GS256 Lecture 3: Hilbert Transformation

1. **The Hilbert Transform or the Allied Function**

   (a) **The Allied function:** Let's consider a Fourier pair $f(t)$ and $F(\omega)$:

   $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \tag{1}$$

   Substituting

   $$F(\omega) = \int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}d\tau \tag{2}$$

   in (8), we can write

   $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(\tau)e^{i\omega(t-\tau)}d\tau \tag{3}$$

   Changing the range of integration for ω only for $0 \leq \omega \leq \infty$, we can write

   $$f(t) = \frac{1}{\pi} \int_{0}^{\infty} d\omega \int_{-\infty}^{\infty} f(\tau)\cos\{\omega(\tau - t)\}d\tau \tag{4}$$

   This should always hold for any fucntion $f(t)$.

   Now we define a similar but different function by switching from cosine to sine in the integrand:

   $$\hat{f}(t) = \frac{1}{\pi} \int_{0}^{\infty} d\omega \int_{-\infty}^{\infty} f(\tau)\sin\{\omega(\tau - t)\}d\tau \tag{5}$$

   This is the definition of **the Allied function** of $f(t)$.

   Note that the allied function can be computed by FFT by adding phase $\pi/2$ to the Fourier spectra $F(\omega)$. This phase will convert cosine into sine, as in equations (4) and (5).

   (b) **Form of the Hilbert transformation:** If we carry out the integration in this formula, we get

   $$\hat{f}(t) = \frac{1}{\pi}P \int_{-\infty}^{\infty} \frac{f(\tau)}{\tau - t}d\tau \tag{6}$$

   This is the definition of the Hilbert transform of $f(t)$. It is the same with the allied function.

   There is actually a subtle point in this derivation. We carry out the integration with respect to ω but we should first write

   $$\begin{aligned} \hat{f}(t) &= \frac{1}{\pi} \lim_{A \to \infty} \int_{-\infty}^{\infty} d\tau f(\tau) \int_{0}^{A} \sin\{\omega(\tau - t)\}d\omega \\ &= \frac{1}{\pi} \lim_{A \to \infty} \int_{-\infty}^{\infty} f(\tau)\frac{1 - \cos\{A(\tau - t)\}}{\tau - t}d\tau \end{aligned} \tag{7}$$

   If $f(t)$ is bounded and continuous, the term with cosine will be zero as $A \to \infty$. Then we are left with the first term that lead to (13).

(c) **Relation to Fourier transform:** Consider a real function defined by

$$f_\epsilon(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t + i\epsilon sgn(\omega)} d\omega \qquad (8)$$

where $sgn(\omega)$ is 1 if ω is positive, -1 if ω is negative and 0 if $\omega = 0$.

After some algebra, we can find

$$f_\epsilon(t) = \cos\epsilon \cdot f(t) + \sin\epsilon \cdot \hat{f}(t) \qquad (9)$$

Note that when $\epsilon = 0$, it is the original Fourier transform and when $\epsilon = \pi/2$, it is the Hilbert transform of the original function $f(t)$.

Problem 1: Show that (9) is true.

(d) How to compute the Hilbert Transform: Computationally, it is very easy to get the Hilbert transform by using the above frequency domain formula.

   i. First, transform $f(t)$ to the frequency domain or get $F(\omega)$.

   ii. Add phase shift $\pi/2$ (for the part from k=0 to $N/2$).

   iii. Fill in the other part by complex conjugates.

   iv. Transform it back to the time domain.

The task is similar to filtering, probably easier.

(e) **Inverse Hilbert transformation:** If we apply the Hilbert transformation twice, we get negative of the original function. This is because we apply the phase shift of $\pi/2$ twice. This means the inverse Hilbert transformation is given by

$$f(t) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\hat{f}(\tau)}{\tau - t} d\tau \qquad (10)$$

(f) **Envelope:** One of the useful application of the Hilbert transform is the calculation of the envelope. For a given time series, if one wants to get its envelope. we calculate $\hat{f}(t)$ and evaluate $\sqrt{f^2(t) + \hat{f}^2(t)}$.

(g) **Why bother ?** One of the reasons why the Hilbert transformation is important in seismology is because we see its examples everywhere. For example, whenever a ray touches a caustic, a wave packet goes through a phase shift of $\pi/2$, which is basically the Hilbert transformation. Many body wave pairs make up a Hilbert transform pair. See an example in Figure 1.

Also Rayleigh waves have $\pi/2$ phase shift between a vertical and a horizontal component. This is not exactly a Hilbert transforma pair because their amplitudes are not the same but in terms of phase they make up a similar Hilbert-transform like pair.
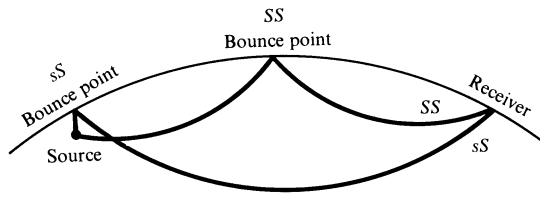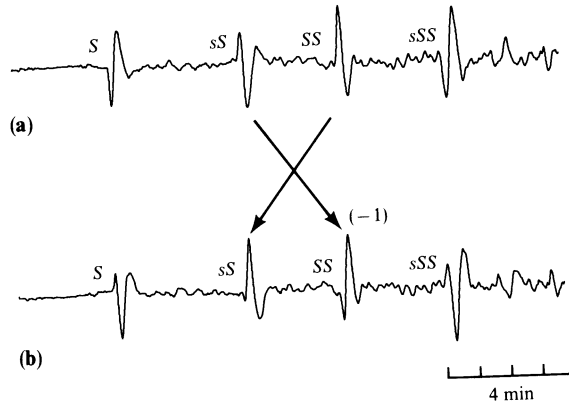
Figure 1: Some body wave pairs make Hilbert transform pairs.

Problem 2: What is the Hilbert transform of $f(t) = \cos(\omega t)$?

Problem 3: On our class website, you will find three synthetic seismograms for Earth model PREM. An assumed earthquake was in the western Pacific and seismograms at GSC (Goldstone) in Southern California are given in ascii format. Each file contains the number of points and the sampling interval (dt=1.0 sec). Figure 2 and 3 are the original plot and filtered plot. Filtering in Figure 3 is done for the frequency range 0.008-0.01-0.012 Hz, which means the peak is at 0.01 Hz (1.0) and signals are tapered toward both lower (0.008) and higher (0.012) frequencies. Specifically, I used cosine taper between 0.008 and 0.01 and also between 0.01 and 0.012 Hz.

(a) First plot the seismograms.

(b) Apply a narrow bandpass filter for the range 0.005-0.01-0.015 Hz. Plot the filtered seismorgams and their envelopes.

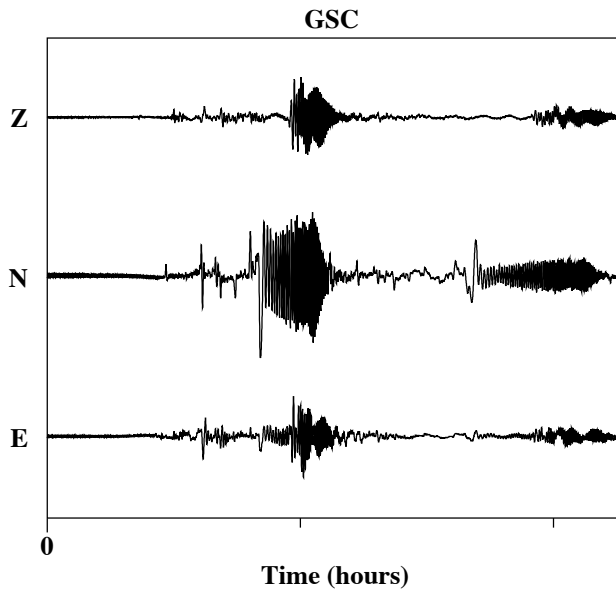(c) In a similar manner, apply a filter for the range 0.025-0.030-0.035 Hz. Plot seismograms and envelopes.

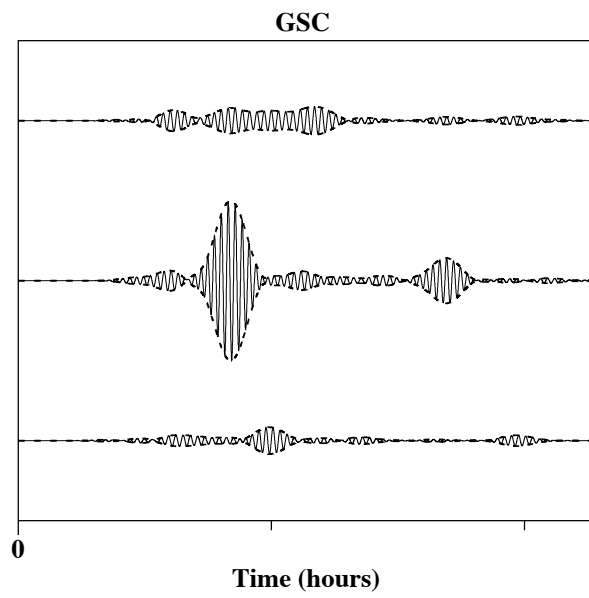Figure 2: Three synthetic seismograms at GSC

GSC

Figure 3: Filtered seismograms with envelopes (0.008-0.01-0.012 Hz)

Time (hours)

0