Probability bounds for series systems with variables constrained by sets of probability measures

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Abstract: The paper first revisits the problem of finding bounds on the probability of failure of a series system when no information is available on the dependencies among modes' failure probabilities. Second, reliability bounds are calculated when input variables that control the modes' probability of failure are given as parameterised probability measures, random sets or *p*-boxes. The different meanings of independence for the input variables are folded into the formulation. It is determined when calculating the failure probability bounds for each failure mode and then calculating bounds for the system is advantageous with respect to directly calculating the system's probability of failure. The paper also shows how discretising given upper and lower CDF envelopes using the outer discretisation method (ODM) leads to validated bounds on the system's probability of failure.

Keywords: probability bounds; series systems; reliability; sets of probability measures; credal sets; random sets; *p*-boxes; independence.

Reference to this paper should be made as follows: Fetz, T. and Tonon, F. (2008) 'Probability bounds for series systems with variables constrained by sets of probability measures', *Int. J. Reliability and Safety*, Vol. 2, No. 4, pp.309–339.

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Fulvio Tonon has more than 13 years of research experience in the fields of geomechanics with emphasis on rock mechanics and rock engineering, numerical modelling in geomechanics, underground excavations and tunnelling; uncertainty modelling, design process and optimisation, and artificial intelligence as applied to civil engineering. He has published two books on tunnelling, 35 papers in peer-reviewed journals and 32 papers in conference proceedings. He has more than 11 years of professional experience in geotechnical and structural engineering with projects completed in the USA, Europe, and Africa.

1 Introduction

The current (civil) engineering approach to design makes use of limit states, which are conditions beyond which a system or some of its components fails to satisfy the provisions for which it was designed (Melchers, 1999). Current codes establish that several limit states be checked even for a simple component (e.g., a structural member, a retaining wall, etc.). When an entire system is designed, many failure modes (each one being a limit state) may arise and additional limit states pertaining to the whole system may originate (e.g., deflections, overturning, etc.). Thus, the design of engineering systems entails checking that several limit states be satisfied.

This paper concentrates on the particular case in which failure is achieved if any of the limit states is violated (series system). If information on the uncertain variables is given in terms of probability distributions, then one can either attempt to directly calculate the probability of failure for the system or revert to the upper and lower bounds on the probability of failure for the system (Melchers, 1999; Ang and Tang, 1981). The latter approach is useful because directly calculating a system's probability of failure may be extremely time consuming and computationally intensive. In addition, when the correlation among the failure modes is unknown, the first-order bounds can still be calculated.

When correlation among failure modes is unknown, system bounds are typically calculated based on the two assumptions of complete independence and complete dependence, respectively, between any two modes of failure. Starting with the pioneering work of Freudenthall et al. (1966) and Cornell (1967), the definition of independence in probability theory has been used in the literature to define the upper system bound.

The subject has been critically reviewed by Riha and Manteufel (2001); in order to improve on the first-order bounds, Kounias (1968) formulated the second-order bounds for a system's probability of failure. These bounds proved difficult to evaluate since they also required the probability of intersection between each pair of events. Ditlevsen (1979) proposed a weakened version of these bounds for Gaussian variates. The approach calculates the joint probability of two events based on a first-order approximation of the failure space (FORM). This method is simpler to evaluate, but may be inaccurate for non-linear functions. The improvements to this first-order approximation proposed by Madsen et al. (1986) and Cruse (1997) may give a better estimate of the joint probability for non-linear functions. Ditlevsen (1994) proposed a method to numerically integrate the joint failure region based on the first-order approximations of the performance functions. This approach yields the joint probability of failure instead of bounds used by Ditlevsen (1979). According to Riha and Manteufel (2001), although there are several textbooks

that cover system reliability (e.g., Ang and Tang, 1981; Cruse, 1997; Ditlevsen and Madsen, 1996; Madsen et al., 1986; Rao, 1996; Thoft-Christensen and Murotsu, 1986), these works include some contradictions in the explanation of the first-order bounds.

Let $g_1,\ldots,g_m$ be the m limit state functions depending on n random variables:

$$X = (X_1,\ldots,X_n), \tag{1}$$

given for m different failure modes of a series system. Then:

$$P(F_i) = P(\{g_i(X) \le 0\}) \tag{2}$$

is the probability of failure in mode i. The probability of failure of the series system is:

$$p_f = P(F_1 \cup \ldots \cup F_m) = P(\{(\min_{i=1,\ldots,m} g_i(X)) \le 0\}). \tag{3}$$

Riha and Manteufel (2001) use the correlation, ρ, among failure modes to show that:

$$\max_{i=1,\ldots,m} P(F_i) \le p_f \le 1 - \prod_{i=1}^{m} (1 - P(F_i)), \text{ for } 0 \le \rho \le 1, \tag{4}$$

$$\sum_{i=1}^{m} P(F_i) - \prod_{i=1}^{m} P(F_i) \le p_f \le \sum_{i=1}^{m} P(F_i), \text{ for } -1 \le \rho \le 0, \tag{5}$$

$$\max_{i=1,\ldots,m} P(F_i) \le p_f \le \sum_{i=1}^{m} P(F_i), \text{ for unknown } \rho. \tag{6}$$

In this paper, it is assumed that the correlation among failure modes is unknown. Using set-theoretic arguments rather than correlation arguments, the first part of the paper derives the largest upper and lower bounds for series system probability of failure. The paper then introduces the basic concepts of independence that apply when input variables are not constrained by a unique probability measure, but rather, by a closed convex set of probability measures (Couso et al., 1999; Ferson et al., 2004; Fetz and Oberguggenberger, 2004; Giron and Rios, 1980; Levi, 1980; Walley, 1991). This situation occurs in practice when epistemic uncertainty (Walley, 1991) cannot be further reduced because of funding and/or time limitations. For example, it occurs when (Bernardini and Tonon, 2007): an expert only provides the upper and lower bounds on event probabilities or intervals of possible values for a given variable; experts do not agree on a single probability measure for the events of interest; intervals (or sets) are measured, e.g., because of instrument imprecision; input variables are the output of a previous calculation whose input variables are random variables and:

1 two or more models are used for carrying out calculations

2 discretisation error bounds are taken into account

3 upper and lower bounds in the limit theory of plasticity are used.

When working with sets of probability measures, only the upper and lower bounds on the probability of an event of interest can be calculated. As a consequence, when the input information is given as sets of probability measures, each series bound becomes a probability interval. Additionally, when working with sets of probability measures, the

concept of independence is not unique. The formulation presented takes into account these complexities.

A comparison is made with the recent results of Utkin and Kozine (2005). They describe a generic algorithm to find an interval-valued reliability assessment of a system given imprecise reliability information concerning the components. Utkin and Kozine (2005) also derived closed-form formulas for special cases, such as a series system with no information on component dependency.

In this paper, sets of probability measures are generated by parameterised probability distributions, random sets or upper and lower cumulative distribution functions. In this latter case, calculations are performed by discretising continuous distribution functions into random sets in such a way that the calculated bounds are automatically verified, i.e., they always contain the true bounds. The Outer Discretisation Method (ODM) introduced by Tonon (2004) independently from the work of Williamson and Downs (1990) is used to carry out the discretisation. A comparison with the Iterative Rescaling Method (IRM) of Hall and Lawry (2004) rounds off the presentation. Concepts are illustrated throughout using an example of a simple portal structure [modified from Melchers (1999)] which is introduced below.

2 Example statement

In this paper, we consider a 'series system'. We assume that there are m failure modes and therefore m limit state functions $g_1,\ldots,g_m$, where $g_i(x_0)\le 0$ means failure in the *i*th failure mode for values x_0 of the input variables.

For the examples given in this paper, it is assumed that these functions are 'linear' and are written as:

$$g:\mathbb{R}^n \to \mathbb{R}^m : x \mapsto \mathbf{A}x, \tag{7}$$

where $g=(g_1,\ldots,g_m)$ and $\mathbf{A}$ is an $m\times n$ matrix. In this case, g_i is a 'monotonic' function in all variables $x_1,\ldots,x_n$; this is an important property that will be used in the computations. However, the limit state functions g_i are not always monotonic in the same direction because a limit state g_i can be an increasing function in x_k, whereas another limit state, $g_j, i\neq j$, may be a decreasing function in x_k. This is the reason why a limit state function g_{syst} for the whole series system is defined by:

$$g_{\text{syst}}:\mathbb{R}^n \to \mathbb{R}: x \mapsto g_{\text{syst}}(x) = \min_{i=1,\ldots,m} g_i(x). \tag{8}$$

[$g_{\text{syst}}(x_0)\le 0$ if there is failure with respect to at least one failure mode] is neither a linear function nor a monotonic function in general. Further, we assume that the basic variables $x=(x_1,\ldots,x_n)$ are independent.

2.1 Example: failure analysis of a rigid-plastic portal frame

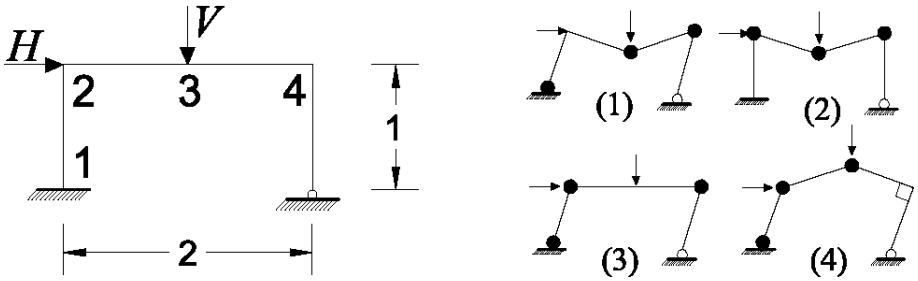
As an accompanying example, we consider, similarly to Melchers (1999), the rigid-plastic portal frame shown in Figure 1, whose limit state functions $g_1,\ldots,g_4$ corresponding to failure modes 1,…, 4 are as follows:

$$\begin{array}{lll} \text{mode 1:} & g_1(x) = M_1 & +2M_3+2M_4-H-V \\ \text{mode 2:} & g_2(x) = & M_2+2M_3+M_4-V \\ \text{mode 3:} & g_3(x) = M_1+M_2 & +M_4-H \\ \text{mode 4:} & g_4(x) = M_1+2M_2+2M_3 & -H+V \end{array} \tag{9}$$

where $x=(M_1,M_2,M_3,M_4,H,V)^T$ is the vector of the independent uncertain variables considered or, for short, $g_i(x)=\mathbf{A}_{i,*}x$ and $g(x)=\mathbf{A}x$ where $\mathbf{A}_{i,*}$ is the *i*th row of $\mathbf{A}$.

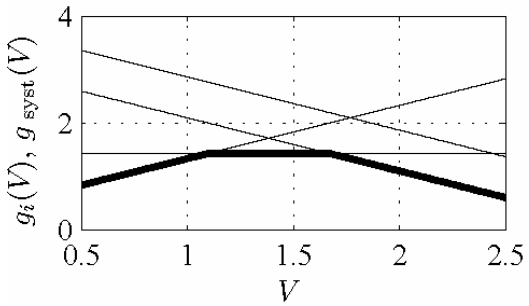
All limit state functions, g_i, are monotonically increasing with respect to all resisting moments M_k and monotonically decreasing with respect to the horizontal force H. As for the vertical force V, only the first three functions g_i are monotonically decreasing in V and the last one, g_4, is increasing with respect to V. As mentioned above and shown in Figure 2, the limit state function g_{syst} for the system is, in general, neither linear nor monotonic.

Figure 1 Example: rigid frame and four collapse modes



Source: Redrawn from Melchers (1999)

Figure 2 $g_i, i=1,\ldots,4$ and g_{syst} as functions of $V\in[0.5,2.5], M_1=1.0,$ $M_2=M_3=0.33, M_4=2.1, H=2.0$



Notes: Thin lines: functions g_i and bold line: function g_{syst}

3 System reliability bounds

With reference to the notation introduced in equations (1) to (3), the lower and upper bounds for p_f are derived using only the single modes' probabilities of failure, $P(F_i), i = 1,\ldots,m$, together with set-theoretic arguments. Fréchet (1935) proved not only that they are the extreme cases, but also that they are the bounds on all possible cases of dependence and moreover, that they are the best possible such bounds in the absence of information about the dependence (Ferson et al., 2004).

3.1 Lower bounds

Let us assume that the sets F_i are nested. They can then be ordered in such way that:

$$F_1 \subseteq F_2 \subseteq \ldots \subseteq F_m. \tag{10}$$

Then:

$$p_f = P(F_1 \cup \ldots \cup F_m) = P(F_m) = \max_{i=1,\ldots,m} P(F_i). \tag{11}$$

In this case, if a realisation, x, of the input parameters is in F_1, then $x \in F_2,\ldots,F_m$ and therefore failure occurs in all modes. In Melchers (1999), this is called 'total dependence', which means that if failure occurs in mode 1, then failure occurs in all other modes. When failure events, F_i, are not nested, the lower bound reads:

$$p_f = P(F_1 \cup \ldots \cup F_m) \geq \max_{i=1,\ldots,m} P(F_i) =: p_f^- \tag{12}$$

because then the event F_i with the greatest failure probability is, in general, only a subset of $F_1 \cup \ldots \cup F_m$.

3.2 Upper bounds

In the opposite extreme case (the events F_i are pairwise disjoint), if failure occurs in one mode, then failure does not occur in all other modes. The probability of failure for the system is:

$$p_f = P(F_1 \cup \ldots \cup F_m) = \sum_{i=1}^{m} P(F_i). \tag{13}$$

For general events F_i, the above equation leads to an upper bound for the probability of failure for the system because the probability of non-empty intersections (e.g., $F_i \cap F_j \neq \varnothing$) are not subtracted from $\sum_{i=1}^{m} P(F_i)$. But then $\sum_{i=1}^{m} P(F_i)$ can be greater than one, so we take:

$$p_f = P(F_1 \cup \ldots \cup F_m) \leq \min\left(\sum_{i=1}^{m} P(F_i), 1\right) =: p_f^+ \tag{14}$$

as an upper bound.

3.3 Remark

The lower bound p_f^- and the upper bound p_f^+ correspond to those derived by Riha and Manteufel (2001) in equation (6), except for the unity cap imposed on the upper bound. In Melchers' (1999) textbook, the expression $1-\prod_{i=1}^{m}(1-P(F_i))$ is used for the upper bound, p_f^+, because it is assumed that the failure modes are independent. This may be true for some components, such as light bulbs, where the correlation may only vary between total correlation and independence. But, in general, two failure modes, i and j, are not independent if their limit state functions, g_i and g_j, share at least one variable. They are correlated in a certain way (positive or negative).

3.4 Example, continued

To better understand the examples that will be introduced later on (when more complicated models of uncertainty are used), let us recall an example similar to Melchers' (1999): each random variable $X_i, X=(M_1,M_2,M_3,M_4,H,V)$, is normally distributed, with parameters (μ_{X_i},σ_{X_i}) where:

$$\mu_X=(1.0,1.0,1.0,2.1,2.0,1.0)^T \tag{15}$$

and

$$\sigma_X=(0.15,0.15,0.15,0.15,0.17,0.80)^T. \tag{16}$$

Then the components $g_i(X)$ of $g(X)$ are again normally distributed with parameters $\mu_{g(X)}$ and $\sigma_{g(X)}$ obtained as:

$$\mu_{g(X)}=\mathbf{A}\mu_X \quad \text{and} \quad \sigma^2_{g(X)}=\mathbf{B}\sigma^2_X, \tag{17}$$

where

$$\mathbf{B}_{ij}=\mathbf{A}^2_{ij}, \quad \sigma^2_X=(\sigma^2_{X_1},\ldots,\sigma^2_{X_6})^T, \quad \sigma^2_{g(X)}=(\sigma^2_{g_1(X)},\ldots,\sigma^2_{g_4(X)})^T. \tag{18}$$

The first failure mode's failure probability, $P(F_1)$, is obtained as:

$$P(F_1)=P(\{g_1(X)\le 0\})=F(0;\mu_{g_1(X)},\sigma^2_{g_1(X)})=F(0;\mathbf{A}_{1,*}\mu_X,\mathbf{B}_{1,*}\sigma^2_X) \tag{19}$$

where F is the value of the normal distribution function with parameters $\mu_{g_1(X)}$ and $\sigma^2_{g_1(X)}$ and evaluated at zero. In our example, the probabilities of failure for failure modes $i=1,\ldots,4$ are:

$$P(F_1)=3.4096\cdot 10^{-6}$$
$$P(F_2)=1.6020\cdot 10^{-6}$$
$$P(F_3)=6.7281\cdot 10^{-12}$$
$$P(F_4)=9.1368\cdot 10^{-6}.$$

The left bound of the failure probability takes the value:

$$p_f^- = \max_{i=1,\ldots,4} P(F_i) = 9.1368 \cdot 10^{-6} \tag{20}$$

and the right bound takes the value:

$$p_f^+ = \min\left(\sum_{i=1}^{4} P(F_i), 1\right) = 1.4148 \cdot 10^{-5}. \tag{21}$$

By using the limit state function itself, g_{syst}, the probability of failure of the system, p_f, is:

$$p_f = P(\{g_{\text{syst}}(X) \le 0\}) = 1.3138 \cdot 10^{-5} \in [9.1368 \cdot 10^{-6},\ 1.4148 \cdot 10^{-5}] \tag{22}$$

where p_f is an exact value because here, the probability distributions are exactly known.

4 Parameterised probability measures and random sets

In this section, we want to show how to describe the uncertainty about the value of a variable, x_k, by combining the widely used approaches of probability measures and intervals, which lead to 'parameterised probability measures' and 'random sets'.

4.1 Sets of probability measures

Let $\mathcal{M}_k$ be a 'credal set' (Giron and Rios, 1980; Levi, 1980), i.e., a convex set of probability measures, P_k, that describes the uncertainty about the value of a variable, x_k (Klir, 2006; Walley, 1991). Since more than one probability measure is used, the probability of an event, *E*, is not a single number, but an interval; indeed, the lower and upper probabilities are, respectively:

$$\underline{P}_k(E) = \inf\{P_k(E) : P_k \in \mathcal{M}_k\} \tag{23}$$

and

$$\overline{P}_k(E) = \sup\{P_k(E) : P_k \in \mathcal{M}_k\}. \tag{24}$$

In this paper, we restrict ourselves to two types of sets of probability measures:

1 sets $\mathcal{M}_k$ of probability measures generated by parameterised probability measures

2 sets $\mathcal{M}_k$ of probability measures generated by random sets.

Remarks: Although the index *k* seems to be meaningless, it is used here to indicate marginal variables, as opposed to joint uncertainty where no index is used.

4.2 Sets $\mathcal{M}_k$ of probability measures generated by parameterised probability measures

Probability distributions of a certain family are defined by some parameters, e.g., μ and σ for a normal distribution. Let P_θ be a probability measure parameterised by $\theta \in \Theta_k$, which defines the corresponding probability distribution. Then, the set of probability measures is:

$$M_k = \{P_\theta : \theta \in \Theta_k\} \tag{25}$$

and the lower and upper probabilities are, respectively:

$$\underline{P}_k(E) = \inf\{P_\theta(E) : \theta \in \Theta_k\}, \quad \overline{P}_k(E) = \sup\{P_\theta(E) : \theta \in \Theta_k\}. \tag{26}$$

Here, uncertainty about the parameter θ is modelled using a set (interval); this is to show how the probability bounds are computed. There are many other possibilities to model one's uncertainty about θ, such as fuzzy and random sets. These cases are covered in Fetz (2003). Using these additional models of uncertainty would only increase the computational effort required, but the ideas and algorithms for computing the bounds remain the same as for intervals.

4.3 Example, continued

Let us assume that the mean value of the normally distributed vertical force, V (or X_6), is not precisely given. The uncertainty about the mean value is modelled by the interval $\Theta_6 = [\mu^L_{X_6}, \mu^R_{X_6}] = [0.95,\ 1.15]$. This leads to the set of probability measures, $\mathcal{M}_6$:

$$\begin{aligned}\mathcal{M}_6 = \Big\{ P_\mu : P_\mu(E) = \int_E \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathrm{d}x, \\ \mu \in \Theta_6 = [\mu^L_{X_6}, \mu^R_{X_6}] = [0.95,\ 1.15], \sigma = \sigma_{X_6} = 0.8 \Big\}.\end{aligned} \tag{27}$$

4.4 Sets $\mathcal{M}_k$ of probability measures generated by random sets

In this case, the uncertainty about the value of variable x_k is modelled by using a set of probability measures, $\mathcal{M}_k$, generated by random sets.

4.4.1 Random sets

A random set, $(\mathcal{A}_k, m_k)$, (Dempster, 1967, 1968; Molčanov, 2005) consists of a finite class of sets (so called focal sets) $A^i_k \subseteq \mathbb{R}, \mathcal{A}_k = \{A^1_k, \ldots, A^{n_k}_k\}$ and a weighting function (or basic probability assignment):

$$m_k : \mathcal{A}_k \to [0,1] : A^i_k \mapsto m_k(A^i_k)$$

with $\sum_{i=1}^{n_k} m_k(A_k^i) = 1$ where n_k is the number of focal sets. Probability assignment $m_k(A_k^i)$ indicates that an event in (subset of) A_k^i has probability $m_k(A_k^i)$, but the available information does not allow for a more precise identification of which event in A_k^i has probability $m_k(A_k^i)$. Random sets occur frequently in engineering. For example, whenever the observation of a random variable is imprecise, i.e., whenever a set of values (and not a precise value) is observed. Or another example: the sets $A_k^1, \ldots, A_k^{n_k}$ are n_k expert meanings about the possible range of the *k*th parameter considered. Then the weights represent the trust in the expert opinions.

For an event, F_k, the upper probability, $\overline{P}_k$, can be easily calculated in closed form because it coincides with the plausibility measure, Pl_k, defined as (Dempster, 1967):

$$\overline{P}_k(F_k) = \mathrm{Pl}_k(F_k) = \sum_{A_k^i \cap F_k \neq \varnothing} m_k(A_k^i). \tag{28}$$

Likewise, the lower probability, $\underline{P}_k$, coincides with the belief measure, Bel_k, defined as (Dempster, 1967):

$$\underline{P}_k(F_k) = \mathrm{Bel}_k(F_k) = \sum_{A_k^i \subseteq F_k} m_k(A_k^i). \tag{29}$$

Remark: The weighting function, m_k, can also be interpreted as a probability measure m_k on a finite set $\Omega_k = \{\omega_k^1, \ldots, \omega_k^{n_k}\}$ together with a multi-valued mapping (Dempster, 1967):

$$\mathfrak{X}_k : \Omega_k \to \mathcal{P}(\mathbb{R}) : \omega_k^i \mapsto \mathfrak{X}_k(\omega_k^i) \tag{30}$$

which maps the singletons $\omega_k^i \in \Omega_k$ onto the focal sets $A_k^i := \mathfrak{X}_k(\omega_k^i)$. Then $m_k(A_k^i)$ is 'more exactly' $m_k(\mathfrak{X}_k^{-1}(A_k^i))$ and $\mathfrak{X}_k$ can be interpreted as a multi-valued random variable (Alvarez, 2006; Molčanov, 2005).

4.4.2 Sets $\mathcal{M}_k$ of probability measures generated by random sets

Let $\mathcal{M}_k^i = \{P_k^i : P_k^i(A_k^i) = 1\}$ be the set of all probability measures, P_k^i, 'on' the corresponding focal set, A_k^i, and equal to zero outside A_k^i. Since the probability assignment, $m_k(A_k^i)$, can be distributed in all possible ways in A_k^i, the set of probability measures generated by $(\mathcal{A}_k, m_k)$ is the set of all weighted sums of $P_k^i, i = 1, \ldots, n_k$, where the weights are the probability assignments, i.e.:

$$\mathcal{M}_k = \left\{ P_k = \sum_{i=1}^{n_k} m_k(A_k^i) P_k^i : P_k^i \in \mathcal{M}_k^i \right\} = \sum_{i=1}^{n_k} m_k(A_k^i) \mathcal{M}_k^i. \tag{31}$$

An element of this set, $\mathcal{M}_k$, is generated as:

$$P_k = \sum_{i=1}^{n_k} m_k(A_k^i) P_k^i, \quad P_k^i \in \mathcal{M}_k^i. \tag{32}$$

4.5 Sets $\mathcal{M}_k$ of probability measures generated by lower and upper cumulative distribution functions

Consider two given upper semi-continuous cumulative distribution functions, $\underline{F}_k$ and $\overline{F}_k$, with $\underline{F}_k \le \overline{F}_k$. $\underline{F}_k$ and $\overline{F}_k$ are the lower and the upper cumulative distribution functions, respectively. The set of probability measures whose cumulative distributions are comprised between $\underline{F}_k$ and $\overline{F}_k$ is also called *p*-box (Ferson, 2001):

$$\mathcal{M}_k = \{P_k : \underline{F}_k(x) \le P_k((-\infty, x]) \le \overline{F}_k(x)\}. \tag{33}$$

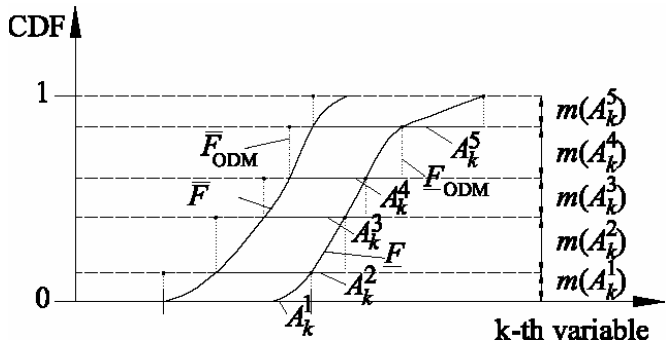
It can also be interpreted as a random set with infinitely many focal sets (intervals):

$$A_k(y) = [\min\{x : \overline{F}_k(x) \ge y\}, \ \min\{x : \underline{F}_k(x) \ge y\}], \quad y \in [0,1] \tag{34}$$

weighted by the uniform distribution on [0, 1].

In order to be computationally practical, sets $\mathcal{M}_k$ (or *p*-boxes) are approximated by a set of probability measures generated by random sets having a finite number of focal sets. This is accomplished via the Outer Discretisation Method (ODM) described by Tonon (2004) and illustrated in Figure 3.

Figure 3 Outer discretisation method



We assume that $\underline{F}$ and $\overline{F}$ are continuous and strictly monotonically increasing. As a first step, the [0, 1] ordinate intervals of $\underline{F}$ and $\overline{F}$ are both discretised into n subintervals of length $m_j > 0 (j = 1, \ldots, n)$; for example, $n = 5$ in Figure 3. By definition, let $m_0 := 0$ and let $\underline{F}^{-1}$ and $\overline{F}^{-1}$ indicate the inverse functions of $\underline{F}$ and $\overline{F}$, respectively. As a second step, the focal elements are calculated: the focal element corresponding to (the weight) m_j is the interval:

$$\left[\overline{F}^{-1}(\sum_{i=0}^{j-1} m_i),\ \underline{F}^{-1}(\sum_{i=1}^{j} m_i)\right], \tag{35}$$

where

$$\overline{F}^{-1}(0) := \lim_{y\to 0+} \overline{F}^{-1}(y), \tag{36}$$

$$\underline{F}^{-1}(1) := \lim_{y\to 1-} \underline{F}^{-1}(y). \tag{37}$$

For example, the focal set A_k^3 corresponding to $m(A_k^3)$ in Figure 3 is given by $\left[\overline{F}^{-1}(m(A_k^1)+m(A_k^2)),\ \underline{F}^{-1}(m(A_k^1)+m(A_k^2)+m(A_k^3))\right]$.

An ODM example is contained in Section 5.4.2. A comparison with the Iterative Rescaling Method (IRM) of Hall and Lawry (2004) is given in the Appendix. In Figure 3, the approximating upper and lower cumulative distribution functions are indicated as $\overline{F}_{\mathrm{ODM}}$ and $\underline{F}_{\mathrm{ODM}}$, respectively. As shown in Tonon (2008), the ODM algorithm ensures that the calculated probability bounds always contain the bounds that could be calculated using $\underline{F}$ and $\overline{F}$.

4.6 Sets of joint probability measures and independence

For notational convenience, let us consider only two variables, x_1 and x_2 and let us assume that their uncertainty is described by sets of probability measures, $\mathcal{M}_1$ and $\mathcal{M}_2$. In order to compute the failure probabilities $P(F_i)$ and $P(F_1 \cup \ldots \cup F_m)$, the joint probability measures, P, are needed. They must be consistent with the available information and thus the marginal probability measures must be in the corresponding sets, $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, i.e.:

$$\mathcal{M} \subseteq \{P : P(\cdot \times \mathbb{R}) \in \mathcal{M}_1 \text{ and } P(\mathbb{R} \times \cdot) \in \mathcal{M}_2\}. \tag{38}$$

If the input variables are independent, the question arises as to what independence means in the context of the sets of probability measures. At difference with precise probabilities, there are several notions of independence; three of them are the most important (Couso et al., 1999):

- epistemic independence
- strong independence
- random set independence (if sets $\mathcal{M}_k$ are generated by random sets).

Only the strong and random set independence will be used in this paper.

- Strong independence: The set $\mathcal{M}$ is defined in a very natural way as the set of all product measures $P_1 \otimes P_2$ of the marginals $P_1 \in \mathcal{M}_1$ and $P_2 \in \mathcal{M}_2$ (Couso et al., 1999):

$$\mathcal{M}_S = \{P : P = P_1 \otimes P_2, P_1 \in \mathcal{M}_1, P_2 \in \mathcal{M}_2\}, \tag{39}$$

where $(P_1 \otimes P_2)\ (A_1 \times A_2) = P_1(A_1)P_2(A_2)$. This type of independence is always used if the sets $\mathcal{M}_k$ are generated by parameterised probabilities.

- Random set independence: It is assumed that the marginal focal sets are chosen in an independent way. The set of joint probability measures is generated by the joint random set with focal sets $A_1^i \times A_2^j,\ i = 1,\ldots,n_1, j = 1,\ldots,n_2$ and weights $m(A_1^i \times A_2^j) = m_1(A_1^i)m_2(A_2^j)$ (Fetz, 2001, 2003):

$$\mathcal{M}_R = \left\{ P : P = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} m_1(A_1^i)m_2(A_2^j)P^{ij}, P^{ij}(A_1^i \times A_2^j) = 1 \right\}. \tag{40}$$

In general, $\mathcal{M}_S \subseteq \mathcal{M}_R$ (Couso et al., 1999) and therefore $\underline{P}_R(E) \le \underline{P}_S(E) \le \overline{P}_S(E) \le \overline{P}_R(E)$ where $\underline{P}_R$ and $\overline{P}_R$ are the lower and upper probabilities corresponding to random set independence and $\underline{P}_S$ and $\overline{P}_S$ are the lower and upper probabilities corresponding to strong independence.

We do not go further into the details, except for the following two results:

1 Let f be a continuous function; let the marginal focal sets A_1^i and A_2^j be intervals and let $B^{ij} = [\underline{b}^{ij}, \overline{b}^{ij}] = f(A_1^i \times A_2^j)$ be all of the images of the joint focal sets. Then we have:

$$\begin{aligned} \overline{P}_R(\{f(x) \le 0\}) &= \sum_{A_1^i \times A_2^j \cap \{f(x) \le 0\} \ne \varnothing} m(A_1^i \times A_2^j) = \\ &= \sum_{B^{ij} \cap (-\infty, 0] \ne \varnothing} m(B^{ij}) = \sum_{\underline{b}^{ij} \le 0} m(B^{ij}) \end{aligned} \tag{41}$$

and

$$\begin{aligned} \underline{P}_R(\{f(x) \le 0\}) &= \sum_{A_1^i \times A_2^j \subseteq \{f(x) \le 0\}} m(A_1^i \times A_2^j) = \\ &= \sum_{B^{ij} \subseteq (-\infty, 0]} m(B^{ij}) = \sum_{\overline{b}^{ij} \le 0} m(B^{ij}) \end{aligned} \tag{42}$$

with $m(B^{ij}) = m(A_1^i \times A_2^j)$.

2 If f is monotonic, then $\underline{P}_R(E) = \underline{P}_S(E)$ and $\overline{P}_R(E) = \overline{P}_S(E)$ (Fetz, 2003). For a general function, f, it is very hard to compute $\underline{P}_S(E)$ and $\overline{P}_S(E)$.

More information can be found in Fetz (2001, 2003) and Fetz and Oberguggenberger (2004).

5 System reliability bounds for sets of probability measures

If sets of probability measures are used to describe the uncertainty of the input variables, the resulting mode probabilities of failure are intervals and so too are the system reliability bounds and the probability of failure of the system itself. In this section, we show how to compute the system reliability bounds for sets of probability measures and we highlight the problems that may arise in these computations.

5.1 Notation

Let us introduce the following notation for the intervals of probabilities resulting for sets of probability measures as input:

$I_{F_i} = [\underline{P}(F_i), \overline{P}(F_i)]$ interval for the ith mode's probability of failure,
$I_f = [\underline{p}_f, \overline{p}_f]$ interval for the system's probability of failure,
$I^-_{f,\mathrm{ex}} = [\underline{p}^-_{f,\mathrm{ex}}, \overline{p}^-_{f,\mathrm{ex}}]$ interval for the lower bound, exact computation,
$I^+_{f,\mathrm{ex}} = [\underline{p}^+_{f,\mathrm{ex}}, \overline{p}^+_{f,\mathrm{ex}}]$ interval for the upper bound, exact computation,
$I^-_f = [\underline{p}^-_f, \overline{p}^-_f]$ interval for the lower bound, interval arithmetics,
$I^+_f = [\underline{p}^+_f, \overline{p}^+_f]$ interval for the lower bound, interval arithmetics.

5.2 Computation of the bounds

Since a set of probability measures, $\mathcal{M}_k$, models the uncertainty of each variable, x_k, the probability of failure for the ith mode is the interval:

$$I_{F_i} = [\underline{P}(F_i), \overline{P}(F_i)] \tag{43}$$

with

$$\underline{P}(F_i) = \inf\{P(F_i) : P \in \mathcal{M}\}, \quad \overline{P}(F_i) = \sup\{P(F_i) : P \in \mathcal{M}\} \tag{44}$$

where $\mathcal{M}$ is the set of joint probability measures of the input generated by the sets $\mathcal{M}_k, k = 1, \ldots, n$.

If these intervals, I_{F_i}, are inserted into the formulas for the lower and upper system reliability bounds, the upper bounds are overestimated and the lower bounds are underestimated (naïve interval computations). This is a very well-known problem in interval analysis, which has devised many algorithms to counter this problem. In our case, since the modes of failure share input variables, X_i, there are interactions between the intervals $I_{F_i}, i = 1, \ldots, m$ and these shared variables are repeated in the formulas. By treating each interval separately, a repeated variable affecting two intervals is treated as if it were two different variables. Cf., the famous example $f(x) = x - x$ evaluated for an interval $[x^L, x^R]$: using naïve interval computations treating both x as different

variables, the result is not [0, 0], but $[x^L - x^R, x^R - x^L]$. As a consequence, the set of the probabilities of failure:

$$S = \{(P(F_1),\ldots,P(F_m)) : P \in \mathcal{M}\} \tag{45}$$

is just a subset of the Cartesian product of the failure probability intervals:

$$S_{\square} = I_{F_1} \times I_{F_2} \times \cdots \times I_{F_m}. \tag{46}$$

In order to obtain the exact lower and upper system reliability bounds:

$$I^-_{f,\mathrm{ex}} = [\underline{p}^-_{f,\mathrm{ex}}, \overline{p}^-_{f,\mathrm{ex}}], \quad I^+_{f,\mathrm{ex}} = [\underline{p}^+_{f,\mathrm{ex}}, \overline{p}^+_{f,\mathrm{ex}}], \tag{47}$$

one has to work on set S:

$$\underline{p}^-_{f,\mathrm{ex}} = \min\left\{\max_{i=1,\ldots,m} P(F_i) : (P(F_1),\ldots,P(F_m)) \in S\right\}, \tag{48}$$

$$\overline{p}^-_{f,\mathrm{ex}} = \max\left\{\max_{i=1,\ldots,m} P(F_i) : (P(F_1),\ldots,P(F_m)) \in S\right\}, \tag{49}$$

$$\underline{p}^+_{f,\mathrm{ex}} = \min\left\{\min(\sum_{i=1}^{m} P(F_i),1) : (P(F_1),\ldots,P(F_m)) \in S\right\}, \tag{50}$$

$$\overline{p}^+_{f,\mathrm{ex}} = \max\left\{\min(\sum_{i=1}^{m} P(F_i),1) : (P(F_1),\ldots,P(F_m)) \in S\right\}. \tag{51}$$

In most cases, this is very difficult to accomplish because it entails solving two min-max optimisation problems on the modes' probabilities of failure. Replacing S by $S_{\square}$ leads to the results of naïve interval computations and to the formulas:

$$\underline{p}^-_f = \max_{i=1,\ldots,m} \underline{P}(F_i), \qquad \overline{p}^-_f = \max_{i=1,\ldots,m} \overline{P}(F_i), \tag{52}$$

$$\underline{p}^+_f = \min\left(\sum_{i=1}^{m} \underline{P}(F_i),1\right), \quad \overline{p}^+_f = \min\left(\sum_{i=1}^{m} \overline{P}(F_i),1\right), \tag{53}$$

which are the outer approximations to the actual probability intervals, i.e.:

$$I^-_{f,\mathrm{ex}} \subseteq I^-_f = [\underline{p}^-_f, \overline{p}^-_f], \quad I^+_{f,\mathrm{ex}} \subseteq I^+_f = [\underline{p}^+_f, \overline{p}^+_f]. \tag{54}$$

Only for the more or less useless upper bound of the lower bound we have $\overline{p}^-_f = \overline{p}^-_{f,\mathrm{ex}}$ because interactions do not play a role in the calculation of max(max(·)). In the following, we will also use the notation p^-_f for the lower bound $\underline{p}^-_f$ of the interval $[\underline{p}^-_f, \overline{p}^-_f]$ and p^+_f for $\overline{p}^+_f$.

The bounds in equations (52) and (53) coincide with the bounds derived by Utkin and Kozine (2005, p.28) when one substitutes $1-P(F_i)$ for the component reliability (a_i) and takes $\sup f = 1$ to turn previsions into probabilities. The same result is obtained when one considers that the reliability of a parallel system [Utkin and Kozine, (2005), p.29] means survival if at least one component survives and this corresponds to equations (52) and (53) for failure of series systems (failure occurs if at least one component fails). Since Utkin and Kozine (2005) assume that their available information consists of the upper and lower previsions ($\overline{a}_i$ and $\underline{a}_i$) on the reliability of each component, they are already operating on set $S_{\square}$ [Utkin and Kozine, (2005), p.26, 29]. The discussion contained in this section puts their work in the proper light.

5.3 Conditions leading to exact bounds

In this section, the expression 'exact bounds' does not refer to the probability of failure for the system $I_f = [\underline{p}_f, \overline{p}_f]$. Rather, it refers to the exact intervals $I^-_{f,\mathrm{ex}}$ and $I^+_{f,\mathrm{ex}}$ for the lower and upper system reliability bounds.

In order to calculate the intervals $I^-_{f,\mathrm{ex}}$ and $I^+_{f,\mathrm{ex}}$, it is not required that $S = S_{\square}$. Indeed, it is sufficient to have:

$$(\underline{P}(F_1),\ldots,\underline{P}(F_m)) \in S \quad \text{and} \quad (\overline{P}(F_1),\ldots,\overline{P}(F_m)) \in S \tag{55}$$

because these are the only values used in the above formulas (52) and (53).

If the sets of probability measures, $\mathcal{M}_k$, are generated by random sets and if the limit state functions g_i are monotonic always in the same direction, then (55) holds because all $\underline{P}(F_i)$ and all $\overline{P}(F_i)$ can be obtained always at the same corners of the joint random sets.

5.4 Example, continued

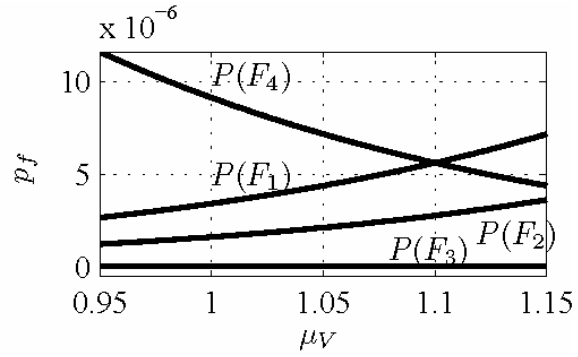
5.4.1 Parameterised probability measures

To visualise the problems arising in computing the system reliability bounds, let us start with the case where the uncertainty on V is described by the set of probability measures given in Section 4.3. All other variables are normally distributed as in Section 3.4. The input (and therefore all results) are parameterised by the mean value of the vertical load $V, \mu_V \in [0.95,\ 1.15]$. Figure 4 depicts the probabilities of failure for modes $i = 1,\ldots,4$ as a function of μ_V. $P(F_1)$, $P(F_2)$ and $P(F_3)$ are increasing functions in μ_V, but $P(F_4)$ is a decreasing function of μ_V.

The images of [0.95, 1.15] through these four functions lead to the following intervals for the mode's probabilities of failure:

$$\begin{aligned}
&P(F_1) \in [\underline{P}(F_1), \overline{P}(F_1)] = [2.64662 \cdot 10^{-6}, 7.17076 \cdot 10^{-6}] \\
&P(F_2) \in [\underline{P}(F_2), \overline{P}(F_2)] = [1.21401 \cdot 10^{-6}, 3.61337 \cdot 10^{-6}] \\
&P(F_3) \in [\underline{P}(F_3), \overline{P}(F_3)] = [6.72815 \cdot 10^{-12}, 6.72815 \cdot 10^{-12}] \\
&P(F_4) \in [\underline{P}(F_4), \overline{P}(F_4)] = [4.38048 \cdot 10^{-6},\ 1.16099 \cdot 10^{-5}].
\end{aligned}$$

Figure 4 Failure probabilities as functions of μ_V



How to obtain these intervals is explained below [equations (56) to (59)] for the most general case.

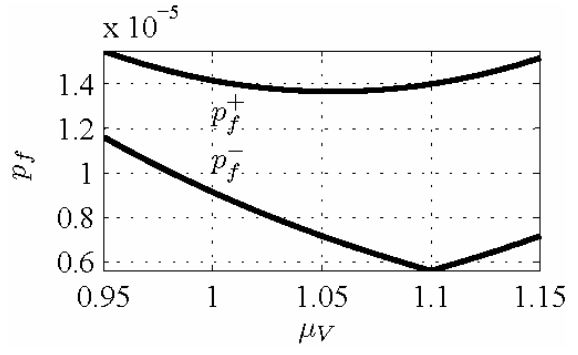
By looking at Figure 4, it is obvious that the upper bounds of $P(F_1)$, $P(F_2)$ and $P(F_3)$ do not correspond to the upper bound of $P(F_4)$. By applying interval arithmetics to the formulas for p_f^- and p_f^+, one does not get exact intervals $I_{f,\mathrm{ex}}^-$ and $I_{f,\mathrm{ex}}^+$, but only $I_f^- \supset I_{f,\mathrm{ex}}^-$ and $I_f^+ \supset I_{f,\mathrm{ex}}^+$:

$$I_f^- = [4.38048 \cdot 10^{-6},\ 1.16099 \cdot 10^{-5}]$$
$$I_f^+ = [8.24112 \cdot 10^{-6},\ 2.23941 \cdot 10^{-5}].$$

In this simple example, we can easily calculate the exact bounds by computing the minimum and maximum of p_f^- and p_f^+ as functions of μ_V in [0.95, 1.15], see also Figure 5:

$$I_{f,\mathrm{ex}}^- = [\underline{p}_{f,\mathrm{ex}}^-, \overline{p}_{f,\mathrm{ex}}^-] = [5.62629 \cdot 10^{-6},\ 1.16099 \cdot 10^{-5}]$$
$$I_{f,\mathrm{ex}}^+ = [\underline{p}_{f,\mathrm{ex}}^+, \overline{p}_{f,\mathrm{ex}}^+] = [1.36547 \cdot 10^{-5},\ 1.54705 \cdot 10^{-5}].$$

Figure 5 p_f^- and p_f^+ as functions of μ_V



As mentioned above, the only exact bound obtained by using interval arithmetics is the (useless) upper bound $\overline{p}_{f,\mathrm{ex}}^{-}$ of the lower bound. In the following, it is assumed that the means μ_{X_i} of all variables are intervals, rather than singletons. The standard deviations remain the same as before. The set Θ for the mean values is given by:

$$\Theta = [0.75,1.05]\times[0.75,1.05]\times[0.75,1.05]\times[1.75,2.2]\times[1.9,2.5]\times[0.75,1.25],$$

where the *i*th interval is the interval for the mean value of X_i. Based on equation (19), the lower and upper probabilities of failure for the *i*th mode, $\underline{P}(F_i)$ and $\overline{P}(F_i)$, are obtained in the following way:

$$\underline{P}(F_i) = F(0;\mathbf{A}_{i,*}\mu_X^{i+},\mathbf{B}_{i,*}\sigma_X^2) \tag{56}$$

$$\overline{P}(F_i) = F(0;\mathbf{A}_{i,*}\mu_X^{i-},\mathbf{B}_{i,*}\sigma_X^2) \tag{57}$$

with

$$\mu_{X_j}^{i-} = \begin{cases} \mu_{X_j}^{L} & \mathbf{A}_{ij} > 0 \\ \mu_{X_j}^{R} & \mathbf{A}_{ij} < 0 \end{cases} \tag{58}$$

and

$$\mu_{X_j}^{i+} = \begin{cases} \mu_{X_j}^{L} & \mathbf{A}_{ij} < 0 \\ \mu_{X_j}^{R} & \mathbf{A}_{ij} > 0, \end{cases} \tag{59}$$

if we assume (as in our example) that all mean values are positive. Then, we get:

$$\begin{aligned} P(F_1) &\in [\underline{P}(F_1),\overline{P}(F_1)] = [7.64097\cdot10^{-8},1.60766\cdot10^{-2}] \\ P(F_2) &\in [\underline{P}(F_2),\overline{P}(F_2)] = [8.69605\cdot10^{-8},8.92689\cdot10^{-4}] \\ P(F_3) &\in [\underline{P}(F_3),\overline{P}(F_3)] = [5.38242\cdot10^{-15},7.85493\cdot10^{-3}] \\ P(F_4) &\in [\underline{P}(F_4),\overline{P}(F_4)] = [4.15900\cdot10^{-7},1.60766\cdot10^{-2}]. \end{aligned}$$

Applying the formulas for the system reliability bounds [equations (52) to (53)], we obtain:

$$\begin{aligned} p_f^{-} &= 4.15900\cdot10^{-7} \\ p_f^{+} &= 4.09007\cdot10^{-2}. \end{aligned}$$

5.4.2 Random sets, ODM

In order to make the uncertain information more realistic, truncated normal distributions (as opposed to normal distributions) are assumed. The cumulative distribution function:

$$F_{\mathrm{trunc}}(x;\mu,\sigma^2) = \frac{F(x;\mu,\sigma^2) - F(x^L;\mu,\sigma^2)}{F(x^R;\mu,\sigma^2) - F(x^L;\mu,\sigma^2)} \tag{60}$$

is the CDF which we get if a normal distribution with parameters μ, σ and CDF $F(x;\mu,\sigma^2)$ is truncated to the interval $[x^L, x^R]$.

As a first step, we start with the lower and upper CDFs, $\underline{F}_i$ and $\overline{F}_i$, for each variable X_i given by:

$$\underline{F}_i(x) = F(x;\mu_{X_i}^R, \sigma_{X_i}^2), \tag{61}$$

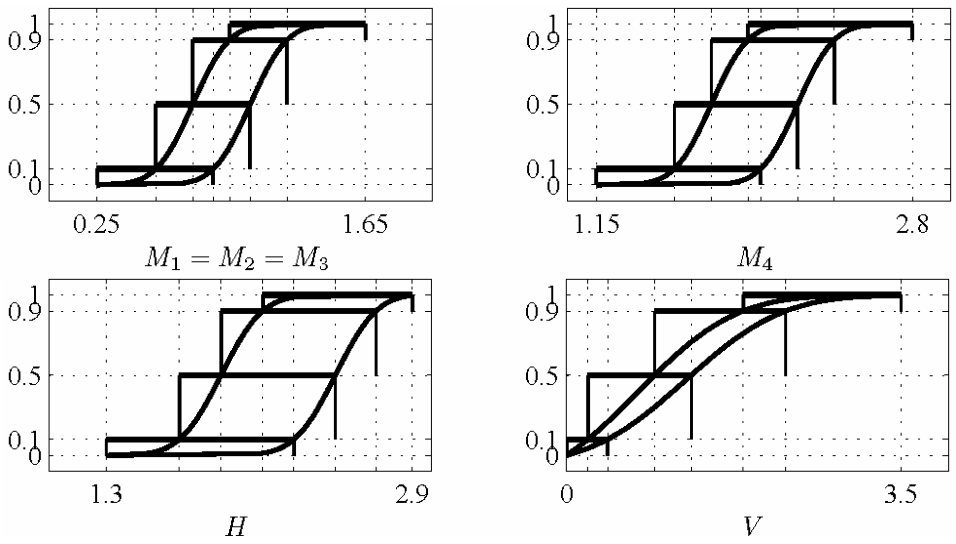
$$\overline{F}_i(x) = F(x;\mu_{X_i}^L, \sigma_{X_i}^2), \tag{62}$$

with means and variances from the previous example. Then, we replace $\underline{F}_i$ and $\overline{F}_i$ (which are CDFs) by the CDF of the corresponding truncated normal distributions. The intervals used for truncation are given in Table 1.

Table 1 Intervals for truncation

Variable	*Interval for truncation of the lower CDF*	*Interval for truncation of the upper CDF*
M_1	[0.25, 1.65]	[0.25, 1.65]
M_2	[0.25, 1.65]	[0.25, 1.65]
M_3	[0.25, 1.65]	[0.25, 1.65]
M_4	[1.15, 2.80]	[1.15, 2.80]
H	[1.30, 2.90]	[1.30, 2.90]
V	[0.00, 3.00]	[0.00, 3.50]

Figure 6 Lower and upper CDFs of truncated normal distributions and the random sets obtained by ODM



As a second step, the resulting *p*-boxes are approximated by random sets using the ODM: the ordinate y (the interval [0, 1]) is discretised by:

$$y^{(0)} = 0.0,\ y^{(1)} = 0.1,\ y^{(2)} = 0.5,\ y^{(3)} = 0.9 \text{ and } y^{(4)} = 1.0,$$

which leads to focal sets (intervals) A_k^i with weights $m_k(A_k^i) = y^{(i)} - y^{(i-1)}$. The lower and upper CDFs and the random sets are depicted in Figure 6.

If random set independence is assumed, we have to compute the images:

$$B_i^j = [\underline{b}_i^j, \overline{b}_i^j] = g_i(A^j) \text{ and } B_{\text{syst}}^j = [\underline{b}_{\text{syst}}^j, \overline{b}_{\text{syst}}^j] = g_{\text{syst}}(A^j) \tag{63}$$

of all $4^6 = 4{,}096$ joint random sets A^j.

For the mode's limit states, g_i, this can be accomplished very easily by using their monotonicity properties. As we have already mentioned, in general, the system limit state function, g_{syst}, is not monotonic. For the lower bounds, $\underline{b}_{\text{syst}}^j$, which are needed to calculate the upper probability [equation (41)], one has:

$$\underline{b}_{\text{syst}}^j = \min_{x \in A^j} g_{\text{syst}}(x) = \min_{x \in A^j} \min_i g_i(x) = \min_i \min_{x \in A^j} g_i(x) = \min_i \underline{b}_i^j, \tag{64}$$

which is also very easy to compute. As for the upper bounds, $\overline{b}_{\text{syst}}^j$, which are needed to calculate the lower probability [equation (42)], one has:

$$\overline{b}_{\text{syst}}^j = \max_{x \in A^j} g_{\text{syst}}(x) = \max_{x \in A^j} \min_i g_i(x) \le \min_i \max_{x \in A^j} g_i(x) = \min_i \overline{b}_i^j. \tag{65}$$

Alternatively, one may calculate the left-hand side of equation (65) by solving the linear optimisation problem:

$$\text{maximise } y, \tag{66}$$

subject to

$$g_i(x) \ge y \quad i = 1, \ldots, m, \tag{67}$$

$$x_k \in I_k \quad k = 1, \ldots, n, \tag{68}$$

where $I_1 \times \cdots \times I_m = A^j$ is the joint focal set generated by the Cartesian product of marginal focal sets (intervals) I_k.

Numerical results

The probabilities of failure for the single failure modes are as follows:

$$P(F_1) \in [0, 2.319 \cdot 10^{-1}]$$
$$P(F_2) \in [0, 4.410 \cdot 10^{-2}]$$
$$P(F_3) \in [0, 2.021 \cdot 10^{-1}]$$
$$P(F_4) \in [0, 4.938 \cdot 10^{-2}].$$

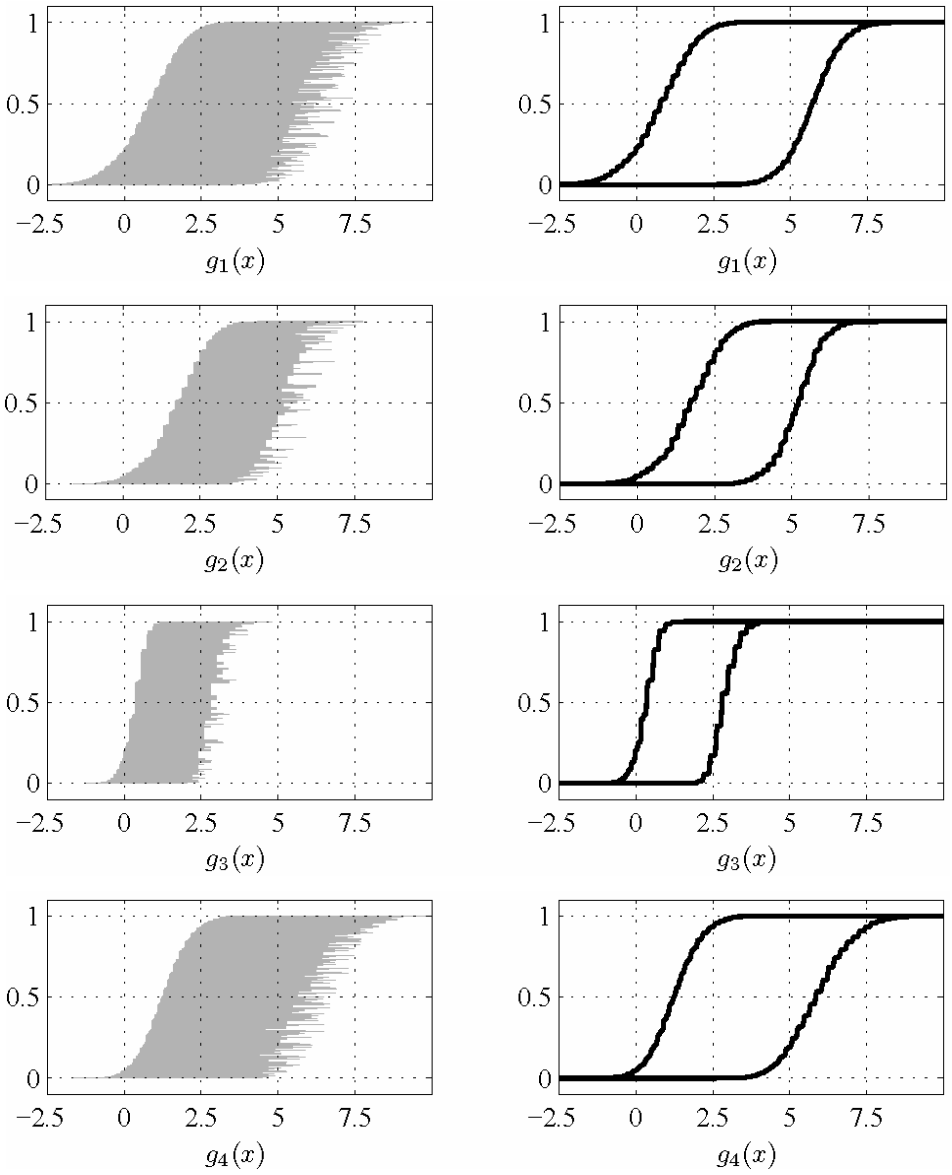
The system reliability bounds yield [equations (52) to (53)]:

$$p_f^- = 0$$
$$p_f^+ = 5.2748 \cdot 10^{-1}.$$

The system's probability of failure obtained using g_{syst} is comprised within the bounds [equations (66) to (68)]:

$$p_f \in [0, 3.65221 \cdot 10^{-1}].$$

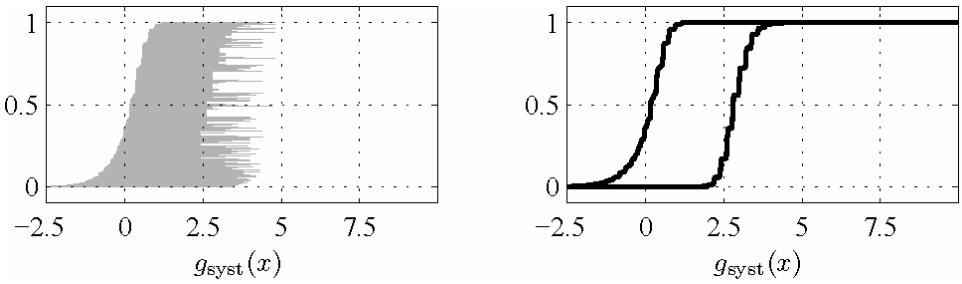
Figure 7 Images of the joint focal sets as stacks and the *p*-boxes of the probabilities of failure for the four failure modes (Section 5.4.2)



For the four failure modes, Figure 7 depicts the images of the joint focal sets (as stacks in the figures to the left) and their relevant *p*-boxes (figures to the right). The images of the joint focal sets are sorted by the left interval bounds and visualised as a stack where the distance of a focal set to the focal set directly below is exactly the joint weight. So the contour of the left side of the stack coincides with the left side of the *p*-box. Figure 8 depicts the results for the system obtained by using g_{syst}.

Some comments on the more interesting upper bounds are in order: in general, the interactions between the probability intervals $[\underline{P}(F_i), \overline{P}(F_i)]$ results in $p_f^+ \neq \overline{p}_{f,\text{ex}}^+$. On the other hand, it is not sure that $\overline{p}_f$ is the upper probability for the strong independence assumption because g_{syst} is not monotonic and therefore, in general, the results for the random set independence are not equal to the results for strong independence.

Figure 8 Images of the joint focal sets as stacks and the *p*-box for the probability of failure of the system (Section 5.4.2)



5.4.3 Random sets, ODM, Monte-Carlo

Using only four focal sets leads to a very rough approximation of the *p*-boxes. If we use a finer discretisation, e.g., ten focal sets, we would get a better approximation, but then we have to compute 10^6 images of joint focal sets. The idea is now not to consider all 10^6 joint focal sets, but only, say, N = 10,000 randomly chosen sets. Notice that the probability bounds are no longer automatically verified. Since the weights, m_k, are probability distributions, the focal sets are chosen as follows:

1 for each variable, x_k, choose N focal sets using the weights m_k

2 the *j*th joint focal set is the Cartesian product of all *j*th chosen marginal focal sets, $j = 1,\ldots,N$

3 the weights of these joint focal sets are $1/N$ for all sets.

In the following, we use ten focal sets (discretisation : $y^{(i)} = i/10, i = 0,\ldots,10$) or 10,000 focal sets (discretisation : $y^{(i)} = i/10{,}000, i = 0,\ldots,10{,}000$) and N = 10,000 Monte-Carlo simulations. The finer discretisation results in ten to 100 times narrower intervals for the probabilities of failure, see Tables 2, 3 and 4. In Figure 9, the coarse discretisation is depicted and in Figures 10 and 11, the corresponding images of the joint focals and the *p*-boxes are depicted.

Table 2 Probabilities of failure for the single failure modes

Discretisation using ten focal sets	*Discretisation using 10,000 focal sets*
$P(F_1) \in [0,\ 1.251 \cdot 10^{-1}]$	$P(F_1) \in [0,\ 1.45 \cdot 10^{-2}]$
$P(F_2) \in [0,\ 2.670 \cdot 10^{-2}]$	$P(F_2) \in [0,\ 3.00 \cdot 10^{-4}]$
$P(F_3) \in [0,\ 9.680 \cdot 10^{-2}]$	$P(F_3) \in [0,\ 6.30 \cdot 10^{-3}]$
$P(F_4) \in [0,\ 1.710 \cdot 10^{-2}]$	$P(F_4) \in [0,\ 1.00 \cdot 10^{-4}]$

Table 3 The system reliability bounds [equations (52) and (53)]

Discretisation using ten focal sets	*Discretisation using 10,000 focal sets*
$p_f^- = 0$	$p_f^- = 0$
$p_f^+ = 2.657 \cdot 10^{-1}$	$p_f^+ = 2.12 \cdot 10^{-2}$

Table 4 The system's probability of failure obtained using g_{syst} [equations (66) to (68)]

Discretisation using ten focal sets	*Discretisation using 10,000 focal sets*
$p_f \in [0,\ 2.042 \cdot 10^{-1}]$	$p_f \in [0,\ 2.04 \cdot 10^{-2}]$

Figure 9 Lower and upper CDFs of truncated normal distributions and the random sets obtained by ODM (Section 5.4.3, coarse discretisation)

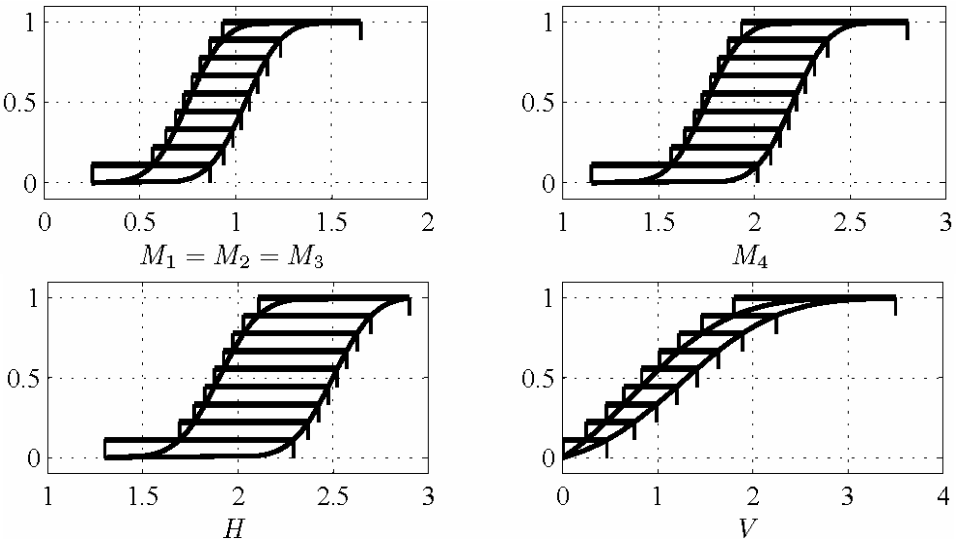


Figure 10 Images of the joint focal sets as stacks and their relevant *p*-boxes of the probabilities of failure for the four failure modes (Section 5.4.3, coarse discretisation)

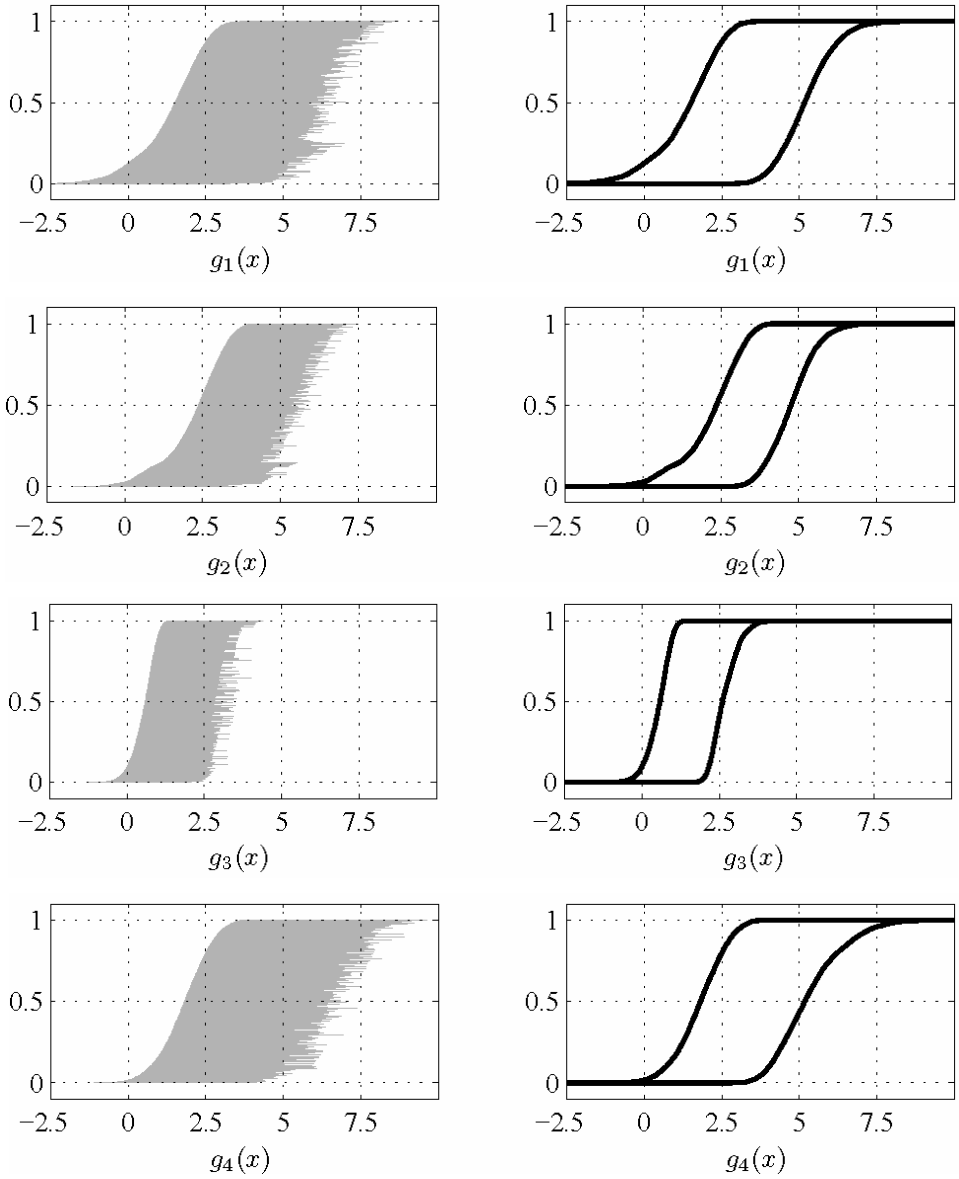
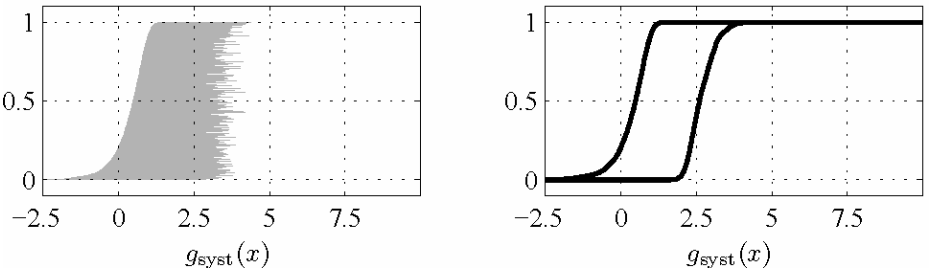


Figure 11 Images of the joint focal sets as stacks and the *p*-box for the probability of failure of the system (Section 5.4.3, coarse discretisation)



5.5 Criteria for using or not using system reliability bounds

As already mentioned in the Introduction, the main reason to use system reliability bounds is the high computational effort to obtain the probability of failure of the system, p_f, when the computation of the probability of failure for each mode, $P(F_i)$, is easier to obtain. Based on the algorithms and numerical examples presented, the following criteria can be offered to the analyst.

5.5.1 Linear g_i, monotonicity always in the same direction

- Parameterised probabilities (normal distribution): Cheap calculation of single mode's probabilities of failure, but expensive calculation of system's probability of failure because of the non-linearity of g_{syst}.
- Random sets: Cheap calculation of single mode's probabilities of failure; exact bounds can be obtained. However, the calculation of the system's probability of failure is also cheap because of the monotonicity of g_{syst} in this case. All of these bounds calculated using the strong independence assumption coincide with those obtained using the random set independence assumption. Therefore, one should compute the probability of failure for the system directly by using g_{syst} and one should not use the system's reliability bounds.

5.5.2 Linear g_i, monotonicity not always in the same direction

- Parameterised probabilities (normal distribution): Cheap calculation of single mode's probabilities of failure, expensive calculation of system's probability of failure.
- Random sets: Cheap calculation of single mode's probabilities of failure; exact bounds cannot be calculated. As for the system, cheap calculation of the upper probability of failure, but more expensive calculation of the lower probability of failure. The assumptions of random set independence and strong independence lead to different bounds.

5.5.3 Non-linear g_i, monotonicity always in the same direction

- Random sets: Cheap calculation of single mode's probabilities of failure; exact bounds can be obtained. However, the calculation of the system's probability of failure is also cheap because of the monotonicity of g_{syst} in this case. All of these bounds calculated using the strong independence assumption coincide with those obtained using the random set independence assumption. Therefore, one should compute the probability of failure for the system directly by using g_{syst} and one should not use the system's reliability bounds.
- Parameterised probabilities (normal distribution): If the g_i are non-linear, the effort to compute one single mode's probability is the same as calculating the system's probability of failure. For the computations, we have to make use of the monotonicity of g_{syst} and of Monte-Carlo simulations.

5.5.4 Non-linear g_i, monotonicity not always in the same direction

- Random sets: Cheap calculation of single mode's probabilities of failure; exact bounds cannot be calculated. As for the system, cheap calculation of the upper probability of failure, but more expensive calculation of lower probability of failure. The hypotheses of random set independence and strong independence lead to different bounds.
- Parameterised probabilities (normal distribution): As in Section 5.5.3, but the effort to compute the system's probability of failure is greater because now, g_{syst} is not monotonic.

5.5.5 Non-linear g_i, not monotonic

- Parameterised probabilities (normal distribution) and random sets: In general, all bounds are expensive to compute and one should directly compute the probability of failure for the system.

6 Conclusions

Algorithms were presented for calculating bounds on the probability of failure of a series system when no information is available on the dependencies among mode's failure probabilities and when input data on the variables that control the components' probability of failure is constrained either by parameterised probability measures or random sets. The assumption of independence for the input variables has been folded into the formulation.

It turns out that calculating the failure probability bounds for each failure mode and then calculating bounds for the system is advantageous (with respect to directly calculating the system's probability of failure) when:

- the system's safety margin is linear in the input variables; monotonicity is always in the same direction and input variables are given as parameterised probabilities
- the system's safety margin is linear in the input variables; monotonicity is not always in the same direction and input variables are given either as parameterised probabilities or as random sets
- the system's safety margin is non-linear in the input variables; monotonicity is not always in the same direction and input variables are given as random sets.

The paper also shows how discretising given the upper and lower CDF envelopes using the ODM leads to validated bounds on the system's probability of failure.

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Appendix

Comparison between ODM and the iterative scaling method

Hall and Lawry (2004) introduced the Iterative Rescaling Method (IRM) with the aim of discretising set $\mathcal{M}_k$ in equation (33) into a random set with a finite number of focal elements. Their approach consists in discretising the abscissa, rather than the ordinate as is done in Figure 3. The interval of interest for variable x, I, is partitioned into non-overlapping segments, I_i and a heuristic algorithm is used to allocate the probability assignment onto the set of all unions of $I_i - \mathrm{s}$.

If the aim of the analysis is to generate bounds on the probability of a quantity being less than (or greater than) some threshold value, as, for example, is the case in reliability analysis, then the ODM is more accurate than the IRM at a given level of discretisation because the ODM approximation touches the lower and upper cumulative curves which it approximates, whereas this is not necessarily the case with IRM. On the other hand, if the aim is to support arbitrary information processing which might, for example, involve calculating the probability of a quantity of interest lying in some arbitrary closed interval, then the ODM gives a poorer approximation than the IRM. This is simply because bounds on cumulative distributions (which ODM discretises) contain much less information than bounds on density functions (which IRM discretises).

In order to elucidate this comparison, let us reconsider the numerical example used by Hall and Lawry (2004), in which the lognormal distribution is used with mean in the range [0.1, 1.0], standard deviation in the range [0.1, 0.5] and $I = [0.172, 17.453]$. A uniform discretisation is used for the ODM; the focal elements are as follows: [0.172, 2.604], [0.891, 3.372], [1.154, 17.453] and all with m = 0.333. The IRM based on the partition obtained by projecting the focal elements extremes onto I yields (Hall and Lawry, 2004):

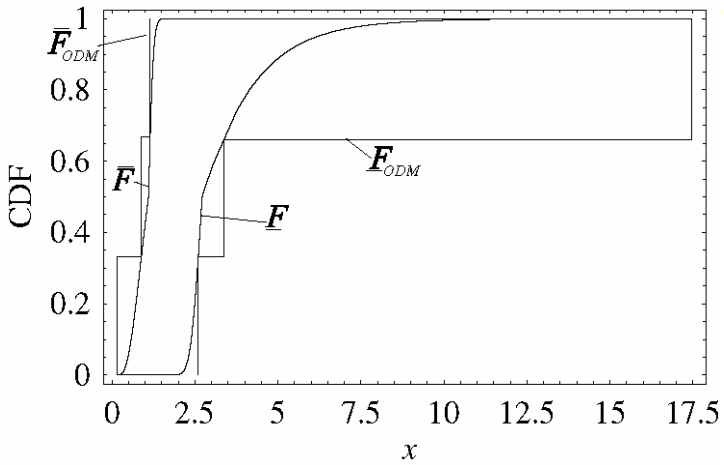
$$
\begin{aligned}
&[1.154, 2.604), && m = 0.333;\\
&[0.172, 0.891) \cup [1.154, 3.372), && m = 0.010;\\
&[0.172, 0.891) \cup [1.154, 2.604) \cup [3.372, 17.453], && m = 0.005;\\
&[0.891, 3.372), && m = 0.318;\\
&[0.891, 17.453), && m = 0.008;\\
&[0.172, 17.453], && m = 0.318.
\end{aligned}
$$

First, notice that three focal elements are used in the ODM discretisation, versus seven in IRM discretisation. However, as one can see in Figure 12, the two discretisations induce the upper (and lower, respectively) CDFs that are undistinguishable from the other at the scale of the figure, with the ODM approximation actually touching the upper and lower CDFs and the IRM approximation getting very close to the same CDFs. Since the upper and lower CDFs uniquely generate a convex set of probability measures by natural extension (Walley, 1991), the convex set of probability measures generated by the CDFs induced by ODM is contained in the convex set generated by IRM.

Since seven focal elements were used in the IRM discretisation and three in the ODM, the IRM leads to higher calculation burden than the ODM. This is especially true when joint random sets must be considered, whose focal elements are the Cartesian products of the marginal focal elements. Figure 13 shows the ODM approximation

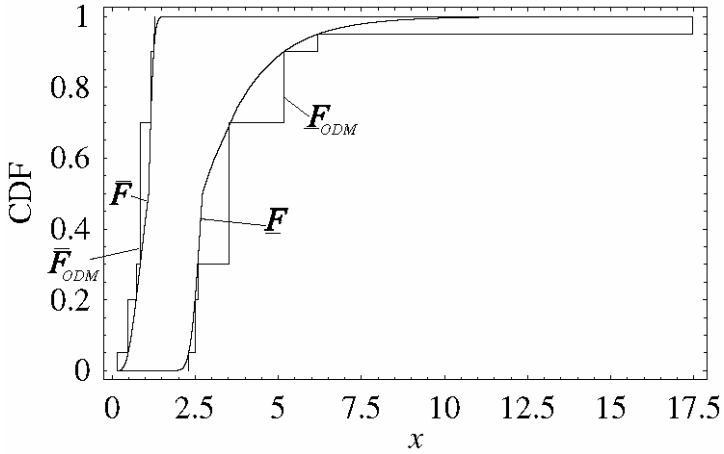
obtained with seven focal elements. By comparing with Figure 12, the improvement over the IRM is noticeable.

Figure 12 IRM and ODM approximation to upper and lower CDFs



Note: IRM uses seven focal elements and ODM uses three focal elements.

Figure 13 ODM approximation to the same CDF as in Figure 12 (seven focal elements)



On the other hand, consider the probability that x lies in a specific interval comprised in [0.172, 17.453]. Table 5 illustrates that IRM yields a better approximation than ODM.

Table 5 Probability bounds for some intervals calculated with IRM and ODM

Interval	*Probability bounds from IRM*	*Probability bounds from ODM*
[0.891, 1.154)	[0.000, 0.651]	[0, 0.667]
[1.154, 2.604)	[0.333, 1.000]	[0, 1.000]
[2.604, 3.372)	[0.000, 0.654]	[0, 0.667]

Call $\mathcal{M}^r$ the set of probability measures induced by the measurable selections of the random set, $S(\Gamma)$, (Miranda et al., 2005). These are the random variables defined on Ω whose values are included in the images of the random set $\mathfrak{X}(\omega)$. Call $\mathcal{M}^p$ the set of probability measures (credal set) generated by the upper and lower probabilities induced by the same random set [equation (33)].

Lemma: Let $G: X \to \mathbb{R}$ be a bounded random variable and let $(C)\int$ indicate the Choquet integral (Choquet, 1953–1954). Let an upper bar indicate closure of a set and let 'Conv(A)' indicate the convex hull of set A. Let $\beta_{\mathbb{R}}$ indicate the Borel σ-field on $\mathbb{R}$. When using ODM:

1 $\overline{P}(H) = \max \mathcal{M}^r(H) \forall H \in \beta_{\mathbb{R}}$

2 $\underline{P}(H) = \min \mathcal{M}^r(H) \forall H \in \beta_{\mathbb{R}}$

3 $(C)\int G\mathrm{d}\overline{P} = \sup\{\int G\mathrm{d}P_V : V \in S(\Gamma)\}$

4 $(C)\int G\mathrm{d}\underline{P} = \inf\{\int G\mathrm{d}P_V : V \in S(\Gamma)\}$

5 $\mathcal{M}^p$ is convex and closed and $\mathcal{M}^p = \overline{\mathrm{Conv}(\mathcal{M}^r)}$

6 $\mathcal{M}^p = \overline{\mathcal{M}^r} \Leftrightarrow \overline{\mathcal{M}^r}$ is convex.

Proof: Propositions 1 and 2: in ODM, the variables are defined on a Euclidean space. The Euclidean space equipped with the Euclidean topology is separable (i.e., has a countable dense subset) because it has the lattice of rational numbers as a countable dense subset and thus every open ball contains a point whose coordinates are all rational. The multi-valued mapping, Γ, is compact and complete because ODM generates compact and complete focal elements (a set, A, is complete if every Cauchy sequence of points in A has a limit that is also in A). Thus, random sets generated by ODM satisfy the hypotheses of Theorem 1 in Miranda et al. (2003), which ensures that Propositions 1 and 2 are true. Propositions 3 and 4: random sets generated by ODM satisfy the hypotheses of Theorem 2 in Miranda et al. (2003), which ensures that Propositions 3 and 4 are true. Proposition 5: the multi-valued mapping of the ODM generates random closed intervals. They constitute a particular case of compact random sets on Polish spaces and Theorem 3.2 in Miranda et al. (2005), states Proposition 5. Proposition 6: Theorem 1 in Miranda et al. (2003) ensures that ODM random sets satisfy the hypotheses of Theorem 4 in Miranda et al. (2003), which Proposition 6 states.

Since, in general, the focal elements generated by IRM are not complete and are neither closed nor open, the results in Miranda et al. (2003) are not applicable. Thus, the question is still open as to whether the upper and lower probabilities induced by an IRM random set can be used to summarise the information on the distribution of the original random variable without a substantial loss of precision. Indeed, all one can say at the moment is that $\mathcal{M}^r \subseteq \mathcal{M}^p$ [Miranda et al., (2003), p.386].