The Alexander Polynomial

The woefully overlooked granddaddy of knot polynomials

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Abstract

Once upon a time (actually in 1928) a polynomial was born, and it was named 'Alexander' after its creator. It caused much excitement, being the first object of its kind to appear in knot theory, and was universally loved and admired by anyone who had the fortune to use it. The more people studied it, the more interesting it became. UNTIL... One day, far away in the deepest darkness of operator algebras and statistical mechanics, another knot polynomial was born. Named the 'Jones polynomial', it quickly became the new favourite for its abilities to distinguish more knots and for its esoteric beginnings. Disaster struck again a year later (1985) with the birth of 'HOMFLY' which achieved the great feat of generalising both previous polynomials.

In this talk I would like to pay a tribute to the greatness of the first ever knot polynomial, and to show that we should not underestimate its continued usefulness in the world of knot theory.

Contents

1 Penetrating the mists of knotty beginnings

What *are* knots, and how can we work with them?

1.1 Formalities

Before we go gallavanting into definitions of knot polynomials, we should first be clear in our minds of the objects we are working with. We all have an intuitive idea of what a knot should be: a closed loop of string in $\mathbb{R}^3$. We also are happy with saying that two knots are the same if one can be wiggled around, stretched and bent (but not cut or glued) until it coincides with the other one. Lastly, we are all quite contented to draw 2-dimensional projections of knots and not feel that we are losing information by doing so. These notions all have to be made rigorous before we can do anything more interesting.

Definition 1.1 (take 1)**.** We define a knot K as an embedding of S^1 in $\mathbb{R}^3$, and we say that two knots are equivalent if there is an isotopy between them (i.e. a homotopy which is an embedding at every step).

Unfortunately this definition means that any knot is equivalent to the unknot by just pulling the string tighter and tighter until the knot disappears!

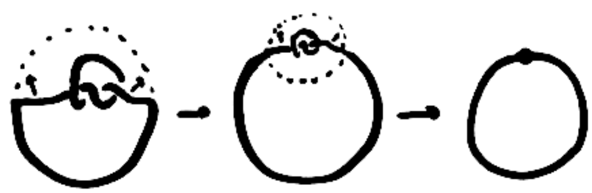


Figure 1: A reason why 'isotopy' is the wrong notion of equivalence.

Definition 1.2 (take 2)**.** We say two knots K_1, K_2 are equivalent if they are *ambient isotopic*, which means there is a homotopy of (orientation-preserving) homeomorphisms $f_t\colon \mathbb{R}^3 \to \mathbb{R}^3$ so that f_0 is the identity and f_1 carries K_1 to K_2.

So an ambient isotopy is like an isotopy, except that instead of distorting the embedding we are distorting all of the ambient space as well. This is enough to take care of bad case we had before in Figure 1, but we still have a problem with *wild knots* which have infinitely complicated knotting going on (Figure 2).

Definition 1.3 (we got there eventually!)**.** A knot is a subset of $\mathbb{R}^3$ homeomorphic to S^1 and expressible as a disjoint union of finitely many points (vertices) and open straight arc-segments (edges). Two knots are equivalent iff they are ambient isotopic.

Figure 2: A wild knot.

Since a knot can only be made up of finitely many straight-line segments, this eliminates the kind of 'wildness' we see in Figure 2.
Now we shall just say a few words on projections of knots.

Definition 1.4. For a knot K in $\mathbb{R}^3$, its projection $\pi(K) \subset \mathbb{R}^2$ is the projection along the z-axis onto the xy-plane. The projection is called *regular* if the preimage of a point of $\pi(K)$ consists of either one or two points of K, in the latter case neither being a vertex of K.

Definition 1.5. A regular projection of a knot, together with information about which arc crosses over/under at each crossing, is called a *knot diagram*. It is possible to reconstruct the 3-dimensional knot uniquely from this information.

Fact: If a knot has an irregular projection then it is possible to perturb it by an arbitrarily small amount to make the projection regular. Thus every knot has a knot diagram.

Generalisations:

- If we consider embeddings $S^n \subset S^{n+2}$ then we are looking at n-dimensional knots.
- An embedding of a disjoint union of circles into $\mathbb{R}^3$ is called a link, and all the following theory goes through for links of more than one component unless otherwise stated.

1.2 Reidemeister Moves

Something which may have occurred to you on reading about knot diagrams is that there are many different way one can represent the same knot. For example, the humble trefoil knot can be drawn in two common ways (see Figure 1.2.
How can we tell if two different diagrams represent the same knot? Actually this is the ultimate question in knot theory and there is no algorithm for finding the answer to it! But Kurt Reidemeister (1893-1971) made a good start on the problem by proving that two diagrams representing the same knot are always related by a sequence of three special moves.

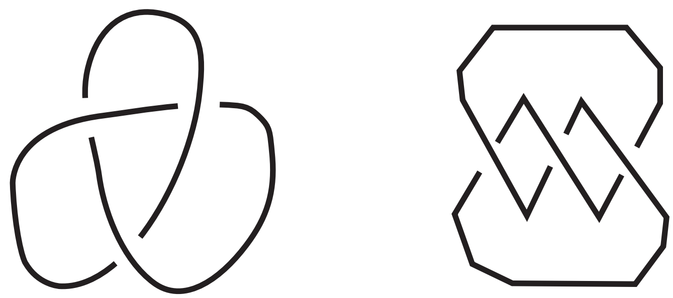


Figure 3: The humble trefoil.

Theorem 1.6 (Reidemeister's Theorem)**.** Two knots K_1, K_2 with diagrams D_1, D_2 are equivalent if and only if their diagrams are related by a finite sequence of intermediate diagrams such that each differs from its predecessor by one of the following three Reidemeister moves:

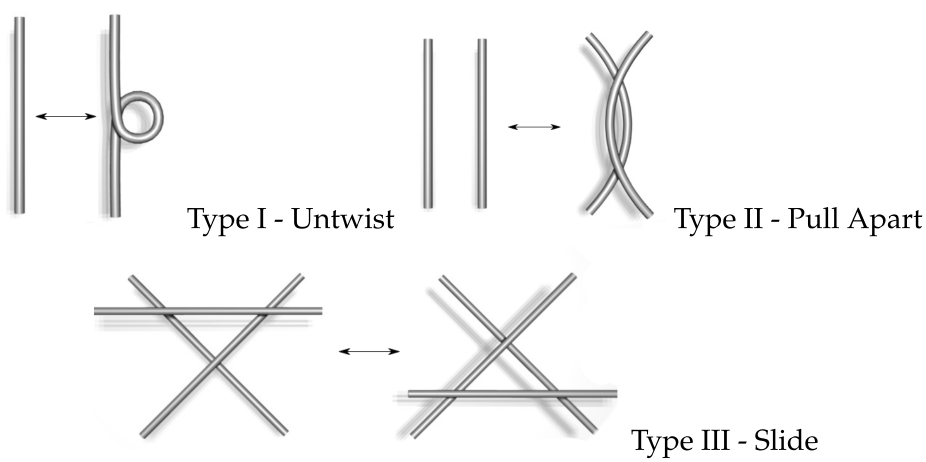


Figure 4: The Reidemeister Moves

We won't give a proof of the theorem as it is a bit tedious and not very enlightening. But having this theorem makes it much easier to find invariants of knots based on diagrams. To check that it is an invariant, we would only need to show that it doesn't change under any single Reidemeister move.

2 Discovering (and rediscovering) the Alexander polynomial

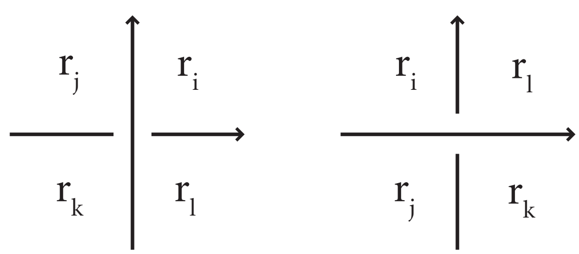
Definition 2.1. A *knot invariant* is any function ϕ of knots which takes the same value on their equivalence classes. So if $\phi(K_1) \neq \phi(K_2)$ then we know that K_1 is not equivalent to K_2.

It is a pretty cool idea to associate an invariant polynomial with a knot, but how on earth should we go about doing it? In 1928, James Alexander came up with the first idea.

2.1 Alexander's Method

1. Take an oriented diagram of a knot K with n crossings, $c_1, \ldots, c_n$. By Euler's theorem there are $n+2$ regions, which we label $r_1, \ldots, r_{n+2}$.

2. Create an $n \times (n+2)$ matrix where the $(i,j)^{th}$ entry corresponds to the c_i^{th} crossing and the r_j^{th} region. If a region does not touch a crossing, the entry is zero. Around the crossing we have the following configurations:



which correspond to entries

Region	r_i	r_j	r_k	r_l
Matrix entry	t	$-t$	1	-1

3. Remove any two columns which correspond to adjacent regions.

4. Find the determinant of the remaining matrix. This is the Alexander polynomial $\Delta_K(t)$.

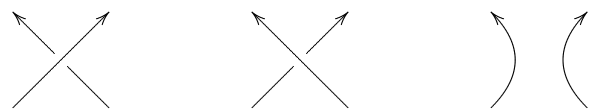
Of course there is going to be some ambiguity in the answer because of the choice we had to make in Step 3 of columns to delete, but it is easily shown that the answer will only fluctuate by a factor of $\pm t^m$.

2.2 Conway's Skein Relation

In 1969 John Conway became famous in knot theory for discovering a polynomial which satisfied a *skein relation* (we will see what this is in a minute). Unfortunately for him, it turned out that he had only rediscovered the Alexander polynomial, and not only that, but Alexander himself had discovered the same thing in the 'Miscellaneous' section of his original paper! Nevertheless, it was a very important paper as it paved the way for the discovery of the Jones polynomial 15 years later.

Definition 2.2. The *Alexander polynomial* for an oriented link L is the Laurent polynomial $\Delta_L(t) \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$ given by the following skein relation:

- $\Delta_{\text{unknot}}(t) = 1.$
- $\Delta_{L_+}(t) - \Delta_{L_-}(t) - (t^{1/2} - t^{-1/2})\Delta_{L_0}(t) = 0,$ where



L_+ L_- L_0

Example 2.3. Find the Alexander polynomial of the disjoint union of two unknots, i.e. $\Delta(\bigcirc\bigcirc)$.
We picture this link as L_0 and then use the skein relation:

$0 = \Delta(L_+) - \Delta(L_-) + (t^{1/2} - t^{-1/2})\Delta(L_0)$

$=$ Δ() + Δ() $+ (t^{1/2} - t^{-1/2})\Delta(L_0)$

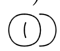
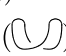
$= \Delta(\bigcirc) - \Delta(\bigcirc) + (t^{1/2} - t^{-1/2})\Delta(L_0)$

$= (t^{1/2} - t^{-1/2})\Delta(L_0)$

$\Rightarrow \Delta(L_0) = 0.$

Example 2.4. Find the Alexander polynomial of the Hopf link, i.e. Δ().
We picture this link as L_+ and use the skein relation (on the top crossing), along with the result from Example 2.3:

$0 = \Delta(L_+) - \Delta(L_-) + (t^{1/2} - t^{-1/2})\Delta(L_0)$

$= \Delta(L_+) -$ Δ() $+ (t^{1/2} - t^{-1/2})$Δ()

$= \Delta(L_+) - 0 + (t^{1/2} - t^{-1/2})\Delta(\bigcirc)$

$= \Delta(L_+) + (t^{1/2} - t^{-1/2})$

$\Rightarrow$ Δ() $= t^{-1/2} - t^{1/2}$

Exercise: Find the Alexander polynomial of the trefoil and check that it agrees with your answer using Alexander's method!

Remark 2.5. The Alexander polynomial for a *knot* found using this method will always be a polynomial in $\mathbb{Z}[t, t^{-1}]$.

Lemma 2.6. It is always possible to transform a knot diagram into the unknot by changing a finite number of crossings (and therefore the skein relation given above is sufficient to allow Δ_L to be computed for any knot (and link)).

Proof. One way to discover a set of changes that will create the unknot is as follows: start at an arbitrary point P on the knot projection and pretend you are an ant walking over the knot. You will encounter each crossing twice, so change the crossings so that on the first encounter you are on the overpass and on the second encounter you are on the underpass. Now use your imagination once more to imagine a z-axis running into the page. Starting at P, which we give z-coordinate $z = 1$, and following the path along the modified knot you will always be decreasing the value of the z-coordinate, until you are almost back at P. This means you have drawn the unknot in 3-space, completing the proof. ■

Remark 2.7. The minimum number of crossing changes required, computed over all possible diagrams of the knot, is called the *unknotting number* of the knot.

The polynomial given by this skein relation is often called the 'Conway-normalised Alexander polynomial' because it gives a polynomial uniquely without ambiguity of sign or powers of t.

2.3 Seifert's Surfaces

This next method of finding the Alexander polynomial may require the most advanced mathematics that we have seen so far, but the rewards are very high. This method is easily generalisable to knots of higher dimension, and it also yields many more invariants than just the Alexander polynomial.
What we are going to do is associate a surface with a knot/link and compute a matrix which contains information about the linking of the homology generators of the surface. A certain determinant of this matrix will then yield the polynomial we want.

Definition 2.8. A *Seifert surface* of a knot (or link) is an oriented surface whose boundary coincides with that of the knot.

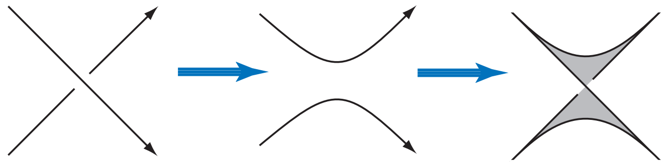
Theorem 2.9 (Frankl, Pontrjagin, 1930)**.** Every knot is the boundary of an orientable surface.

Proof. (Seifert, 1934)

1. Fix an oriented diagram for the knot.

2. At each crossing of the projection, two strands come in and two go out. Eliminate the crossings by swapping which incoming strand is connected to which outgoing strand. The result is a set of non-intersecting oriented topological circles called *Seifert circles*, which, if they are nested, we imagine being at different heights perpendicular to the plane with the z-coordinate changing linearly with the nesting.

3. Fill in the circles, giving discs. Now connect the discs together by attaching twisted bands where the crossings used to be. The twist corresponds to the direction of the crossing in the knot.



Clearly this gives us a surface with the knot as its boundary. It remains to prove that it is orientable, i.e. that it has two distinct sides. Give each Seifert circle the orientation that it inherits from the knot. If it is clockwise, we paint its upward face blue and its downward face red; vice versa for an anticlockwise orientation. If two discs are adjacent then they will have opposite orientations, and the twists in the bands allow us to extend the painting consistently. If one disc is contained in another, then they have the same orientation and one is above the other (in 3D-space), so the twist in the bands still allows the colouring to be extended consistently. ■

Remark 2.10. The result of this algorithm is sometimes a little hard to visualise for people approaching the subject for the first time. Thankfully there is an excellent program available on the web called Seifertview [11] which draws Seifert surfaces obtained using Seifert's algorithm. (Requires Windows to run. Check out the 'rollarcoaster' function under the Miscellaneous tab!)

Of course, the Seifert surface of a link is not unique in any way, and even Seifert's algorithm applied to different link projections will result in different surfaces. Since the surface itself cannot be an invariant of the link, we need to look for other information given in the Seifert surface.

Definition 2.11. Suppose we have an oriented link L of more than one component, with D a corresponding diagram. Assign to each crossing of D a sign according to the following rule:

$+1$ $\qquad$ -1

The *linking number* $\mathrm{lk}(D)$ is half of the sum of the signs of the crossings at which the strands are from different components of the link.

Definition 2.12. Let L be an oriented link of n components and Σ a Seifert surface for L. Take a basis $\{[f_i]\}$ for $H_1(\Sigma;\mathbb{Z})$. The orientation of Σ determines a normal direction, which we will think of as being the 'top' of the surface. Now, given any simple oriented curve f on Σ, we can form the

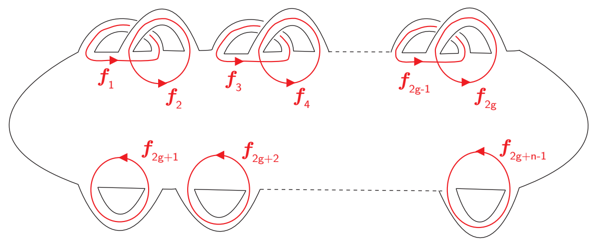


Figure 5: A standard basis for a genus g surface with n boundary components.

positive push off of f, denoted f^+, which runs parallel to f and lies just above F. The *Seifert matrix* of L is the integer matrix M with $(i,j)^{th}$ entry

$$m_{ij} = \mathrm{lk}(f_i, f_j^+).$$

Remark 2.13. For a link of n components and a Seifert surface of genus g the size of M will be $2g+n-1$. In particular, for a knot we get a $2g \times 2g$ matrix. Imagine a genus g surface with one boundary component. Around each of the g 'holes' there are two homology generators, because a torus is generated by two circles $S^1 \times S^1$. Then for each boundary component we add in, we get an extra homology generator. (See Figure 5.)

So the Seifert matrix encapsulates much of the information of how the knot twists about itself. It is not an invariant because it depends on the surface chosen for the knot, and also on the homology basis used in calculating the matrix. But knot theorists know how these actions influence the resulting matrix (e.g. see [6] pg 81) and so it is easy to construct real invariants from it. The first of these is our much beloved polynomial.

Definition 2.14. The *Alexander polynomial* of a link L with Seifert matrix M coming from a connected Seifert surface is given by the formula

$$\Delta_L(t) \doteq \det(M - tM^T)$$

where M^T denotes the transpose of M, and "$\doteq$" means equal up to multiplication by $\pm t^n$. For a link which has a disconnected Seifert surface we define the Alexander polynomial to be zero.

There are still more definitions of the Alexander polynomial, one involving elementary ideals of presentation matrices and another using a kind of formal calculus on the fundamental group of the knot complement. But we shall content ourselves with these three for now and see what we can learn about knots from them.

3 Pretty Properties

In this section we'll quickly look at some of the nice things that make the Alexander polynomial so much better than the average run-of-the-mill polynomial.

Proposition 3.1. **(i)** $\Delta_K(1) = \pm 1$ for any knot K.

(ii) $\Delta_L(t) \doteq \Delta_L(t^{-1})$ for any link L.

Proof.

(i) Consider the standard homology basis for a genus g Seifert surface as shown in Figure 5 (with $n = 1$). Now $(M - M^T)_{ij} = \mathrm{lk}(f_i, f_j^+) - \mathrm{lk}(f_j, f_i^+)$, and a small amount of calculation will show you that $(M - M^T)$ consists of blocks of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ along the diagonal and zeros elsewhere. Thus $\Delta_K(1) = \pm\det(M - M^T) = \pm 1$.

(ii) Suppose M is an $n \times n$ Seifert matrix for L. Then

$$\Delta_L(t) \doteq \det(M - tM^T) = \det(M^T - tM) = (-t)^n \det(M - t^{-1}M^T) \doteq \Delta_L(t^{-1})$$

■

Actually we can get a unique Alexander polynomial for each knot by insisting that $\Delta_K(1) = 1$ and $\Delta_K(t) = \Delta_K(t^{-1})$.

A kind of converse to Proposition 3.1 is the following.

Proposition 3.2. Given any polynomial $p \in \mathbb{Z}[t, t^{-1}]$ satisfying $p(t) = p(t^{-1})$ and $p(1) = \pm 1$, we can find a knot K such that $p = \Delta_K$.

Proof. Look in, for example [2], Theorem 8.13 (pg 113). ■

Proposition 3.3. The Alexander polynomial is multiplicative under connected sum of knots. I.e. $\Delta_{K_1+K_2}(t) \doteq \Delta_{K_1}(t)\Delta_{K_2}(t)$.

Proof. If M_1 and M_2 are Seifert matrices for K_1 and K_2 respectively, then $\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$ is a Seifert matrix for $K_1 + K_2$. ■

So, for example, to find the Alexander polynomial of the granny knot (composite of two left-handed trefoils), we just square the polynomial of the trefoil.

The Alexander polynomial has some nice properties when applied to *alternating knots*. These are knots which have a diagram whose crossings alternate between 'over' and 'under' as you follow it around.

Proposition 3.4. The Alexander polynomial of an alternating knot is alternating, i.e. the signs of the coefficients alternate between '+' and '-'.

Proposition 3.5. For an alternating knot, the breadth of the Alexander polynomial (i.e. the difference between the largest and smallest powers of t) is equal to twice the genus of the knot. Here the genus of a knot is the minimal genus of all possible Seifert surfaces of the knot.

These last two propositions are quite deep and proofs will not be generally found in textbooks on knot theory. They were proved by R. Crowell in 1959.

4 Sure it looks pretty, but how useful is it?

The Alexander polynomial had great success on its inception, being able to distinguish all knots of eight crossings or less. Nowadays we can tabulate all knots of up to 16 crossings, and although it's still a pretty good invariant there are classes of knots which it cannot distinguish.

Firstly, and quite importantly, the Alexander polynomial cannot distinguish between knots and their mirror images.

Lemma 4.1. Let L be an oriented link. Then $\overline{L}$ and rL, the reflection and reverse of L, have the same Alexander polynomial. (The reflection of a knot is obtained by changing all over-crossings to under-crossings and vice versa; the reverse is obtained by changing the orientation of the knot.)

Proof. If M is a Seifert matrix for L, then $-M$ is a matrix for $\overline{L}$ and M^T is a matrix for rL. ■

Thus, for example, the Alexander polynomial cannot distinguish between the granny knot (two left handed trefoils) and the reef knot (one right and one left handed trefoil).

More seriously, there exist non-trivial knots with Alexander polynomial 1, so that the Alexander polynomial cannot distinguish the unknot. A famous example of such a knot is the Kinoshita-Terasaka knot, which is actually inscribed on the gates of the Maths department in Cambridge.

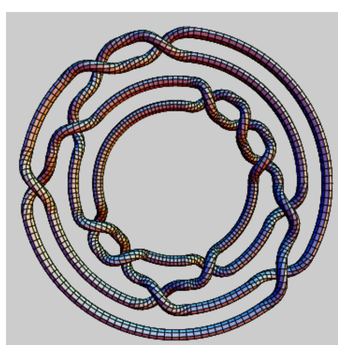


Figure 6: The Kinoshita-Terasaka knot, which has 11 crossings and trivial Alexander polynomial.

A more general problem is the phenomenon of *mutation*. This does not mean that knots morph into strange evil monsters, but that they differ from each other only in a very tiny way.

Definition 4.2. Given a diagram of a knot D, draw a disc in the diagram whose boundary intersects the knot in exactly four places. If the part of the knot inside the disc is rotated or reflected, then the resulting knot is called a *mutant* of the original knot.

Example 4.3. Here is an example of mutation of the Kinoshita-Terasaka knot. The mutant is called the Conway knot.

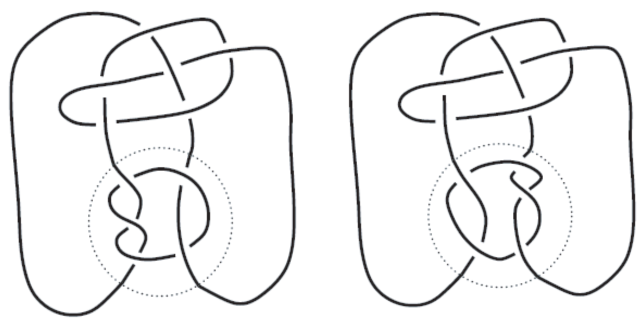


Figure 7: Mutation!

Theorem 4.4. Any knots which are related by mutation have the same Alexander polynomial.

5 Bigger and better polynomials?

We will end by having a look at the other polynomials which appeared in the 1980's, stealing the limelight from poor Alexander.
Recall again the skein relation for the Alexander polynomial:

Definition 5.1. The Alexander polynomial for links L is distinguished by the following relations:

- $\Delta_{\text{unknot}}(t) = 1.$
- $\Delta_{L_+}(t) - \Delta_{L_-}(t) - (t^{1/2} - t^{-1/2})\Delta_{L_0}(t) = 0$

The skein relation for the Jones polynomial looks so similar that you may wonder why (a) it took so long to discover, and (b) why it is so much better!

Definition 5.2. For oriented links L, the *Jones polynomial* $V_L \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ is given by the following skein relation:

- $V_{\text{unknot}}(t) = 1.$
- $t^{-1}V_{L_+}(t) - tV_{L_-}(t) - (t^{1/2} - t^{-1/2})V_{L_0}(t) = 0$

The Jones polynomial has the great advantage over Alexander that it can distinguish knots from their mirror images. It is also better at distinguishing knots in general, and it is an open problem whether there exist any non-trivial knots with Jones polynomial 1.

A year after the Jones polynomial arrived on the scene, a group of knot theorists got together and completely generalised it to a two-variable polynomial. It was named HOMFLY, after the initials of its creators: Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter.

Definition 5.3. For oriented links L, the *HOMFLY polynomial* $P(L) \in \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ is given by the following skein relation:

- $P(\text{unknot}) = 1$.
- $lP(L_+) - l^{-1}P(L_-) - mP(L_0) = 0$

We can recover our other polynomials from HOMFLY as follows:

- $\Delta_L(t) = P(L)(l = 1, m = t^{1/2} - t^{-1/2})$
- $V_L(t) = P(L)(l = t, m = t^{1/2} - t^{-1/2})$

Sadly, even the mighty HOMFLY cannot distinguish mutant knots, and so it not a perfect knot invariant. The search is on to find something even better! In the meantime, we should not forget the important role that the Alexander polynomial has had in shaping knot theory for the past 90 years.

Stay tuned for the next instalment of Geometry Club Knot Theory! Next semester I will talk about either Braid Theory or Slice Knots & Knot Cobordism, depending on which one I have learnt the most about over the summer...

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