Intermittency and Clustering in a System of Self-Driven Particles

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Intermittent behavior is shown to appear in a system of self-driven interacting particles. In the ordered phase, most particles move in the same approximate direction, but the system displays a series of intermittent bursts during which the order is temporarily lost. This intermittency is characterized and its statistical properties are found analytically for a reduced system containing only two particles. For large systems, the particles aggregate into clusters that play an essential role in the intermittent dynamics. The study of the cluster statistics shows that both the cluster sizes and the transition probability between them follow power-law distributions. The exchange of particles between clusters is shown to satisfy detailed balance.

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Intermittency is one of the many interesting behaviors that can be observed in systems far from equilibrium. It appears in several different systems, including biological ones [1–4]. In the context of fluid dynamics, intermittency can be observed in the transition from a laminar to a turbulent regime, where the stationary flow is interrupted by chaotic bursts occurring at irregular time intervals [5]. As the Reynolds number is increased, these bursts appear more and more often until the flow becomes fully turbulent. In spite of its ubiquity, a detailed theory of the intermittent behavior exists only in models with few degrees of freedom, and its full understanding in complex systems has not yet been achieved.

In this Letter we report the existence of intermittent dynamics in the self-driven particle model (SDPM) introduced in 1995 by Vicsek *et al.* [6]. This model was proposed as a minimal description of the collective motion of large groups of organisms such as herds of quadrupeds or groups of migrating bacteria. It displays a nonequilibrium phase transition from an ordered state in which all particles head in approximately the same direction to a disordered state where they move randomly. While more realistic models for swarming have been developed (see [7] and references therein), the SDPM remains an important referent as a simple model displaying a phase with self-organized collective motion. It therefore has become increasingly important to understand all the nonequilibrium properties of the SDPM dynamics. In particular, we show that the aggregation of particles into clusters plays an essential role in the intermittent dynamics.

The SDPM is defined for N point particles with positions $\{\vec{x}_i(t)\}_{i=1}^N$ and on-plane velocities $\{\vec{v}_i(t)\}_{i=1}^N$ in a 2D periodic square box of sides L. Their self-driven character is imposed by fixing the magnitude of all velocities to a constant $v_0 = |\vec{v}_i(t)|$ (taken equal to 1 for all i). At every time t, the angle $\theta_i(t)$ of each $\vec{v}_i(t)$ is updated by $\theta_i(t + \delta t) = \text{angle}[\sum_{|\vec{x}_i - \vec{x}_j| < R} \vec{v}_j(t)] + \xi_i(t)$. The first term gives the direction of the mean velocity of all particles located within an interaction range R of $\vec{x}_i$. The second term $\xi_i(t)$ is a uniformly distributed random variable in the interval $[-\eta/2, \eta/2]$. Each $\vec{x}_i$ is then updated through the kinematic rule $\vec{x}_i(t + \delta t) = \vec{x}_i(t) + \vec{v}_i(t + \delta t)\delta t$. The dynamics of the system can thus be set from deterministic to fully random by changing the value of the noise intensity η from 0 to 2π.

The degree of order in the system at time t is measured by the magnitude of the system-average velocity, denoted as $\psi(t)$. When all the $\vec{v}_i$ are randomly oriented $\psi(t) = 0$, and when all are aligned $\psi(t) = 1$. By averaging $\psi(t)$ over large enough times, it has been shown that $\Psi(\eta) \equiv \langle \psi(t) \rangle_t$ undergoes a second order phase transition at a critical value of $\eta = \eta_c$ [6]. One has $\Psi(\eta) > 0$ for $\eta < \eta_c$(ordered state), and $\Psi(\eta) = 0$ for $\eta \geq \eta_c$(disordered state). While this has been one of the main results of the SDPM up to now [6,8–11], it hides important information about the system dynamics since the fluctuations of $\psi(t)$ turn out to be nontrivial.

Figures 1(a) and 1(b) show the value of $\psi(t)$ as a function of time for a SDPM with $R = 1$, $v_0 \delta t = 0.1$, $\eta = 1$, mean density $\rho = N/L^2 = 0.4$, and two different system sizes: $N = 5000$ and 500. For these values of the parameters, the system is subcritical ($\eta_c \approx 1.6$) and $\langle \psi(t) \rangle_t$ converges to $\Psi \approx 0.62$ as we average over longer and longer time intervals. However, it is apparent that $\psi(t)$ exhibits strong intermittent fluctuations. Figure 1(d) displays the probability density function (PDF) $P(\psi)$ of Figs. 1(a) and 1(b). Only the small fluctuations about the mode of the $P(\psi)$ distribution follow a Gaussian behavior (indicated by the solid curves), which we associate with a laminar flow. The large fluctuations in $\psi(t)$ produce the exponential behavior observed in $P(\psi)$,

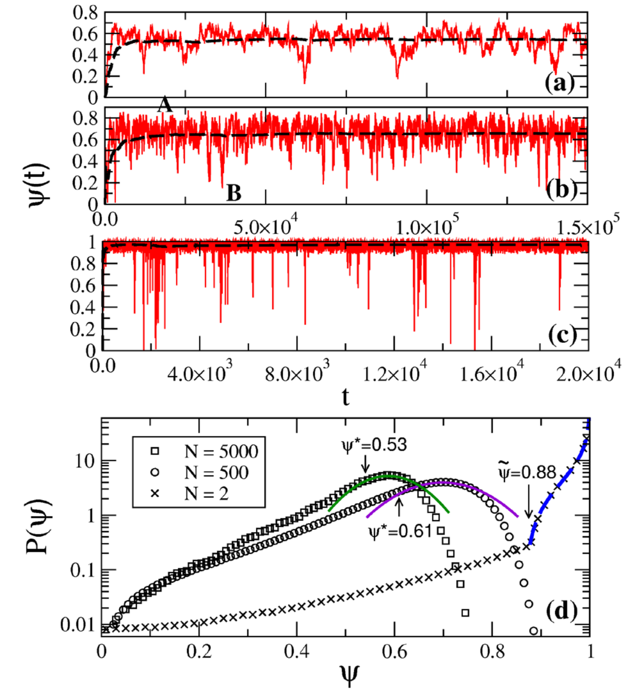

FIG. 1 (color online). Intermittent behavior of $\psi(t)$ for three different system sizes: (a) $N = 5000$, (b) $N = 500$, and (c) $N = 2$. The curves in bold show the cumulative time average converging to $\Psi(\eta)$. (d) $P(\psi)$ for the systems in the previous three graphs. The solid curves are Gaussian fits around the mode of each distribution. The ψ^*'s indicate the points at which $P(\psi)$ deviates 10% from a Gaussian. Since large fluctuations mainly shift ψ towards the $\psi = 0$ (disordered) bound, $P(\psi)$ displays a strong skewness. $\tilde{\psi}$ is the lower bound of the laminar region for the $N = 2$ case. The dashed curve is the analytic expression (1) for the two-particle laminar regime.

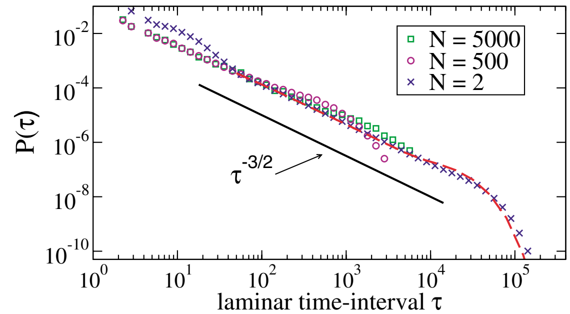

FIG. 2 (color online). Numerical PDF of the duration of the laminar flow intervals τ for the systems in Fig. 1. The dashed curve is the inverse Laplace transform of Eq. (2).

which is characteristic of an intermittent burst. It resembles the one obtained for a confined turbulent hydrodynamic flow driven at a constant Reynolds number in the intermittent regime [1,12]. We have observed this intermittent behavior for a wide range of subcritical values of the parameters. In all the tested cases the power spectrum of $\psi(t)$ displays a $1/f^2$ law, with f the frequency mode. However, as we decrease the noise amplitude η and increase the density ρ, the amplitude of the Gaussian fluctuations becomes smaller and the laminar time intervals between intermittent bursts grow. A standard intermittent signal analysis consists of obtaining the statistics of the duration τ of these intervals [5]. We measure each τ as the time interval during which $\psi(t) > \psi^*$ continuously, with the laminar-behavior threshold ψ^* set where $P(\psi^*)$ deviates 10% from the Gaussian distribution [see Fig. 1(d)]. The following results, however, are not critically sensitive to the exact choice of ψ^*. Figure 2 shows that the PDF of τ behaves as $P(\tau) \sim \tau^{-3/2}$ for the $N = 500$ and 5000 systems. All other tested values of the parameters yield the same $\tau^{-3/2}$ behavior.

An important insight into the origin of the intermittent behavior is obtained by considering a system with only two particles. Figure 1(c) shows that intermittency is also present for $N = 2$, but with different characteristics. The intermittency in this case can be understood by analyzing the temporal evolution of the distance r between the two particles. When $r < R$, the two particles are in a *bound* state (corresponding to the laminar flow) in which the direction of their velocities fluctuates (due to the noise) within the range $[-\eta/2, \eta/2]$ around the common angle $\Theta(t) = \text{angle}[\vec{v}_1(t) + \vec{v}_2(t)]$. In contrast, when $r \geq R$ the particles are in an *unbound* state (producing an intermittent burst) in which they move independently of each other following a persistent random walk [13]. Within this picture, the PDF of $\psi(t)$ in the laminar region ($\psi \geq \tilde{\psi} \approx 0.88$) is found to be

$$P(\psi) = \frac{4}{\eta^2}\left(\frac{\eta - \arccos(2\psi^2 - 1)}{\sqrt{1 - \psi^2}}\right) + P_u, \qquad (1)$$

where P_u is the unbound state contribution estimated by $P_u = P(\tilde{\psi})$. Figure 1(d) shows that this result perfectly matches the simulation data.

To compute $P(\tau)$, we note that τ is equivalent to the first-passage time needed for r to exit the $r \leq R$ region. The dynamics of r can be approximated by a one-dimensional random walk with a reflecting boundary at $r = 0$ and an absorbing one at $r = R$. For $v_0 \delta t \ll R$ this can be described by a diffusion equation for r with constant $D = \Delta r^2/(2\delta t)$. We take Δr as the typical step size: $\Delta r \sim v_0 \delta t \sin(\eta/2)$. Using standard techniques [14] we solve this equation with initial condition $r_0 = R - \Delta r$ (the typical distance when the two particles bind). The Laplace transform of $P(\tau)$ is found to be

$$\hat{P}(s) = \cosh(r_0\sqrt{s/D})/\cosh(R\sqrt{s/D}), \qquad (2)$$

where s is the conjugate time variable. In Fig. 2, we show that this result provides an excellent approximation to the $P(\tau)$ computed numerically. Both the numerical and theoretical curves display an exponential cutoff at $\tau \sim 10^5$ and a bump in the $10^4 < \tau < 10^5$ interval (features produced by the trajectories that are reflected on the $r = 0$ boundary before escaping). Both also present the same $\tau^{-3/2}$ behavior (equal to the escape flux from a half plane) for

$\tau < 10^4$. Surprisingly, this is the same behavior followed by the larger systems with $N = 500$ and 5000, suggesting that these cases could be described with similar analytic tools.

For $N > 2$, though, the description is more complicated. The particles aggregate into clusters of different sizes [15]. (Dynamical clusterization was predicted in Ref. [10] as density fluctuations for a continuous field swarming model and in Ref. [16] for "granular gases.") Figure 3 presents two snapshots of the system corresponding to the points labeled A [$\psi(t) \approx 0.8$] and B [$\psi(t) \approx 0.2$] in Fig. 1(b). In both, all particles within a given cluster move approximately in the same direction (indicated by the big arrows). However, in snapshot A all clusters move in a similar direction, whereas in snapshot B, they head in different directions.

While for $N = 2$ a particle can be in only a bound or an unbound state, for $N > 2$ it can be in any of multiple states given by all the possible sizes of the cluster it belongs to [17]. Figure 3 shows the time-averaged cluster size distribution computed numerically for the $N = 500$ and the $N = 5000$ systems. The curves follow a power-law distribution that breaks down as n approaches N due to finite size effects. (The value of the exponent depends on the system parameters, becoming more negative when η is increased or ρ is reduced.) This power-law behavior indicates the lack of a typical cluster size, confirming the need for a multiple-state description of the particle dynamics.

Although a two-state description of individual particles is no longer valid, each cluster can be interpreted as a single big particle that still follows a bind-unbind dynamics. To support this view, we present in Figs. 4(a)–4(c) the magnitude of the velocity of the whole system [$\psi(t)$], of the largest cluster [$\psi_L(t)$], and of the second largest cluster [$\psi_{SL}(t)$] for the $N = 500$ system. It can be seen that $\langle\psi_L(t)\rangle_t$ and $\langle\psi_{SL}(t)\rangle_t$ are bigger than $\langle\psi(t)\rangle_t$, which shows that a cluster is a coherent collective structure that displays, on average, a larger amount of internal order than the entire system. Within this picture, the intermittent bursts of $\psi(t)$ would be mainly due to changes in the relative direction of motion of the different clusters. This is confirmed by Fig. 4(d), where we plot the cosine of the angle between the velocities of the largest and second largest clusters. It shows that the intermittent bursts in $\psi(t)$ coincide mainly with the times at which these clusters move in opposite directions (the cosine approaches -1). The correlation between these two quantities is further highlighted by the fact that their corresponding power spectra are almost identical [Figs. 4(a) and 4(d), right panels].

Nevertheless, the picture in which each cluster is considered as a single big particle presents a richer dynamics. First, as Figs. 4(b) and 4(c) show, each cluster exhibits an internal intermittent dynamics (which appears in our simulations to be mainly related to changes in their

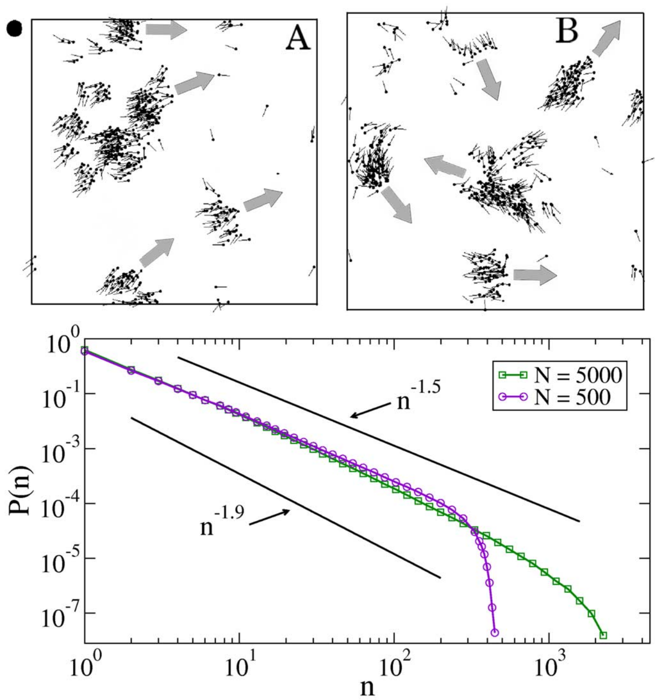

FIG. 3 (color online). Snapshots of the SDPM corresponding to the points labeled A and B of Fig. 1(b). The solid dots represent the particles and the tails indicate the direction of motion. The black circle in the upper-left corner shows the size of the interaction vicinity. The arrows show the direction of motion of the clusters. Bottom: Probability $P(n)$ of having a cluster with n particles for the $N = 5000$ (squares) and the $N = 500$ (circles) systems. For $10 < n \ll N$ the curves follow the power law $P(n) \sim n^{-\beta}$ with $1.5 < \beta < 1.9$.

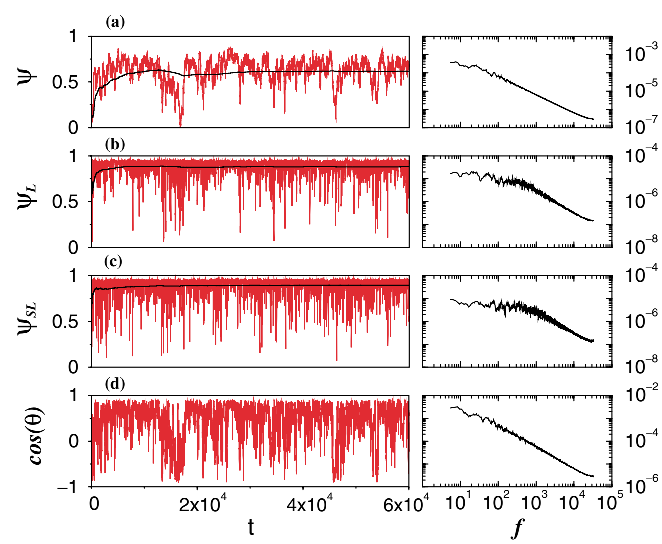

FIG. 4 (color online). (a) Graph of $\psi(t)$ for a system with $N = 500$. (b) Magnitude of the velocity of its largest cluster. (c) Magnitude of the velocity of its second largest cluster. (d) Cosine of the angle between the velocities of the largest and second largest clusters. The panels on the right are the power spectra of the data plotted on the respective left panels.

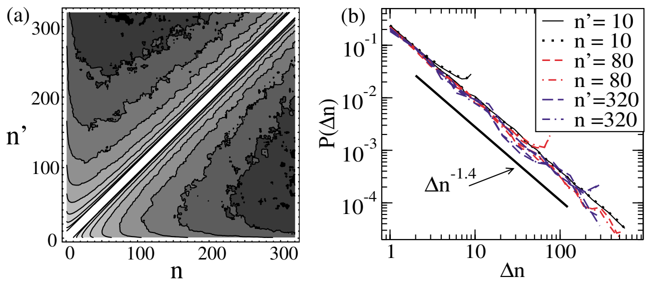

FIG. 5 (color online). Transition probability $P(n, n'; t, t + \delta t)$ between different cluster sizes for the $N = 5000$ system. (a) Contour plot displaying with lighter tones the higher values of $P(n, n'; t, t + \delta t)$. Note that clusters mostly gain or lose a few particles and that the process obeys detailed balance. (b) Plots of selected rows and columns of (a) as functions of $\Delta n = |n' - n|$. All curves follow the same approximate power law $P(\Delta n) \sim \Delta n^{-1.4}$.

size). Second, in order to properly describe a cluster, one has to weigh all its relevant properties by the number of particles it contains. And third, the simple binding and unbinding dynamics appearing for $N = 2$ becomes here the dynamics of the transition between any two different cluster sizes. We study below the dynamics of these transitions as a first step in describing the intermittency in a many-particle system.

In Fig. 5 we present the transition probability between different cluster sizes for the $N = 5000$ system. In order to describe all possible cluster evolutions as a transition between two states, the contour plot on Fig. 5(a) displays the probability $P(n, n'; t, t + \delta t)$ that a particle is contained at time t in a cluster of size n and at time $t + \delta t$ in one of size n'. It shows that events involving the loss or gain of a few particles are the most common. Its symmetry with respect to the $n = n'$ diagonal indicates that (within the precision of our statistics) the process obeys detailed balance. Figure 5(b) presents the probability for a particle to be in a cluster that changes its size by $\Delta n = |n' - n|$ particles for clusters that become (or cease to be) of sizes $n = 10$, 80, and 320, by plotting the data in the corresponding rows (or columns) of Fig. 5(a). Surprisingly, it shows that there is an equal probability for a cluster to lose or to gain Δn particles, with a $P(\Delta n) \sim \Delta n^{-1.4}$ behavior for all cluster sizes.

The ubiquity of simple power laws in the complex cluster dynamics suggests that a theory for the intermittency in the SDPM could be developed by using a renormalization approach in which each cluster is considered as a single big weighted particle. This formulation, however, would require a model for the cluster behavior that is not provided by the existing field equation description of the SDPM [10]. Instead, a promising new approach consists of mapping the particle interactions to a dynamic network [18,19]. The analysis presented in this Letter shows that intermittency is a generic phenomenon in the SDPM. This brings into question its presence and role in more realistic swarming models and in experimental systems. Our results are compatible with the few existing experiments able to measure intermittency or clustering in biological swarms [20]. Our tools could be used to understand these behaviors in this and other contexts, starting from a microscopic description. We hope that our work will trigger further experimental research.

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