

Probability Bounds Analysis as a general approach to sensitivity analysis in decision making under uncertainty

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ABSTRACT

Engineers often perform sensitivity analyses to explore how changes in the inputs of a physical process or a model affect the outputs. This type of exploration is also important for the decision-making process. Specifically, engineers may want to explore whether the available information is sufficient to make a robust decision, or whether there exists sufficient uncertainty—i.e., lack of information—that the optimal solution to the decision problem is unclear, in which case it can be said to be *sensitive* to information state. In this paper, it is shown that an existing method for modeling and propagating uncertainty, called Probability Bounds Analysis (PBA), actually provides a general approach for exploring the global sensitivity of a decision problem that involves both probabilistic and imprecise information. Specifically, it is shown that PBA conceptually generalizes an approach to sensitivity analysis suggested in the area of decision analysis. The global nature of the analysis theoretically guarantees that the decision maker will identify any sensitivity in the formulated problem and information state. However, a tradeoff is made in the numerical implementation of PBA; a particular existing implementation that preserves the guarantee of identifying existing sensitivity is overly conservative and can result in “false alarms.” The use of interval arithmetic in sensitivity analysis is discussed, and additional advantages and limitations of PBA as a sensitivity analysis tool are identified.

INTRODUCTION

Engineers are often interested in how the conclusions of an analysis might change if the inputs or assumptions are changed. In general, such a process of exploring how changes in inputs affect outputs is called a *sensitivity analysis*. However, that specific phrase is used to describe several different methods in engineering applications.

In this paper, sensitivity analysis is defined such that its primary goal is to examine whether the optimal solution to the decision problem is sensitive to the current information state, focusing on not only what information the decision maker (hereafter abbreviated as DM) has available, but also on what information the decision maker lacks. By performing a sensitivity analysis, the DM is in essence checking whether his or her lack of knowledge affects what he or she decides.

In this paper, two approaches to uncertainty modeling and sensitivity analysis are compared. Specifically, the performance of Probability Bounds Analysis, or PBA [1], as a sensitivity analysis tool is examined via comparison with a decision analysis approach to sensitivity analysis [2-5], which will be referred to as DASA for brevity. This comparison will reveal advantages and disadvantages of both approaches.

A high-level view of PBA as a sensitivity analysis tool was presented by Ferson and coauthors [6, 7], and other existing work has compared the conclusions of DASA and PBA in the context of a specific environmentally benign design and manufacture example problem [8]. In this paper, the underlying, specific mathematical structure of the two approaches is compared.

UNCERTAINTY IN ENGINEERING DESIGN

Researchers in engineering design generally recognize two types of uncertainty: *imprecision* and *variability* (see [9] for an overview). Variability is a description of naturally random behavior in a physical process or property, and it is also referred to as aleatory uncertainty, objective uncertainty, and irreducible uncertainty in the literature. Probability theory is a natural model for this type of uncertainty.

Imprecision, sometimes called epistemic uncertainty, incertitude, reducible uncertainty, or subjective

uncertainty, is due to a lack of knowledge or information. There are many sources of imprecision in engineering, including simplified behavioral models, limited data regarding environmental factors, incompletely elicited preferences, and unknown physical relationships [10].

The distinction between variability and imprecision is not always clear philosophically, but in practice it is often useful to make such distinctions when modeling uncertainty [11]. More generally, it is useful to distinguish cases in which information is plentiful from cases in which information is scarce. For example, how can one manage the efficient collection of information without explicitly acknowledging an existing lack of information?

Engineers often recognize their lack of knowledge implicitly, even when they use mathematical formalisms such as precise probabilities that do not recognize this imprecision. For example, they often perform “what-if” analyses that compare the assumed scenario to different, but closely related scenarios.

Some mathematical formalisms, such as imprecise probabilities [e.g. 12] and evidence theory [13], express imprecision explicitly in addition to modeling variability, while other formalisms, such as classical, precise probabilities, do not distinguish the two types. The goal of this paper is to demonstrate how the explicit, simultaneous consideration of imprecise and probabilistic aspects of uncertainty can improve the sensitivity analysis process.

SUMMARY OF PROBABILITY BOUNDS ANALYSIS (PBA)

It is possible to represent both variability and imprecision separately using the formalisms of credal sets of probabilities [14-16] or *imprecise probabilities*, a theory suggested by various authors [12, 17-20] that extends traditional probability theory by allowing for intervals or sets of probabilities.

In general, imprecise probabilities present computational challenges. By imposing some additional restrictions, Ferson and Donald [1] have developed a formalism called *Probability Bounds Analysis* (PBA) that facilitates computation; Berleant and collaborators independently developed a similar approach [21, 22], and related methods were developed earlier for Dempster-Shafer representations of uncertainty [23]. Although PBA is not quite as expressive as imprecise probabilities, it can still represent both variability and imprecision, and it has been shown to be useful in engineering design [11, 24].

PBA essentially allows a DM to explore whether the optimal solution changes as the imprecisely characterized parameters are varied across some neighborhood of the DM's best-guess, or base values.

This is done while still preserving any well-founded probabilistic information about the problem.

PBA represents uncertainty using a structure called a probability-box, or p-box. Essentially, a p-box is an imprecise cumulative distribution function (CDF). Upper and lower CDF curves represent the bounds between which all possible probability distributions might lie. The commercially available software RAMAS Risk Calc 4.0 [25] provides one implementation of PBA computations by discretizing the p-box and then using algorithms, called *dependency bounds convolutions* (DBC) developed by Williamson and Downs [26] for the binary mathematical operations of addition, subtraction, multiplication, and division.

These algorithms are based on interval arithmetic [27] and result in bounds on the true probability distribution. They handle various dependence relationships between the uncertain quantities, including independence and unknown dependence. Methods for computing with and propagating p-boxes are compared, summarized, and extended by Bruns and Paredis [28]. In this paper, PBA is assumed to be performed using the DBC algorithms and the conclusions are intended specifically for PBA with DBC unless clearly stated otherwise.

TRADITIONAL SENSITIVITY ANALYSIS APPROACHES

At the highest level of abstraction, a sensitivity analysis is the study of how certain things influence other things. More concretely, Leamer [29] has defined a *global sensitivity analysis* as a systematic study in which “a neighborhood of alternative assumptions is selected and the corresponding interval of inferences is identified.” More specific to the context of engineering design, a sensitivity analysis can be viewed as the quantitative study of how the inputs to a model affect the outputs, where a model is defined broadly and includes all functions, calculations, and simulations.

There are two fundamental reasons for conducting a sensitivity analysis: to understand the reliability of conclusions and inferences drawn from an analysis (which will be called *sensitivity analysis for decision robustness*) and to focus future information collection efficiently on those aspects to which the problem is most sensitive (which will be called *sensitivity analysis for information prioritization*). Most of this paper deals with decision robustness, while information prioritization is considered in the discussion section.

PARTIAL DERIVATIVE APPROACHES

One form of sensitivity analysis looks at the partial derivatives of a function. For example, if a model is represented by Equation (1), it is possible to express the

sensitivity of the result to each input parameter in terms of partial derivatives, as shown in Equation (2):

$$f(a,b,c) = ab + a^2c \quad (1)$$

$$\frac{\partial f}{\partial a} = b + 2ac; \quad \frac{\partial f}{\partial b} = a; \quad \frac{\partial f}{\partial c} = a^2 \quad (2)$$

The partial derivatives represent the local sensitivity of the function to the three independent variables around some point $\{a,b,c\}$. Note that the partial derivatives can in general have different units, which makes direct comparisons question. More importantly, they fail to consider the quantity of available information.

For example, if the available knowledge is only that $a=[6,17]$, $b=2$, and $c=[1,2]$, then it is pretty clear from Equation (1) that the output is very sensitive to the imprecision in a , but the partial derivatives, shown in Equations (3)-(5) (evaluated using interval arithmetic), make it appear as if the greatest sensitivity is to c .

$$\frac{\partial f}{\partial a} = b + 2ac = [12, 68] \quad (3)$$

$$\frac{\partial f}{\partial b} = a = [6, 17] \quad (4)$$

$$\frac{\partial f}{\partial c} = a^2 = [36, 289] \quad (5)$$

However, this sensitivity is really a result of the lack of information about a . For a complete analysis of sensitivity, a DM must consider both the inherent system dynamics and the information state.

DETERMINISTIC APPROACHES

In the context of DASA, the word *deterministic* implies a lack of probability information, but it can still imply significant uncertainty. A typical deterministic sensitivity analysis is *nominal range sensitivity analysis* (NRSA), such as described by Frey and Patil [30]. In a NRSA, a single, best-guess value is made for an uncertain quantity, and then a bounded range of values in the neighborhood of the best-guess value is defined. The behavior of the system or decision problem is then explored by varying all of the parameters across these neighborhoods, or *nominal ranges* of values.

PROBABILISTIC APPROACHES

Other research in engineering design has focused on probabilistic sensitivity analysis [31-35]. These methods focus on identifying the largest contributors to probabilistic effects in the output. For example, one type of approach attempts to determine how much each input contributes to the statistical variance of the output. These methods are not exploring sensitivity to a lack of

information, but rather sensitivity to a physical reality of the problem or process being modeled. This analysis can provide important insight during the design process, such as indicating that a manufacturing process needs to be improved so that the variance of the thickness of parts is reduced to an acceptable level. However, such analyses do not help a DM explore the robustness of his or her decisions to the available information.

SENSITIVITY ANALYSIS IN DECISION ANALYSIS

Decision analysis [2-5] is a name given to a specific discipline and body of work that studies procedures, tools, and frameworks for transforming problems that are difficult to understand, solve, or explain, into problems that are more readily understood and solved. One goal of decision analysis is to model the impact of uncertain quantities on the decision, where an *uncertain quantity* is any parameter that involves uncertainty.

One of the motivations for decision analysis is that decision problems are difficult to formulate and solve. It is therefore necessary to simplify the decision problem by including and collecting detailed information about only those quantities that most significantly impact the decision outcome. To accomplish this, sensitivity analysis is applied in two phases: a deterministic phase and a probabilistic phase.

During the deterministic phase of DASA, a DM essentially performs a NRSA to identify which quantities have the greatest impact on the decision. Those that are found to be unimportant are simply replaced with best-guesses, and those that are important are explored further by defining probability distributions. One of the motivations for limiting the number of quantities for which probabilities are assigned is the significant difficulty and cost associated with eliciting such distributions [36]. However, no explicit allowance is made in decision analysis for the partial specification of these probabilities; it is assumed that the difficult and costly process of probability elicitation can be completed perfectly, a consequence of the philosophy of DASA that uncertainty can only be modeled with traditional probabilities [5, 37].

During the probabilistic phase, the DM explores the behavior of the decision. For example, some alternatives may be stochastically dominated, and therefore they may be eliminated. If there is no clear optimal decision at this stage, the DM continues to the informational phase of decision analysis. During this phase, the DM explores how reducing the variance in the probabilistic quantities would improve the value of the decision. It is the view of the authors that this type of information modeling confounds imprecision and variability [11], and that therefore it is more effective to consider information management [24] and sensitivity

analysis using the more general formalism of imprecise probabilities.

DEFINITIONS AND NOTATION

In this paper, uncertain quantities are divided into two categories. First, *imprecise quantities* will be denoted as x_i , with the set of all m imprecise quantities defined as $X = \{x_i\}_{i=1}^m$. Imprecise quantities are those about which the DM has limited information and consequently specifies as intervals. The intervals are defined such that the DM is highly confident that the true values are contained in the intervals.

Probability distributions can be defined for uncertain quantities that the DM believes are well modeled probabilistically, such as environmental noise fluctuations. These *probabilistic quantities* will be denoted as y_j , with the set of all n probabilistic quantities defined as $Y = \{y_j\}_{j=1}^n$.

It is also recognized that some uncertain quantities involve both variability and imprecision. Some of these quantities can be expressed as “nested” quantities. For example, assume that y_1 is normally distributed, such that $y_1 \sim N(\mu, \sigma^2)$. Thus, y_1 is a probabilistic quantity. Next assume that the mean μ of this normal distribution is not known precisely, such that μ is an imprecise quantity. Letting $x_1 = \mu$ one can rewrite y_1 as $y_1 \sim N(x_1, \sigma^2)$.

CHOOSING THE BEST ALTERNATIVE

The DM’s goal in decision analysis is to select the best (according to some objective function) action a^* from a set of k possible actions $A = \{a_i\}_{i=1}^k$. One way to identify an optimal action is to ignore all imprecision. This is accomplished by replacing all imprecise parameters with best guesses, or *base values*. For example, X is replaced with the vector of base values $\hat{X} = \{\hat{x}_1, \dots, \hat{x}_m\}$. Nested uncertainties, such as the example $y_1 \sim N(x_1, \sigma^2)$ from the previous section, are replaced with precise, probabilistic quantities, such as $y_1 \sim N(\hat{x}_1, \sigma^2)$, where \hat{x}_1 is the best guess or base value for the imprecise quantity x_1 . The replacement of X by \hat{X} removes all imprecision from the problem formulation, so the problem can be solved using standard approaches such as expected utility maximization. For example, if the objective function for a problem is $f(X, Y, a)$, then the optimization can be formulated as in Equation (6), where a^* is the identified “optimal” action.

$$a^* = \arg \max_{a \in A} [f(\hat{X}, Y, a)] \quad (6)$$

Note that for utility analysis, $f(X, Y, a)$ would be the expected utility function. Naturally, a DM should be concerned with the sensitivity of the “optimality” of a^* to

the state of available information, a question for deterministic, or nominal range sensitivity analysis.

ONE-WAY DETERMINISTIC SENSITIVITY ANALYSIS

During the deterministic phase of DASA, it is assumed that the DM can specify a nominal range for the imprecise parameters, such as given by Equation (7).

$$x_i = [\underline{x}_i, \bar{x}_i] \text{ for } i = 1 \dots m \quad (7)$$

The sensitivity analysis step of the decision process explores how varying the values of the x_i ’s across their ranges affects the optimal decision—a nominal range sensitivity analysis (NRSA). One convenient way of performing a NRSA for a selection decision is to evaluate the sensitivity of the decision outcome graphically using tornado diagrams, as in Figure 1 [3, 4, 38]. A tornado diagram allows a DM to perform a one-way NRSA—to explore the effects of uncertain parameters one at a time.

A simple tornado diagram, such as shown in Figure 1, compares a single action (for example a_1) to another action (for example a_2), where it is assumed that action a_2 is entirely robust to the imprecision; that is, the expected utility of a_2 is completely independent of $X = \{x_1, \dots, x_m\}$. The perfectly known expected utility of action a_2 is represented by the large vertical, dotted line on the tornado plot.

Also displayed on the tornado diagram is a bar for each imprecise quantity, where each bar represents the range of the expected utility of action a_1 across the range of an imprecise quantity. Each bar is created by plotting the output while taking one imprecise quantity x_i and varying it from its lower limit \underline{x}_i to its upper limit \bar{x}_i with all other quantities held constant at their base values. The distinction “one-way sensitivity analysis” is made to emphasize that the sensitivity is examined one quantity at a time. This is repeated for each imprecise quantity. The bars are then reordered from largest (at the top) to smallest (at the bottom), thus creating the shape that gave rise to the name “tornado diagram.”

If any of the bars of the tornado plot intersect the dotted

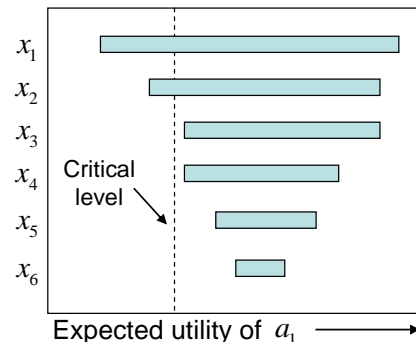


Figure 1. Sample tornado diagram, one imprecisely known alternative

line corresponding to the expected utility of a_2 , then the decision is sensitive to the imprecision. Recall that each bar represents the range of expected utilities that can be realized for action a_1 depending on where in the imprecise region $x_i = [\underline{x}_i, \bar{x}_i]$ the true value of x_i lies. When a bar crosses the dotted line, it means that depending on where the true x_i lies, either a_1 or a_2 can yield a higher expected utility.

MORE ADVANCED ONE-WAY ANALYSIS

Tornado diagrams can be generalized to situations with more than two alternative actions, including cases in which there are multiple alternatives whose expected utilities are functions of imprecise parameters, although this is not addressed explicitly in the existing literature. One direct way to accomplish this is to consider the differences in expected utility between alternatives.

For example, assume that two alternatives (a_1 and a_2) are available. Let x_3 represent the ambient temperature in which the system will operate, and assume that this temperature is known imprecisely, such that $x_3 = [\underline{x}_3, \bar{x}_3]$. Next assume that the expected utility of each action (a_1 and a_2) depends on temperature. The changes in the expected utilities is in a sense coordinated with the changes in x_3 . Assuming that the action chosen has no effect on the resolution of the imprecision in x_3 , then whichever action is chosen, it would face the same true value of x_3 —i.e., the same ambient temperature. The expected utilities of two actions should therefore be compared pair-wise, using the same values for x_3 . In the ambient temperature example, this means that the expected utility for a_1 at low temperatures (e.g. $x_3 = \underline{x}_3$) should be compared to the expected utility of a_2 at low temperatures (e.g. $x_3 = \underline{x}_3$).

More generally, the tornado diagram should be constructed by comparing the differences of expected utility across the range of the imprecise parameters, written mathematically as $E[u(a_1) - u(a_2) | x_3]$. When these tornado diagrams are analyzed, the critical value for comparison is a difference in expected utility of zero, as shown in Figure 2. If a bar of the diagram crosses

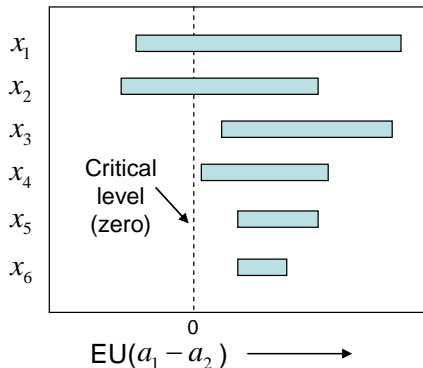


Figure 2. Sample tornado diagram for comparing alternatives

zero for some x_i , then given the existing imprecision in that x_i , the decision is one-way sensitive to that imprecise quantity. This means that the difference in expected utility could be positive or negative; it is not clear which alternative yields the higher expected utility.

BEYOND ONE-WAY SENSITIVITY ANALYSIS

A one-way sensitivity analysis can be generalized to a 2-way, 3-way, and so on up to a m -way (or an all-way, global sensitivity) analysis. A m -way analysis considers all m imprecise quantities at the same time, effectively considering all possible combinations of resolutions of the imprecision. However, due to computational costs and visualization challenges, decision-makers are often limited to performing one-way sensitivity analyses in decision analysis. The mathematical formulation of one-way and m -way (global) sensitivity analyses are presented and compared in the following section.

MATHEMATICAL FORMULATION OF GLOBAL SENSITIVITY ANALYSIS

In the tornado diagram approach to sensitivity analysis, intervals for each imprecise quantity were defined explicitly in terms of their bounds (or endpoints), as in Equation (7). It will now be convenient to define these intervals with reference to the base values. For a particular imprecise quantity, the interval of possible values that are consistent with the available information (denoted \bar{x}_i) is given by Equation (8).

$$\bar{x}_i = \hat{x}_i + \Delta x_i \text{ where } \Delta x_i = [\underline{x}_i - \hat{x}_i, \bar{x}_i - \hat{x}_i] \quad (8)$$

For the entire vector of imprecise quantities, the *consistent region* \bar{X} is given by Equation (9).

$$\bar{X} = \hat{X} + \Delta X, \text{ where } \Delta X = \{\Delta x_1, \dots, \Delta x_m\} \quad (9)$$

It is assumed that the objective G is to be maximized during a design decision, and G is a function of the chosen design action a (where a can be considered some alternative or some set of design variables) and the imprecise quantities, e.g. $G = f(X, a)$.

During the first stage of decision analysis, the DM considers only the base values for the imprecise quantities and therefore solves the optimization problem given in Equation (10).

$$a^* = \arg \max_a (f(\hat{X}, a)) \quad (10)$$

During the sensitivity analysis stage, the DM's goal is to explore how this optimal solution may change as the imprecise quantities vary from their best guess values of \hat{X} . The optimal solution a^* found in Equation (10) will be insensitive to the imprecision if it is the optimal for all $X \in \bar{X}$. Because the sensitivity is considered for all

points in the consistent region, this type of sensitivity analysis is called *global* sensitivity analysis.

A global sensitivity analysis can be performed in the following way. First, the DM defines a new quantity $\Delta G_{*,j}(X)$, the difference in objective value between the optimal solution a^* from Equation (10) and the objective value for a particular alternative a_j , as in Equation (11).

$$\Delta G_{*,j}(X) = f(X, a^*) - f(X, a_j) \quad (11)$$

The optimal solution a^* is insensitive to the imprecision if $\Delta G_{*,j}(X)$ is non-negative for all $X \in \bar{X}$ and all $a_j \in A$. This leads to the global optimization problem formulated in Equation (12).

$$\Delta_{\min} G_{*}(X) = \min_{\substack{a_j \in A, \\ X \in \bar{X}}} (f(X, a^*) - f(X, a_j)) \quad (12)$$

For future comparisons, it is useful to note that the global analysis defined in Equation (12) can be broken into two steps. First, the optimization shown in Equation (13) can be performed for each of the actions $a_j \in A$.

$$\Delta_{\min} G_{*,j}(X) = \min_{X \in \bar{X}} (f(X, a^*) - f(X, a_j)) \quad (13)$$

In the second step, the minimum is taken across all of these values as in Equation (14), where each value corresponds to some particular alternative a_j .

$$\Delta_{\min} G_{*}(X) = \min_{a_j \in A} (\Delta_{\min} G_{*,j}(X)) \quad (14)$$

This is equivalent to the formulation in Equation (12).

More generally, it is useful to define the difference in objective value between two alternatives as in Equation (15), with bounds given in Equations (16) and (17).

$$\Delta G_{i,j}(X) = f(X, a_i) - f(X, a_j) \quad (15)$$

The minimum and maximum of this function are defined in the following equations.

$$\Delta_{\min} G_{i,j}(X) = \min_{X \in \bar{X}} (f(X, a_i) - f(X, a_j)) \quad (16)$$

$$\Delta_{\max} G_{i,j}(X) = \max_{X \in \bar{X}} (f(X, a_i) - f(X, a_j)) \quad (17)$$

Since it is required that $\Delta_{\min} G_{i,j}(X) \leq \Delta_{\max} G_{i,j}(X)$, there are three scenarios¹ of sensitivity:

1. $\Delta_{\min} G_{i,j}(X) \geq 0$: selection of a_i over a_j is robust.
2. $\Delta_{\min} G_{i,j}(X) < 0$ and $\Delta_{\max} G_{i,j}(X) > 0$: the selection of a_i or a_j is sensitive to the lack of information.
3. $\Delta_{\max} G_{i,j}(X) \leq 0$: selection of a_j over a_i is robust.

¹ Note that if $\Delta_{\max} = \Delta_{\min} = 0$ then the alternatives yield identical, precise performance and it does not matter which is chosen.

COMPARING ONE-WAY AND GLOBAL ANALYSIS

The goal of this section is to compare the formulation of global sensitivity analysis described in the previous section with the one-way, tornado diagram analysis.

We next develop the mathematical equivalent of the general one-way sensitivity analysis using tornado diagrams. For example, consider a comparison between just two alternatives, a_1 and a_2 , as was shown in Figure 2 for the scenario of six imprecise quantities, or $X = \{x_1, \dots, x_6\}$. The question of interest is whether $\Delta G_{1,2}(X)$ is always positive, always negative, or overlaps zero.

Unlike in global analysis, the imprecise quantities are considered individually, one at a time. Consequently, each bar in the tornado diagram is only an estimate of $\Delta G_{1,2}(X)$, and as such the endpoints of each bar are also only estimates of the true upper and lower bounds.

For example, when the bar in the tornado plot for x_1 is computed, only the imprecision in x_1 is considered. This means that all the other imprecise quantities are fixed at their best-guess values ($x_i = \hat{x}_i$ for all $i \neq 1$), while x_1 is allowed to take on all values in the consistent region \bar{x}_1 . Mathematically, the lower and upper endpoints of the bar are given in Equations (18) and (19) respectively for the general case of m imprecise quantities.

$$\min_{x_1 \in \bar{x}_1} (f(x_1, \hat{x}_2, \dots, \hat{x}_m), a_1) - f((x_1, \hat{x}_2, \dots, \hat{x}_m), a_2) \quad (18)$$

$$\max_{x_1 \in \bar{x}_1} (f(x_1, \hat{x}_2, \dots, \hat{x}_m), a_1) - f((x_1, \hat{x}_2, \dots, \hat{x}_m), a_2) \quad (19)$$

For simplicity of notation, we now consider the minimization problem for the case with $X = \{x_1, x_2\}$. In this case, the lower bound on x_1 is given by a reduced form of Equation (18), shown in Equation (20).

$$\min_{x_1 \in \bar{x}_1} (f(x_1, \hat{x}_2), a_1) - f((x_1, \hat{x}_2), a_2) \quad (20)$$

Similarly, the lower bound for x_2 would be given by Equation (21).

$$\min_{x_2 \in \bar{x}_2} (f(\hat{x}_1, x_2), a_1) - f((\hat{x}_1, x_2), a_2) \quad (21)$$

Leaving one-way analysis and returning to the global sensitivity analysis problem, the general form in Equation (13) reduces to Equation (22) in this example.

$$\min_{\substack{x_1 \in \bar{x}_1, \\ x_2 \in \bar{x}_2}} (f(x_1, x_2), a_1) - f((x_1, x_2), a_2) \quad (22)$$

The following question is now posed: is there any way to get from a one-way sensitivity analysis to a global sensitivity analysis? Mathematically, this question

translates to: is it possible to solve Equation (22) using just the results from Equations (20) and (21)?

To see that the answer is *no*, consider the following example. It is entirely possible that for some function $f(x_1, x_2)$, the optimal in the global problem of Equation (22) occurs at the point (x_1, x_2) where $x_1 = \hat{x}_1 + \delta_1, x_2 = \hat{x}_2 + \delta_2, \delta_1 \neq 0, \delta_2 \neq 0, \hat{x}_1 + \delta_1 \in \bar{x}_1,$ and $\hat{x}_2 + \delta_2 \in \bar{x}_2$. However, neither Equation (20) nor Equation (21) ever considers this point. As such, a one-way analysis can fail to identify the true lower bound of difference. It is a trivial extension (replace the minimizations with maximizations) to show the same for the upper bound. Consequently, it is quite possible that a one-way analysis fails to identify a global sensitivity, as shown in the following, specific example.

NUMERICAL AND GRAPHICAL EXAMPLE

We now consider a specific example. First, assume that the decision involves two imprecise quantities $x_1 = [3,5]$ and $x_2 = [2,7]$, with best-guess base values of $\hat{x}_1 = 4$ and $\hat{x}_2 = 5$. Assume that the difference in expected utility between the two possible alternatives, a_1 and a_2 , is given by Equation (23).

$$\Delta G_{1,2}(X) \equiv g(x_1, x_2) = 2x_1 - 3x_2 - 3 \quad (23)$$

The goal of sensitivity analysis is to determine if the preference of one alternative over another is sensitive to the existing the lack of information. This requires the DM to find the interval of difference in expected utility that is consistent with available information. This is equivalent to finding the minimum and maximum of the expected utility in the consistent region—an optimization-like problem.

The consistent region for this decision is shown in Figure 3. Also shown (by the solid lines) is the set of points that a one-way analysis would consider. Because one-way NRSA only considers points on the lines, it will only reveal the extrema of the objective that fall on these lines.

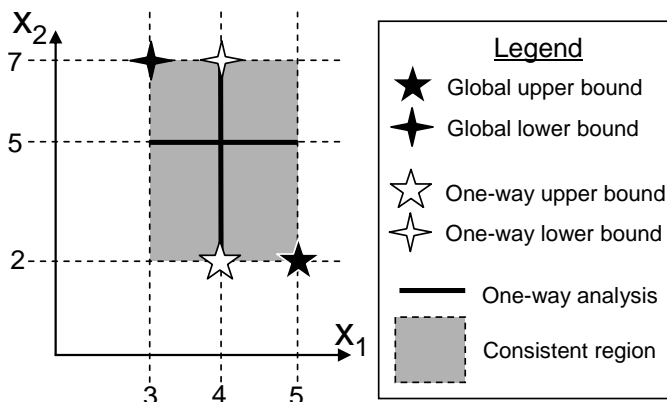


Figure 3. Two dimensional imprecise parameter space example problem

consistent region do not fall on these lines, one-way NRSA will not identify the correct intervals. This can lead to an incorrect conclusion about the sensitivity of the problem to imprecision.

For example, along the one-way lines, the minimum of $g(x_1, x_2)$ is found to be -16 at the point (4,7). The maximum is found to be -1 at (4,2). Since the maximum is less than zero, the one-way sensitivity analysis concludes that a_2 is better than a_1 , and that this difference is *insensitive* to the imprecision.

However, the global minimum across the entire consistent region is found to be -18 at the point (3,7), and the global maximum is 1 at the point (5,2). Since the true maximum is greater than zero, the range crosses zero and the decision actually *is* sensitive to the imprecision, which contradicts the conclusion of the one-way analysis.

SUMMARY OF ONE-WAY ANALYSIS RESULTS

The preceding simple example illustrates that the use of only one-way analysis can lead to an incorrect conclusion. The results and examples presented in this section may not be that astonishing in and of themselves; if one does not search the full range of possibilities, one could miss a significant pointing that space. In the next section, the framework established in this section will be used to show how interval arithmetic makes it possible to search the entire consistent region, thereby avoiding this problem. Subsequently, the result will be extended to a case that involves imprecision and variability.

INTERVAL ARITHMETIC AS AN APPROACH TO SENSITIVITY ANALYSIS

Intervals are actually a specific case of a p-box, and PBA using DBC is based on interval arithmetic. Consequently, it is useful to consider interval arithmetic as a sensitivity analysis.

INTRODUCTION TO INTERVAL ARITHMETIC

Interval arithmetic refers to arithmetic operations that are defined on sets of intervals rather than on sets of real numbers [27]. An interval is a bounded set of real numbers, such that the interval $[a,b]$ refers to the set of all $x \in \mathbb{R}$ such that $a \leq x \leq b$. The basic operations of interval arithmetic are straightforward to define, such as shown in Equations (24)-(27).

$$[a,b] + [c,d] = [a + c, b + d] \quad (24)$$

$$[a,b] - [c,d] = [a - d, b - c] \quad (25)$$

$$[a,b] * [c,d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)] \quad (26)$$

$$1/([a,b]) = [1/b, 1/a] \text{ if } b < 0 \text{ or } a > 0 \quad (27)$$

The more difficult problem is how to implement interval arithmetic on computers. Modern techniques for interval arithmetic began with the work of R. E. Moore [39]. A desirable characteristic of interval methods is that they be rigorous, where the term *rigorous* describes the bounds of the intervals; bounds on a quantity are *rigorous* if the true value of the quantity actually lies between the bounds.

For example, let ε be a small positive number, and let $x = [a,b] + [c,d]$. Equation (24) gives the true bounds on x as $[a+c, b+d]$, meaning that $x \geq a+c$ and $x \leq b+d$. However, the interval $[a+c-\varepsilon, b+d+\varepsilon]$ is also a rigorous description of x , because the relationships $x \geq a+c-\varepsilon$ and $x \leq b+d+\varepsilon$ are also true. However, these bounds are not the *best-possible* (or tightest) bounds, because they include other points that actually are not in the true output interval.

INTERVAL ARITHMETIC AS A SENSITIVITY ANALYSIS

One goal of interval arithmetic algorithms is to ensure that when given correct input intervals, the calculated output intervals are rigorous (given the input intervals) without being unnecessarily large. Ideally, interval arithmetic algorithms would find the exact minimum and maximum of a function given particular interval inputs. For example, given two interval inputs $x_1 = [\underline{x}_1, \bar{x}_1]$ and $x_2 = [\underline{x}_2, \bar{x}_2]$, interval arithmetic can be used to evaluate bounds on the value of a function, such as $g(x_1, x_2) = 2x_1 - 3x_2 - 3$, which is the same function given in Equation (23). Finding the bounds on $g(x_1, x_2)$ given the interval inputs is equivalent to solving the optimization problems formulated in Equations (28) and (29) for the lower bound (\underline{g}) and upper bounds (\bar{g}), respectively.

$$\underline{g} = \min_{\substack{x_1 \in [\underline{x}_1, \bar{x}_1] \\ x_2 \in [\underline{x}_2, \bar{x}_2]}} (2x_1 - 3x_2 - 3) \quad (28)$$

$$\bar{g} = \max_{\substack{x_1 \in [\underline{x}_1, \bar{x}_1] \\ x_2 \in [\underline{x}_2, \bar{x}_2]}} (2x_1 - 3x_2 - 3) \quad (29)$$

The formulation of interval arithmetic in Equations (28) and (29) is equivalent to the first step of global sensitivity: optimizing over the consistent region, as formulated in Equations (16) and (17). One merely needs to recognize that the intervals $x_1 = [\underline{x}_1, \bar{x}_1]$ and $x_2 = [\underline{x}_2, \bar{x}_2]$ define a consistent region \underline{X} as in Equation (22).

Using the input intervals defined in the previous example ($x_1 = [3,5]$ and $x_2 = [2,7]$), the sensitivity analysis for $g(x_1, x_2)$ can now be repeated using interval arithmetic rather than an explicit optimization formulation. Referring back to the basic interval operations defined in

Equations (24)-(27), the problem can be solved as follows.

$$\begin{aligned} \Delta G_{1,2}(X) &= g(x_1, x_2) = 2x_1 - 3x_2 - 3 = \\ &= 2 \cdot [3,5] - 3 \cdot [2,7] - 3 \\ &= [6,10] - [6,21] - 3 \\ &= [6-21, 10-6] - 3 = [-15, 4] - 3 \\ &= [-18, 1] \end{aligned} \quad (30)$$

The bounds found in Equation (30) are identical to those found using the global sensitivity analysis approach in the previous section, and they clearly indicate that the interval of expected utility crosses zero. This means that depending on where in the input intervals the true values fall, the "optimal" alternative can change.

LIMITATIONS OF INTERVAL ARITHMETIC

Interval arithmetic algorithms are adept at finding rigorous bounds on a function with interval inputs. However, the bounds are not necessarily best-possible. Recall that if the true bounds on a quantity are $[a,b]$, then bounds such as $[a-\varepsilon, b+\varepsilon]$ for $\varepsilon > 0$ are rigorous (since they contain the true interval), but they are not best-possible because they include points—namely $[a-\varepsilon, a)$ and $(b, b+\varepsilon]$ —that are not in the true interval.

The bounds found using interval arithmetic may not be best-possible due to what is often referred to as the *repeated variable* problem. For example, consider a functions shown in Equations (31) and (32).

$$f_1(x_1, x_2) = x_1 \cdot x_2 - x_1 \quad (31)$$

$$f_2(x_1, x_2) = x_1 \cdot (x_2 - 1) \quad (32)$$

In actuality, $f_1(x_1, x_2)$ is equivalent to $f_2(x_1, x_2)$, and traditional operations over real numbers preserve this equivalence via the distributive property. However, in interval arithmetic, this equivalence may not be preserved.

For example, let $x_1 = [1,2]$ and $x_2 = [3,5]$. Applying the relationships of Equations (24) and (26), the following results are found.

$$\begin{aligned} f_1(x_1, x_2) &= x_1 \cdot x_2 - x_1 \\ &= [1,2] \cdot [3,5] - [1,2] \\ &= [3,10] - [1,2] \\ &= [1,9] \end{aligned} \quad (33)$$

$$\begin{aligned} f_2(x_1, x_2) &= x_1 \cdot (x_2 - 1) \\ &= [1,2] \cdot ([3,5] - 1) \\ &= [1,2] \cdot [2,4] \\ &= [2,8] \end{aligned} \quad (34)$$

A quick, constrained optimization approach reveals that the true bounds for both functions is [2,8]. For function $f_1(x_1, x_2)$, which contains the repeated variable, interval arithmetic yields bounds that are rigorous (since $[2, 8] \subset [1, 9]$), but not best-possible.

Because interval arithmetic can lead to bounds on functions that are correct but wider than necessary, inferences drawn during sensitivity analysis can be incorrect. However, these errors are always in the direction of over-conservativeness, meaning that the DM can conclude that the decision is sensitive when it is not, but the DM will never conclude that the decision is not sensitive when it actually is. This point is revisited in the discussion section.

SENSITIVITY ANALYSIS WITH IMPRECISE AND PROBABILISTIC QUANTITIES

In the previous sections, it was assumed that there were no probabilistic parameters. In this section, that assumption is lifted. It is in the context of having both imprecision and probabilistic uncertainty that the value of PBA is clearest. This combined analysis is not performed in traditional DASA, which assumes that precise probabilities are the only description of uncertainty, as noted earlier. This scenario of analysis is first presented as an extension to the deterministic, NRSA phase of DASA, and then it is shown how PBA generalizes this approach.

EXTENSION OF NRSA TO COMBINED ANALYSIS

From one perspective, sensitivity analysis for decision robustness with probabilistic quantities is fundamentally equivalent to sensitivity analysis without probabilistic quantities. Note that the analysis intended in this section is different from probabilistic sensitivity analysis approaches referenced earlier that seek to identify the largest contributors to probabilistic effects in the output. Here the emphasis is on the sensitivity to lack of knowledge (i.e. imprecision) about the probabilistic components. However, the presence of probabilistic quantities does not change the basic process; it only changes the function that needs to be evaluated in the decision problem, since now mathematical expectations need to be calculated over the probability distributions.

For example, consider again the example $y_1 \sim N(x_1, \sigma^2)$. This means that y_1 is a probabilistic quantity that is normally distributed with imprecise parameter x_1 as the mean and the known σ^2 as the variance.

One can now think of applying an NRSA for x_1 ; when a NRSA is performed, a single value from the consistent region of each imprecise quantity is considered at a time. Consequently, for each combination of imprecise parameters (whether one-way, two-way, or more

analysis), a particular precise distribution is specified and the expected value can be calculated in the standard manner. This process is then repeated for multiple (discretized) values, and the results of these calculations are aggregated into an interval of expected utility.

Another way of viewing this process is to consider a function $\Delta G_{1,2}(X, Y)$ to be the difference in expected utility (or other objective) between alternatives a_1 and a_2 , given the imprecise variables X and probabilistic quantities Y . Assuming there is one probabilistic quantity and one imprecise quantity such that $y_1 \sim N(x_1, \sigma^2)$, and assuming that the utility function is $f(y_1, a_i)$, one finds Equation (35).

$$\Delta G_{1,2}(X, Y) = \int [f(y_1, a_1) - f(y_1, a_2)] \cdot p_{\bar{x}_1}(y_1) \cdot dy \quad (35)$$

Note that $p_{\bar{x}_1}(y_1)$ is not a joint probability density function, but rather a set of density functions; specifically, the set of normal (Gaussian) density functions with variance σ^2 and means $\mu \in \bar{x}_1$ for this example. A NRSA approach considers all of the values $x_1 \in \bar{x}_1$, just as it would for any other function $\Delta G_{1,2}(X, Y)$. The added wrinkle here is that each value of x_1 results in a different density function being used for the expectation.

Mathematically, this does not change the solution process; $\Delta G_{1,2}(X, Y)$ is still just a function of the inputs. However, it does make a difference when formulating the problem, because this extended NRSA approach requires the form of $p_{\bar{x}_1}(y_1)$ to be known; without specifying $p_{\bar{x}_1}(y_1)$, there is no way to evaluate Equation (35). While the NRSA could be repeated for various distributions, this would add to the already burdensome computational cost of performing a brute force analysis over the entire consistent region. Fortunately, PBA provides a more efficient and flexible approach to sensitivity analysis with imprecise and probabilistic quantities.

PBA AS A GENERALIZED SENSITIVITY ANALYSIS FOR DECISION ROBUSTNESS

By modeling and propagating uncertainty using PBA, a DM can achieve an all-way sensitivity analysis for decision robustness (identifying whether the decision is sensitive to the existing imprecision), in essence combining aspects of the deterministic, probabilistic, and informational phases of decision analysis.

PBA as a sensitivity analysis

In the previous section, the case of an imprecisely characterized probabilistic parameter was considered. The presented model, $y_1 \sim N(x_1, \sigma^2)$, is actually a p-box. Specifically, it is the family of normal distributions with means $x_1 \in \bar{x}_1$ and standard deviation σ^2 . For

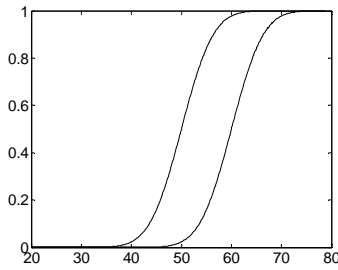


Figure 4. Example p-box for y_1

example, if $x_1 = [50, 60]$ and $\sigma^2 = 5$, then the uncertainty in y_1 can be expressed with the p-box shown in Figure 4.

Calculations based on PBA preserve both the interval and probabilistic forms of uncertainty. The dependency bounds convolutions (DBC) methods for implementing PBA allow for rigorous calculations with p-boxes for the principle binary operators of addition, subtraction, multiplication, and division (as long as the value zero is not in the consistent region of the denominator) [1, 21, 26]. Similar to rigorous interval calculations, a p-box calculation is rigorous if the following is true: given that the true input distributions are contained in the input p-boxes, then the true output distribution is included in the output p-box.

In this way, PBA generalizes sensitivity analysis for decision robustness. The output p-box contains all distributions that are consistent with the available information, and the calculation of the expected value over a p-box leads to an interval of expected utilities. This interval comprises the individual expected values that correspond to each distribution in the p-box.

Because the output p-box for utility is guaranteed (using DBC) to include all of the distributions consistent with the available information (assuming the inputs are correctly defined), the output interval of expected utility is guaranteed to include the true interval. Consequently, if the true interval crosses the critical value², the output interval will, too. Thus, any sensitivity of the optimal action to the imprecision will be detected. If the chosen action's output expected utility interval does not cross the critical value and all inputs were rigorously defined, then the optimality of the chosen action is robust to imprecision.

Example analysis with probabilistic and imprecise quantities

A new model is considered now, with $y_1 \sim N(x_1, (x_2)^2)$, where $x_1 = [50, 60]$, $x_2 = [3, 7]$, $\hat{x}_1 = 55$, and $\hat{x}_2 = 5$. In other words, y_1 is normally distributed with imprecisely

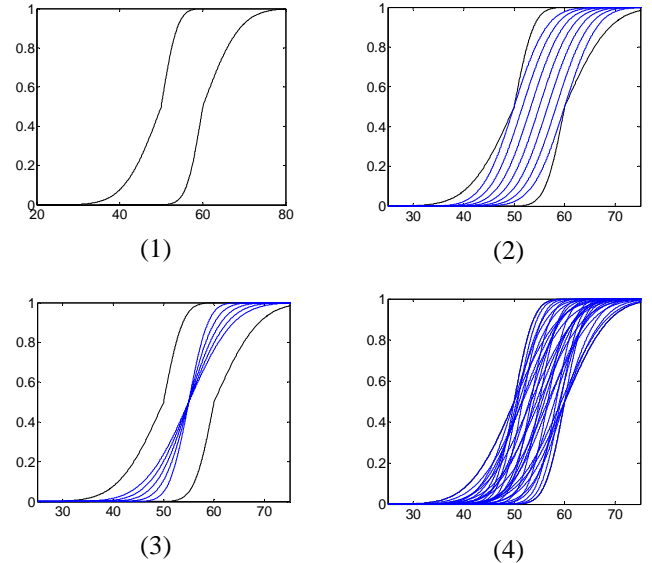


Figure 5. Distributions considered in sensitivity analysis

known mean x_1 and standard deviation x_2 . The corresponding p-box is shown in panel (1) of Figure 5. A one-way sensitivity analysis will consider all distributions $y_1 = N([50, 60], 5^2)$, such as the examples shown in panel (2) of Figure 5, and all of the distributions $y_1 = N(55, [3, 7]^2)$, such as the examples shown in panel (3) of Figure 5. For a global sensitivity analysis, all of these distributions, plus many others, such as the example shown in panel (4) of Figure 5, must be considered. The p-box contains all of these distributions, and hence all are considered when PBA is used as a sensitivity approach.

A general p-box may contain distributions that do not satisfy the complete uncertainty model. For example, the p-box in Figure 4 contains step functions that are not consistent with $y_1 \sim N(x_1, \sigma^2)$. Depending on the implementation of PBA chosen (see the discussion of parameterized p-boxes in [28] for more information), these distributions can be included in or excluded from the propagation. The DBC algorithm implements the calculations by considering only the bounds, and therefore includes these extra distributions. The consequences of this are described in first sub-section of the following discussion.

DISCUSSION

In the preceding, it was shown that PBA provides a convenient way to perform a global sensitivity analysis for decision robustness, even when both imprecise and probabilistic quantities are present. In this section, the primary limitations of PBA as a sensitivity analysis tool are presented, and then a few additional advantages over DASA are described.

² The critical value is known. For comparison between alternatives it is zero. Alternatively it can be defined by some known benchmark.

BOUNDS ARE RIGOROUS BUT NOT NECESSARILY BEST POSSIBLE

In the previous sections, it was emphasized that the results of PBA computations using dependency bounds convolutions (DBC) [1, 21, 26] are rigorous, meaning that the true interval is contained in the output interval assuming the inputs are correctly defined. Note that it was *not* said that the output intervals are the *true* intervals, because the output intervals actually can be larger than the true distributions for three reasons.

First, it can be a consequence of the repeated variable limitation of interval arithmetic described earlier in this paper, since DBC is based on interval arithmetic. Second, DBC involves a discretization of the p-box. This discretized p-box contains the original p-box, but is actually slightly larger. Consequently, the resultant calculated p-boxes will also be slightly larger than the theoretical output.

Third, DBC considers the p-box to be the absolute representation of uncertainty. In truth, a p-box is a conservative abstraction of the information that is truly available: the p-box includes all of the distributions that are consistent with the available information, but it may also include some distributions that are excluded from the truth by the available information. To avoid this abstraction, one would have to sacrifice the computational advantages of DBC and use the broader formalism of imprecise probabilities.

Since the bounds found using DBC are rigorous, the optimal decision will never appear to be robust when in fact it is highly sensitive to the information state. However, if the bounds are much larger than the best possible bounds (a situation referred to as overly conservative), then there may appear to be significant sensitivity when in fact the real problem may involve none. This could lead to the conclusion that the decision is not robust when in fact it is robust. This is the opposite of the problem faced by less than all-way sensitivity analysis, which ignores dependencies and higher order interactions and can lead to results that are non-rigorous, i.e., that are inconsistent with the truth.

Type I and Type II Error Performance

If a selection decision problem is recast as an exercise in hypothesis testing, the types of errors made with the PBA and sensitivity analyses can be discussed in standard statistical terms of Type I and Type II errors via an analogy³. The decision problem involves the selection of the best action $a^* \in \{a_1, \dots, a_k\}$. Without loss of generality, we can assume that after solving the basic,

precise optimization problem of Equation (6), the alternatives are renumbered such that $a_1 = a^*$

In this scenario, the null hypothesis is that any one of the $a_i \in \{a_2, \dots, a_k\}$ is actually the optimal action. The alternative hypothesis is that a_1 is the optimal action. Formulated this way, the burden of proof is on showing that a_1 is optimal.

A less than all-way NRSA (the deterministic phase of DASA) may *underestimate* the true imprecision and indicate that there is enough evidence to reject the null hypothesis in favor of the alternative when there really is not sufficient evidence to do so. In this situation, the null hypothesis would be rejected when it is actually true, a Type I error.

Conversely, PBA may *overestimate* the uncertainty and lead to acceptance of the null hypothesis when it is actually false, a Type II error. This can also be viewed as a “false alarm,” in that sensitivity was detected when it did not exist. A Type II error is an error in the sense that an opportunity to make a decision is lost; the null hypothesis could have been rejected, but was not. Consequently, a DM may waste resources or make an arbitrary decision while trying to reduce indeterminacy that does not exist in the actual problem. However, PBA with DBC will not lead to a Type I error (assuming the imprecision in the inputs is sufficiently characterized).

Which is preferable, a Type I or Type II error? A Type II error may be preferable in high-risk applications; when the cost of failure is high, one is often more willing to be conservative and spend additional resources to reduce uncertainty further. In other applications, the cost of delaying a decision or collecting more information may exceed any potential benefit from waiting. There is no general answer; the analyst must assess the situation and make his or her own choice. However, one can conclude that PBA with DBC leads to a more rigorous sensitivity analysis for robustness in that it avoids a Type I error; it will always detect a lack of robustness.

Quality of inputs

In the preceding discussion, it was noted that the rigor of the PBA method depends on the accuracy of the characterization of the inputs. This is a somewhat obvious but crucial limitation. For example, if a DM assumes that a quantity is known precisely when actually significant imprecision exists (as is required by traditional probability theory), then any subsequent analysis will underestimate sensitivity. The characterization of inputs is not a perfect science, so underestimation can occur. Consequently, the “guarantee” of rigor does not apply universally, but only when the inputs are correct. Naturally, this is an inherent difficulty that all analyses encounter. We believe that by allowing for the independent expression

³ The alignment is not exact, but is illustrative. As such, this hypothesis testing procedure is meant for discussion only; it is not proposed as an actual approach to engineering design.

of probabilistic and imprecise aspects of the problem, the use of PBA improves the ability of an engineer to adequately recognize and reason with these uncertainties.

Summary of performance

The key conclusion is that if the same assumptions are made for DASA and PBA, then PBA with DBC will conclude sensitivity in all cases that DASA does and possibly in more cases, including all cases that the decision (given the defined inputs) actually is sensitive to the available information. In other words, if the imprecision in the inputs is underestimated, both methods can yield results that underestimate the imprecision in the output, and thus may fail to identify sensitivity. However, there is no case in which DASA avoids a Type I error and PBA does not, as summarized in Table 1.

LACK OF RIGOROUS BLACK-BOX METHODS

The basic DBC methods for p-box computations require an open, operationally defined model (e.g. algebraic) of the problem. Consequently, they cannot be used to analyze differential equations or so-called black-box models, such as simulations and finite element analysis in which the underlying equations cannot be expressed in the appropriate form. Ongoing, unpublished research in PBA by other investigators is making advances in propagating p-boxes through ordinary differential equations, so there is promise that the applicability of PBA will continue to increase.

Methods for black-box analysis are discussed in Bruns and Paredis [28], but these generally sacrifice the guarantee of rigor in the calculations. Optimization-based methods often lead to bounds that are close to being both rigorous and best possible, but no guarantee can be made with these heuristic or sampling based methods. Consequently, the value of PBA for *guaranteed* rigorous sensitivity analysis for robustness is limited currently to non-black box models for which DBC algorithms are applicable.

SENSITIVITY ANALYSIS FOR INFORMATION PRIORITIZATION

Heretofore, the discussions were focused on what was described as sensitivity analysis for decision robustness—that is, identifying whether there is enough information to guarantee the optimality (with respect to a specific objective function) of a particular solution. Another major use of sensitivity analysis is for identifying to which uncertainties the decision is most sensitive, assuming a decision is sensitive (i.e. lacks robustness) given the existing imprecision.

Table 1. Summary of veracity of analyses.

Truth	DASA NRSA conclusion	PBA using DBC conclusion
Sensitive	Sensitive Not Sensitive ¹	Sensitive
Not Sensitive	Not Sensitive	Sensitive Not Sensitive ²

¹ possible if extrema are in a region not searched by a less than all-way analysis

² possible due to repeated variable limitation of interval arithmetic

NRSA-based DASA analysis gives a direct answer to this question; heuristically speaking, the decision is most sensitive to those imprecise parameters for which the bars in the tornado diagram cross the critical value (e.g. zero) by the largest amount. Assuming equal cost in information collection, resources should be directed first towards these imprecise quantities. However, DASA does not include a process for the joint consideration of imprecise and probabilistic quantities that PBA enables.

Ferson and Tucker [7] propose a meta-level sensitivity analysis in PBA for identifying where future empirical efforts (or information collection) might be most productive. This analysis is similar to an approach in probabilistic sensitivity analysis in which the variance of a parameter is reduced, e.g. “pinched,” to zero, and the resulting reduction in the output variance is measured.

The idea of this meta-level sensitivity analysis is to pinch the p-box of a particular input quantity and to compare the resultant p-box using this pinched input to the results from the original input p-box. The goal is to identify for which input quantity the pinching reduced the uncertainty in the output p-box the most. Unlike a precise distribution, a p-box has two dimensions of uncertainty. This leads to two important questions: (1) *To which distribution should the p-box be pinched?* (2) *How can the uncertainty in two p-boxes be compared?*

There are many possible answers to the first question, such as the examples shown in Figure 6 for the p-box from Figure 4: (1) pinching the mean to a single value such as the midpoint or other best guess value; (2) reducing the mean to a smaller interval; (3) reducing the mean and variance to point estimates (i.e. reducing the p-box to a single, precise distribution); (4) some arbitrary set of distributions inside the p-box.

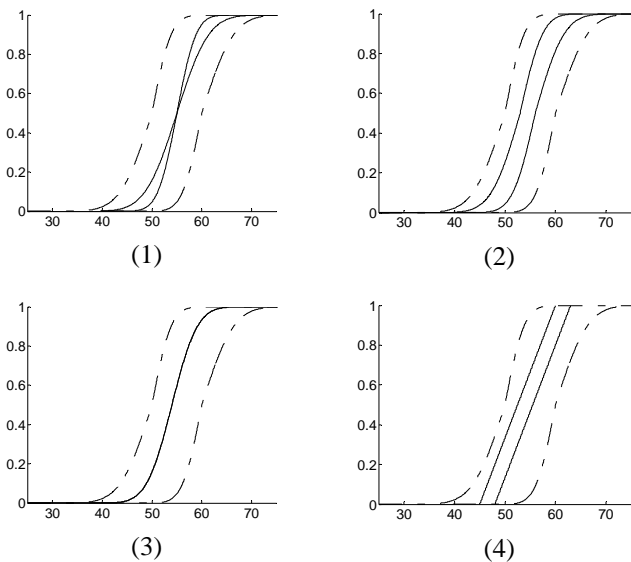
Which approach is the best? The answer is not obvious because by its very nature, the p-box contains all of the distributions that are deemed consistent with the available information. Ferson and Tucker [7] suggest considering the entire set of possible pinchings. This

leads to an interval of uncertainty reductions, since each pinching leads to a particular reduction in uncertainty. In some cases, these intervals will provide a reasonable ranking of the importance of reducing uncertainty in specific inputs. For example, if the total uncertainty can be reduced 30-50% by pinching input A, but only 10-15% by pinching input B, then it is probably more valuable to spend resources on reducing the uncertainty in input A than on reducing the uncertainty in input B.

The question regarding comparing the uncertainty in two p-boxes is also difficult to answer. In probabilistic sensitivity analysis, the difference between two assumptions is measured in uncertainty reduction in terms of variance reduction, or sometimes in terms of some entropy measure. Ideally, a meta-level sensitivity analysis also would compare the reduction in uncertainty using some single metric.

Ferson and Tucker [7] use the area of a p-box (i.e., the difference between the integrals of the upper bound and lower bound of the p-box). This number decreases as the imprecision is reduced, reaching zero in the case of a precisely known distribution. However, there is no consensus on single “best” scalar measure that captures the combined effect of both the imprecise and probabilistic dimensions of uncertainty, as other researchers have proposed alternative metrics [40, 41]. It may be that different definitions of total uncertainty are appropriate and valuable in different contexts.

The ability of PBA to evaluate the potential changes in total uncertainty (however measured), makes it a more powerful formalism for prioritizing information collection than measures, such as variance, that only capture one aspect of uncertainty. However, calculating the various “pinchings” is much more computationally expensive



(original in dotted line, pinched p-boxes in solid lines)

Figure 6. Pinching a p-box

than the direct, NRSA-based DASA result. It may be most valuable to consider both methods, depending on the required tradeoffs, interests, and resources of the problem at hand.

ADDITIONAL ADVANTAGES OF PBA AS A SENSITIVITY ANALYSIS

In addition to providing an all-way, rigorous sensitivity analysis for decision robustness, PBA with DBC is more flexible than traditional DASA methods. Specifically, PBA allows for unknown distribution types and various conditions of dependency between uncertain parameters.

Unknown distribution types

It was suggested earlier that PBA can also handle cases of unknown distribution type. This is possible because a p-box can be constructed that captures various distributions. For example, a p-box can be constructed and propagated with only knowledge of the mean and variance [42]; no assumption of distribution type (e.g. normal, lognormal, gamma, or Weibull) is necessary. For example, a p-box for a mean of 15 and variance of 20 is shown in Figure 7. Such flexibility is useful when, for example, a decision maker has estimates of the mean and variance of a probabilistic parameter, but no theoretical or empirical evidence about the distribution family. Using methods based on DBC, such p-boxes can be propagated into output p-boxes, which in turn can be used for the analysis described earlier in the paper.

Known or unknown dependencies

Traditional statistical methods require that the joint probability density functions between all uncertain quantities be known precisely. Since information regarding joint density functions is often scarce, it is convenient to assume independence, thus making the joint density functions simple functions of the marginal density functions. While assumptions of independence are in some cases justifiable, it is desirable to limit the number of assumptions that are made merely for computational convenience.

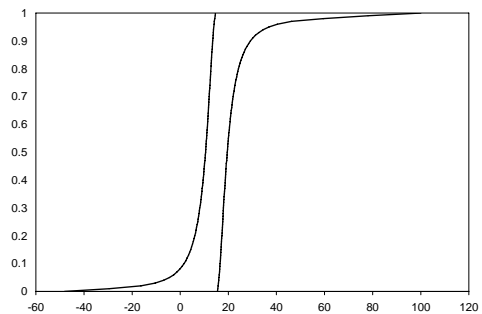


Figure 7. Example p-box for a ranges of mean and variance with no knowledge of distribution

DBC methods can determine probability bounds in the case of unknown dependence between the inputs [1, 25]. Essentially, a p-box can be computed that covers all possible dependency scenarios; this allows the DM to avoid making unwarranted assumptions about dependencies. On the other hand, DASA (in both the deterministic and probabilistic phases), often ignores dependencies and higher order interactions due to excessive computational and elicitation effort, and the probabilistic analysis also requires a known distribution type. Consequently, the class of problems that can be accurately explored with PBA (using DBC) is more inclusive than the class of problems that sensitivity analysis can explore.

SUMMARY: GENERAL USEFULNESS OF PBA

In this paper, it was shown that PBA generalizes a global (or all-way) sensitivity analysis for identification of decision robustness. When implemented using the DBC algorithm, PBA provides a means for performing a sensitivity analysis for robustness identification that avoids a Type I error; it will never lead to the conclusion that the decision is not sensitive to the existing lack of information when in fact the decision is sensitive, assuming the imprecision in the inputs is properly characterized. In other words, PBA avoids giving a false sense of security. PBA can also handle a wider array of problems than a traditional sensitivity analysis. It allows for a sensitivity analysis even when limited probabilistic information is available, and can be used to propagate uncertainties without making assumptions about the dependencies between uncertain quantities.

However, PBA also has important limitations. First, it has a much higher computational cost than DASA. Second, PBA (when implemented using DBC algorithm) is subject to a Type II error, which means concluding that the decision is sensitive to the existing lack of information when in fact it is not sensitive. This could result in missed opportunities or unnecessary expenditures on information collection.

A decision maker must consider multiple criteria when choosing a sensitivity analysis strategy. This paper helps to guide this selection by revealing the nature of two approaches and discussing the tradeoffs between Type I errors, Type II errors, computational expense, identification of sensitivity, and information prioritization that must be considered.

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