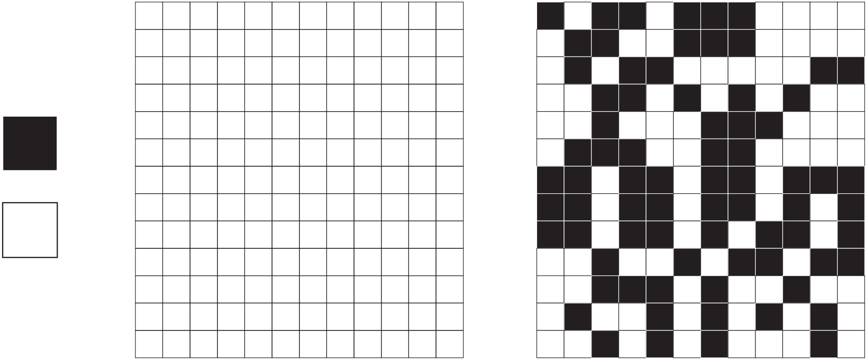
Aperiodic Tilings

Chaim Goodman-Strauss
Univ Arkansas
strauss@uark.edu

Black and white squares can tile the plane *non-periodically*, but can also tile periodically. They are not, then **aperiodic**.



Aperiodicity would be a truly amazing condition! Somehow, *at all scales*, periodicity has to be disrupted.

The mere *non-periodicity* of a tiling is not particularly special from this point of view.

Aperiodicity would be a truly amazing condition! Somehow, *at all scales*, periodicity has to be disrupted.

The mere *non-periodicity* of a tiling is not particularly special from this point of view.

In particular, in the Euclidean plane, a set of tiles is *aperiodic* if it admits *only* non-periodic tilings.

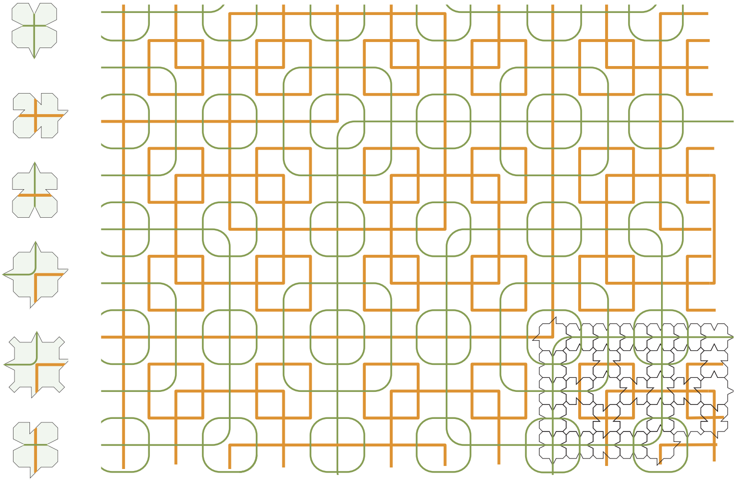
In yesterday’s lecture, we saw that, in a particular setting, say tilings in the Euclidean plane, if the Domino Problem is undecidable, then there must exist an aperiodic set of tiles.

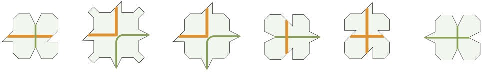
And Robert Berger gave the first such set ca. 1966, as a kind of scaffold for his proof that the Domino Problem is in fact undecidable in $\mathbb{E}^2$.

His set of tiles force a particular kind of hierarchical structure in every tiling they admit, and still today, most known aperiodic sets of tiles have this property.

Berger's construction was highly complex; within a few years, R. Robinson produced a greatly simplified aperiodic set of just six tiles:

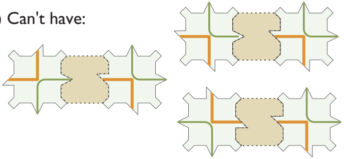
Berger's construction was highly complex; within a few years, R. Robinson produced a greatly simplified aperiodic set of just six tiles: Not only does the set admit non-periodic tilings, it *only* does so and is thus *aperiodic*.



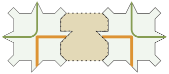


1) Every tile is either a  or incident to 

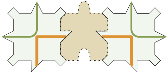
2) Can't have:



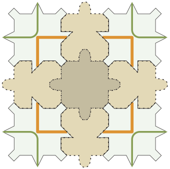
Only:



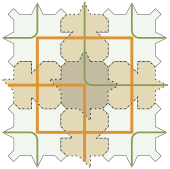
Hence:



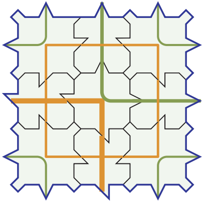
3) So each  is part of:



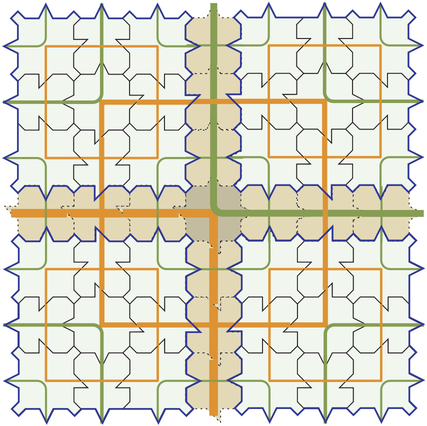
Hence, up to rotation,
every tile is in or next to:



4) These 3x3 blocks act like large  's



& up to rotation,
every tile is in or next to
a 7x7 block:



& up to rotation, every tile is in or next to a 15x15 block, a 31x31 block, etc...

Consider a tiling by the Robinson tiles. Any translation has a finite magnitude and will translate some giant block onto itself. But this will not leave the tiling invariant. Hence every tiling by the Robinson tiles is non-periodic and the tiles themselves are aperiodic.

Even today, this remains the most common style of proof that a set of tiles is aperiodic:

Even today, this remains the most common style of proof that a set of tiles is aperiodic:

- Begin with some non-periodic, hierarchical structure

Even today, this remains the most common style of proof that a set of tiles is aperiodic:

- Begin with some non-periodic, hierarchical structure
- somehow (magic?) produce tiles

Even today, this remains the most common style of proof that a set of tiles is aperiodic:

- Begin with some non-periodic, hierarchical structure

- somehow (magic?) produce tiles

- that fit together to form, in effect, larger tiles with the same combinatorial properties

Even today, this remains the most common style of proof that a set of tiles is aperiodic:

• Begin with some non-periodic, hierarchical structure

• somehow (magic?) produce tiles

• that fit together to form, in effect, larger tiles with the same combinatorial properties

• giving an inductive proof: Every tile must lie in a *unique* infinite hierarchy of larger and larger “supertiles"

Even today, this remains the most common style of proof that a set of tiles is aperiodic:

• Begin with some non-periodic, hierarchical structure

• somehow (magic?) produce tiles

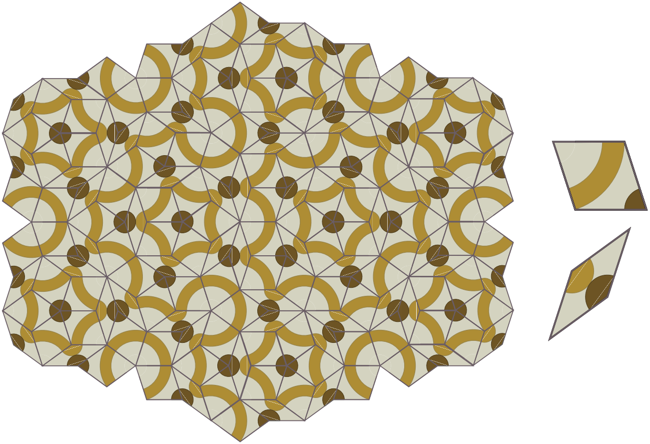
• that fit together to form, in effect, larger tiles with the same combinatorial properties

• giving an inductive proof: Every tile must lie in a *unique* infinite hierarchy of larger and larger “supertiles"

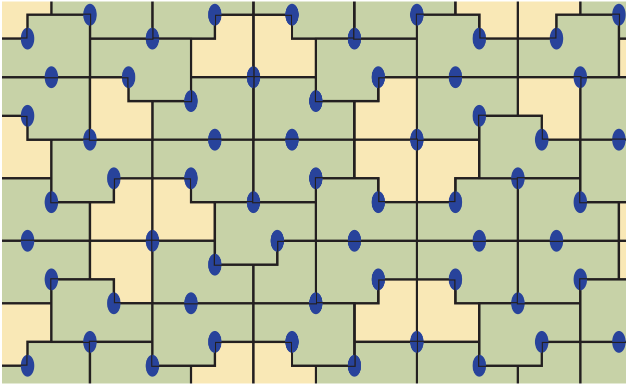
• Thus the tiles *do* admit a tiling, and *only* non-periodic tilings.

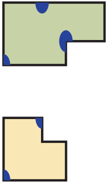
Other interesting small sets of tiles include:

The Penrose (-Ammann-Conway) tiles (1972-78):

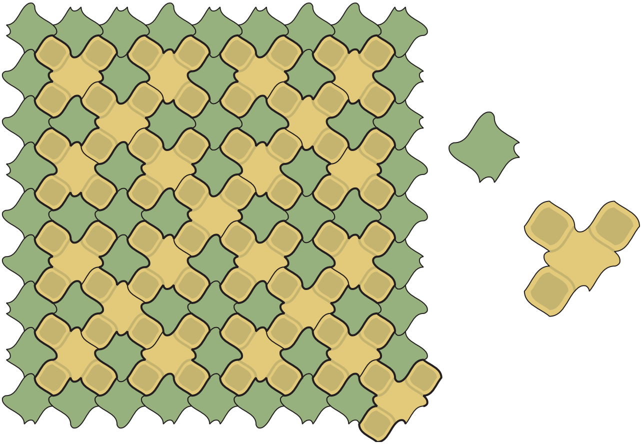


Other interesting small sets of tiles include:
Ammann gave several examples (ca. 1978), including:

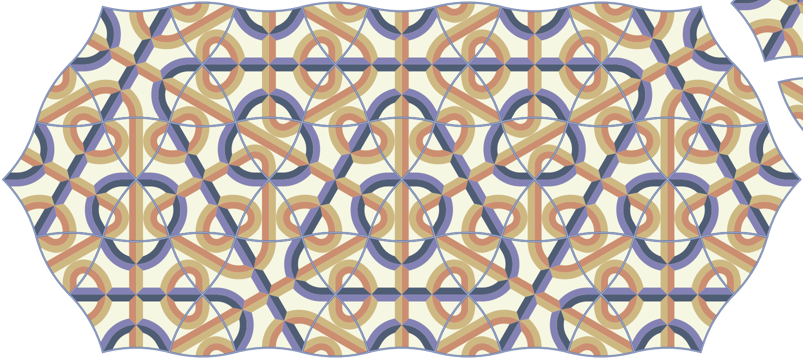


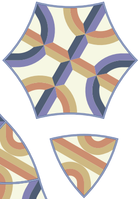


Other interesting small sets of tiles include:
The trilobite and crab (GS, 1994):



Other interesting small sets of tiles include:
The Penrose (-Socolar-GS) tiles (1994-96):

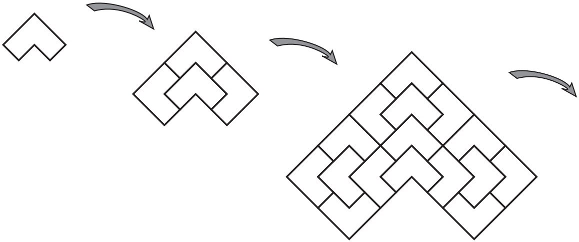




But each of these is somewhat by magic.

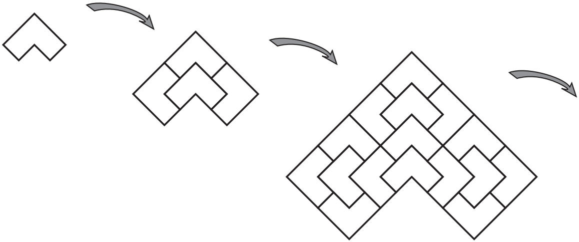
Remarkably, forty-five years after Berger’s construction, very few aperiodic sets of tiles are known– and only a handful of techniques for their construction!

Substitutions on tiles give hierarchical tilings:

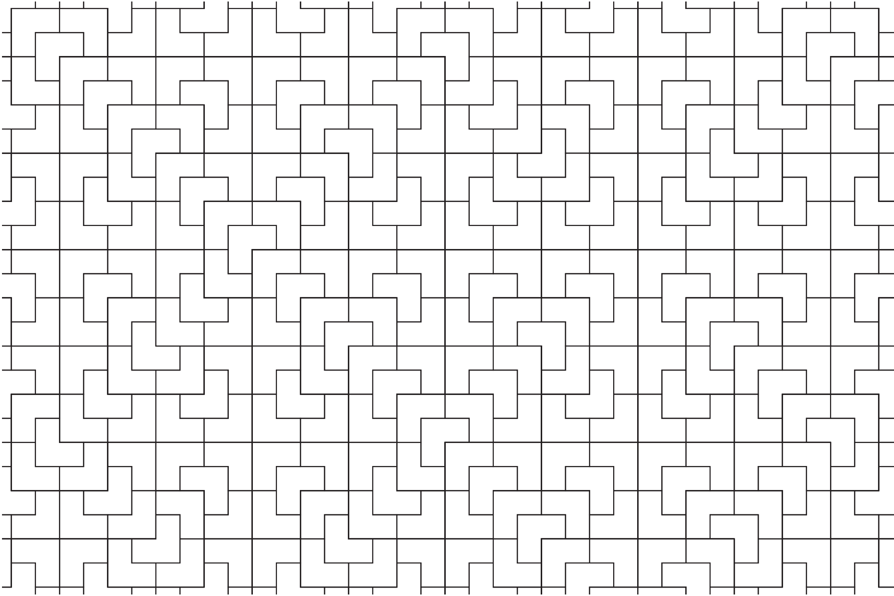


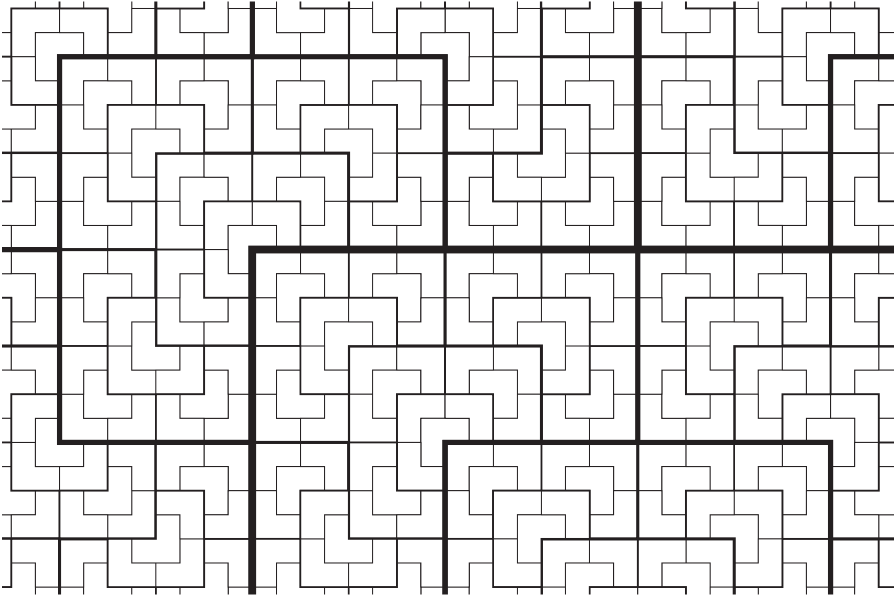
We substitute, producing larger and large supertiles

Substitutions on tiles give hierarchical tilings:

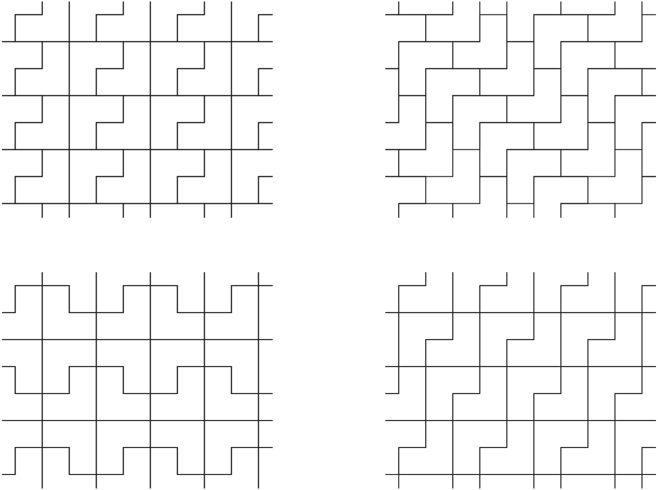


We substitute, producing larger and large supertiles, and ultimately, tilings of the plane.

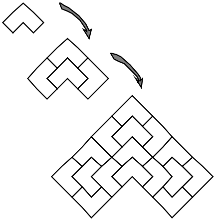


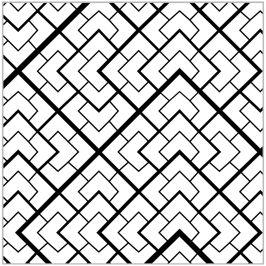


These substitution tilings are *non-periodic* but the L-tiles themselves are *not* aperiodic. They can tile non-periodically, but they also admit periodic tilings.

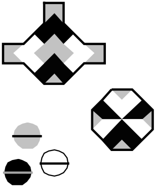


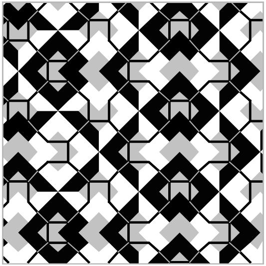
Thus we have *globally* defined, non-periodic hierarchical tilings



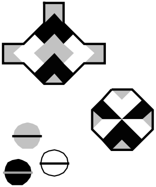


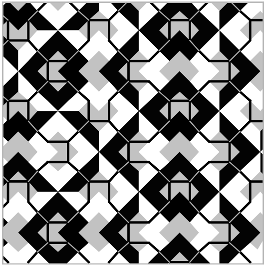
... and *locally* defined matching rule tilings.





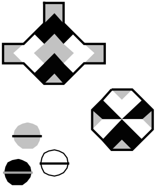
... and *locally* defined matching rule tilings.

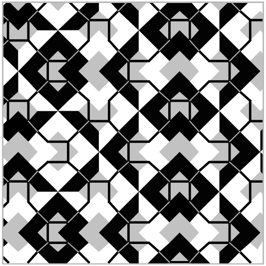




These tiles *enforce* the hierarchical substitution structure: any* of the matching rule tilings they admit is locally reducible to a substitution tiling. And so the tiles are themselves *aperiodic*.

... and *locally* defined matching rule tilings.

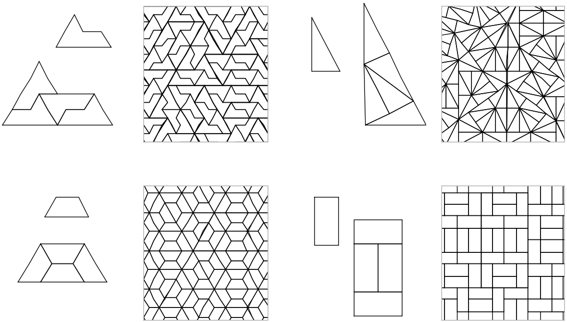




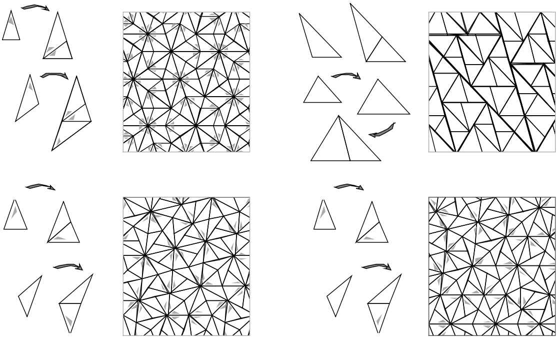
These tiles enforce the hierarchical substitution structure: any* of the matching rule tilings they admit is locally reducible to a substitution tiling. And so the tiles are themselves *aperiodic*.

The game is: *Given a substitution, produce a set of tiles that enforces it*

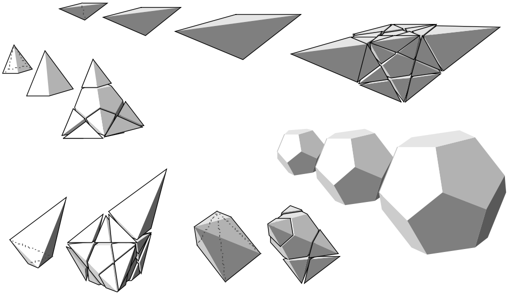
There are a bewildering array of substitution tilings



There are a bewildering array of substitution tilings



There are a bewildering array of substitution tilings



Which substitution tilings can be enforced by matching rules?

Is there a general method for their construction?

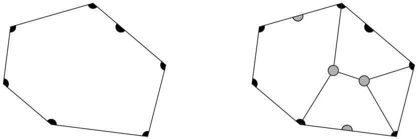
Theorem (GS, 1998) Every substitution tiling (*) can be enforced by matching rules.

All the tile sets produced by this theorem are aperiodic.

Theorem (GS, 1998) Every substitution tiling (*) can be enforced by matching rules.

All the tile sets produced by this theorem are aperiodic.

(*) Our tiles must have well-defined vertices, edges, etc, and the substitution must be “sibling-vertex-to-vertex" and have “hereditary vertices".



But the result holds:

But the result holds:

- in any dimension

But the result holds:

- in any dimension
- for generalized substitutions

But the result holds:

- in any dimension
- for generalized substitutions
- for fractal tiles

But the result holds:

- in any dimension
- for generalized substitutions
- for fractal tiles
- for non-finite local complexity substitutions

But the result holds:

- in any dimension
- for generalized substitutions
- for fractal tiles
- for non-finite local complexity substitutions
- for non-affine substitutions

But the result holds:

- in any dimension
- for generalized substitutions
- for fractal tiles
- for non-finite local complexity substitutions
- for non-affine substitutions
- even for substitutions with periodic substitution tilings (non-unique hierarchy).

How can we accomplish this? In some manner, this seems paradoxical:

• information must be locally finite (how is the full hierarchy to be stored/encoded?)

• information must be transmitted arbitrarily far (over self-organizing transmission lines!)

• and just what is the desired structure anyway??

How can we accomplish this? In some manner, this seems paradoxical:

• information must be locally finite (how is the full hierarchy to be stored/encoded?)

• information must be transmitted arbitrarily far (over self-organizing transmission lines!)

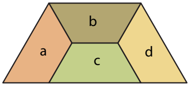
• and just what is the desired structure anyway??

The key, of course, is the self-similar nature of the construction.

Each “piece" only needs to know its role in the hierarchy.

That is:

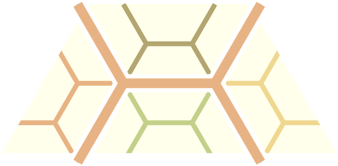
a) Each supertile will “know" its type and position relative to its parent.



That is:

a) Each supertile will “know" its type and position relative to its parent.

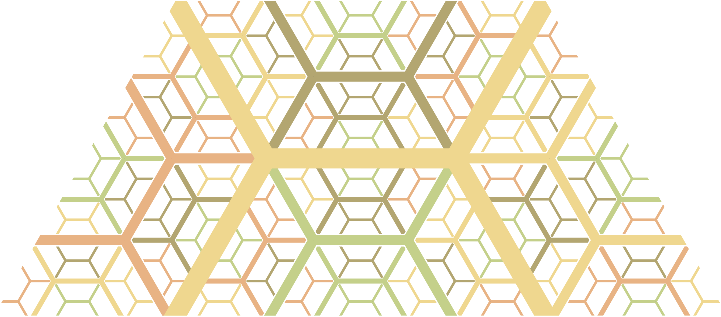
b) The edges of the tiling will be partitioned into “skeletons"; each skeleton holds information for a particular supertile.



That is:

a) Each supertile will “know" its type and position relative to its parent.

b) The edges of the tiling will be partitioned into “skeletons"; each skeleton holds information for a particular supertile.

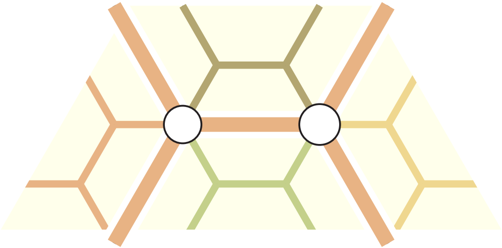


That is:

a) Each supertile will “know" its type and position relative to its parent.

b) The edges of the tiling will be partitioned into “skeletons"; each skeleton holds information for a particular supertile.

c) The vertices of the tiling will mediate between skeletons of parent and child.



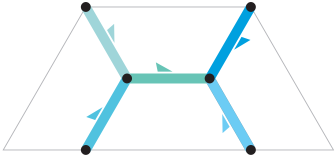
That is:
a) Each supertile will “know" its type and position relative to its parent.

b) The edges of the tiling will be partitioned into “skeletons"; each skeleton holds information for a particular supertile.

c) The vertices of the tiling will mediate between skeletons of parent and child.

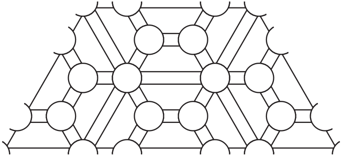
d) Structures at any scale will be combinatorially the same as those at any other scale.

We name every bit of structure in the hierarchy we might need to refer to. For example, we give individual names to each edge in the skeletons:



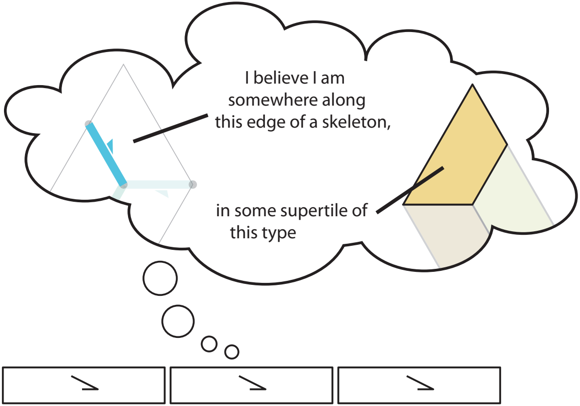
We name every bit of structure in the hierarchy we might need to refer to. For example, we give individual names to each edge in the skeletons:

Our new tiles will thus be individual vertices, edges, and tile-interiors.

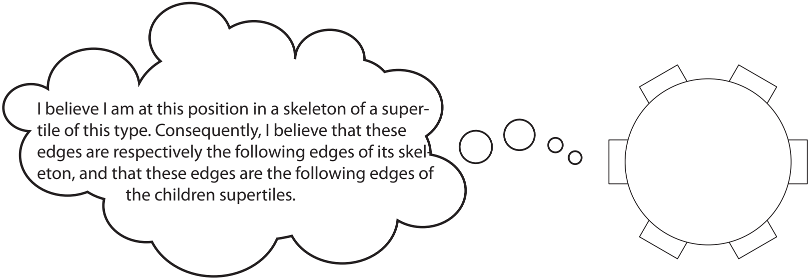


Each piece will have a "fantasy" of where it belongs in some supertile at some level of the hierarchy.

For example, a typical edge tile might fantasize:

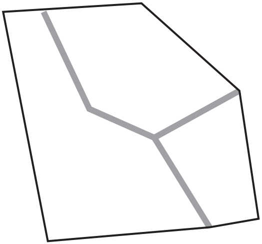


A typical vertex tile might believe:

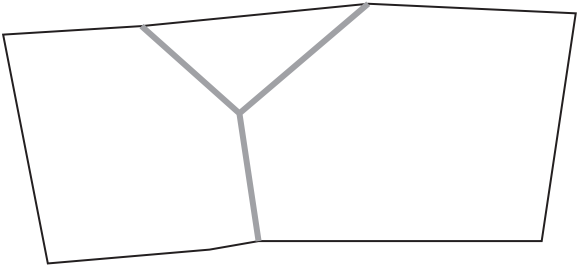


The matching rules are simple: tiles are marked by these fantasies and neighbors must agree.

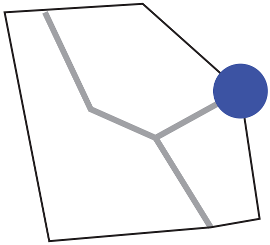
We proceed by induction, defining “well-marked supertiles": our marked tiles, knowing their position relative to their parent are well-marked. Assume every 1st level tile lies in a kth level well-marked supertile. Show it lies in well-marked $(k+1)$st level supertile:



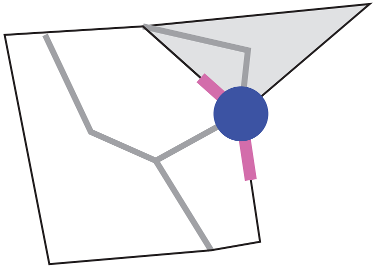
We proceed by induction, defining “well-marked supertiles": our marked tiles, knowing their position relative to their parent are well-marked. Assume every 1st level tile lies in a kth level well-marked supertile. Show it lies in well-marked $(k+1)$st level supertile:



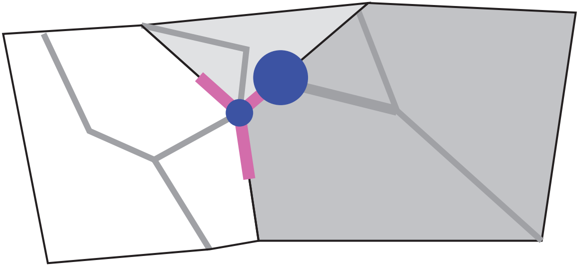
We proceed by induction, defining “well-marked supertiles": our marked tiles, knowing their position relative to their parent are well-marked. Assume every 1st level tile lies in a kth level well-marked supertile. Show it lies in well-marked $(k+1)$st level supertile:



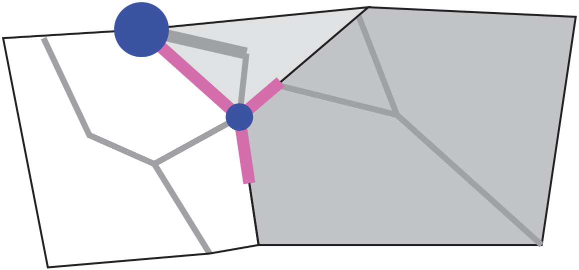
We proceed by induction, defining “well-marked supertiles": our marked tiles, knowing their position relative to their parent are well-marked. Assume every 1st level tile lies in a kth level well-marked supertile. Show it lies in well-marked $(k+1)$st level supertile:



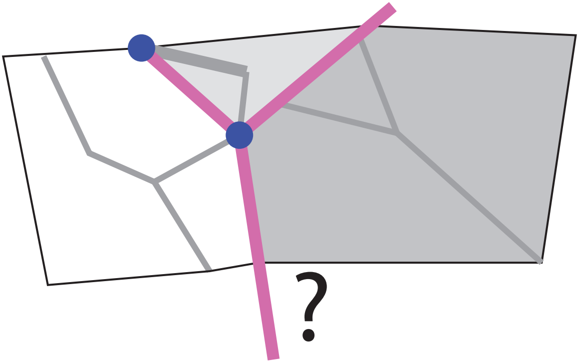
We proceed by induction, defining “well-marked supertiles": our marked tiles, knowing their position relative to their parent are well-marked. Assume every 1st level tile lies in a kth level well-marked supertile. Show it lies in well-marked $(k+1)$st level supertile:



We proceed by induction, defining “well-marked supertiles": our marked tiles, knowing their position relative to their parent are well-marked. Assume every 1st level tile lies in a kth level well-marked supertile. Show it lies in well-marked $(k+1)$st level supertile:

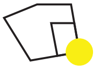


We proceed by induction, defining “well-marked supertiles": our marked tiles, knowing their position relative to their parent are well-marked. Assume every 1st level tile lies in a kth level well-marked supertile. Show it lies in well-marked $(k+1)$st level supertile:

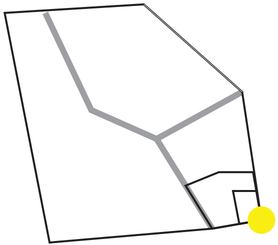


This subtlety is handled by one additional set of markings, “vertex wires": successive supertiles fantasize that a particular vertex will lie at a particular point on the skeleton of some giant supertile:

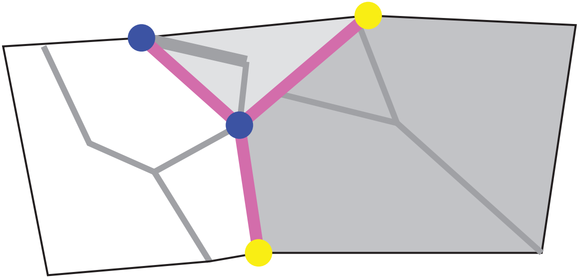
This subtlety is handled by one additional set of markings, “vertex wires": successive supertiles fantasize that a particular vertex will lie at a particular point on the skeleton of some giant supertile:



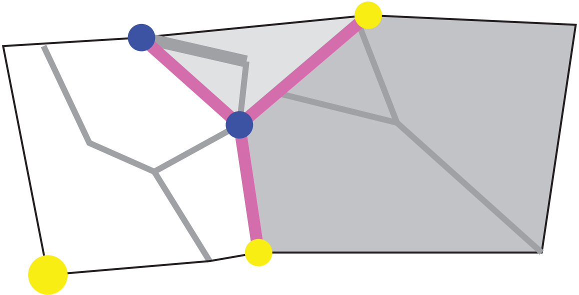
This subtlety is handled by one additional set of markings, “vertex wires": successive supertiles fantasize that a particular vertex will lie at a particular point on the skeleton of some giant supertile:



This subtlety is handled by one additional set of markings, “vertex wires": successive supertiles fantasize that a particular vertex will lie at a particular point on the skeleton of some giant supertile:

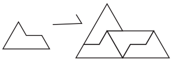


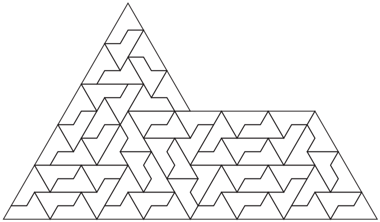
This subtlety is handled by one additional set of markings, “vertex wires": successive supertiles fantasize that a particular vertex will lie at a particular point on the skeleton of some giant supertile:



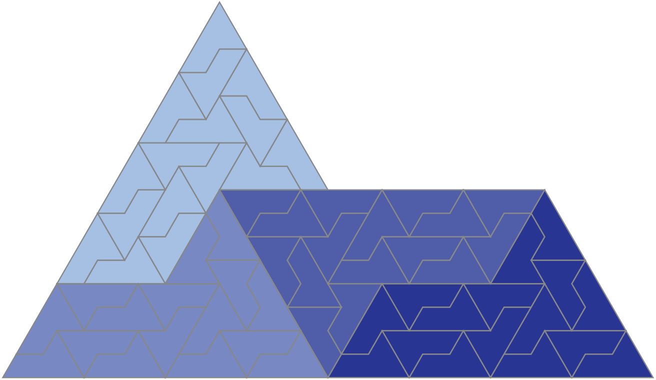
Of course this gives atrocious tile sets!

An Example: The Sphinx

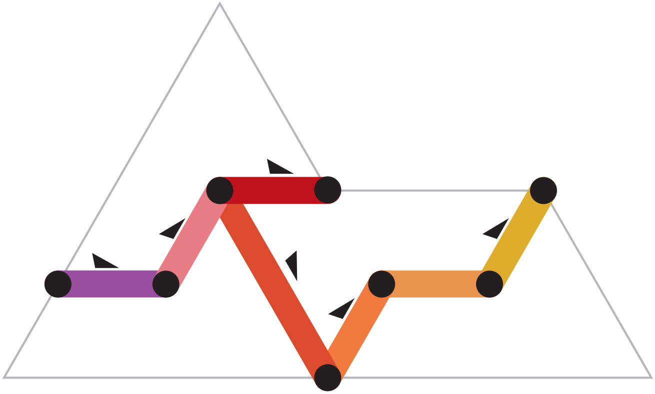




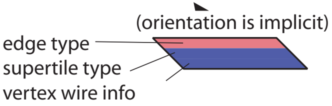
We name the four positions of child within parent (with colors)



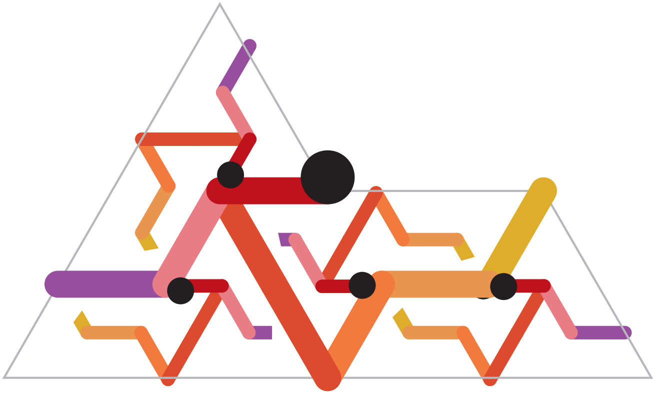
and orient and name all of the edges.



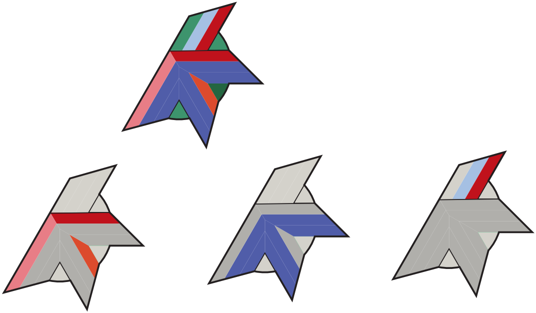
Each edge contains information about its position within a skeleton, the type of the supertile, and, possibly, the supertiles position in a vertex wire.



Vertices connect the skeletons of the children to skeletons of the parents.



The arrangements of markings on the vertices mediate these connections.

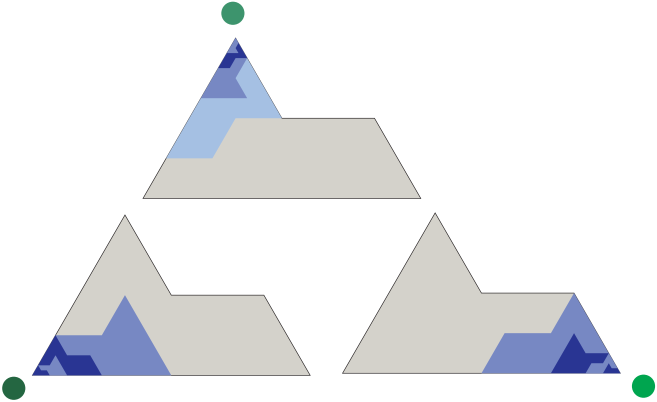


ensures skeleton is assembled correctly

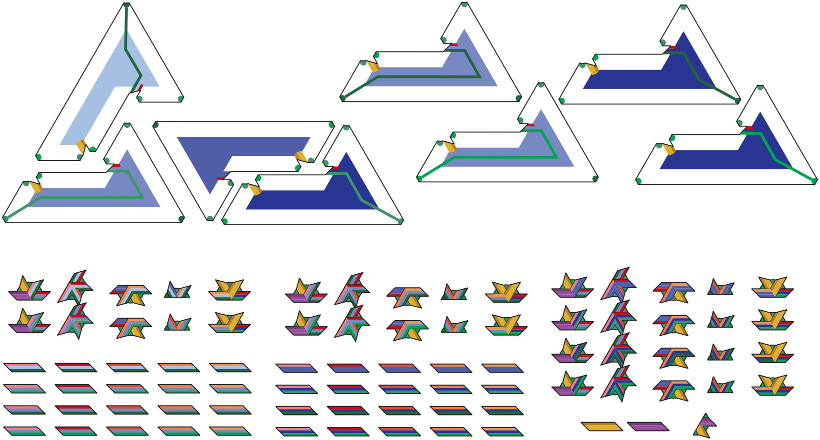
ensures skeleton has consistent information

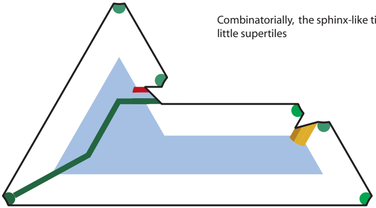
ensures correct child, meeting correctly, at this location

Three different vertex wires are needed; each supertile will lie in at most one such wire, delivering a “stop" signal to some higher level skeletal edge.

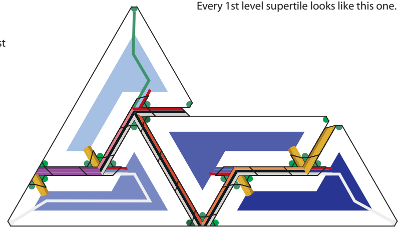


The full set of tiles





Combinatorially, the sphinx-like tiles are just
little supertiles



Every 1st level supertile looks like this one.

An Aside: Remarkably, forty-five years after Berger’s construction, very few aperiodic sets of tiles are known– and only a handful of techniques for their construction!

An Aside: Remarkably, forty-five years after Berger's construction, very few aperiodic sets of tiles are known– and only a handful of techniques for their construction!

• Berger, Knuth, Läuchli, Robinson, Penrose, Amman, Socolar, Danzer, Mozes, Radin and GS have each produced tiles that use the hierarchical structures we will discuss today.

An Aside: Remarkably, forty-five years after Berger's construction, very few aperiodic sets of tiles are known– and only a handful of techniques for their construction!

• Berger, Knuth, Läuchli, Robinson, Penrose, Amman, Socolar, Danzer, Mozes, Radin and GS have each produced tiles that use the hierarchical structures we will discuss today.

• In 1981, De Bruin opened another avenue, noting that Penrose tilings arise via a cut-and-project method; ultimately Le T.T.Q. gave sharp conditions for the construction of such aperiodic tiles.

An Aside: Remarkably, forty-five years after Berger’s construction, very few aperiodic sets of tiles are known– and only a handful of techniques for their construction!

• Berger, Knuth, Läuchli, Robinson, Penrose, Amman, Socolar, Danzer, Mozes, Radin and GS have each produced tiles that use the hierarchical structures we will discuss today.

• In 1981, De Bruin opened another avenue, noting that Penrose tilings arise via a cut-and-project method; ultimately Le T.T.Q. gave sharp conditions for the construction of such aperiodic tiles.

• Kari-Culik and Hanf each gave unique constructions.

An Aside: Remarkably, forty-five years after Berger's construction, very few aperiodic sets of tiles are known– and only a handful of techniques for their construction!

• Berger, Knuth, Läuchli, Robinson, Penrose, Amman, Socolar, Danzer, Mozes, Radin and GS have each produced tiles that use the hierarchical structures we will discuss today.

• In 1981, De Bruin opened another avenue, noting that Penrose tilings arise via a cut-and-project method; ultimately Le T.T.Q. gave sharp conditions for the construction of such aperiodic tiles.

• Kari-Culik and Hanf each gave unique constructions.

• And in more exotic settings, Mozes, GS, Margenstern and Kari have given some techniques.

An Aside: Remarkably, forty-five years after Berger’s construction, very few aperiodic sets of tiles are known– and only a handful of techniques for their construction!

• Berger, Knuth, Läuchli, Robinson, Penrose, Amman, Socolar, Danzer, Mozes, Radin and GS have each produced tiles that use the hierarchical structures we will discuss today.

• In 1981, De Bruin opened another avenue, noting that Penrose tilings arise via a cut-and-project method; ultimately Le T.T.Q. gave sharp conditions for the construction of such aperiodic tiles.

• Kari-Culik and Hanf each gave unique constructions.

• And in more exotic settings, Mozes, GS, Margenstern and Kari have given some techniques.

But that’s it! Basically— very little is known, still.

An Aside:

Interestingly, the undecidability of the Domino Problem tells us that our understanding of aperiodic sets of tiles must inherently be “measure-zero"— given any list of *checkable* or *provable* conditions, such as

An Aside:

Interestingly, the undecidability of the Domino Problem tells us that our understanding of aperiodic sets of tiles must inherently be “measure-zero"— given any list of *checkable* or *provable* conditions, such as

“not tiling"

An Aside:

Interestingly, the undecidability of the Domino Problem tells us that our understanding of aperiodic sets of tiles must inherently be “measure-zero"— given any list of *checkable* or *provable* conditions, such as

“not tiling"

“tiling periodically"

An Aside:

Interestingly, the undecidability of the Domino Problem tells us that our understanding of aperiodic sets of tiles must inherently be “measure-zero"— given any list of *checkable* or *provable* conditions, such as

“not tiling"

“tiling periodically"

“tiling in a hierarchical manner"

An Aside:

Interestingly, the undecidability of the Domino Problem tells us that our understanding of aperiodic sets of tiles must inherently be “measure-zero"— given any list of *checkable* or *provable* conditions, such as

“not tiling"

“tiling periodically"

“tiling in a hierarchical manner"

there must be infinitely many tile sets that satisfy *none* of the conditions. So any understanding we have is *intrinsically* incomplete.

An Aside:

Interestingly, the undecidability of the Domino Problem tells us that our understanding of aperiodic sets of tiles must inherently be “measure-zero"— given any list of *checkable* or *provable* conditions, such as

“not tiling"

“tiling periodically"

“tiling in a hierarchical manner"

there must be infinitely many tile sets that satisfy *none* of the conditions. So any understanding we have is *intrinsically* incomplete. What great job security!

An Aside:

As many participants here are fully aware, there is a large and growing body of theory constructing and classifying substitution tilings, in somewhat broader settings. This is a very exciting area:

Ultimately, will we have a fully complete theory, along the following lines?

An Aside:

As many participants here are fully aware, there is a large and growing body of theory constructing and classifying substitution tilings, in somewhat broader settings. This is a very exciting area:

Ultimately, will we have a fully complete theory, along the following lines?

A description of the possible combinatorial structures and

An Aside:

As many participants here are fully aware, there is a large and growing body of theory constructing and classifying substitution tilings, in somewhat broader settings. This is a very exciting area:

Ultimately, will we have a fully complete theory, along the following lines?

A description of the possible combinatorial structures and

Given a combinatorial structure, a description of how to produce geometric rules.

An Aside:

As many participants here are fully aware, there is a large and growing body of theory constructing and classifying substitution tilings, in somewhat broader settings. This is a very exciting area:

Ultimately, will we have a fully complete theory, along the following lines?

A description of the possible combinatorial structures and

Given a combinatorial structure, a description of how to produce geometric rules.

Or could the geometric satisfiability of such a combinatorial structure be undecidable?

The progress of recent years is extremely encouraging!