# ON ADDITIVE AND MULTIPLICATIVE MAGIC CUBES

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**Abstract.**

An additive magic cube is a cubical array containing different natural numbers such that the sum of the numbers along every row and every diagonal is the same. A multiplicative magic cube is cubical array containing mutually different natural numbers such that the product of the numbers along each row and diagonal is the same. In this paper we give several ways to construct additive and multiplicative magic cubes.

## 1. Introduction

Magic squares have fascinated people for centuries. An $n \times n$ *additive magic square* contains the natural numbers $1, 2, \ldots, n^2$, such that the sum of every rows, column and diagonal is the same. Figure 1 depicts a $3 \times 3$, a $4 \times 4$ and a $5 \times 5$ table each containing a different set of natural numbers in such a way that the product of the numbers in every row, column and diagonal is the same. (In the first table the product is $6^3$, in the second one is 7! and in the third one is 9!.) Such tables are called *multiplicative magic squares* (See [3]).

| 12 | 1 | 18 |
|----|----|----|
| 9 | 6 | 4 |
| 2 | 36 | 3 |

| 1 | 24 | 14 | 15 |
|----|----|----|----|
| 21 | 10 | 4 | 6 |
| 20 | 7 | 18 | 2 |
| 12 | 3 | 5 | 28 |

| 1 | 15 | 42 | 16 | 36 |
|----|----|----|----|----|
| 14 | 32 | 9 | 3 | 30 |
| 27 | 6 | 10 | 28 | 8 |
| 20 | 7 | 24 | 54 | 2 |
| 48 | 18 | 4 | 5 | 21 |

Fig. 1. Multiplicative magic squares

A *magic cube* is a natural generalization of a magic square. (In this paper we will call it an *additive magic cube*.) The first magic cube probably appeared about 1640 in a letter of *Pierre de Fermat* (see [1, p. 365]). Information and many interesting results about magic squares and cubes can be found in the references and web-pages.

An *additive magic cube* of order $n$ is a cubical array (3-dimensional matrix of order $n$)

$$\mathbf{M}_n = |\mathbf{m}_n(i,j,k); \quad 1 \leq i,j,k \leq n|$$

containing natural numbers $1, 2, 3, \ldots, n^3$ such that the sum of the numbers along every row and diagonal is the same, i.e. $\frac{n(n^3+1)}{2}$. By a row of a magic cube we mean an $n$-tuple of elements having the same coordinates in two places. (Note: We use the same term for a row or a column or a pilar.) Every additive magic cube of order $n$ has exactly $3n^2$ rows and 4 diagonals connecting the eight corners of the cube.

In [4] there are depicted additive magic cubes $\mathbf{M}_3$ and $\mathbf{M}_4$.

A *multiplicative magic cube* of order $n$ is a cubical array

$$\mathbf{Q}_n = |\mathbf{q}_n(i,j,k); \quad 1 \leq i,j,k \leq n|$$

containing $n^3$ mutually different natural numbers such that the product of the numbers along each row and every one of its four diagonals is the same. We call this product the *magic constant* and denote $\sigma(\mathbf{Q}_n)$.

Figure 2 shows $\mathbf{Q}_3$ with the magic constant $(2.3.5)^3$. The element $\mathbf{m}_3(1,1,1) = 18$ is contained in the rows $\{18, 20, 75\}$, $\{18, 300, 5\}$, $\{18, 60, 25\}$ and in the diagonal $\{18, 30, 50\}$.
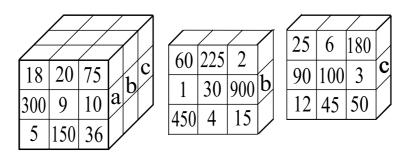


Fig. 2. Multiplicative magic cube $\mathbf{Q}_3$

In [4] it is proved that an additive magic cube $\mathbf{M}_n$ of order $n$ exists for every $n \neq 2$. If we know a construction of $\mathbf{M}_n = |\mathbf{m}_n(i,j,k)|$, then we can easily make a multiplicative magic cube

$$\mathbf{Q}_n = |\mathbf{q}_n(i,j,k) = 2^{\mathbf{m}_n(i,j,k)-1}; \ 1 \leq i,j,k \leq n|$$

with the magic constant $\sigma(\mathbf{Q}_n) = 2^{\frac{n(n^3-1)}{2}}$. This paper contains formulas for construction of magic cubes $\mathbf{M}_n$ and $\mathbf{Q}_n$ for all $n \neq 2$. Moreover, the constructed cubes $\mathbf{Q}_n$ have a significantly smaller magic constant than cubes constructed using (1).

We construct an additive magic cube $\mathbf{M}_n = |\mathbf{m}_n(i,j,k); 1 \leq i,j,k \leq n|$ of order $n$ and a multiplicative magic cube $\mathbf{Q}_n = |\mathbf{q}_n(i,j,k); 1 \leq i,j,k \leq n|$ of order $n$ for all $n \neq 2$ using the following formulas. We consider three cases ($n$ is an odd integer; if $n$ is an even integer, then we distinguish whether $n$ is or is not divisible by four.) The correctness of formulas for additive magic cubes follow immediately from the proofs in [4, 6].

We use the following notation:

$x \pmod{n}$    is the remainder in the division of $x$ by $n$,

$\overline{x} = n + 1 - x$,

$x^* = \min\{x, \overline{x}\}$,

$\widetilde{x} = \begin{cases} 0 & \text{for} \quad 1 \leq x \leq \frac{n}{2}, \\ 1 & \text{for} \quad \frac{n}{2} < x \leq n. \end{cases}$

**1.** If $n \equiv 1 \pmod 2$, then

$$\mathbf{m}_n(i,j,k) = \alpha n^2 + \beta n + \gamma + 1, \tag{1.1}$$

$$\mathbf{q}_n(i,j,k) = 2^\alpha . 3^\beta . 5^\gamma, \tag{1.2}$$

where

$\alpha = (i - j + k - 1) \pmod n$,

$\beta = (i - j - k) \pmod n$,

$\gamma = (i + j + k - 2) \pmod n$.

| 99 | 182 | 300 | 408 | 16 |
|---|---|---|---|---|
| 1456 | 75 | 102 | 4 | 792 |
| 600 | 816 | 1 | 198 | 364 |
| 204 | 8 | 1584 | 91 | 150 |
| 2 | 396 | 728 | 1200 | 51 |

| 26 | 540 | 952 | 80 | 33 |
|---|---|---|---|---|
| 135 | 238 | 20 | 264 | 208 |
| 1904 | 5 | 66 | 52 | 1080 |
| 40 | 528 | 13 | 270 | 476 |
| 132 | 104 | 2160 | 119 | 10 |

| 180 | 136 | 144 | 77 | 130 |
|---|---|---|---|---|
| 34 | 36 | 616 | 1040 | 450 |
| 9 | 154 | 260 | 360 | 272 |
| 1232 | 65 | 90 | 68 | 72 |
| 520 | 720 | 17 | 18 | 308 |

    *1st* layer           *2nd* layer           *3rd* layer

| 680 | 48 | 11 | 234 | 420 |
|---|---|---|---|---|
| 12 | 88 | 1872 | 105 | 170 |
| 22 | 468 | 840 | 1360 | 3 |
| 117 | 210 | 340 | 24 | 176 |
| 1680 | 85 | 6 | 44 | 936 |

| 112 | 55 | 78 | 60 | 1224 |
|---|---|---|---|---|
| 440 | 624 | 15 | 306 | 28 |
| 156 | 120 | 2448 | 7 | 110 |
| 30 | 612 | 56 | 880 | 39 |
| 153 | 14 | 220 | 312 | 240 |

     *4th* layer             *5th* layer

Fig. 3. Multiplicative magic cube $\mathbf{Q}_5$

If $n \not\equiv 0 \pmod 3$, then in every row and also in every diagonal $\mathbf{Q}_n$ constructed by (1.2) there is exactly one number which is divisible by the $z$th power but is not divisible by the $(z+1)$th power of the number 2 (3 or 5, respectively). We obtain a multiplicative magic cube $\mathbf{Q}_n$ with a smaller magic constant $\sigma(\mathbf{Q}_n)$ if in formula (1.2) we replace powers of 3 by the numbers $(2\beta+1)$ for $\beta = 1, 2, \ldots, n-1$ and the powers of 5 by the numbers $(2n+2\gamma-1)$ for $\gamma = 1, 2, \ldots, n-1$. Figure 3 shows five layers of $\mathbf{Q}_5$. (By a different substitution we can obtain a $\mathbf{Q}_5$ with a smaller magic constant).

**2**. If $n \equiv 0 \pmod 4$, then

$$\mathbf{m}_n(i,j,k) = \begin{cases} (i-1)\, n^2 + (j-1)\, n + k & \text{if} \quad \mathcal{F}(i,j,k) = 1, \\ (\overline{i}-1)\, n^2 + (\overline{j}-1)\, n + \overline{k} & \text{if} \quad \mathcal{F}(i,j,k) = 0; \end{cases} \tag{2.1}$$

$$\mathbf{q}_n(i,j,k) = \begin{cases} 2^{(i-1)}.3^{(j-1)}.5^{(k-1)} & \text{if} \quad \mathcal{F}(i,j,k) = 1, \\ 2^{(\overline{i}-1)}.3^{(\overline{j}-1)}.5^{(\overline{k}-1)} & \text{if} \quad \mathcal{F}(i,j,k) = 0, \end{cases} \tag{2.2}$$

where $\mathcal{F}(i,j,k) = (i + \widetilde{i} + j + \widetilde{j} + k + \widetilde{k}) \pmod 2$.

If $n \equiv 0 \pmod 4$, we construct a cube $\mathbf{Q}_n$ with a smaller magic constant $\sigma(\mathbf{Q}_n)$ using a method which we demonstrate in the following example. In Figure 4 there are depicted the four layers of $\mathbf{M}_4$ (constructed by (2.1)) whose numbers are the binary representation of the numbers $\mathbf{m}_4(i,j,k) - 1$.

| | | | |
|---|---|---|---|
| 000000 | 111110 | 111101 | 000011 |
| 111011 | 000101 | 000110 | 111000 |
| 110111 | 001001 | 001010 | 110100 |
| 001100 | 110010 | 110001 | 001111 |

| | | | |
|---|---|---|---|
| 101111 | 010001 | 010010 | 101100 |
| 010100 | 101010 | 101001 | 010111 |
| 011000 | 100110 | 100101 | 011011 |
| 100011 | 011101 | 011110 | 100000 |

<center>1<em>st</em> layer            2<em>nd</em> layer</center>

| | | | |
|---|---|---|---|
| 011111 | 100001 | 100010 | 011100 |
| 100100 | 011010 | 011001 | 100111 |
| 101000 | 010110 | 010101 | 101011 |
| 010011 | 101101 | 101110 | 010000 |

| | | | |
|---|---|---|---|
| 110000 | 001110 | 001101 | 110011 |
| 001011 | 110101 | 110110 | 001000 |
| 000111 | 111001 | 111010 | 000100 |
| 111100 | 000010 | 000001 | 111111 |

<center>3<em>rd</em> layer            4<em>th</em> layer</center>

<center>Fig. 4</center>

By closely examining Figure 4 you can find out that in every 4-tuple of numbers in any row or diagonal it holds that on the $z$th position,

$z = 1, 2, \ldots, 6$, there are exactly two ones and two zeroes. We use this fact in the construction. If $b_1 b_2 b_3 \ldots b_6$ is the representation of the number $\mathbf{m}_4(i, j, k)$ lowered by 1 in binary code, then

$$\mathbf{q}_4(i, j, k) = 2^{b_1} 3^{b_2} 4^{b_3} 5^{b_4} 7^{b_5} 9^{b_6}.$$

We have chosen the set $\{2, 3, 4, 5, 7, 9\}$ in such a way that it does not contain two nonempty subsets of numbers whose product is the same. The magic constant of the cube $\mathbf{Q}_4$ (see Figure 5) is $\sigma(\mathbf{Q}_4) = (2.3.4.5.7.9)^2 = 57\,153\,600$.

| 1 | 840 | 1080 | 63 |
|---|---|---|---|
| 1512 | 45 | 35 | 24 |
| 1890 | 36 | 28 | 30 |
| 20 | 42 | 54 | 1260 |

1*st* layer

| 2520 | 27 | 21 | 40 |
|---|---|---|---|
| 15 | 56 | 72 | 945 |
| 12 | 70 | 90 | 756 |
| 126 | 540 | 420 | 2 |

2*nd* layer

| 3780 | 18 | 14 | 60 |
|---|---|---|---|
| 10 | 84 | 108 | 630 |
| 8 | 105 | 135 | 504 |
| 189 | 360 | 280 | 3 |

3*rd* layer

| 6 | 140 | 180 | 378 |
|---|---|---|---|
| 252 | 270 | 210 | 4 |
| 315 | 216 | 168 | 5 |
| 120 | 7 | 9 | 7560 |

4*th* layer

Fig. 5. Multiplicative magic cube $\mathbf{Q}_4$

Because binary representation of elements (lowered by one) of the magic cube $\mathbf{M}_n$ fulfills the condition about the same number of ones and zeroes in the corresponding positions, we can generalize the given construction for all $n \equiv 0 \pmod 4$.

**3.** If $n \equiv 2 \pmod 4$ (in this case $\frac{n}{2}$ is odd and let $t = \frac{n}{2}$), then

$$\mathbf{m}_n(i, j, k) = \mathbf{d}(u, v)t^3 + \mathbf{m}_t(i^*, j^*, k^*), \tag{3.1}$$

$$\mathbf{q}_n(i, j, k) = 7^{\mathbf{d}(u,v)}.\mathbf{q}_t(i^*, j^*, k^*), \tag{3.2}$$

where

$\quad \mathbf{q}_t(i^*, j^*, k^*)$ is constructed using (1.2),

$\quad u = (i^* - j^* + k^*) \pmod t + 1$,

$\quad v = 4\widetilde{i} + 2\widetilde{j} + \widetilde{k} + 1$,

$\quad \mathbf{d}(u, v)$ for $1 \le u \le t$, $1 \le v \le 8$ is defined by the table in Fig. 6,
$\qquad (a = 1, 2, \ldots, \frac{n-6}{4})$.

|  | $\mathbf{d}(u,1)$ | $\mathbf{d}(u,2)$ | $\mathbf{d}(u,3)$ | $\mathbf{d}(u,4)$ | $\mathbf{d}(u,5)$ | $\mathbf{d}(u,6)$ | $\mathbf{d}(u,7)$ | $\mathbf{d}(u,8)$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{d}(1,v)$ | 7 | 3 | 6 | 2 | 5 | 1 | 4 | 0 |
| $\mathbf{d}(2,v)$ | 3 | 7 | 2 | 6 | 1 | 5 | 0 | 4 |
| $\mathbf{d}(3,v)$ | 0 | 1 | 3 | 2 | 5 | 4 | 6 | 7 |
| $\mathbf{d}(2a+2,v)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $\mathbf{d}(2a+3,v)$ | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

Figure 6

**Problem 1.** Using analogous formulas it is easy to make a computer program which constructs an additive and a multiplicative magic square for every $n \neq 2$.

**Problem 2.** Find a smaller magic constant for a multiplicative magic cubes of order $n$.

*Remark.* By the end of the 19-th century (see [1]) mathematicians began to consider also 4-dimensional magic cubes. But only in 2001 the following result was published:

**Theorem.** *An additive magic d-dimensional cube of order n exists if and only if $d > 1$ and $n \neq 2$ or $d = 1$.*

Similarly we can consider the existence of multiplicative magic $d$-dimensional cubes for any natural $d$. The constructions given in [6] allow us to experiment with magic cubes also in higher dimensional spaces.

# References

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