

INCENTIVES, DECENTRALIZED CONTROL, THE ASSIGNMENT OF JOINT COSTS AND INTERNAL PRICING*

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1. The Problems of Control and Cost Accounting

Cost accounting in the modern corporate economy is recognized as a tool with many purposes. It must serve, to a greater or lesser extent, financial and legal requirements, the technical needs of branch managers of industrial plants, and at the same time is used by top management as a basis for policy decisions. The growing recognition of the importance of cost accounting control^{1, 2} has highlighted several problems which belong jointly to the fields of concern of managers, engineers, cost accountants, and economists.

Information which is portrayed and set out with one purpose in mind may be worse than useless when used for another purpose. The detailed reports and statistics which are of the utmost necessity to a branch manager concerned with technical problems may confuse and mislead a controller interested in investment policy and other areas where finance and tax structure come to the fore and technical, physical details are of secondary importance. The boiled down aggregate information used by a board of directors may seem to be a travesty of the facts to an operating engineer, but in a world where information is expensive, time dear and decisions cannot be postponed, abbreviations and condensations must be made.

A goal of good management should be to design a reward system for those who take risks in making decisions in such a manner that the rewards to the individual correlate positively with the worth of the decision to the organization (taking into account the attitude of the top management to variance as well as to expected gain). In many organizations cost accounting supplies much of the information used for control at several levels. In this paper we examine some of the control problems that arise if joint costs are assigned by various cost accounting and some internal pricing conventions.

A method for assignment of costs which has desirable incentive and organization properties is then discussed. This method is based upon a result in the theory of games obtained by L. S. Shapley.³ A self-contained exposition of the features of game theory required for this paper is given in section 2 following.⁴

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¹ Lang, T., W. B. McFarland, and M. Schiff, *Cost Accounting*, New York: Ronald Press, 1953, Chapters 3, 27.

² Brummet, R. L., *Overhead Costing*, Ann Arbor, Michigan: University of Michigan Press, 1957.

³ Shapley, L. S., "The Value of an N-Person Game," in Kuhn, H. W. and Tucker, A. W.,

2. Basic Concepts Relevant to the Study of the Assignment of Joint Costs

The theory of cooperative games, as developed by von Neumann and Morgenstern depends upon a measure of the interrelatedness and increase in joint rewards obtained by a group of individuals who are willing to act together, as compared to acting individually. The profitability of a corporation may be viewed as depending upon the sum of the joint rewards which can be obtained by the optimum coordination of all branches. This analogy will be specified even more closely in section 3. The *players* in a von Neumann and Morgenstern game may be regarded as the branches or departments of a corporation or the sections in a factory.

The measure of the complementarity in a game (i.e., the worth of joint coordinated action) is given by its *characteristic function*.⁵ This function is a *super-additive set function*, and although its technical name may at first frighten the non-mathematician, the meaning of it is relatively easy to explain. It is called a set function as it is defined, i.e., it takes on values for a set of entities. In this case the entities consist of every possible combination of departments in a corporation. For example suppose that a corporation consisted of a central office and two departments; if each of these were regarded as an independent entity denoted by 1, 2, and 3 then the characteristic function would be defined for seven values. These are values for 1, 2, and 3 individually; the pairs (1, 2) (1, 3) and (2, 3) and the firm as a whole (1, 2, 3). For completeness we assign a value of zero to a coalition consisting of no one. This gives eight values to the characteristic function of a firm considered as three entities.

The characteristic function is called *super-additive* because the value of the amount obtained by any grouping of participants is always as much or more than they can obtain by individual action. For instance, a coat may be worth more than two halves of the same coat. The characteristic function provides a handy way in which complementarity can be described between different objects or groups.

Consider a firm with several branches, say different plants. They share the common overheads of the firm, and the actions of one branch may affect the direct profits of another (vertical integration might cause this, or there may even be competition in the market between differentiated products of the same corporation; for instance the different automobiles produced by General Motors). One way in which an index of the importance of any branch can be measured is by calculating the effect upon profits if it closed down, and the optimum alternative use were made of the resources it relinquished. In a similar way we can evaluate the importance of any set of branches to the corporation as a whole.

Eds., *Contributions to the Theory of Games*, Vol. II, Princeton: Princeton University Press, 1953, pp. 307-317.

⁴ von Neumann, J., and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton: Princeton University Press, 3rd ed., 1953.

⁵ *Ibid.*, p. 238 ff.

Let $v(\{i, j\})$ stand for the profit that branches i and j of a firm can make on the assumption that the remaining branches have closed down.⁶ In general, $v(S)$, the characteristic function, describes the profit made by the set S , of departments or other separate components of the firm which are to be considered as acting in unison.

As a simple example consider a factory consisting of two departments, 1 and 2. The only cost that they share in common is a joint overhead for the factory. Furthermore suppose that if either department closes down, there is no alternative use that can be made of the excess plant facilities. Assume that the net receipts for Department 1, leaving out the joint cost assessment, are x , and for Department 2, are y . Let the joint cost be C . The set consisting of the two departments is denoted by $\{1, 2\}$. The value of the profits they can obtain together is:

$$v(\{1, 2\}) = x + y - C.$$

The amount that the firm obtains if the second department is closed is:

$$v(\{1\}) = x - C, \text{ with the first department closed it can obtain}$$

$$v(\{2\}) = y - C.$$

We note that:

$$v(\{1\}) + v(\{2\}) = x + y - 2C,$$

hence:

$$v(\{1, 2\}) > v(\{1\}) + v(\{2\}).$$

Although formally the inequality above can be defined, in this example care must be taken in interpreting the meaning of $v(\{1\}) + v(\{2\})$. Both departments cannot simultaneously realize their own survivor's value. If instead of two departments, the example had been that of a husband and wife deciding to file a joint tax return or to both "go it alone" then the above sum would have a direct physical counterpart.

In order to avoid difficulties such as the one above, it is necessary to divide the firm into separate decision-making entities and to specify the powers of the various decision makers to close down a plant, to go out of business or to "secede from the union." This is discussed in more detail in section 3.

By utilizing the characteristic function, the von Neumann and Morgenstern theory of games leads to a concept of solution in which all players act in a manner to jointly maximize profits and then use their bargaining power as represented in the possible coalitions to arrive at an imputation of the proceeds. The method suggested for splitting the profits⁷ is somewhat complicated and does not concern us here. No unique imputation is given, although certain bounds are placed upon the shares received by each individual.

⁶ We are implicitly assuming that the strategy space of a top manager is limited to such a manner that he has the choice of closing down or producing optimally with those branches which do not close down. This assumption is discussed further in the text.

⁷ von Neumann, J., and O. Morgenstern, *op. cit.*, p. 264.

The method for assigning a portion of joint proceeds to each player which has been advanced by Lloyd Shapley does provide for a unique division. Furthermore it will be shown that this method satisfies a certain set of properties which an accounting system should have if decentralized decisions are to be based upon the internal imputation of profits to semi-autonomous sections of an organization.

3. Incentives, Control and Cost Accounting

Broadly speaking it is often deemed desirable to be able to delegate as many decisions as possible to the branches of a firm. In many organizations of large size the exponential growth of messages and red tape cause diseconomies in centralized decision-making for those decisions which depend heavily upon on-the-spot knowledge. If decision-making power is to be delegated it is preferable to have an organization which is designed to encourage initiative. One way of doing this is to have the reward structure designed so that the selection of choices which are best for the individual decision-maker will always coincide with those which are best for the organization. For instance, a branch manager may be aware of a change which may have the effect of increasing corporate profits, but decreases the size of his own department and may even reduce the profits assigned to it by the accounting system. If his success and income are measured and determined by the accounting profits assigned to his department then it may not be in his interest to select the decision optimal for the firm.

Of course there are many sociological and psychological aspects to an incentive structure in a corporation, church, university or commissariat. Thus gold medals, memberships in golf clubs, prestige, pride in workmanship and so forth all play an important role.⁸ Furthermore in even the most impersonal and mechanized systems single number measures of the performance of an individual are rarely used. For purposes of this paper, however, the sociological, psychological and psychiatric aspects of the individuals are taken as given. As bonuses and "incentive compensation" to executives in many corporations are based upon the profits imputed to their operations the economic and accounting problem may be of interest by itself.

There are many technical and conceptual difficulties to be faced in the accounting treatments of fixed costs, variable costs and joint costs. There is a wide variety of practice in accounting methods. Lee Brummet notes five for example:

- (1) Complete absorption costing
- (2) Expected or average activity standard costing
- (3) Practical capacity standard costing
- (4) Direct standard costing
- (5) Prime standard costing.⁹

⁸ Argyris, C., "The Individual and Organization; Some Problems of Mutual Adjustment," *Administrative Science Quarterly*, June 1957.

⁹ Brummet, R. L., *op. cit.*, p. 32.

Joint costs have been assigned as a percentage related to the direct labor costs of each operation; charged as a rate per direct labor hour; a rate per unit of product; a percentage of direct material cost or a percentage of prime cost (direct labor + direct material cost).¹⁰ No exegesis of accounting methods is to be presented here. Many important and vexatious accounting problems are ignored. However viewing one of the roles of accounting as helping: "to provide management with cost information necessary to business decisions and related policy",¹¹ it is observed that under several of the methods above it is possible that a department be assigned costs which make its "paper profits" negative even though it may be a vital and efficient part of the firm. It is also possible that an improvement in the efficiency of a department may damage its individual profit statement even though it increases the over-all profitability of a firm.

There should always be an incentive for a manager to implement an efficiency or report a new idea if it benefits the firm as a whole no matter what changes it may cause to take place in his own operation. Under some methods of cost assignment, for example, if the decision to discontinue a product line rests with individual department heads it is possible that individual rational action based upon the cost assignment may add up to corporate idiocy. A simple example of this is given in section 5.

Ideally the assignment of joint costs to individual products or departments is not necessary from a purely economic point of view if the decision to maximize for the company as a whole is made in a single office. This is usually impossible in practice, hence a cost accounting and internal pricing scheme can serve as an administrative device in the design of a viable and economic decision-making system.

4. Decentralization, Decisions and Information

The concept of decentralization deals with the possibility of delegating decision making to more than one location in an organization. An optimally decentralized system will have the property that the net effect of all individual actions will be more favorable to the firm than the actions selected by any other array of decision centers. This must take into account costs of messages and organization and the possibilities of committing errors when decisions which appear to be locally optimal are not of benefit to the organization as a whole.

The limiting case for the possibilities for decentralization comes when all decision centers or units are independent. This is merely another way of saying that an action by anyone or any group has no effect on any other unit or combination of units. This is true for small numbers in a purely competitive market which may be viewed as a decentralized organization. It is not completely true as can be seen by problems in agriculture and other "chronically competitive markets." If the characteristic function of an organization is *flat*, i.e., if

¹⁰ Lang, T., W. B. McFarland, and M. Schiff, *op. cit.*, Chapter 13.

¹¹ "Report of the Committee on Cost Concepts and Standards," *The Accounting Review*, April 1952, p. 175.

the sum of the amounts which can be obtained by any two coalitions acting together is precisely the same as the amounts that they can obtain by acting independently, then obviously there is no need whatsoever to coordinate their actions as each unit is an autarky and neither gains from nor adds to any joint venture sufficiently to merit other than individual action.

Interesting and important cases for decentralization arise when the joint welfare is influenced by individual action, or the action of coalitions. The degree of influence is reflected in the characteristic function; which, if its values are appropriately defined display both the technological and decision structure of the firm.

In a game, a player is characterized as an individual decision-maker with some degree of free choice. By analogy we may consider a general manager in a corporation as a player in a position to choose among a set of actions pertaining to his department or part of the corporation. He is a "dummy player" if, in fact, his actions are irrelevant to the functioning of the organization. This happens when some other individual is in a position to over-rule and change any of his decisions. This is so in a completely centralized organization; or is apparently so until we consider the information conditions. In theory as well as in practice the selection of what type of message to send up to the decision-maker is a decision in itself and gives the individual a degree of power which varies as the difference between his knowledge and the knowledge of his superior, and the importance of this to the decision.

Effective complete centralization requires either that the central office is completely informed and merely uses the remainder of the organization as an instrument for execution and not for information gathering; or that all individuals are assumed to be unbiased gatherers of data. In other words the central office, if it is not totally informed, must assume that individuals within the organization will not be motivated to distort the information they send or to take actions based on goals which do not correlate with those of the central office. This calls for a concept of an organization as a *team*¹² rather than a series of arrangements between individuals with possibly differing goals.¹³ The former can be regarded as a limiting case of the latter. Our interests here are concerned with simple problems arising from the latter concept.

The specification of a characteristic function as a model of the potentials of sectors of a firm contains within it both considerations of the decision structure of the firm and the potential worth of the resources. This can be seen when an attempt is made to assign a worth to what can be achieved by a subset of departments. In order to do this several questions must be posed concerning the

¹² Marschak, J., R. Radner, "Economic Theory of Teams," Cowles Foundation Discussion Paper No. 59, 1958.

¹³ An organization may be regarded as a game with restraints on the players and the sending of messages as one of the major weapons of control. A concentration camp fits this model better than does a team.

location of responsibility for key decisions. A partial list of relevant decisions is given below:

1. The decision on major investment
2. The liquidation of a department
3. The abolition of a product line
4. The introduction of a new product
5. The introduction of other innovations (such as a change in distribution)
6. The merger of several departments
7. The splitting of a department into several independent entities
8. Pricing, purchase of raw materials and sales of final product

If, for example, the managers of each department had decision responsibility for all of the above (which might be the case if the organization being described were a weak cartel rather than a corporation) then the meaning of the value attached to any subset would be the value that a subgroup of participants in the cartel agreement could obtain by acting together by themselves outside of the cartel agreement.

If only some of the decisions are to be delegated while others remain under the control of an executive or central office, then it may be desirable to introduce the office as a player. Returning to and reworking the example of a factory with two departments given in section II; it can be regarded as an organization with three participants. Suppose that there is a president and executive office which has delegated decisions 2, 3, 4, 5 and 8 above to the two managers of the departments, but maintains its decision-making power on the others. Furthermore suppose that the managers are instructed to maximize the profit assigned to their departments under the accounting system used by the firm. We assume that the central office has dictated a method of accounting which calls for all costs and revenues to be imputed. As the managers are in a position to liquidate their departments unilaterally and to discontinue product lines, they can guarantee themselves individually a profit of not less than zero. Let the central office be player 1 and the departments be 2 and 3. The characteristic function for this firm with structure shown in Figure 1 is:

$$\begin{aligned}
 v(\{\emptyset\}) &= 0 \\
 v(\{1\}) &= 0 \quad v(\{2\}) = 0 \quad v(\{3\}) = 0 \\
 v(\{1, 2\}) &= x - c \quad v(\{1, 3\}) = y - c \quad v(\{2, 3\}) = 0 \\
 v(\{1, 2, 3\}) &= x + y - c
 \end{aligned}$$

We assume that the central office can obtain a value of zero by liquidating and employing the proceeds elsewhere, hence $v(\{1\}) = 0$.

A good decentralized system should have the property that each decision center will make a decision which is optimal for the whole with a minimum of cost for coordination and information and message costs. In the example above,

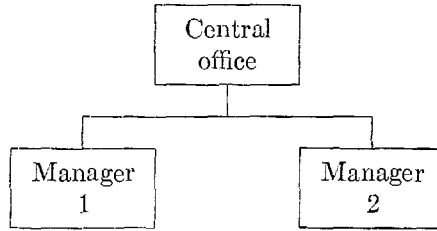


FIG. 1

the role of the executive or central office is to assign joint costs. It must do so in a manner that will guarantee that if a department should exist for the good of the firm as a whole, then it will not get an assessment that makes its net revenue negative. For example, suppose $c = 10$, $x = 4$ and $y = 7$, then an assignment of costs of 5 each to the two operating departments will motivate player 2 to shut down even though his operation is of value to the business, hence this is not a good assignment.

If $c = 10$ and $x = y = 4$, then the assignment of costs should be such that the firm should be motivated to liquidate (or otherwise change drastically).

Looking at this firm as an administrative system, the only information needed by the central office from the departments is their individual net revenues, and the only information that it will send them is the size of their assessments (it is presumed that the executive office has some other economic, financial or service function which it renders to the firm as a whole).

For another example we consider a firm without overheads or other joint costs, but with two departments producing the same item at costs $C_1(x)$ and $C_2(y)$ which is then sold by the central office which acts as a marketing agency for the firm as a whole. Here the problem is to assign shadow prices to be paid by the central office to the departments and to impute the remaining profit. We assume that the only decisions which are decentralized are production levels and individual technology. The characteristic function will be:

$$\begin{aligned}
 v(\{\Theta\}) &= 0 \\
 v(\{1\}) &= 0, \quad v(\{2\}) = 0, \quad v(\{3\}) = 0 \\
 v(\{1, 2\}) &= \underset{x}{\text{Max}} [x\varphi(x) - C_1(x)] \\
 v(\{1, 3\}) &= \underset{y}{\text{Max}} [y\varphi(y) - C_2(y)] \\
 v(\{2, 3\}) &= 0 \\
 v(\{1, 2, 3\}) &= \underset{x}{\text{Max}} \underset{y}{\text{Max}} [(x + y)\varphi(x + y) - C_1(x) - C_2(y)]
 \end{aligned}$$

where $p = \varphi(q)$ is the final demand schedule for the product. It has been shown by Dantzig and Wolfe that for the appropriate limitations on technology, a firm decentralized in the manner above need only send messages concerning shadow prices and outputs in order to reach a joint optimum. This is the "de-

composition" principle for solving a linear program, and although it is primarily a computational device it may be viewed as an administrative arrangement.¹⁴

The two examples given above are treated in detail in section 6. We turn in section 5 to the development of the general method for imputing costs and assigning prices to satisfy incentive criteria.

Although the topic is not pursued further at this time, it should be noted that the problem to which this paper is addressed is closely related to the more general problems of organization and stability in an economy.¹⁵ This in turn is related to the role and effectiveness of group action in administrative systems.¹⁶

5. An Incentive System for Decentralized Control

A corporation is characterized as a set of n decision centers. The characteristic function of a corporation reflects not only the technological features of complementarities between products, common overheads, joint costs and other technological interrelationships, but also the decision structure of the various centers.

We limit ourselves to considering only firms which should not completely liquidate. A firm should not liquidate if there is at least one subset of decision centers S which can earn as much or more than the income obtained from investing the proceeds of liquidation.

Let the set of decisions of the i^{th} center be denoted by D_i . An individual decision is $d_i \in D_i$. In general the characteristic function is calculated for all values as follows. We define $\psi_{S_j}(d_i, d_j, \dots, d_m)$ a function of s variables which represents the payoff to the firm as a whole on the assumption that a particular set of s centers denoted by S_j are active and the remainder have been dissolved.

$$v(S_j) = \max_{d_i \in D_i}, \max_{d_j \in D_j}, \dots, \max_{d_m \in D_m} \psi_{S_j}(d_i, d_j, \dots, d_m)$$

In particular, $v(I) = \max_{d_1 \in D_1}, \max_{d_2 \in D_2}, \dots, \max_{d_n \in D_n} \psi_1(d_1, d_2, \dots, d_n)$.

For many large corporations with diversified businesses some of the structure of the functions ψ_{S_j} can be specified simply. For instance, if a firm sells two products which share no joint variable costs, incur no joint economies in marketing and have negligible influence on each others' markets, an example is diesel locomotives and Christmas tree lights, the function $v(S_j)$ where S_j consists of department 1 making locomotives and department 2 making lights, can be written as:

$$v(S_j) = \max_{d_1 \in D_1} \max_{d_2 \in D_2} [\psi_1(d_1) + \psi_2(d_2) - C].$$

The only connection between the departments is a joint fixed cost.

¹⁴ Dantzig, G. B., and P. Wolfe, "Decomposition Principle for Linear Programs," *Operations Research*, Vol. 8, No. 1, January 1960, pp. 101-111.

¹⁵ Arrow, Kenneth J. and Gerard Debreu, "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, Vol. 22, July 1954, pp. 265-290; Arrow, Kenneth J. and L. Hurwicz, "On the Stability of Competitive Equilibrium," *Econometrica*, Vol. 26, October 1958, pp. 522-552.

¹⁶ Shubik, M., "Extended Edgeworth Bargaining Games and Competitive Equilibrium," Cowles Foundation Discussion Paper No. 107, 1961.

The more obvious forms of interconnection also serve to enable us to specify the calculation for the characteristic functions without great difficulty. These include vertical integration, aspects of horizontal integration, joint variable costs, such as transportation or the use of a commonly owned computing machine. Interconnectivity in the market is also reflected in the characteristic function. For example, many consumer durables may compete with each other in the market.

The ascribing of a value to the one decision unit acting by itself, $v(\{i\})$ depends upon whether the decision system allows the manager to close his plant or production.

We assume that it is desirable not to assign negative profits to any decision center whose existence is of value to the firm as a whole. This can be achieved by using a characteristic function where for any i

$$v(\{i\}) = \max [0, \max \psi_{1_i}(d_i)].$$

This is tantamount to allowing a manager to close production or dissolve his unit if he is assigned a negative profit. If the system only assigns him a negative profit when in fact, the liquidation of his activity is for the good of the firm, this has a desirable property for a well decentralized system.

We present the five properties or *axioms* for a good assignment of the proceeds of a joint profit (and hence, implicitly the imputation of joint costs, internal prices and revenues) to different decision centers of a firm. A verbal statement of each axiom is given first, this is followed with the precise mathematical formulation.

Axiom 1: The profit assigned to a given center depends at most upon the various revenues which can be earned by all alternative uses of all centers or combinations of centers.

Symbolically, if we use the notation ϕ_i to stand for the profit assigned to the i^{th} center, we can write:

$$\phi_i = F(v(\Theta), \dots, v(S), \dots, v(I))$$

where $v(S)$ is the characteristic function which portrays all complementarities inherent in the optimal use of any combination of the facilities of the firm.

Axiom 2: The profit assigned to a center depends symmetrically upon all centers in a firm. In other words, if two firms are identical except that their departments or decision centers are called by different names, then the accounting system will assign the same profit to the centers which are physically the same despite the difference in names.

Symbolically if we let Γ stand for the game characterized by $v(S)$ and Γ' the game such that

$$v'(S) = v(S^*),$$

where S^* is like S but with i replacing j and j replacing i , then

$$\phi'_i = \phi_j \quad \phi'_j = \phi_i$$

Axiom 3: The accounting system imputes all the profits earned by the firm.

$$\sum_{i \in I} \phi_i = v(I) \text{ where } I \text{ is the set of all decision centers.}$$

Axiom 4: A homogeneous expansion of fixed costs, variable costs and profits will result in a homogeneous rise in the accounting profits imputed to all processes.

$$\text{If } \Gamma' = \beta\Gamma, \quad \beta > 0, \quad \text{then } \phi' = \beta\phi$$

For example, if the currency unit were changed so that one new franc is worth one hundred old, the new profit assignment ϕ' if measured in francs is such that $\phi' = \frac{1}{100}\phi$.

The fifth axiom envisions a strange situation which might arise if two independent firms jointly share a facility. For instance suppose that each rents a certain plant and each have managers to run it, one for the day shift and the other for a night shift! Furthermore, let us imagine that neither firm has any use for more than one shift from the facility they both rent. If we were confronted with the strange arrangement then:

Axiom 5: If two independent firms are considered as a unit, the profit imputed to the operations utilizing this facility will be the sum of the profits that each firm imputes to its own operation which utilizes the facility separately. The profits imputed to any department or decision center which is not jointly used by each of the firms will not be changed by the consideration of both firms as a unit.

If Γ consists of the game obtained by considering the games Γ' and Γ'' together, then:

$$\phi_i = \begin{cases} \phi'_i + \phi''_i & \text{for } i \in I' \cap I'' \\ \phi'_i & \text{for } i \in I - I'' \\ \phi''_i & \text{for } i \in I - I' \end{cases}$$

The proof that these five axioms lead to a unique formula based on the characteristic function is given by L. S. Shapley.¹⁷ We will not be concerned with this mathematical problem here, but rather with the interpretation of the result. The formula is:

$$(1) \quad \phi_i = \frac{1}{n!} \sum_i (s-1)!(n-s)! [v(S) - v(S - \{i\})].$$

It assigns a share of the joint profits to each center (and hence automatically imputes joint costs). The rationale behind the formula can be seen in terms of addition to productivity. The addition to profits caused by a center acting jointly under all possible conditions with the other centers (i.e., every possible arrange-

¹⁷ Shapley, L. S., *op. cit.* The axioms used in this paper are an earlier formulation by Shapley in a RAND paper RM-670 which are equivalent to those used in his latter publication.

ment with some shut down and others operating) is evaluated and an average is taken.

The economist will recognize that this amounts to assigning a profit to each center according to its expected marginal or incremental value productivity. This can be seen immediately by examining the terms in (1). First,

$$[v(S) - v(S - \{i\})]$$

is simply the contribution which department i makes to a coalition S if it is a member of S . Second, the term

$$(s - 1)! \quad (n - s)!$$

is the number of orderings of the remaining departments of S and $I - S$, where the latter is the set of all departments of the firm excluding S . $n!$ is the total number of permutations of all members of I .

We now show that the method of imputation obtained by using the Shapley Value defined in (1) upon the characteristic function of the firm has desirable incentive properties.

Theorem 1: The profit assigned to a Department which should be in operation if resources are efficiently allocated by the firm will never be less than $v(\{i\})$ for the i^{th} Department.

This is trivially proved. In the formula given in (1) the sign of ϕ_i depends upon the terms $[v(S) - v(S - \{i\})]$, but as the characteristic function is super-additive all terms are at least as large as $[v(\{i\}) - v(\Theta)] = v(\{i\})$. This completes the proof.

Theorem 2: An increase in efficiency or flexibility (see example 3, section 6 for a definition of flexibility) or any action taken by a center which is of value to the firm as a whole will never cause the profits assigned to that center to fall.

This is easily demonstrated. If the game Γ is defined by $v(S)$ and the new game is defined by $v'(S)$ where $v(S) = v'(S)$ for all S not containing i , $v(S) \leq v'(S)$ for all S containing i , then for all S :

$$[v'(S) - v'(S - \{i\})] = [v'(S) - v(S - \{i\})] \geq [v(S) - v(S - \{i\})].$$

This completes the proof.

6. Some Examples Calculated and Interpreted

Example 1. Common Fixed Overhead

As a first example we take the first case presented in section 4, consider a factory that produces two products which use all the same facilities with the same intensity. Each product is under an independent manager. Each process takes up one half of the factory floor space, railyard, etc. The same number of man hours are used on each production method. An apparently natural way to assign joint fixed and variable costs between the two decision centers is to charge one half of the costs to each as they all utilize one half of the resources of the factory. If we assume that the costs of the raw materials are the same for

the products, then all the cost accounting methods noted would assign overhead equally. The characteristic function for this example is given below:

$$\begin{aligned} v(\{\Theta\}) &= 0 \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0 \\ v(\{1, 2\}) &= \text{Max}((x - c, 0)), \quad v(\{1, 3\}) = \text{Max}((y - c, 0)), \\ v(\{2, 3\}) &= 0, \quad v(\{1, 2, 3\}) = x + y - c \end{aligned}$$

If $x + y > c$ then the firm runs at a profit. Suppose, however, that $x < c/2$. Standard accounting in this instance would compute the overhead evenly, giving $\phi_2 = x - c/2 < 0$, $\phi_3 = y - c/2$.

The first manager would be motivated to close down. To be fanciful let us suppose that this firm were highly decentralized in communication, that $c/2 < y < c$ and that there is no alternative use for the closed plant. On the next assignment, all the costs will be put on the second manager (who, after all, is using the plant himself). This gives $\phi_3 = y - c < 0$, hence he is motivated to close down even though the plant as a whole with the two products could make a profit.

Applying the Shapley value we obtain:

$$\begin{aligned} \phi_i &= \frac{1}{3!} [(2!)(1!)(v(\{i\}) - v(\{\Theta\})) + (v(\{i, j\}) - v(\{j\})) \\ &\quad + (v(\{i, k\}) - v(\{k\})) + (1!)(2!)(v(\{i, j, k\}) - v(\{j, k\}))] \end{aligned}$$

which gives:

$$\begin{aligned} \phi_1 &= \frac{1}{6} [2(0) + 2(x + y - c)] \\ &= \frac{1}{3} (x + y - c) \\ \phi_2 &= \frac{1}{6} [2(0) + 2(x + y - c)] \\ &= \frac{1}{3} (x + y - c) \\ \phi_3 &= \frac{1}{3} (x + y - c) \end{aligned}$$

Suppose $c = 10$, $x = 4$, $y = 7$. This gives $\phi_1 = \phi_2 = \phi_3 = \frac{1}{3}$ thus the assessments are $3\frac{2}{3}$ and $6\frac{2}{3}$ respectively.

If the values had been $y > x > c$ then we would have had

$$\phi_1 = \frac{3x + 3y - 4c}{6}, \quad \phi_2 = \frac{3x - c}{6} \quad \text{and} \quad \phi_3 = \frac{3y - c}{6}$$

For $c = 10$, $x = 16$, $y = 17$ this gives $\phi_1 = \frac{59}{6}$, $\phi_2 = \frac{38}{6}$, $\phi_3 = \frac{41}{6}$ giving assessments of $\frac{59}{6}$ and $\frac{61}{6}$. In both instances the two operating departments are assessed more than the total overhead. They pay a levy to the central office, but their net revenues are always positive. The central office requires only one number from each of them, their net profit before assessment.

Example 2. Common Marketing and Technological Improvement

In the second example in section 4 we considered a centralized sales operation with two decentralized factories. We will modify the example to a trivially simple linear program which will nevertheless be useful in demonstrating the appropriate decentralization properties.

Suppose the sales operation handles two products, 1 and 2, and the market will buy up to 10 units of each at prices π_1 and π_2 . Both factories produce both items. Factory 1 has technology coefficients α_1 and α_2 (both $< \pi_1$ and π_2 respectively). Factory 2 has technology coefficients β_1 , $\pi_1 > \beta_1 > \alpha_1$ and $\beta_2 < \alpha_2$. There is some limit larger than 10 on their productions.

$$\begin{aligned} v(\{\emptyset\}) &= 0 \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0 \\ v(\{1, 2\}) &= 10\pi_1 + 10\pi_2 - 10\alpha_1 - 10\alpha_2 \\ v(\{1, 3\}) &= 10(\pi_1 + \pi_2 - \beta_1 - \beta_2) \\ v(\{2, 3\}) &= 0 \\ v(\{1, 2, 3\}) &= 10(\pi_1 + \pi_2 - \alpha_1 - \beta_2). \end{aligned}$$

Suppose that the marketing board sends out shadow prices p_1 and p_2 and gets back information on production possibilities. By merely solving three local linear programs production will be optimally allocated. In particular the prices $p_1 = \alpha_1$ and $p_2 = \beta_2$ will satisfy. They cause the correct specialization and give the market operation a profit of $10(\pi_1 + \pi_2 - \alpha_1 - \beta_2)$ and the others obtain profits of zero.

Suppose there is a potential shift in technology which can be installed by the manager of the first plant. It replaces α_1 by $\alpha'_1 \ll \alpha_1$. If he puts this in, then in the optimum production search via shadow-pricing, the prices $p_1 = \alpha'_1$ and $p_2 = \beta_2$ will serve to allocate production. The accounting profits of the first plant are still zero. Is there a measure which will more or less automatically reflect the worth of the action of the first manager? Calculating the set of values we obtain:

$$\begin{aligned} \phi_1 &= \frac{1}{6} [(\pi_1 + \pi_2 - \alpha_1 - \alpha_2) + (\pi_1 + \pi_2 - \beta_1 - \beta_2) \\ &\quad + 2(\pi_1 + \pi_2 - \alpha_1 - \beta_2)] \\ &= \frac{1}{6} [4\pi_1 + 4\pi_2 - 3\alpha_1 - 3\beta_2 - \alpha_2 - \beta_1] \\ \phi_2 &= \frac{1}{6} [(\pi_1 + \pi_2 - \alpha_1 - \alpha_2) + 2(\beta_1 + \beta_2 - \alpha_1 - \beta_2)] \\ &= \frac{1}{6} [\pi_1 + \pi_2 - 3\alpha_1 + 2\beta_1 - \alpha_2] \\ \phi_3 &= \frac{1}{6} [(\pi_1 + \pi_2 - \beta_1 - \beta_2) + 2(\alpha_1 + \alpha_2 - \alpha_1 - \beta_2)] \\ &= \frac{1}{6} [\pi_1 + \pi_2 - 3\beta_2 + 2\alpha_2 - \beta_1] \end{aligned}$$

We observe that if α_1 is replaced by $\alpha'_1 \ll \alpha_1$, both ϕ_1 and ϕ_2 rise in value. There

is an extra information cost implicit in this method however, inasmuch as extra computations were needed to obtain the value of subsets such as $v(\{1, 2\})$.

The ϕ_i can be used to calculate shadow prices or awards which are both consistent with the optimal production under current technology and provide an incentive for improvement. It should be noted that throughout this paper the discussion switches from costs to prices and profit allocations. If information and computation were free and all men had the same goal there would be no need to allocate many joint costs or revenues. It is suggested here that allocation, whether involving costs or revenues, is part of the same problem which is the utilization of these imputations for the appropriate incentives in a decentralized decision system.

Example 3. Incentives for Flexibility

Suppose a firm has two identical departments. One say, produces pink refrigerators, the other white ones. Let us furthermore suppose that they each have the same costs and face identical inelastic markets and each can more than cover total overheads. Thus:

$$\begin{aligned} v(\{\emptyset\}) &= v(\{1\}) = v(\{2\}) = v(\{3\}) = 0 \\ v(\{1, 2\}) &= v(\{1, 3\}) = x - c, & v(\{2, 3\}) &= 0 \\ v(\{1, 2, 3\}) &= 2x - c. \end{aligned}$$

As everything is symmetric for the two departments we expect and find that the imputation to both centers is the same. Suppose that there were a probability p that demand for both products would decrease, leaving both with excess capacity. Thus expected revenues are down symmetrically. Suppose, however, that new product entry has been decentralized; if one of the departments has a business plan ready to utilize the expected excess capacity while the other does not, the general managers know that the imputation scheme will acknowledge this, immediately any change in state occurs.

Example 4. Cost-Plus Internal Pricing

Under some methods of dividing joint profit an improvement instigated by one operation may not only not improve its own profit imputation but can actually have an adverse effect. An example is provided by "cost plus" pricing in a vertically integrated organization. Suppose there is a sales office and a factory. The factory produces a product not produced elsewhere, hence there is no "lowest priced alternative supply" method for establishing a price. A common prac-

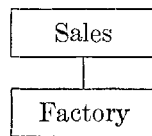


FIG. 2

tice is to use a cost plus formula. Suppose the sales office faces an inelastic demand for its product, hence at any (sufficiently low price) it will buy the same number from the factory. Say its selling price is π and that it has a fabricating, packaging or selling cost of k per unit. The cost at the factory is c per unit. The markup is $(1 + \Theta)$. Then if q units are sold

$$P_1 = (\pi - k - c(1 + \Theta))q$$

$$P_2 = c\Theta q.$$

Now suppose that the factory has a technological breakthrough which halves costs $c' = c/2$, the new imputation is:

$$P_1 = \left(\pi - k - \frac{c}{2} (1 + \Theta) \right) q$$

$$P_2 = \frac{c\Theta q}{2}$$

The innovator is penalized for his action. A manager whose bonus depends on the "profits" of his department might think twice before acting here.

$$v(\{\Theta\}) = v(\{1\}) = v(\{2\}) = 0$$

$$v(\{1, 2\}) = (\pi - k - c)q$$

$$\phi_1 = \phi_2 = \frac{(\pi - k - c)q}{2}$$

Any improvement by either is shared in this scheme.

Example 5. Inferior Goods

The next two examples envisage relatively complex relations between the components of the firm. If such relations exist they have to be known and their effects on profits coordinated for optimum behavior of the firm.

Consider a firm with three centers which produce and market, and with a headquarters whose expenses vary directly as the volume of business. Suppose that the first two sell products to the third, say potatoes, rice and meat. Suppose that a drop in the price of either of the first two will be more than compensated by the rise in revenues from the third. The initial characteristic function could be as follows:

$$v(\{\Theta\}) = v(\{i\}) = 0 \quad i = 1, \dots, 4$$

$$v(\{1, 2\}) = R_2 - C(q_2),$$

$$v(\{1, 3\}) = R_3 - C(q_3) \quad v(\{1, 4\}) = R_4 - C(q_4)$$

$$v(\{1, 2, 3\}) = R_2 + R_3 - C(q_2 + q_3)$$

$$v(\{1, 2, 4\}) = R_2 + R_4 - C(q_2 + q_4)$$

$$v(\{1, 3, 4\}) = R_3 + R_4 - C(q_3 + q_4)$$

$$v(\{1, 2, 3, 4\}) = R_2 + R_3 + R_4 - C(q_2 + q_3 + q_4).$$

All other coalitions not noted have a value of zero.¹⁶

Suppose that there is an important improvement in the technology for producing potatoes. If the manager of the potato board is in control of both technology and pricing he has a choice. He can introduce the efficiency and maintain his price. This replaces R_2 by \tilde{R}_2 where $\tilde{R}_2 > R_2$ and all other costs, levels of production and revenues remain the same. We can see from the calculation of the ϕ_i below that this is of benefit to the manager. He also can reduce the price of his product. This will reduce his individual net revenue vis-a-vis the market, however, if the executive office is able to gauge the overall effect of his action his assessment will be such that this will constitute his most profitable course. In a decentralized system we can imagine that, at least as a first approximation he can send a message stating that unless his minimal estimates of his effect on the values of the characteristic function are regarded as reasonable, he will merely maintain his price.

If there are strange complementarities or complex relationships between departments which are present, then it is reasonable to suspect that at least those most concerned will attempt to evaluate them. In general, such an attempt is not going to call for the re-evaluation of $2^n - 1$ values for coalitions, but for observations on a very limited number.

$$\begin{aligned} \phi_i = \frac{1}{4!} & [(0)!(3)!(v(4) - v(4 - \{2\})) + (1)!(2)!(v(\{i, j, k\}) - v(\{j, k\})) \\ & + (1)!(2)!(v(\{i, j, l\}) - v(\{j, l\})) + (1)!(2)!(v(\{i, k, l\}) - v(\{k, l\})) \\ & + (2)!(1)!(v(\{i, j\}) - v(\{j\})) + (2)!(1)!(v(\{i, k\}) - v(\{k\})) \\ & + (2)!(1)!(v(\{i, l\}) - v(\{l\})) + (0)!(3)!(v(\{i\}) - v(\{\Theta\}))] \\ \phi_1 = \frac{1}{2^4} & [6(R_2 + R_3 + R_4 - C(q_2 + q_3 + q_4)) + 2(R_2 + R_3 - C(q_2 + q_3)) \\ & + 2(R_2 + R_4 - C(q_2 + q_4)) + 2(R_3 + R_4 - C(q_3 + q_4)) \\ & + 2(R_2 - C(q_2)) + 2(R_3 - C(q_3)) + 2(R_4 - C(q_4))] \end{aligned}$$

In this example we will assume that

$$C(\sum_i q_i) = \sum_i q_i, \text{ hence:}$$

$$\begin{aligned} \phi_1 &= \frac{1}{2} (R_2 + R_3 + R_4 - (q_2 + q_3 + q_4)), \\ \phi_2 &= \frac{1}{2} (R_2 - q_2), \quad \phi_3 = \frac{1}{2} (R_3 - q_3), \quad \phi_4 = \frac{1}{2} (R_4 - q_4) \end{aligned}$$

Suppose the manager does not change price, then the values become:

$$\begin{aligned} \phi_1 &= \frac{1}{2} (\tilde{R}_2 + R_3 + R_4 - (q_2 + q_3 + q_4)), \\ \phi_2 &= \frac{1}{2} (\tilde{R}_2 - q_2), \quad \phi_3 = \frac{1}{2} (R_3 - q_3), \quad \phi_4 = \frac{1}{2} (R_4 - q_4) \end{aligned}$$

Now we consider the case where he cuts price. This changes his revenue by ΔR_2 . Suppose it has no effect on player 3 but sends up the revenue of 4 by ΔR_4 (where $|\Delta R_4| > |\Delta R_2|$), furthermore we assume that the output of player 2 is reduced by Δq_2 and the output of player 4 is raised by Δq_4 , these affect costs. The new value for ϕ_2 is given by:

$$\phi_2 = \frac{1}{2} (R_2 - q_2) + \frac{1}{24} [8(\Delta R_4 - \Delta R_2) + 8(\Delta q_2 - \Delta q_4) + 4(\tilde{R}_2 - R_2)]$$

The first term in the square bracket represents the effect of the overall changes in revenue upon the imputation to the second manager. The second term (which is negative here) measures the change in the structure of joint costs; and the third term takes account of the value of improvement even under conditions of absence of the fourth player (in which case price should not be cut and the second player would take in a revenue of \tilde{R}_2).

An example for which a price cut is marginally better is given. Suppose:

$$R_2 = 10, \quad q_2 = 4, \quad \tilde{R}_2 = 15, \quad \Delta R_2 = 5$$

$$\Delta q_2 = 1, \quad \Delta q_4 = 2 \quad \text{and} \quad \Delta R_4 = 12.$$

Initially $\phi_2 = \frac{1}{2}(10 - 4) = 3$.

If he puts in his improvement but does not cut price

$$\phi_2 = \frac{1}{2} (15 - 4) = 5\frac{1}{2}$$

If he cuts price:

$$\phi_2 = 3 + \frac{1}{24} [8(7) + 8(-1) + 4(5)] = 5\frac{5}{8}.$$

In the three instances the overall profits to the firm are respectively $v(\{1, 2, 3, 4\})$, $v(\{1, 2, 3, 4\}) + 5$ and $v(\{1, 2, 3, 4\}) + 6$.

The ϕ_i represent the final allotments, hence the actual assessments are obtained by subtracting the net revenues collected by each decision center from the ϕ_i .

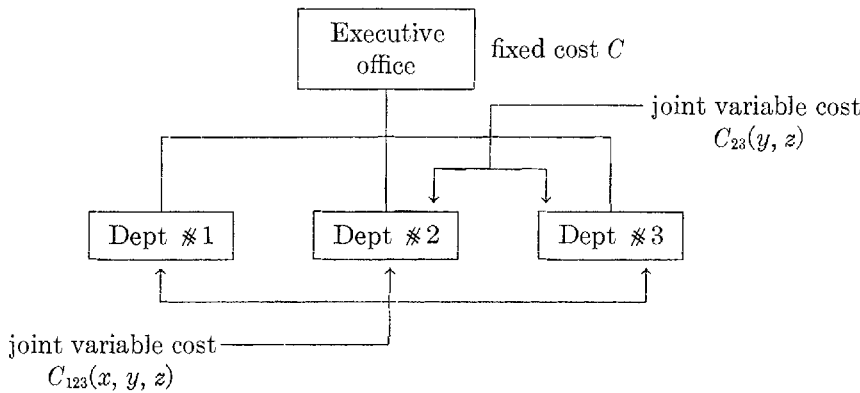


FIG. 3

Example 6. Joint Cost Upon Joint Costs

A further example where, if an attempt to impute joint costs might easily lead to an undesirable incentive system is indicated in Figure 3.

This example is not developed further here. It involves a straightforward application of the Shapley value to the characteristic function that can be easily written down.

It must be emphasized that in this case because of the complexity of the inter-relationship more joint knowledge of the characteristic function is needed. The firm is basically less decentralizable than others.

7. Conclusions

This paper has attempted to emphasize a decision-making point of view to any scheme designed to impute joint costs or interrelated revenues. These problems are not separable without a loss in terms of the use of a system for internal imputation as a means for control.

Although it does not appear that the computations required to calculate the necessary information concerning the characteristic function needed at various levels present a major problem; it is desirable that methods be designed to do so and that the costs and information flows involved in doing this be included in the model.