MIXMASTER UNIVERSE*

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The generic, nonrotating, homogeneous cosmological model for a closed space (Bianchi type IX) has a very complex singularity which can, however, be described in detail. It appears that only the exceptional (previously studied) cases will have particle horizons. Thus these models may lead to some insight into how the broad-scale homogeneity of the universe may have been produced at very early times.

Particle horizons[1] in cosmological models are limits on the possibilities of causal interactions between different parts of the universe in the time available since the initial singularity. In the standard metric $ds^2=\eta^2\{-d\eta^2+dx^2+dy^2+dz^2\}$ for the radiation-dominated early phase of a Robertson-Walker (RW) cosmological model, it is clear that the coordinate time $\Delta\eta$ required for a light signal ($ds^2=0$) to connect two regions of spatial-coordinate separation Δx is $\Delta\eta=|\Delta x|$. Thus at a fixed epoch $\eta_0>0$, no causal interactions subsequent to the singularity at $\eta=0$ have occurred between regions of coordinate separation $|\Delta x|>\eta_0$. In observational terms this effect says, for example, that if the 3°K background radiation[2] were last scattered at a redshift $z=7$, then the radiation coming to us from two directions in the sky separated by more than about 30° was last scattered by regions of plasma whose prior histories had no causal relationship. These Robertson-Walker models therefore give no insight into why the observed microwave radiation from widely different angles in the sky has[2] very precisely ($\lesssim 0.2\,\%$) the same temperature.

We will describe a model of a closed (type-IX)[2] universe which has a very different singularity behavior than the RW models, but which could evolve into the closed RW model at the present epoch. Several aspects of the description parallel the much simpler behavior of a type-I universe with metric

$$ds_{\mathrm{I}}^2=-dt^2+\sum_k(l_k)^2dx_k^2 \tag{1}$$

which is closed artificially by assuming that each space coordinate x_k is periodic with, say, period 4π. Near the singularity the matter or radiation density terms in the Einstein equations can be neglected, and one finds the Kasner[3] solutions $l_k=t^{2p_k}$ with $\sum(p_k)^2=1=\sum p_k$. The model with $p_k=\delta_k{}^1$ then has $ds^2=-dt^2+t^2dx^2+dy^2+dz=e^{2\eta}\times(-d\eta^2+dx^2)+dy^2+dz^2$, where $\eta=\ln t$. Evidently light rays ($ds=0$) can completely circle the universe in the x direction ($\Delta x=4\pi$) in a coordinate-time interval $\Delta\eta=4\pi$ for this metric. Since the singularity is at $\eta=-\infty$ here, this much coordinate time has preceded every nonsingular epoch in this model, and there exist no horizons for causal propagation in the x direction.[4] To compare later with the type-IX model, note that this interval $\Delta\eta=4\pi$ corresponds to a volume expansion ratio of $\Delta\ln(l_1l_2l_3)=4\pi$. In the course of our description of the type-IX model, we will see that it closely approximates this model during periods involving large expansion ratios, but does this infinitely many times with different directions having the open channels of communication each time. On this basis we expect that the absence of horizons in one direction only in this particular Kasner metric corresponds to a total absence of horizons in the generic nonrotating, type-IX metric.

The Bianchi type-IX metric is

$$ds_{\mathrm{IX}}^2=-dt^2+\sum_k(l_k)^2\sigma_k^2, \tag{2}$$

where $\sigma_z=-(d\psi+\cos\theta\,d\varphi)$, $\sigma_x=\sin\psi\,d\theta-\cos\psi\sin\theta\times d\varphi$, and $\sigma_y=\cos\psi\,d\theta+\sin\psi\sin\theta\,d\varphi$ satisfy $d\sigma_i=\frac{1}{2}\epsilon_{ijk}\sigma_j\wedge\sigma_k$ and are differential forms on the three-sphere (covering group of the rotation group) parametrized by Euler angles $\psi\theta\varphi$ with $0\leqslant\psi\leqslant4\pi$, $0\leqslant\theta\leqslant\pi$, and $0\leqslant\varphi\leqslant2\pi$. To distinguish between expansion (volume change) and anisotropy (shape change) we write $l_k=R\exp\beta_k$, where

$$R\equiv e^{-\Omega}=(l_1l_2l_3)^{\frac{1}{3}} \tag{3}$$

governs the volume, and the shape parameters β_k then satisfy $\sum \beta_k = 0$. As two independent shape parameters choose[5]

$$\beta_+ = \beta_1 + \beta_2 = -\beta_3 = -\ln(l_3/R)$$

and

$$\beta_- = 3^{-1/2}(\beta_1 - \beta_2) = 3^{-1/2}\ln(l_1/l_2). \qquad (4)$$

The closed ($k = +1$) RW models are the special case $\beta_+ = 0 = \beta_-$.

The evolution of this universe is described by giving $\beta_\pm$ as functions of Ω, i.e., by giving its shape as a function of its volume. We concern ourselves only with the behavior near the singularity $\Omega \to \infty$, $R \to 0$. Then the empty-space case $R_{\mu\nu} = 0$ is sufficient since the terms due to a matter or radiation fluid are negligible near the singularity. The Einstein equations include $(dt/d\Omega) = -2e^{-3\Omega}(\Lambda - e^{-4\Omega})^{-\frac{1}{2}}$ which would give $t(\Omega)$ were we interested in t. Since Λ (see below) is nearly constant, we approximate this as $(dt/d\Omega) = -2\Lambda^{-\frac{1}{2}}e^{-3\Omega}$ henceforth to obtain simpler equations near the $\Omega \to \infty$ singularity. Then the entire problem is governed by a function $V(\beta)$ which enters in an energylike equation

$$4 = (d\beta_+/d\Omega)^2 + (d\beta_-/d\Omega)^2 + 4\Lambda^{-1}e^{-4\Omega}V(\beta) \quad (5)$$

and the equation

$$d(\ln\Lambda)/d\Omega = -4\Lambda^{-1}e^{-4\Omega}V(\beta) \qquad (6)$$

governing the changes in Λ. [Equation (5) serves as a definition of Λ.] The basic Einstein equations for $\beta_\pm(\Omega)$ are summarized by $\delta\int \mathcal{L} d\Omega = 0$, where Λ is treated as a known function of Ω in the Lagrangian[5]:

$$\mathcal{L} = \tfrac{1}{2}\Lambda^{\frac{1}{2}}(\beta_+'^2 + \beta_-'^2) - 2\Lambda^{-\frac{1}{2}}e^{-4\Omega}V(\beta), \qquad (7)$$

and a prime means $d/d\Omega$. Thus the evolution of this universe is described by the motion of a point $\beta \equiv (\beta_+, \beta_-)$ as a function of the time coordinate Ω using this time-dependent Lagrangian.

The curvature anisotropy potential $V(\beta)$ in these equations is sketched in Fig. 1 and arises from terms in the Einstein equations due to the anisotropy of the curvature of the three-dimension space sections of the universe. The definition is

$$V(\beta) = \tfrac{1}{3}e^{-4\beta_+} - \tfrac{4}{3}e^{-\beta_+}\cosh(\sqrt{3}\beta_-) + \tfrac{2}{3}e^{2\beta_+}[\cosh 2(\sqrt{3}\beta_-) - 1] + 1. \qquad (8)$$

This function has the symmetry of an equilateral

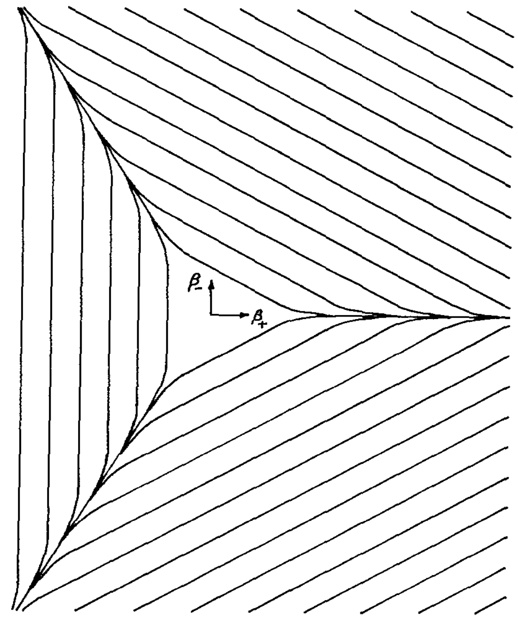

FIG. 1. Equipotentials of the function $V(\beta)$ are sketched here in the β plane from the asymptotic forms of Eqs. (9) and (10). (Equipotentials near the origin, not shown, are closed curves for $V < 1$.) Between successive equipotentials on this diagram, which have separations $\Delta\beta = 2$, V increases by a factor of $e^8 \approx 3\times10^3$. From Eq. (5), the system point $\beta(\Omega)$ moves with velocity $|d\beta/d\Omega| = 2$ except when it approaches a limiting equipotential $V = \Lambda e^{4\Omega}$. This limiting equipotential moves outward with velocity $|d\beta_{\rm wall}/d\Omega| = 1$ except during the brief period when the system point β bounces against it. The "velocity" $d\beta/d\Omega$ changes its direction in what appears to be an ergodic way as a result of these bounces; and whenever it is closely parallel to one of the three corner axes, horizons along the corresponding one of the three expansion axes of the universe approach (or perhaps exceed) the circumference of the universe.

triangle reflecting the equivalence of three axes in the metric [cf. Eq. (4)]. For $\beta_+ \to -\infty$ the asymptotic form

$$V \sim \tfrac{1}{3}e^{-4\beta_+} \qquad (9)$$

shows one of the three exponentially steep walls on which the equipotentials are straight lines (e.g., $\beta_+ =$ neg. const). The corners of this triangular potential are flared open; for instance, if $\beta_+ \to +\infty$ with $|\beta_-| \ll 1$ one finds

$$V(\beta) \sim 4\beta_-^2 e^{2\beta_+} + 1; \qquad (10)$$

so the equipotentials, $\beta_- \propto e^{-\beta_+}$, narrow down exponentially. The potential satisfies $V(\beta) \geq 0$ and vanishes only at the origin, where $V \approx 2(\beta_+^2 + \beta_-^2)$.

Because the potential rises so steeply for large β, little Ω time is spent with the β point bouncing against the potential wall [second term of Eq. (5) large] and most of the time is spent in free motion when V can be neglected. This latter condition then gives $\beta' \equiv (\beta_+'^2 + \beta_-'^2)^{1/2} = 2$ from Eq. (5) and $\Lambda = \text{const}$ from Eq. (6). But $V=0$ just reproduces the Einstein equations $R_{\mu\nu}=0$ for Bianchi type I; so these epochs parallel Kasner solutions using $\Omega = -\frac{1}{3}\ln t + \text{const}$ as the independent variable.

The system velocity $d\beta/d\Omega$ has maximum magnitude 2, but the potential is time dependent and also moves. In Eq. (5) let the value of β_+ at which the energy level "4" and the potential well $4\Lambda^{-1}e^{-4\Omega}V$ intersect be called β_{wall}. Then asymptotically from Eq. (9) we find

$$\beta_{\text{wall}} = -\Omega - \tfrac{1}{4}\ln(3\Lambda) \qquad (11)$$

showing a velocity[5] $(d\beta_{\text{wall}}/d\Omega) = -1$, which is outward or expanding as Ω increases, during the free motions when Λ is constant. The β point can therefore overtake the receding potential walls and make repeated collisions, thereby shifting from one Kasner-like model to another at each collision.

The equilateral-triangle geometry means that $|d\beta_{\text{corner}}/d\Omega| = 2|d\beta_{\text{wall}}/d\Omega|$. Thus if the system point finds itself running toward a corner rather than a wall of the potential, the velocities β' and β_{corner}' are in first approximation equal, and Kasner behaviors with these parameters will last a long Ω time. But these directions of β motion are just those required to remove horizons in a particular direction; so the long duration $(\Delta\Omega \gg \frac{4}{3}\pi)$ allows us to anticipate a similar effect here. The Kasner solution labels (p_1, p_2, p_3) and the velocity $d\beta/d\Omega$ are related by

$$\begin{aligned} d\beta_+/d\Omega &= 3p_3 - 1 = 2(3-s^2)/(s^2+3) \\ &= (2u^2+2u-1)/(u^2+1), \\ d\beta_-/d\Omega &= 3^{1/2}(p_2-p_1) = +4(3^{1/2}s)/(s^2+3) \\ &= 3^{1/2}(2u+1)/(u^2+u+1) \end{aligned} \qquad (12)$$

which introduces labelings s and u for later use. If a communication phase $(u=-1, 0, \infty)$ persisted as $\Omega \to \infty$, then causal influence could circumnavigate the universe only in one direction, as in the Kasner models or the Taub-NUT metric[6] [which is type IX as in Eq. (2), but with $\beta_-=0$, and has $u=+\infty$ asymptotically]. To see that this cannot occur we set $\beta_+ = \beta_0 + 2\Omega$, assume $|\beta_-| \ll 1$, and use Eq. (10); so the Lagrangian for the β_- motion becomes

$$\mathcal{L}_- = \tfrac{1}{2}\Lambda^{\frac{1}{2}}\beta_-'^2 - 8\Lambda^{-\frac{1}{2}}e^{2\beta_0}\beta_-^2,$$

giving simple harmonic motion for β_- with a frequency $\omega_- = 4\Lambda^{-\frac{1}{2}}e^{\beta_0}$. Recognizing then that β_0 can change slowly, we use the adiabatic invariant

$$\Sigma = E_-/\omega_- = \tfrac{1}{8}\Lambda e^{-\beta_0}(\beta_-'^2 + \omega_-^2\beta_-^2)$$

and Eq. (5) for $d\beta_0/d\Omega$ to deduce the behavior $\beta_0 = \ln(\Omega_0 - \Omega) + \text{const}$. This shows that as Ω increases toward Ω_0 (an integration constant) the β point drifts away from the corner of the potential ($\beta_0 \to -\infty$) to resume bouncing on the flat walls as the small β_- approximation breaks down. [The adiabatic analysis also shows $\langle\beta_-^2\rangle \propto (\Omega_0 - \Omega)^{-1}$ and $\omega_-^2 = \text{const}$ for as long as β_- remains small.]

It remains to study the bounce when β collides with a flat face of the potential wall, say the one at negative β_+. Then, using the asymptotic form (9) for V, the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\Lambda^{\frac{1}{2}}(\beta_+'^2 + \beta_-'^2) - \tfrac{2}{3}\Lambda^{-\frac{1}{2}}e^{-4(\beta_+ + \Omega)}.$$

Since β_- does not appear, $p_- \equiv \partial\mathcal{L}/\partial\beta_-' = \Lambda^{1/2}\beta_-'$ is a constant of motion. Another constant is found by comparing the equations $p_+' = (8/3)\Lambda^{-\frac{1}{2}} \times e^{-4(\beta_+ + \Omega)}$ from this Lagrangian and

$$(\Lambda^{\frac{1}{2}})' = -\tfrac{2}{3}\Lambda^{-\frac{1}{2}}e^{-4(\beta_+ + \Omega)}$$

from Eq. (6) with the result $(p_+ + 4\Lambda^{1/2})' = 0$. The constant $(p_+ + 4\Lambda^{1/2})/p_-$ can be expressed simply in terms of $\beta_\pm'$:

$$\begin{aligned} (p_-)^{-1}(p_+ + 4\Lambda^{1/2}) &= (\beta_+' + 4)/\beta_-' \\ &= \tfrac{1}{2}\sqrt{3}(\tfrac{1}{3}s + 3/s). \end{aligned} \qquad (13)$$

The final form uses the parametrization of $\beta_\pm'$ introduced in Eq. (12). This parametrization is only possible during the Kasner epochs when $\beta_+'^2 + \beta_-'^2 = 4$, i.e., for the initial and final states of the "bounce." These states must, by Eq. (13), have the same values of $\frac{1}{3}s + 3/s$; so the bounce from one Kasner-like solution to another is described by the operation $B_3: \frac{1}{3}s \to 3/s$. I looked for this simple way to restate the bounce law in terms of a parametrization of the Kasner models after Wheeler[7] suggested that studies of singularities by Belinsky and Khalatnikov had also found alternating Kasner-like epochs but with very simple description in terms of a related parameter

$u=(3-s)/2s$. A preprint[8] of their work has recently appeared and a portion of it applied to this type-IX metric. Their qualitative picture of the solution and ours are consistent, and they provide a corresponding numerical example; however, their techniques are entirely different [they make no use of the Lagrangian (7) nor the potential (8)], and they do not consider the significance of the results for the questions of horizons and the establishment of large scale homogeniety in the early universe.

The Lifshitz-Khalatnikov[8] bounce law

$$B: u \to u-1 \tag{14}$$

is much easier to use than B_3 above. The two are related by $B=P_{23}B_3P_{12}P_{13}$, where the P's are operators permuting the Kasner exponents p_1, p_2, p_3 namely $P_{23}: u \to 1/u$, $P_{12}: u \to -(1+u)$, and $P_{13}: u \to -u/(1+u)$. It appears that almost all solutions come arbitrarily close to the values $u=-1,\ 0,\ \infty$ ($s=-3, +3, 0$) which we recognize as the states giving communication along the three corresponding expansion axes of the universe. To see this, start with $u>1$ by using the permutations to put u in the standard interval $u \geqslant 1$ (one of the six permutation-equivalent intervals with end points $u=-\infty, -2, -1, -\frac{1}{2}, 0, 1, \infty$). Then after a finite number of steps $u \to u-1$, which correspond to $\beta(\Omega)$ rattling back and forth between the walls leading to one fixed corner of the potential, one will find $0 \leqslant u \leqslant 1$. The permutation $u \to 1/u$ then resets $u \geqslant 1$ again, to let β begin rattling in another corner. On the basis of this analysis, a countable set of initial u values could avoid $u=0$ by a finite amount; these would be fixed points of operators $P_{12}B^{n_1}P_{12}B^{n_2}P_{12}\cdots B^{n_k}$. The simplest of these is $P_{12}B$ with fixed point $u_f=\frac{1}{2}(1+5^{1/2})$, for which β bounces off every wall of the triangular potential in turn without ever heading toward a corner. This behavior is unstable, however, and some of the exponentially small terms neglected in Eq. (9) when deriving (14) will no doubt divert the corresponding exact solutions toward the $u \to 0$ stages which open up horizon limits.

A full account of this work will be submitted for publication elsewhere.

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