mathrm{der}}(\mathbf{A})$ such that

$$y^{n_2} = t \cdot \alpha$$

As $\mathbf{H}^{\mathrm{der}}(\mathbf{A})/\rho\widetilde{\mathbf{H}^{\mathrm{der}}}(\mathbf{A})$ is killed by a uniform integer n_3, the images of $y^{n_2 n_3}$ and t^{n_3} in $\pi(\mathbf{H})$ coincide.

Consider the exact sequence

$$W \longrightarrow \mathbf{T} \xrightarrow{\nu} \mathbf{C}$$

where $W = \mathbf{T} \cap \mathbf{H}^{\mathrm{der}}$ and hence is of order n_2. As $\overline{y}$ (and hence $\overline{y^{n_2 n_3}}$) is in the kernel of p, the image of t^{n_3} in $\mathbf{C}(\mathbf{A})$ is in $\mathbf{C}(\mathbb{Q})$ (note that $\pi(\mathbf{C}) = \mathbf{C}(\mathbf{A})/\mathbf{C}(\mathbb{Q})$).

A n_2-th power of any element of $\mathbf{C}(\mathbb{Q})$ is in the image of $\mathbf{T}(\mathbb{Q})$ hence there exists a q in $\mathbf{T}(\mathbb{Q})$ such that

$$\nu(t^{n_3 n_2}) = \nu(q)$$

It follows that

$$t^{n_3 n_2} = qw$$

where w is in $W(\mathbf{A})$. As $W(\mathbf{A})$ is killed by n_2, we see that $t^{n_3 n_2^2} = q^{n_2} \in \mathbf{T}(\mathbb{Q})$. The image of $t^{n_3 n_2^2}$ in $\pi(\mathbf{H})$ equals the image of $y^{n_3 n_2^3}$ and therefore a uniform power of y is in the kernel. $\square$

We have proved the following:

Lemma 7.2.5. *There is a uniform integer h such that the image of m^h in $\overline{\pi_0}\pi(\mathbf{H})$ is in $r_{(\mathbf{H},X_{\mathbf{H}})}(\mathrm{Gal}(\overline{\mathbb{Q}}/F))$.*

Proof. Take $h = n_1 n$ with n the integer from 7.2.4. $\square$

It remains to see that some Galois conjugate (and therefore the whole of the Galois orbit) of V is in $T_{m^h}V$. The variety V is the image of $(X_{\mathbf{H}}^+, 1)$ in $\mathrm{Sh}_K(\mathbf{G}, X)$. Let σ be the

element of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ as above. By definition of the Galois action on the set of connected components of a Shimura variety, we get

$$\sigma(V) = \overline{(X_{\mathbf{H}}^+, m^h)} \subset T_{m^h} V$$

where $\overline{(X_{\mathbf{H}}^+, m^h)}$ stands for the image of $(X_{\mathbf{H}}^+, m^h)$ in $\mathrm{Sh}_K(\mathbf{G}, X)$. $\qquad\square$

7.3. Proof of theorem 7.1.

As V is non-strongly special, the torus $\mathbf{T}_V^{\mathrm{ad}} := \lambda(\mathbf{T}_V)$ is a non-trivial torus in $\mathbf{G}^{\mathrm{ad}}$, where $\lambda : \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ denotes the natural morphism. Let $\mathbf{A}^{\mathrm{ad}}$ be a maximal split torus of $\mathbf{G}_{\mathbb{Q}_l}^{\mathrm{ad}}$ containing $\mathbf{T}_{V,\mathbb{Q}_l}^{\mathrm{ad}}$. Let C be the unique chamber of the Bruhat-Tits building $\mathcal{X}$ of $\mathbf{G}_{\mathbb{Q}_l}^{\mathrm{ad}}$ fixed by I_l and x_0 a special vertex in the closure of C such that the intersection of its stabilizer with $\mathbf{T}_V^{\mathrm{ad}}(\mathbb{Q}_l)$ is maximal compact in $\mathbf{T}_V^{\mathrm{ad}}(\mathbb{Q}_l)$. Choose a minimal parabolic subgroup $\mathbf{P}^{\mathrm{ad}}$ of $\mathbf{G}_{\mathbb{Q}_l}^{\mathrm{ad}}$ whose Levi subgroup is the centraliser $\mathbf{M}^{\mathrm{ad}}$ of $\mathbf{A}^{\mathrm{ad}}$.

We use the notations of section 7.1 applied to $\mathbf{G}_{\mathbb{Q}_l}^{\mathrm{ad}}$. By lemma 2.4 of [37] and the proposition 7.4.3 of [17] there exists a uniform constant k_1 and an element $m \in \mathbf{T}_V(\mathbb{Q}_l)$ such that $\lambda(m)$ has a non-trivial image in $\Lambda^+ \subset X_*(\mathbf{M}^{\mathrm{ad}})$ and $[K_l : K_l \cap mK_lm^{-1}] < l^{k_1}$. By proposition 7.2.2, there is a uniform constant h such that for some $\sigma \in \mathrm{Gal}(\overline{F}/F)$, one has $\sigma(V) \subset T_{m^h} V$.

The uniform constant $k = k_1 h$ and the element m^h satisfies the conditions of the theorem :

From $\sigma(V) \subset T_{m^h} V \subset T_{m^h} Z$ and as $T_{m^h} Z$ is defined over F, we deduce $V \subset T_{m^h} Z$. As $V \subset Z$ we obtain condition (1).

As $\lambda(m)$ has a non-trivial image in $\Lambda^+ \subset X_*(\mathbf{M}^{\mathrm{ad}})$, $\lambda(m^h)$ too. By proposition 7.1.4, for any k_1, k_2 in K_l, the image of $k_1 \cdot m \cdot k_2$ in $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$ generates an unbounded subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$: this is condition (2).

As $\deg T_m = [K_l : K_l \cap mK_lm^{-1}] < l^{k_1}$ and $T_{m^h} \subset (T_m)^h$ as algebraic correspondences, $[K_l : K_l \cap mK_lm^{-1}] = \deg T_{m^h} \leq (\deg T_m)^h \leq l^k$: this is condition (3).

This finishes the proof of theorem 7.1.

8. CONDITION ON THE PRIME l

In this section, we use theorem 2.4.3, theorem 6.1, theorem 7.1 to show (under one of the assumptions of theorem 3.1.1) that the existence of a prime number l satisfying certain conditions forces a subvariety Z of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing a non-strongly special subvariety V to contain a special subvariety V' containing V properly.

8.1. Passing to an Iwahori subgroup.

In the process of constructing V' we will encounter one minor technical difficulty : we will have to lift the situation to an Iwahori

level in order to apply theorem 7.1. The following lemma introduces an absolute constant f which controls this phenomenon.

Lemma 8.1.1. *Let $\mathbf{G}$ be a reductive $\mathbb{Q}$-group.*

a) For any prime l, any $\mathbb{Q}_l$-split torus $\mathbf{T} \subset \mathbf{G}$ and any special maximal compact subgroup $K_l \subset \mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}$, there exists an Iwahori subgroup I_l of K_l in good position with respect to $\mathbf{T}$.

b) There exists an integer f such that for any reductive $\mathbb{Q}$-subgroup $\mathbf{H} \subset \mathbf{G}$, any prime l such that $\mathbf{H}_{\mathbb{Q}_l}$ is not $\mathbb{Q}_l$-anisotropic, and any special maximal compact subgroup K_l of $\mathbf{H}(\mathbb{Q}_l)$, any Iwahori subgroup $I_l \subset K_l$ is of index $|K_l/I_l|$ smaller than l^f.

Proof. To prove $a)$ let l, $\mathbf{T}$ and K_l be as in the statement. Choose a maximal split torus $\mathbf{A}$ of $\mathbf{G}_{\mathbb{Q}_l}$ containing $\mathbf{T}_{\mathbb{Q}_l}$, denote by $\mathbf{M}$ the centraliser of $\mathbf{A}$ in $\mathbf{G}_{\mathbb{Q}_l}$ and choose any minimal parabolic $\mathbf{P}$ with Levi $\mathbf{M}$. By construction the Iwahori subgroup I_l defined by $\mathbf{P}$ and K_l (c.f. section 7.1) satisfies that $I_l \cap \mathbf{A}(\mathbb{Q}_l)$ is the maximal compact open subgroup of $\mathbf{A}(\mathbb{Q}_l)$. In particular $I_l \cap \mathbf{T}(\mathbb{Q}_l)$ is the maximal compact open subgroup of $\mathbf{T}(\mathbb{Q}_l)$.

To prove $b)$: notice that the index $[K_l : I_l]$ coïncide with $\sum_{w \in W_0} q_w$ where W_0 denotes the finite Weyl group of $\mathbf{H}_{\mathbb{Q}_l}$ and q_w denotes $[I_l w I_l : I_l]$ for $w \in W_0$. With the notations of [33, section 3.3.1] for a reduced word $w = r_1 \cdots r_j \in W$ one has $q_w = l^d$ with $d = \sum_{i=1}^{j} d(\nu_i)$, where ν_i denotes the vertex of the local Dynkin diagram of $\mathbf{H}_{\mathbb{Q}_l}$ corresponding to the reflection r_i. As the cardinality of W_0 and its length function are bounded when $\mathbf{H}$ ranges through reductive $\mathbb{Q}$-subgroups of $\mathbf{G}$ and l through prime numbers we are reduced to prove that for any positive integer r there exist a positive integer s such that $d(\nu_i) \leq s$ for any local Dynkin diagram of rank at most r. This follows immediately from inspecting the tables in [33, section 4]. $\qquad\square$

8.2. **Notation.** In the following we will consider the following set of assumptions and data :

Definition 8.2.1. *Assume the GRH.*

Let $(\mathbf{G}', X')$ be a Shimura datum with $\mathbf{G}'$ semi-simple of adjoint type, $K' = \prod_{p \text{ prime}} K'_p$ a neat compact open subgroup of $\mathbf{G}'(\mathbf{A}_{\mathrm{f}})$. We suppose that the group K_3 is the principal congruence subgroup of level three.

Let F a number field containing the reflex field $E(\mathbf{G}', X')$. Let N be a positive integer, let B and $C(N)$ be as in the theorem 2.4.3, k the constant defined in theorem 7.1, and f the constant defined in lemma 8.1.1.

Let $(\mathbf{G}, X)$ be a Shimura subdatum of $(\mathbf{G}', X')$ with $\mathbf{G}$ the generic Mumford-Tate group on X and let $K = K' \cap \mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let $V \subset S_K(\mathbf{G}, X)_{\mathbb{C}}$ be a special subvariety which is not strongly special.

Let l be a prime splitting $\mathbf{T}_V$ such that K_l is contained in a special maximal compact subgroup $K_l^{\max}$ of $\mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}_V$.

The following remark will be especially important :

Lemma 8.2.2. *In the situation of definition 8.2.1 the real torus $\mathbf{T}_V(\mathbb{R})$ is compact.*

Proof. As $\mathbf{T}_V$ is the connected center of the generic Mumford-Tate group $\mathbf{H}_V \subset \mathbf{G} \subset \mathbf{G}'$ of V it fixes some point x' of X'. As $\mathbf{G}'$ is semisimple of adjoint type the stabilizer of x' in $\mathbf{G}'(\mathbb{R})$ is compact. $\square$

8.3. **The criterion.** We can now state the main result of this section :

Theorem 8.3.1. *In the situation of definition 8.2.1 let Z be a Hodge generic F-irreducible F-subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ containing V and satisfying*

$$(8.1) \qquad l^{(k+2f) \cdot 2^{a(r+1)}} \cdot (\deg_{L_K} Z)^{2^{a(r)}} < C(N)\alpha_V \beta_V^N \ ,$$

where $r = \dim Z - \dim V$ and $a : \mathbb{N} \longrightarrow \mathbb{N}$ is the function defined by $a(n) = \frac{n(n+1)}{2}$.

Then Z contains a special subvariety V' that contains V properly.

Moreover if one considers only the subvarieties V such that the associated tori $\mathbf{T}_V$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class, then the assumption of the GRH can be dropped.

8.4. **An auxiliary proposition.** In addition to theorem 2.4.3, theorem 6.1, and theorem 7.1, the main ingredient in the proof of theorem 8.3.1 is the following :

Proposition 8.4.1. *In the situation of definition 8.2.1 let Z be a Hodge generic F-irreducible F-subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ containing V and satisfying*

$$(8.2) \qquad l^{k \cdot 2^{r-1}} (\deg_{L_K} Z)^{2^r} < C(N)\alpha_V \beta_V^N$$

for $r = \dim Z - \dim V$.

Let m be an element of $\mathbf{T}_V(\mathbb{Q}_l)$ satisfying the conclusion of theorem 7.1 with respect to Z.

Then one of the following holds :

(a) $Z \subset T_m Z$.

(b) *there exists an F-irreducible subvariety Y of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ with the following properties :*

- $\mathrm{Gal}(\overline{F}/F) \cdot V \subsetneq Y \subset Z \cap T_m Z \subsetneq Z$.
- $\deg_{L_K} Y \leq l^{k \cdot 2^{r-1}} \cdot (\deg_{L_K} Z)^{2^r}$.
- *Let Y_1 be an irreducible component of Y containing V. We denote by $\mathbf{G}_Y \subset \mathbf{G}$ the generic Mumford-Tate group on Y_1, by $(\mathbf{G}_Y, X_{\mathbf{G}_Y})$ the corresponding Shimura subdatum of $(\mathbf{G}, X)$ and by K_Y the intersection $K \cap \mathbf{G}_Y(\mathbf{A}_{\mathrm{f}})$. Let $\widetilde{V}$*

be the special subvariety of $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_{\mathbf{G}_Y})$ preimage of V. Then $\widetilde{V}$ is not
strongly special.

Moreover if one considers only the subvarieties V such that the associated tori $\mathbf{T}_V$ lie in
one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class, then the assumption of the GRH can be dropped.

8.4.1. We start with the following auxiliary lemma :

Lemma 8.4.2. *In the situation of definition 8.2.1 let Y be a subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$
defined over $\overline{\mathbb{Q}}$ such that an irreducible component Y_1 of Y contains V. We denote by
$\mathbf{G}_Y \subset \mathbf{G}$ the generic Mumford-Tate group on Y_1, by $(\mathbf{G}_Y, X_{\mathbf{G}_Y})$ the corresponding Shimura
subdatum of $(\mathbf{G}, X)$ and by K_Y the intersection $K \cap \mathbf{G}_Y(\mathbf{A}_{\mathrm{f}})$. Let $\widetilde{V}$ and $\widetilde{Y_1}$ be the
preimages of V and Y_1 respectively in $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_{\mathbf{G}_Y})_{\mathbb{C}}$.*

Suppose that $\widetilde{V}$ is strongly special in $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_{\mathbf{G}_Y})_{\mathbb{C}}$ and that

$$\deg_{L_{K_Y}}(\mathrm{Gal}(\overline{F}/F) \cdot \widetilde{Y_1}) \leq C(N)\alpha_V \beta_V^N \ \ .$$

Then

    (a) $\mathbf{T}_V = Z(\mathbf{G}_Y)^0$, where $Z(\mathbf{G}_Y)^0$ denotes the connected centre of $\mathbf{G}_Y$. In particular
       $\mathbf{T}_V(\mathbf{A}_{\mathrm{f}})$ acts on $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}}$.
    (b) There exists $\sigma \in \mathrm{Gal}(\overline{F}/F)$ such that

$$r_{\mathbf{T}_V}(\sigma) \cdot Y_1 \not\subseteq \mathrm{Gal}(\overline{F}/F) \cdot Y_1 \ \ .$$

*Moreover if one considers only the subvarieties V such that the associated tori $\mathbf{T}_V$ lie
in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class, then the assumption of the GRH can be dropped.*

Proof. As $\widetilde{V}$ is strongly special in $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}}$, the connected centre $\mathbf{T}_V$ of $\mathbf{H}_V$ is
contained in the connected centre $Z(\mathbf{G}_Y)^0$ of $\mathbf{G}_Y$. By lemma 5.2, one obtains the equality :

$$\mathbf{T}_V = Z(\mathbf{G}_Y)^0 \ \ .$$

This proves (a).

The idea to prove (b), is to obtain a lower bound for the degree of $r_{\mathbf{T}_V}(\mathrm{Gal}(\overline{F}/F)\widetilde{Y_1})$
orbit of $\widetilde{Y_1}$ by proceeding *exactly* as in [37], section 2.2. This lower bound will contradict
the assumption made on the degree of $\mathrm{Gal}(\overline{F}/F) \cdot \widetilde{Y_1}$. For simplicity of notations, write Y
(resp. $\widetilde{Y}$) instead of Y_1 (resp. $\widetilde{Y_1}$).

Assume for contradiction that

$$r_{\mathbf{T}_V}(\mathrm{Gal}(\overline{F}/F)) \cdot Y \subset \mathrm{Gal}(\overline{F}/F) \cdot Y \ \ ,$$

thus also

$$r_{\mathbf{T}_V}(\mathrm{Gal}(\overline{F}/F)) \cdot \widetilde{Y} \subset \mathrm{Gal}(\overline{F}/F) \cdot \widetilde{Y} \ \ .$$

Write $K_Y = \prod_p K_{Y,p}$, product of compact open subgroups of $\mathbf{G}_Y(\mathbb{Q}_p)$. Define the compact open subgroup $K_Y^m \subset \mathbf{G}_Y(\mathbf{A}_f)$ as the product $K_{\mathbf{T}_V}^m \cdot K_Y$ and, as in the section 2.2 of [37], we make K_Y^m neat by replacing it, by the neat open compact subgroup $(K_{Y,3}^m \cap K_3) \prod_{p \neq 3} K_{Y,p}^m$. This is a neat subgroup of uniformly bounded index. Consider the diagram deduced from the inclusion $K_Y \subset K_Y^m$:

$$
\begin{array}{ccccc}
\widetilde{V} & \hookrightarrow & \widetilde{Y} & \hookrightarrow & \mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}} \\
\downarrow & & \downarrow & & {\scriptstyle \pi}\downarrow \\
\widetilde{V}^m & \hookrightarrow & \widetilde{Y}^m & \hookrightarrow & \mathrm{Sh}_{K_Y^m}(\mathbf{G}_Y, X_Y)_{\mathbb{C}}
\end{array}
$$

where $\widetilde{V}^m$ is $\pi(\widetilde{V})$ and $\widetilde{Y}^m$ is $\pi(\widetilde{Y})$.

The group $K_{\mathbf{T}_V}^m/K_{\mathbf{T}_V}$ acts freely on $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}}$ and the morphism π is finite of degree $|K_{\mathbf{T}_V}^m/K_{\mathbf{T}_V}|$. As $\mathbf{T}_V(\mathbb{R})$ is compact one can follow the proof of the lemma 2.8 of [37] with Y instead of V (there the assumption that the subvariety V is special is not used in the proof, and the assumption that the group $\mathbf{G}$ is semisimple of adjoint type is only used to imply that $\mathbf{T}_V(\mathbb{R})$ is compact). We obtain that $\deg_{L_K}(\mathrm{Gal}(\overline{F}/F)) \cdot \widetilde{Y}$ is at least the degree of $\mathrm{Gal}(\overline{F}/F) \cdot \widetilde{V} \cap \pi^{-1}(\widetilde{Y}^m)$ times the number of $\mathrm{Gal}(\overline{F}/F)$ conjugates of $\widetilde{Y}^m$.

We now use that

$$
r_{\mathbf{T}_V}(\mathrm{Gal}(\overline{F}/F)) \cdot \widetilde{Y} \subset \mathrm{Gal}(\overline{F}/F) \cdot \widetilde{Y}
$$

Following [37], prop 2.11 and 2.12 again with Y instead of V, we obtain that the degree of $\mathrm{Gal}(\overline{F}/F) \cdot \widetilde{V} \cap \pi^{-1}(\widetilde{Y}^m)$ is at least α_V and by following prop 2.10 of [37] we obtain that the number of $\mathrm{Gal}(\overline{F}/F)$ conjugates of $\widetilde{Y}^m$ is at least $C(N)\beta_V^N$. This contradicts the assumption that $\deg_{L_{K_Y}} \widetilde{Y} \leq \deg_{L_K} Y \leq C(N)\alpha_V \beta_V^N$. $\qquad \square$

8.4.2. *Proof of proposition 8.4.1.*

Proof. Suppose we are not in the case (a). We need to construct a subvariety Y satisfying the conditions of (b).

Step 1 : As $V \subset Z \cap T_m Z$, there exists a geometric irreducible component Y_1 of $Z \cap T_m Z$ containing V. Notice that Z and $T_m Z$ do not have any geometric irreducible component in common as Z and $T_m Z$ are defined over F, Z is F-irreducible and $Z \not\subset T_m Z$. In particular $\dim Y_1 < \dim Z$.

Lemma 8.4.3. $V \subsetneq Y_1$

Proof. Otherwise $V = Y_1$ and $\mathrm{Gal}(\overline{F}/F) \cdot V$ is a union of geometrically irreducible components of $Z \cap T_m Z$. Thus

$$
\deg_{L_K}(\mathrm{Gal}(\overline{F}/F) \cdot V) \leq \deg_{L_K}(Z \cap T_m Z) \leq (\deg_{L_K} Z)^2 [K_l : K_l \cap m K_l m^{-1}] \ .
$$

The last inequality is the consequence of the theorem 7.2 of [16] and its proof. As m satisfies the conclusion of theorem 7.1, $[K_l : K_l \cap mK_lm^{-1}] < l^k$.

As $\deg_{L_K}(\mathrm{Gal}(\overline{F}/F) \cdot V) \geq C(N)\alpha_V \beta_V^N$ by theorem 2.4.3, we finally obtain the inequality :

$$C(N)\alpha_V \beta_V^N \leq (\deg_{L_K} Z)^2 l^k \ .$$

This contradicts the inequality 8.2 on page 43.   □

Let Y be the $\mathrm{Gal}(\overline{F}/F)$-orbit of Y_1. We obtain $\mathrm{Gal}(\overline{F}/F) \cdot V \subsetneq Y \subset Z \cap T_m Z \subsetneq Z$. Moreover $\deg_{L_K} Y \leq (\deg_{L_K} Z)^2 l^k < C(N)\alpha_V \beta_V^N$.

Step 2 : Let $\mathbf{G}_1$ be the generic Mumford-Tate group on Y_1, $(\mathbf{G}_1, X_1) \subset (\mathbf{G}, X)$ the Shimura sub-datum it induces, K_{Y_1} the compact open subgroup $K \cap \mathbf{G}_{Y_1}(\mathbf{A}_\mathrm{f})$ of $\mathbf{G}_1(\mathbf{A}_\mathrm{f})$. Let $\widetilde{V}$ be special subvariety of $\mathrm{Sh}_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}}$ preimage of V.

If $\widetilde{V}$ is non-strongly special in $\mathrm{Sh}_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}}$ then Y satisfies the condition (b) of proposition 8.4.1 and we are done.

Thus we can assume that $\widetilde{V}$ is strongly special in $\mathrm{Sh}_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}}$. As $V \subsetneq Y_1$ and $\deg_{L_K} Y \leq C(N)\alpha_V \beta_V^N$, by lemma 8.4.2 applied to Y_1 there exists $\sigma \in \mathrm{Gal}(\overline{F}/F)$ such that $r_{\mathbf{T}_V}(\sigma) \cdot Y_1 \not\subset \mathrm{Gal}(\overline{F}/F) \cdot Y_1$.

As $\sigma(V) = r_{\mathbf{T}_V}(\sigma)V$, we have $\sigma(V) \subset \sigma(Y_1) \cap r_{\mathbf{T}_V}(\sigma) \cdot Y_1$. Thus

$$\mathrm{Gal}(\overline{F}/F) \cdot V \subset Y \cap r_{\mathbf{T}_V}(\sigma)(Y) \ .$$

Let Y_2 be a geometric irreducible component of $Y \cap r_{\mathbf{T}_V}(\sigma)(Y)$ containing V. We obtain

$$\mathrm{Gal}(\overline{F}/F) \cdot V \subset \mathrm{Gal}(\overline{F}/F) \cdot Y_2 \subsetneq Y \ .$$

Moreover $\deg_{L_K}(\mathrm{Gal}(\overline{F}/F) \cdot Y_2) \leq \deg_{L_K}(Y \cap r_{\mathbf{T}_V}(\sigma)(Y)) \leq ((\deg_{L_K} Z)^2 l^k)^2$. Once again the inequality 8.2 on page 43 implies that V is a proper subvariety of Y_2.

We now iterate step 2, replacing Y_1 by Y_2. As $\dim V < \dim Y_2 < \dim Y_1 < \dim Z$, in at most $r = \dim Z - \dim V$ iterations we obtain the variety Y as in (b).

                                                                                      □

8.5. **Proof of theorem 8.3.1.** We prove theorem 8.3.1 by induction on $r = \dim Z - \dim V$.

8.5.1. *Case $r = 1$ i.e. V is of codimension one in Z.* In the situation of definition 8.2.1 let Z be a Hodge-generic F-irreducible subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ containing V as a hypersurface.

We denote $d_Z := \deg_{L_K} Z$. Suppose that the inequality 8.1 on page 43 holds for $r = 1$:

(8.3) $$l^{8(k+2f)} \cdot d_Z^2 < C(N)\alpha_V \beta_V^N \quad .$$

In order to apply the theorem 7.1 to produce V', we first lift to an Iwahori-level at the prime l.

Let $I \subset K$ be the compact open subgroup $K^l I_l$ of $\mathbf{G}(\mathbf{A}_f)$ where I_l denotes the intersection of K_l and an Iwahori subgroup of $K_l^{\max}$ as in the lemma 8.1.1. As K is neat its subgroup I is also neat. We get a finite morphism of Shimura varieties

$$\pi_F \colon \mathrm{Sh}_I(\mathbf{G}, X)_F \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_F \quad,$$

of degree bounded above by l^f by lemma 8.1.1,b).

Let $\widetilde{Z}_F$ be an irreducible component of $\pi_F^{-1} Z_F$. Its base change $\widetilde{Z} := \widetilde{Z}_F \times_F \mathbb{C}$ is the union of the $\mathrm{Gal}(\overline{F}/F)$-conjugates of an irreducible component of $\pi^{-1}(Z)$. The image of $\widetilde{Z}$ in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is Z and

$$\deg_{L_I} \widetilde{Z} \leq l^f \cdot \deg_{L_K} Z \quad .$$

Let $\widetilde{V}$ be an irreducible component of the preimage of V in $\widetilde{Z}$, this is a non-strongly special subvariety of $\mathrm{Sh}_I(\mathbf{G}, X)_{\mathbb{C}}$ contained in $\widetilde{Z}$. We have the inequality

$$\deg_{L_I}(\mathrm{Gal}(\overline{F}/F) \cdot \widetilde{V}) \geq \deg_{L_K}(\mathrm{Gal}(\overline{F}/F) \cdot V) \quad .$$

As the morphism $\pi \colon \mathrm{Sh}_I(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is finite and preserves the property of a subvariety of being special, exhibiting a special subvariety V' such that $V \subsetneq V' \subset Z$ is equivalent to exhibiting a special subvariety $\widetilde{V'}$ such that $\widetilde{V} \subsetneq \widetilde{V'} \subset \widetilde{Z}$.

Thus by replacing K by I, Z by $\widetilde{Z}$, V by $\widetilde{V}$, we can (and we will from now on) assume that K_l is contained in an Iwahori-subgroup of $\mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}_V$ up to the modification $\deg_{L_K} Z \leq d_Z \cdot l^f$.

As K_l is contained in an Iwahori-subgroup of $\mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}_V$, we can apply theorem 7.1. Let m satisfying the conclusion of theorem 7.1. By condition (1) of theorem 7.1, $\mathrm{Gal}(\overline{F}/F) \cdot V \subset Z \cap T_m Z$. If Z and $T_m Z$ have no common (geometric) irreducible component, then any $\sigma(V)$, $\sigma \in \mathrm{Gal}(\overline{F}/F)$ is an irreducible component of $Z \cap T_m Z$ for dimension reasons. By Bezout's theorem, we get

$$C(N)\alpha_V \beta_V^N \leq \deg_{L_K}(\mathrm{Gal}(\overline{F}/F) \cdot V) \leq \deg_{L_K}(Z \cap T_m Z)$$
$$\leq (\deg_{L_K} Z)^2 [K_l : K_l \cap mK_l m^{-1}] < l^{k+2f} \cdot d_Z^2 \quad .$$

Contradiction to the inequality (8.3). Thus we are in case (a) of proposition 8.4.1 : $Z \subset T_m Z$. As m also satisfies condition (2) of theorem 7.1, we can apply theorem 6.1 to this m : there exists V' special subvariety of Z containing V properly.

8.5.2. *The induction.* Fix $r > 1$ an integer and suppose by induction that theorem 8.3.1 holds for $\dim Z - \dim V < r$. In the situation of definition 8.2.1 let Z be a Hodge generic F-irreducible F-subvariety of $S_K(\mathbf{G}, X)$, containing V with $\dim Z - \dim V = r$. Let $d_Z := \deg_{L_K} Z$ and suppose the inequality 8.1 on page 43 is satisfied :

$$l^{(k+2f)\cdot 2^{a(r+1)}} \cdot d_Z^{2^{a(r)}} < C(N)\alpha_V \beta_V^N \quad .$$

As in the case $r = 1$, we can assume that K_l is contained in an Iwahori-subgroup of $\mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}_V$ up to the modification : $\deg_{L_K} Z \leq d_Z \cdot l^f$. Choose $m \in \mathbf{G}(\mathbb{Q}_l)$ satisfying the conclusion of theorem 7.1. As condition 8.1 on page 43 implies condition 8.2 on page 43, one can apply proposition 8.4.1.

If we are in case (a) of proposition 8.4.1, once more as in the case $r = 1$ we are done by the theorem 6.1.

Thus we can assume we are in case (b) : there exists an F-irreducible subvariety Y of $\mathrm{Sh}_K(G, X)$ satisfying the following properties :

- $\mathrm{Gal}(\overline{F}/F) \cdot V \subsetneq Y \subset Z \cap T_m Z \subsetneq Z$.
- $\deg_{L_K} Y \leq l^{(k+2f)\cdot 2^{r-1}} d_Z^{2^r}$.
- V is not strongly special in $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, , X_{\mathbf{G}_Y})_{\mathbb{C}}$, where $\mathbf{G}_Y \subset \mathbf{G}$ denotes the generic Mumford-Tate group of a component Y_1 of Y containing V, $(\mathbf{G}_Y, X_{\mathbf{G}_Y}) \subset (\mathbf{G}, X)$ is the corresponding Shimura sub-datum and K_Y denotes the intersection $K \cap \mathbf{G}_Y(\mathbf{A}_{\mathrm{f}})$.

We obtain a finite morphism of Shimura varieties $\pi : \mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, which is generically of degree one ([37]). Let $E(\mathbf{G}_Y, X_Y)$ be the reflex field of the Shimura datum $(\mathbf{G}_Y, X_Y)$ and let F' be the composite field

$$F' = F \cdot E(\mathbf{G}_Y, X_Y) \quad .$$

The variety Y_1 contains the non-strongly special subvariety V. Let Y' be the $\mathrm{Gal}(\overline{F}/F')$-orbit of Y_1 in $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}}$, Y' is an F'-irreducible F'-subvariety of $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}}$.

Let us check that $\mathbf{G}_Y$, X_Y, K_Y, F', V, l and Y' satisfy the assumptions of theorem 8.3.1. The compact open subgroup $K_Y = K \cap \mathbf{G}_Y(\mathbf{A}_{\mathrm{f}})$ is a product $\prod_{p \text{ prime}} K_{Y,p}$, with $K_{Y,p} = K_p \cap \mathbf{G}_Y(\mathbb{Q}_p)$. As K_l is contained in a special maximal compact open subgroup $K_l^{\max}$ of $\mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}_V$, $K_{Y,l}$ is contained in the compact open subgroup $K_l^{\max} \cap \mathbf{G}_Y(\mathbb{Q}_l)$, which is still in good position with respect to $\mathbf{T}_V$ as $\mathbf{T}_V \subset \mathbf{G}_Y$. It remains to check that

$$l^{(k+2f)\cdot 2^{a(r_Y+1)}} \cdot (\deg_{L_{K_Y}} Y')^{2^{a(r_Y)}} < C(N)\alpha_V \beta_V^N \quad ,$$

where $r_Y = \dim Y' - \dim V$.

As

$$\deg_{L_{K_Y}} Y' \le \deg_{L_K} Y' \le \deg_{L_K} Y \le l^{(k+2f) \cdot 2^{r-1}} \cdot d_Z^{2^r} \quad ,$$

we are reduced to checking the inequality

$$l^{(k+2f) \cdot (2^{a(r_Y+1)} + 2^{r-1+a(r_Y)})} \cdot d_Z^{2^{r+a(r_Y)}} < C(N) \alpha_V \beta_V^N \quad .$$

As Z satisfies the condition 8.1 on page 43, it is enough to check that

$$\begin{cases} 2^{a(r_Y+1)} + 2^{r-1+a(r_Y)} & \le & 2^{a(r+1)} \\ 2^{r+a(r_Y)} & \le & 2^{a(r)} \end{cases} \quad .$$

The second inequality is obviously satisfied because the function a is increasing, $r_Y \le r - 1$ and $r + a(r-1) = a(r)$.

For the first one, notice that $r - 1 + a(r_Y) \le r + a(r-1) = a(r)$, thus :

$$2^{a(r_Y+1)} + 2^{r-1+a(r_Y)} \le 2 \times 2^{a(r)} = 2^{a(r)+1} \le 2^{a(r+1)}$$

and we are done.

As $\dim Y' - \dim V < \dim Z - \dim V = r$, we can by induction apply the theorem 8.3.1 to $\mathbf{G}_Y, X_Y, K_Y, F', V, l$ and Y' : there exists a special subvariety V_Y' of $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)$ such that $V \subsetneq V_Y' \subset Y'$. Let V' denote the special subvariety $\pi(V_Y')$ of $\mathrm{Sh}_K(\mathbf{G}, X)$. As $\pi(Y') \subset Y \subset Z$ and π is finite, we obtain $V \subsetneq V' \subset Z$ and we are done. This finishes the induction and the proof of theorem 8.3.1.

9. THE CHOICE OF A PRIME l

9.1. **Effective Chebotarev.** The choice of a prime l satisfying all of the conditions of the theorem 8.3.1 will be made possible by the effective Cebotarev theorem, which we now recall.

Definition 9.1.1. *Let L be a number field of degree n_L and absolute discriminant d_L. Let x be a positive real number. We denote by $\pi_L(x)$ the number of primes p such that p is split in L and $p \le x$.*

Proposition 9.1.2. *Assume the Generalized Riemann Hypothesis (GRH). There exists a constant A such that the following holds. For any number field L Galois over $\mathbb{Q}$ and for any $x > \max(A, 2\log(d_L)^2(\log(\log(d_L)))^2)$ we have*

$$\pi_L(x) \ge \frac{x}{3 n_L \log(x)} \quad .$$

Furthermore, if we consider number fields such that d_L is constant, then the assumption of the GRH can be dropped.

Proof. The first statement (assuming the GRH) is proved in the Appendix N of [16] and the second is a direct consequence of the classical Cebotarev theorem. □

9.2. **Proof of the theorem 3.2.1.**

Proof. Let $(\mathbf{G}, X)$ be a Shimura datum with $\mathbf{G}$ semisimple of adjoint type. Recall that this assumption implies that for all Shimura subdata $(\mathbf{H}, X_{\mathbf{H}})$ with $\mathbf{H}$ generic Mumford-Tate group on $X_{\mathbf{H}}$ of $(\mathbf{G}, X)$, the centre $\mathbf{T}$ of $\mathbf{H}$ is such that $\mathbf{T}(\mathbb{R})$ is compact.

Let K be a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let F be a number field containing the reflex field $E(\mathbf{G}, X)$. Let Z be a Hodge generic F-irreducible F-subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Suppose that Z contains a Zariski dense set Σ, which is a union of special subvarieties V, $V \in \Sigma$, all of the same dimension $n(\Sigma)$ such that for any modification Σ' of Σ the set $\{\alpha_V \beta_V, \ V \in \Sigma'\}$ is unbounded. We want to show, under each of the two assumptions of theorem 3.2.1 separately, that for every V in Σ there exists a special subvariety V' such that $V \subsetneq V' \subset Z$ (possibly after replacing Σ by a modification).

Lemma 9.2.1. *Without any loss of generality we can assume that :*

(1) *The group K is a product of compact open subgroups K_p of $\mathbf{G}(\mathbb{Q}_p)$, p prime.*
(2) *There is a prime number p_0 such that K_{p_0} is sufficiently small so that the group K is neat.*
(3) *After possibly replacing Σ by a modification, Σ consists of non-strongly special subvarieties.*

Proof. To fulfill the first condition, let $\widetilde{K} \subset K$ be a compact open subgroup which is a product. Let $\widetilde{Z}$ be an F-irreducible component of the preimage of $f^{-1}(Z)$, where $f : \mathrm{Sh}_{\widetilde{K}}(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is the canonical finite morphism. The Hodge generic F-irreducible F-subvariety $\widetilde{Z}$ of $\mathrm{Sh}_{\widetilde{K}}(\mathbf{G}, X)_{\mathbb{C}}$ contains a Zariski-dense set $\widetilde{\Sigma}$, which is a union of special subvarieties V, $V \in \widetilde{\Sigma}$, all of the same dimension $n(\Sigma)$: $\widetilde{\Sigma}$ is the set of all irreducible components $\widetilde{V}$ of $f^{-1}(V)$ contained in $\widetilde{Z}$ as V ranges through Σ. Notice that for any modification $\widetilde{\Sigma}'$ of $\widetilde{\Sigma}$ the set $\{\alpha_{V'} \beta_{V'}, \ V' \in \widetilde{\Sigma}'\}$ is unbounded : $\beta_{V'} = \beta_{f(V')}$ and $\alpha_{V'}$ is equal to $\alpha_{f(V')}$ up to a factor independent of V'. Thus $\widetilde{Z}$ satisfies the assumptions of theorem 3.2.1. As a subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is special if and only if some (equivalently any) irreducible component of its preimage by f is special, theorem 3.2.1 for $\widetilde{Z}$ implies theorem 3.2.1 for Z.

To fulfill the second condition, replace K_{p_0} by a smaller subgroup satisfying lemma 4.1.2. The same argument as above shows that it is safe to do this.

Concerning the last condition : otherwise there is a modification Σ' of Σ consisting only of strongly special subvarieties. Contradiction with the assumption that the set $\{\alpha_V \beta_V, V \in \Sigma'\}$ is unbounded. □

From now on, we fix a faithful rational representation $\rho : \mathbf{G} \hookrightarrow \mathbf{GL}_n$ such that K is contained in $\mathbf{GL}_n(\hat{\mathbb{Z}})$. In the case of the assumption (2) in theorem 3.2.1, we take for ρ the representation which has the property that the centres $\mathbf{T}_V$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class (possibly replacing K by $K \cap \mathbf{GL}_n(\hat{\mathbb{Z}})$) as V ranges through Σ.

For almost all primes l, K_l is a special maximal compact open subgroup of $\mathbf{G}(\mathbb{Q}_l)$ and furthermore $K_l = \mathbf{G}(\mathbb{Z}_l)$, where the $\mathbb{Z}$-structure on $\mathbf{G}$ is given by taking the Zariski closure in in $\mathbf{GL}_{n,\mathbb{Z}}$ via ρ. Moreover, if the group $\mathbf{T}_{V,\mathbb{F}_l}$ is a split torus then K_l is in good position with respect to $\mathbf{T}_V$.

By theorem 8.3.1, it is then enough (for the purposes of proving theorem 3.2.1) to show that for any V in Σ (up to a modification), there exists a prime l satisfying the following conditions :

(1) the prime l splits $\mathbf{T}_V$.
(2) $\mathbf{T}_{V,\mathbb{F}_l}$ is a split torus.
(3) $l^{(k+2f) \cdot 2^{a(r+1)}} \cdot (\deg_{L_K} Z)^{2^{a(r)}} < C(N) \alpha_V \beta_V^N$.
(4) $K_l = \mathbf{G}(\mathbb{Z}_l)$.

Proposition 9.1. *For every $D > 0$, $\epsilon > 0$ and every integer $m \geq \max(\epsilon, 6)$, there exists an integer M such that (up to a modification of Σ) : for every V in Σ with $\alpha_V \beta_V$ larger than M there exists a prime l satisfying the following conditions*

(1) $l < D\alpha_V^\epsilon \beta_V^m$.
(2) $(\mathbf{T}_V)_{\mathbb{F}_l}$ is a split torus.

Moreover the number of such primes goes to infinity as $\alpha_V \beta_V$ goes to infinity.

Proof. For V in Σ recall that n_V is the degree of the splitting field L_V of $\mathbf{C}_V = \mathbf{H}_V / \mathbf{H}_V^{\mathrm{der}}$ over $\mathbb{Q}$. By [41, Lemma 4.2], n_V is bounded when V ranges through Σ.

Fix $D > 0$, $\epsilon > 0$ and $m \geq 6$. For V in Σ, let

$$x_V := D\alpha_V^\epsilon \beta_V^m$$

As we are assuming either the GRH, or that the connected centres $\mathbf{T}_V$ of the generic Mumford-Tate groups $\mathbf{H}_V$ of V lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class under ρ as V ranges through Σ, in which case d_{L_V} is independent of V, we can apply proposition 9.1.2 :

$$\pi_{L_V}(x_V) \geq \frac{x_V}{3n \log(x_V)}$$

provided that x_V is larger than some absolute constant and β_V^3.

If $x_V \geq 4$ (which is true if $\alpha_V \beta_V$ is large enough), then

$$\sqrt{x_V} \geq \log(x_V)$$

and it follows that

$$\pi_{L_V}(x_V) \geq \frac{\sqrt{x_V}}{3n} = \frac{(D\alpha_V^{\epsilon}\beta_V^m)^{\frac{1}{2}}}{3n}$$

Thus to prove the proposition we have to show that $\pi_{L_V}(x_V) > i_V := i(\mathbf{T}_V)$ if $\alpha_V\beta_V$ is large enough. Indeed, this will yield a prime l which is split in L_V and such that $K_{\mathbf{T}_V,l} = K_{\mathbf{T}_V,l}^m$. These conditions imply that $\mathbf{T}_{V,\mathbb{F}_l}$ is a split torus. We refer to the proof of lemma 3.12 of [37] for the proof of this fact.

Write $i_V = i_V' + i_V''$ where i_V' denotes the number of primes *unramified* in L_V and such that $K_{\mathbf{T}_V,p}^m \neq K_{\mathbf{T}_V,p}$ and i_V'' denotes the number of primes *ramified* in L_V such that $K_{\mathbf{T}_V,p}^m \neq K_{\mathbf{T}_V,p}$.

Lemma 9.2.2. $\alpha_V \geq B^{\max\{i_V',1\}}i_V'!$ and $\beta_V \geq \max(i_V'',1)$.

Proof. The first inequality follows from the following facts :

(1) by the proof of lemma 3.12 of [37], for p unramified in L_V and such that $K_{\mathbf{T},p}^m \neq K_{\mathbf{T},p}$ we have

$$|K_{\mathbf{T}_V,p}^m/K_{\mathbf{T}_V,p}| \geq p .$$

(2) in general, for a prime p such that $K_{\mathbf{T},p}^m \neq K_{T,p}$ we have

$$B|K_{\mathbf{T}_V,p}^m/K_{\mathbf{T}_V,p}| \geq 1 .$$

(3) The pth prime in $\mathbb{N}$ is at least p.

The second inequality follows from the definition of $\beta_V = \log(d_{L_V})$. $\square$

Definition 9.2.3. *Given a positive real number t we denote by Σ_t (resp. Σ_t') the set of V in Σ with $i_V > t$ (resp. with $i_V' > t$).*

We proceed by dichotomy :

- Suppose that for any t the set Σ_t' is a modification of Σ. In particular the function i_V' (thus also i_V) is unbounded as V ranges through Σ. Recall the well-known inequality: for every integer $n > 1$,

$$en^n e^{-n} < n! < en^{n+1}e^{-n} .$$

That gives :

$$\alpha_V > e\left(\frac{Bi_V'}{e}\right)^{i_V'} > \left(\frac{Bi_V'}{e}\right)^{i_V'} .$$

Hence :

$$\alpha_V^{\frac{\epsilon}{2}} > \left(\frac{Bi_V'}{e}\right)^{\frac{\epsilon i_V'}{2}} .$$

For $i_V' > \frac{4}{\epsilon}$ we obtain :

$$\alpha_V^{\frac{\epsilon}{2}} > \left(\frac{Bi_V'}{e}\right)^2 .$$

Using the lower bound for $\pi_{L_V}(x_V)$ we obtained above and as $m \geq 6$, we see that

$$\pi_{L_V}(x_V) > \frac{D^{\frac{1}{2}}B^2}{3ne^2} \cdot {i_V'}^2 \cdot (\max\{i_V'', 1\})^2 \geq \frac{D^{\frac{1}{2}}B^2}{12ne^2} \cdot i_V^2 \quad,$$

where we used that

$$ {i_V'}^2 \cdot (\max(i_V'', 1))^2 \geq \frac{i_V^2}{4} $$

because $i_V = i_V' + i_V''$ and $i_V' \geq 1$. Hence, whenever

$$ i_V' > t = \max(\frac{4}{\epsilon}, \frac{12e^2n}{D^{\frac{1}{2}}B^2}) $$

we obtain

$$ i_V \geq i_V' > \frac{12e^2n}{D^{\frac{1}{2}}B^2} $$

and thus $\pi_{L_V}(x_V) > i_V$. As the set Σ_t' is a modification of Σ we get the proposition 9.1.

- Otherwise there exists a positive number c such that $\Sigma \setminus \Sigma_c'$ is a modification of Σ. Replacing Σ by $\Sigma \setminus \Sigma_c'$ the function i_V' is bounded by c as V ranges through Σ. In particular $i_V'' \geq i_V - c$ as V ranges through Σ. Let r be a real positive number such that $r \leq \alpha_V$ for all $V \in \Sigma$.

  - Suppose that for any t the set Σ_t is a modification of Σ. In particular the function i_V is unbounded as V ranges through Σ. This time the lower bound obtained on $\pi_{L_V}(x_V)$ gives :

    $$ \pi_{L_V}(x_V) > \frac{D^{\frac{1}{2}}\alpha_V^{\frac{\epsilon}{2}}}{3n} \cdot \beta_V^2 > C(i_V - c)^2 \quad, $$

    where $C = \frac{D^{\frac{1}{2}}r^{\frac{\epsilon}{2}}}{3n}$. Thus one obtains

    $$ \pi_{L_V}(x_V) > i_V $$

    as soon as i_V is larger than the largest root t of the quadratic polynomial $C(i_V - c)^2 - i_V$. As the set Σ_t is a modification of Σ we get the proposition 9.1.

  - Otherwise there exists a positive number t such that $\Sigma \setminus \Sigma_t$ is a modification of Σ. Replacing Σ by $\Sigma \setminus \Sigma_t$ the function i_V is bounded by t as V ranges through Σ and $\pi_L(x_V)$ will be larger than i_V when $\pi_L(x_V) \geq t$. The inequality we want to prove then is

    $$ \alpha_V^{\frac{\epsilon}{2}}\beta_V^{\frac{m}{2}} > 3ntD^2 \quad. $$

  The inequality $\alpha_V^{\epsilon/2}\beta_V^{m/2} \geq (\alpha_V\beta_V)^{\epsilon/2}$ shows that in this case M can be taken to be $(3nHD^2)^{2/\epsilon}$.

$\square$

Let $r := \dim Z - n(\Sigma)$. Let N be a positive integer, at least $6(k + 2f)2^{a(r+1)}$. Let $\varepsilon < \frac{1}{(k+2f)\cdot 2^{a(r+1)}}$, $D = \left(\frac{C(N)}{(\deg_{L_K} Z)^{2^{a(r)}}}\right)^{\frac{1}{(k+2f)\cdot 2^{a(r+1)}}}$ and $m = \frac{N}{(k+2f)\cdot 2^{a(r+1)}}$. Let M be the integer provided by the proposition 9.1.

We apply the proposition 9.1 with ϵ, m and D : up to a modification of Σ, for every $V \in \Sigma$ we can choose a prime $l \neq p_0$ such that l splits $\mathbf{T}_V$, $\mathbf{T}_{V, \mathbb{F}_l}$ is a split torus and $l < D\alpha_V^\epsilon \beta_V^m$. This last inequality is exactly condition 8.1 on page 43 of theorem 8.3.1.

Finally for every V in Σ we can apply theorem 8.3.1 to Z, V and l and we are done. $\square$

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