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FIRST PART

On the general study of iteration of a rational substitution.
The set of the singular points of the iteration.

7. For all this Memoir, we have defined φ(z) the rational fraction whose iteration will be studied; then we will define

$$z_1 = \varphi(z)$$
$$z_2 = \varphi[\varphi(z)] = \varphi_2(z), \;\; .... \quad z_n = \varphi(z_{n-1}) = [\varphi_{n-1}(z)]$$

The points $z_1, z_2, \ldots, z_n, \ldots$ are defined the *images* of rank 1, 2, n, ... of z.
Inversely, z is *preimage* of rank 1 of z_1 or of rank n for z_n.
If φ is a fraction of degree k, any point has k preimages.
The rational functions $\varphi_2(z)$, ... $\varphi_n(z)$ are defined the *iterations of* φ(z). The problem of the iteration consists in studying the sequence $z, z_1, z_2, \ldots, z_n, \ldots$ and mainly in defining what is the *derived* set from the previous countable set and *how this derived set depends on the choice of* z, *the initial point*.
Up to present, this problem has been studied only from a local view point; obviously, this restriction is explained by the usual definition of the analytic function: by a power sequence converging to the neighbourhood of a point: if one wants that the images of a point z can be progressively defined, it is necessary that the values z_1, assumed by $f(z)$ in the domain where the function itself is defined, do not escape from the same domain.
Together with almost all authors engaging this question up to present, my interest is to study the images in the neighbourhood of only the ζ points, roots of $z = f(z)$ and satisfying $| f'(z) | < 1$; it happens that the values of $| z_1 - \zeta |$ are $< | z - \zeta |$, when $| z - \zeta |$ is small enough so that the images remain in the neighbourhood of ζ, where $f(z)$ is defined ([1]). In order to study the iteration of z, whatever z is in the analytic plane, it is necessary that $f(z)$ and all its iterations would be perfectly defined for any point of the plane. It is necessary that only the point at infinity may be an essential point of $f(z)$.
It induces us to suppose that f is a *rational function* or *entire transcendental*.

Only in these two cases, any z has well defined image points at a finite distance and the problem assumes a sense.
Therefore, for the general study we are going to do, we will suppose that the function to be iterated is a *rational* $z_1 = \varphi(z)$.
Our conclusions will be extended to the iteration of entire transcendental functions.

When φ is rational, *the point at infinity will be compared to any ordinary point*, that's to say by considering the *extended plane* ([2]).

([1]) The points z = f(z) satisfying | f'(z) | < 1 are called limit point with uniform convergence.
([2]) Sometimes it is preferred to take the Riemann sphere as a support for the variable z.

The first remark to do is the following: all iterations $\varphi_2(z)$, …, $\varphi_n(z)$, ... are rational, and if one supposes (except when φ has first degree: this case is very easy and well known) that φ is has degree $k > 1$, then their related degrees are k^2, k^3, ..., k^n ... .

None is the same as z. Simply, an *arbitrary point* of the plane has k preimages of rank 1, k^2 preimages of rank 2, ... k^n preimages of rank n, ...

Local studies showed the importance of circular groups of points $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$, roots of equations $z = \varphi_n(z)$, when, for one of these groups, one gets

$$|\varphi'_n(\zeta)| = |\varphi'_n(\zeta_1)| = \ldots = |\varphi'_n(\zeta_{n-1})| = |\varphi'(\zeta)\varphi'(\zeta_1)\ldots\varphi'(\zeta_{n-1})| < 1$$

Such group is called *periodic cycle*.
One gets

$$\zeta_1 = \varphi(\zeta),\ \zeta_2 = \varphi(\zeta_1),\quad \ldots\quad \zeta = \varphi(\zeta_{n-1})$$

But one won't find any publication about that subject referring for example to a root of $z = \varphi(z)$ when $|\varphi'(\zeta)| > 1$.

In this case the authors, studying from a local view point, immediately go to the inverse function $z = \psi(z_1)$, satisfying $z = \psi(\zeta)$ and $|\psi'(\zeta)| < 1$ and so they come back to the previous case (limit point or limit group), but *for* $\psi(z)$ *and not for* $\varphi(z)$; so one gets informations about some *preimages* of a point z, in the neighbourhood of ζ, but not about its *images*.

The sequence will reveal us important and extremely interesting properties of such points. Let us examine the set of equations

$$z = \varphi(z),\quad z = \varphi_2(z)\ ,\quad \ldots, z = \varphi_n(z),\quad \ldots ;$$

None of them is an identity. Each equation has a finite number of roots; Kœnigs defined what it is necessary to intend as the primitive root ζ ([1]) of an equation

$$z = \varphi_n(z)$$

as well as for the circular group of roots related to ζ. We'll call $\zeta, \zeta_1, \zeta_2, \ldots, \zeta_{n-1}$, the n roots of a such group.
They are distinct and one gets

$$\zeta_1 = \varphi(\zeta),\qquad \zeta_2 = \varphi(\zeta_1),\qquad \ldots,\quad \zeta_{n-1} = \varphi(\zeta_{n-2})\ ,\qquad \zeta = \varphi(\zeta_{n-1})\ ,$$
$$\varphi'_n(\zeta) = \varphi'_n(\zeta_1) = \ldots = \varphi'_n(\zeta_{n-1}) = \varphi'(\zeta)\varphi'(\zeta_1)\ldots\varphi'(\zeta_{n-1})$$

The substitution $[z, \varphi(z)]$ permutes circularly the roots of a group.

([1]) Sometimes Kœnigs writes that one of those roots « belongs to index n », because it is not a root for no equation $z = \varphi_p(z)$ with an index $p < n$.

8. *The set* E. – Let us define E *a countable set* consisting of the primitive roots ζ of all equations:

$$z = \varphi_n(z) \quad (^1) \quad (n = 1, 2, ..., \infty) \quad [\varphi_1(z) = \varphi(z)] \, ;$$

all the roots satisfy

$$| \varphi'_n(\zeta) | > 1 \, .$$

9. *1° Existence.* - For further discussion, it is primarily needed to prove that E always includes some points. Let us take therefore $z = \varphi(z) = \frac{P(z)}{Q(z)}$, where both P and Q two polynomials of degree k (2).

Its roots $\zeta_1, \zeta_2, \dots, \zeta_{k+1}$ are roots of the following equation :

$$P - zQ = 0$$

If $Q = a^0 z^k + \dots$, one gets

$$R = P - zQ = -a^0 z^{k+1} + \dots.$$

Now,

$$\varphi'(\zeta_i) = \left(\frac{P}{Q}\right)'_{\zeta_i} = 1 + \left(\frac{R}{Q}\right)'_{\zeta_i} \quad .$$

And since ξ_i are the roots of $R(z) = 0$, one gets

$$\left(\frac{R}{Q}\right)'_{\zeta_i} = \frac{R'(\zeta_i)}{Q'(\zeta_i)} \quad .$$

Therefore,

$$\textbf{(1)} \qquad \varphi'(\zeta_i) = 1 + \frac{R'(\zeta_i)}{Q(\zeta_i)}$$

Anyway, we recall that due to :

$$\frac{Q(z)}{R(z)} = \frac{a^0 z^k + ...}{-a^0 z^{k+1} + ...} \quad ,$$

one obtains, by decomposing $\frac{Q}{R}$ in simple fractions, the well known relation

$$\textbf{(2)} \qquad \sum \frac{Q(\zeta_i)}{R(\zeta_i)} = -1$$

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(1) Instead of starting from the substitution $z_1 = \varphi(z)$ to define the set E of $z = \varphi_p(z)$, where $|\varphi'_p(z)| > 1$, one may start from the substitution

$$z_n = \Phi(z) = \varphi_n(z)$$

to generate the set ξ of $z = \Phi_p(z)$ satisfying $| \Phi_p(z) | > 1$, then the two sets E and ξ are the same because roots of $z = \varphi_p(z)$ where $| \varphi'_p(z) | > 1$ and satisfying

$$z = \varphi_{np}(z) = \Phi_p(z) \, ,$$

and so one gets

$$\Phi'_p(z) = | \varphi'_p(z) |^n$$

So $| \Phi'_p(z) | > 1$, for that root; each equation $z = \Phi_p(z)$ is $z = \varphi_k(z)$ $(k = np)$ et $| \Phi'_p(z) | = | \varphi'_{np}(z) |$.

(2) After making on z and z_1 the same homographic substitution in case of need, one may always suppose that P(z) and Q(z) have same degree k, degree of the rational fraction.

A point M_i with affixe $\varphi(\zeta_i)$, satisfying $|\varphi'(\zeta_i)| < 1$ is mapped, by **(1)**, to a point N_i, with affixe $\frac{Q(\zeta_i)}{R'(\zeta_i)}$ and lying in the half-plane,

$$\mathrm{Re}(z) = \text{real part of } z < -\frac{1}{2}$$

if $|\varphi'(\zeta_i)| > 1$, then N_i is in the half-plane $\mathrm{Re}(z) > -\frac{1}{2}$ and mutually.

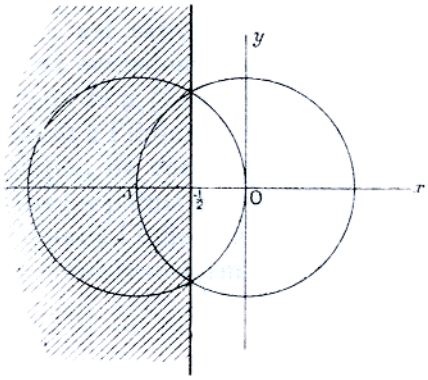

Fig. 11

The relation **(2)** proves that the $k+1$ points $N_1, N_2, \ldots, N_{k+1}$ have, for *gravity centre*, the point with affixe $-\frac{1}{k+1}$, which is in the half-plane $\mathrm{Re}(z) > -\frac{1}{2}$; therefore, at least one of the points N_i lies in this half-plane but not on its boundary ([1]).
Therefore there is at least one of values $\varphi(\zeta_i)$ whose module > 1: so E includes surely some points ([2]).

Remarks. - When P and Q have the same degree k, as it may be always supposed, the point to the infinity cannot be a root of

$$z = \varphi(z) = \frac{P(z)}{Q(z)}$$

When the set of roots of equations $z = \varphi_n(z)$ is countable, then one may suppose that the point at infinity of the plane does not belong to this set: for the proof, it suffices to send a point, not belonging to the previous set, to infinity by a proper homographic substitution on z and z_1.
Anyway it is not so important in our study because the point at infinity does not differ for nothing from any other point of the plane (when φ is rational); the formal advantage, coming from the previous hypothesis, is the power to find, without any possible ambiguity, the value of $\varphi'_n(\zeta)$ in every point ζ (the root of $z = \varphi_n(z)$);

([1]) The same relation shows that all the N_i may lie in the half-plane $\mathrm{Re}(z) > -\frac{1}{2}$, that's to say *it may happen to have no limit points with uniform convergence*.
([2]) *Read* the additional note at the end of this Memoir.

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if ζ could be the point at infinity, then some precautions are necessary in the language: for example, one would agree that $\varphi'_n(\zeta)$ assumes in that point the same value as in all the points of the circular group to $\zeta, \zeta_1, \dots, \zeta_{n-1}$, where the point at infinity belongs to.

This language convention is natural.

If one applies an homographic mapping to the z plane:

$$\textbf{(S)} \qquad z = \frac{\alpha Z + \beta}{\gamma Z + \delta} \; , \quad z_1 = \frac{\alpha Z + \beta}{\gamma Z_1 + \delta} \; ,$$

then the new form of the substitution to iterate will be easily deduced from $z_1 = \varphi(z)$; it will be $Z_1 = \Phi(Z)$; the circular groups, retrieved by $\varphi(z)$, become by the mapping **(S)** the groups retrieved by $\Phi(Z)$, and the value of $\varphi'_n(\zeta)$ for a group $\zeta, \zeta_1, \dots, \zeta_{n-1}$ *is the same as the value of* $\Phi_n(Z)$ for the group $Z, Z_1, \dots, Z_{n-1}$ corresponding to the previous group.

All comes immediately out from the classical formula

$$\textbf{(3)} \qquad \frac{dZ_1}{dZ} = \frac{dZ_1}{dz_1}\frac{dz_1}{dz}\frac{dz}{dZ}$$

which is not useful to insist on.
After these explanations, the convention, defining $\varphi'_n(\zeta)$ the point at infinity ζ in the case where this point is root of

$$z = \varphi_n(z) \; ,$$

is natural.

For the study of the iteration of $\varphi(z)$, one may see how such homographic substitution **(S)** can be applied to the z plane so that one gets *the transformed of the substitution* $[z,\varphi(z)]$ by S ([1]) which is more convenient for a study than $\varphi(z)$.

In the further pages, we will often use homographic transformations.
We will study the iteration in the *extended plane* or, if one prefers, on the Riemann sphere which is a proper support of the variable z.

10. *2°. Properties of points of* E. There is always a point of E satisfying

$$z = \varphi(z) \qquad\qquad |\, \varphi'(z) \,| > 1 \; .$$

A proper homographic transformation allows to suppose that the point above is the origin $z = 0$.

We will study the iteration of a small domain surrounding O, and all we say can be applied to the iteration of the substitution $z_n = \varphi_n(z)$ around a point ζ satisfying

$$\zeta = \varphi_n(\zeta) \qquad\qquad |\, \varphi'_n(\zeta) \,| > 1 \; ;$$

([1]) If one denotes, by Σ , the operation to move from z to $\varphi(z)$, then $S^{-1}\Sigma S$ will be used to move from Z to $\Phi(Z)$, as it is well-known in the groups theory.

the iterations of φ_n belong to the iterations of φ; if one already knows the iteration of this domain for φ_n, one knows well the iteration of a small domain surrounding ζ by φ.

By hypotheses φ(z) develops around the origin in a Taylor series

(4) $z_1 = \varphi(z) = a_1 z + a^2 z^2 + \dots$ with $| a_1 | = | \varphi'(0) | > 1$.

Notice that, among all preimages of 0 of rank 1, there is *only one preimage* confusing with zero since $\varphi'(0) \neq 0$.

When z_1 moves around the origin, even the well determined preimage z varies around the origin; then one can get a Taylor development for $z,$ as a function of z_1, by **(4)**,

(5) $z = \frac{1}{a_1} z_1 + a_2 z^2 + \dots$.

The results obtained by Kœnigs prove that, if z_1 describes a circle C_1 around the origin and with a small enough radius, then z, defined by **(5),** describes a curve C that lies *entirely in the interior of the circle*: that's to say, if $| z_1 | < \delta$, where δ is enough small, then one gets also:

$$| z | < k | z_1 | \qquad \text{with} \qquad k < 1$$

and if z_1 describes the area D bounded by C_1, then z describes the area D bounded by C.

One may say as well: if z describes an area D bounded by a C circle (I say *circle* just to fix ideas in mind) centered at O and of small enough radius, then z_1 describes a area D bounded by a curve C_1 (transformed of C); and D_1 *includes* D *in its interior*.

D_1 is the first iteration of the area D; let us consider the iterations of D.

As z_1 describes D_1, z_2 describes an area D_2 *including all* D_1 *in its interior* ; D_2 will be mapped to itself if, for two distinct values of z in D_1, φ assumes the same value: one realizes that D_2 consists of two or several superimposed layers like the Riemann surface; as z describes D_2 (all the layers of D_2), φ(z) describes a certain area D_3, iterated from D_2 and including D_2 in its interior; in the same way as D_2, D_3 will be a Riemann surface with several layers.

It is worth noticing that all the images D_i are simply connected like the initial area D.

Each one of the areas D_i is completely included in the following area D_{i+1}; it is clear that both D_i and D_{i+1} have only one layer; if they would have several layers, then each layer of D_i would belong to a layer of D_{i+1} and, *on that layer*, *every interior or boundary point of* D_i *will be inside* D_{i+1}.

The rising question is the following: every point of the plane is inside an area D_i with a big enough index ([1]); what happens for the points (of the plane) that are not in the interior of any area D_i ?
The answer is particularly simple and interesting:

11. *Fundamental Theorem*. – Only three cases are may occur:

1) Just only two distinct points of the extended plane are outside all the image areas D_i ([2]).
If we send them respectively to a and to infinity by a proper homographic mapping, then $z_1 = \varphi(z)$ is reduced to one of the two simple forms

$$Z_1 - a = (Z - a)^k \qquad \text{or} \qquad Z_1 - a = \frac{1}{(Z-a)^k}$$

Then every finite area of the plane, not including the point a, is inside an area D_i with an big enough index and in all the iterations of D_i.

2) Just only one point of the extended plane is outside all the areas D_i.
By sending the point at infinity by a proper homographic mapping, $\varphi(z)$ cannot be anything else but a polynomial. [this conclusion is still true if $\varphi(z)$ is a entire transcendental.] For this hypothesis every area (of the plane) lies at a finite distance and it is inside a D_i with an big enough index i and it is inside all the iterations of D_i.

3) Every point of the extended plane is in the interior of a D_i, which has big enough index i ([3]) and it is a closed set like the extended plane ([4]); there will be a D_i with a big enough index and corresponding to the entire extended plane; the same applies to all the iterations of this D_i. This is the case of a general rational fraction $\varphi(z)$.

In fact, suppose that three distinct points of the extended plane are outside all areas D_i and notice that the set of points of the plane, included in D_i, represents the set of all possible values that may be assumed by the function $\varphi_i(z)$ which is an iteration of $\varphi(z)$ in the initial area D.
One concludes that there are at least three values that cannot be assumed by the following family of functions

$$\varphi(z),\ \varphi_2(z),\ \ldots,\ \varphi_i(z),\ \ldots\ .$$

This family consists of rational functions; therefore, it would be *normal in* D, in the same sense as Montel stated; that's to say, one can extract a subsequence from

$$\varphi_{n1}(z),\ \varphi_{n2}(z),\ \ldots,\ \varphi_{ni}(z),\ \ldots$$

([1]) That's to say: there's a layer of D_i covering that point.
([2]) That's to say: they are not covered by any layer of the D_i .
([3]) That's to say is covered by a layer belonging at least to one D_i with a proper index.
([4]) Or the Riemann sphere, which is the stereographic projection.

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converging uniformly in the whole interior of D to a meromorphic function in D, which could be infinite everywhere in D.
This last eventuality is impossible, since all $\varphi(z)$ vanish at the origin.
Therefore the limit function $f(z)$ vanishes at the origin and it is meromorphic in D.
So the origin is an ordinary point of $f(z)$ which is holomorphic in a small area Δ surrounding the origin and included in D.
In this area Δ, all φ_{ni} converge uniformly to $f(z)$.

In particular the values of derivatives of $\varphi_n(z)$ at the origin $\varphi'_n(0)$, converge to a determined and finite value of the derivative $f'(0)$ of f at the origin.
Here a contradiction evidently arises; the Taylor development of $\varphi_n(z)$ around the origin is

$$\varphi_n(z) = a_1^{ni} z + \ldots \quad , \quad \varphi'_{n_i}(0) = a_1^{ni} \quad ,$$

if

$$\varphi(z) = a_1 z + \ldots$$

and while n_i increases to infinity, then a_1^{ni} diverges to infinity since also $| a_1 | > 1$ by hypotheses.

Only two points can escape from all D_i: just three cases follow.

First case. - There are two distinct points so that each one of them coincides with its own preimages; *or* each one of them coincides with all preimages with the rank of the other point; as we send the two points respectively to a and to infinity by a proper homographic mapping, the relation $z_1 = \varphi(z)$ turns (1) either into

$$Z_1 - a = (Z - a)^k ,$$

or into

$$Z_1 - a = \frac{1}{(Z-a)^k}$$

One sees immediately that every area (at finite distance) does not include the point a, inside all D_j of index $j \geq i$ when i is properly chosen. One easily recognizes that in this case, there are only two points ζ_1 and ζ_2 where $\varphi'(z) = 0$ so that (2)

$$\zeta_1 = \varphi(\zeta_1),\ \varphi'(\zeta_1) = 0,\ \varphi''(\zeta_1) = 0, \quad \ldots ,\ \varphi^{(k-1)}(\zeta_1) = 0 ,\quad \varphi^{(k)}(\zeta_1) \neq 0,$$

with

$$\zeta_2 = \varphi(\zeta_2),\ \varphi'(\zeta_2) = 0,\ \varphi''(\zeta_2) = 0, \quad \ldots ,\ \varphi^{(k-1)}(\zeta_2) = 0,\quad \varphi^{(k)}(\zeta_2) \neq 0,$$

(1) To shorten, we say that the fraction $\Phi(Z)$, retrieved by the transformation of $z_1 = \varphi(z)$ in $Z_1 = \Phi(Z)$, with the help of the auxiliary homographic mapping, is the homographic transformed of $\varphi(z)$.
(2) Each one of these two points is a limit point with uniform convergence and it will be showed further that neither a point of E, nor a limit of the points of E.

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or well ([1])

$$\zeta_2 = \varphi(\zeta_1),\ \varphi'(\zeta_1) = 0,\ \varphi''(\zeta_1) = 0\ ,\quad \ldots,\ \varphi^{(k-1)}(\zeta_1) = 0\ ,\quad \varphi^{(k)}(\zeta_1) \neq 0,$$

with

$$\zeta_1 = \varphi(\zeta_2),\ \varphi'(\zeta_2) = 0,\ \varphi''(\zeta_2) = 0,\quad \ldots,\ \varphi^{(k-1)}(\zeta_2) = 0,\quad \varphi^{(k)}(\zeta_2) \neq 0,$$

Second case. - There is only one point escaping from all areas D_i; necessarily it coincides with all its preimages. If one sends this point to infinity by a proper homographic mapping, then the relation $z_1 = \varphi(z)$ turns into a relation $Z_1 = \Phi(Z)$, since $\Phi(Z)$ is a rational fraction whose all poles are at infinity (because a pole is a preimage of the point at infinity), that's to say it is an entire polynomial in Z.

Given a rational form $\varphi(z)$, one will recognize easily when this second case occurs.

The fraction is an homographic transformation of a polynomial if and only if one finds a point ζ coinciding with all its preimages.
This point is critical for the function $\psi(z)$, inverse of $\varphi(z)$ and it is a point satisfying $\varphi'(z) = 0$ since, between the preimages of a critical point of $\psi(z)$, there is one point satisfying $\varphi'(z) = 0$.

Therefore, this is a point ζ, so that the equation

$$\varphi(z) - \zeta = 0\ ,$$

has its k roots equalling to ζ. There is a point $\zeta = \varphi(\zeta)$ where all $\varphi'(\zeta), \varphi''(\zeta), \ldots, \varphi^{(k-1)}(\zeta)$ vanish and but where $\varphi^{(k)}(\zeta) \neq 0$: this will be easily recognized ([2]).

If this case happens, it is easy to turn $\varphi(z)$ in a polynomial; and then one will recognize immediately that, whatever the area Δ of the plane z is at a finite distance, one may find an index i so that the iterations of D, of rank $\geq i$, cover completely Δ: it results from any point of Δ which is a bounded and closed set ([3]) and it is inside one D_i; therefore, since every D_i is included in the next one, there's a determined area including Δ in its interior.

Once more *there exists a layer of* D_i so that every point, inside the area Δ or on its contour, is an interior point of D_i.

Third case. - If the given rational fraction $\varphi(z)$ does not belong to any of the types explained in the two previous cases (and the method to recognize when they occur), that's to say if there are no points coinciding with all its preimages (*1st case*), or there is not

([1]) They will generate a periodic cycle of rank 2 and one will see that they are neither points of E, nor limit points of E.
([2]) Therefore there is a limit point with uniform convergence. It won't belong either to E, nor to E′.
([3]) Here, I recall the lemma of the set theory, called as *the Borel-Lebesgue Theorem* : « If every point of a closed set, bounded at two dimensions, is inside an area, then one can find a *finite* number of areas including, in their interior, all points of the set. » But this is not essentially relevant.

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a system of two points where each point coincides with the k preimages belonging to the rank of the other point (*2nd case*), then one can assert that every point of the extended plane is inside an area D_i an of big enough index i : that area, like the *extended plane* ([1]), is a closed set.
There is an index i so that all the iterations of D with an index $\geq i$ *cover the extended plane*.

Thus our fundamental theorem is demonstrated.

Remarks. - 1. We assume that the initial area D is bounded by an enough small circle centered at O, that's to say whose radius is less than a determined number r_0.
Whatever and how small is the area D surrounding O where one starts from, all our conclusions are still true since such area always includes *in its interior* a small enough circular area D centered at O; and it is evident that each iteration D_i of D includes the iteration D_i of D in its interior.

2. In an equivalent language we wrote in the previous theorem, it can be said that: whatever is the arbitrarily small area D surrounding O, there are at most two complex values (including infinity) that cannot be assumed in D by $\varphi_i(z)$, the iterations of the function $\varphi(z)$.

First case. - There area two values, bringing back to a and *infinity*, when $\varphi(z)$ is reduced, by homographic mapping, to

$$Z_1 - a = (Z - a)^k \qquad \text{or} \qquad Z_1 - a = \frac{1}{(Z-a)^k}$$

If one considers any area Δ (of the extended plane), not including these two points in its interior or on its border, then, starting from a certain index i, each one of the iterations $\varphi_j(z)$, for $j \geq i$, assumes in D all the affixes values of the points belonging to the area Δ.

Second case. - Suppose that the value is *infinity*, so that $\varphi(z)$ is reduced, by an homographic mapping, to an *ordinary polynomial*.
Then there is an index i so that each iteration $\varphi_j(z)$ of an index $j \geq i$ assumes in D all the affixes values of the points belonging to any bounded domain of the plane, already fixed previously.

Third case. - For any fraction $\varphi(z)$ which is not the transformed of a polynomial, neither the transformed of $Z_1 = \frac{1}{Z^k}$ by an homographic mapping, there is an index i so that every iteration $\varphi_j(z)$, of the index $j \geq i$, assumes in D *all the finite or infinite complex values*.

([1]) Or the Riemann Sphere, considered as the support of the variable z.

12. *Corollaries*. 1) - The previous conclusions come true if the point O, the root of the equation $z = \varphi(z)$ where $| \varphi'(z) | > 1$, is replaced by any point of E, root of $z = \varphi_n(z)$ where $| \varphi'_n(z) | > 1$, whatever the index n is.

Test it just by making a substitution of $\varphi(z)$ with $\varphi_n(z)$ for generating the images of the area D that surrounds the considered point of E; the study of D_n, D_{2n}, D_{3n}, …, D_{pn}, … may verify the previous theorem along the whole sequence D_1, D_2, …, D_i.

2) *Every point of* E *is a limit point for the set of preimages of any point of the plane*, except at most for two points of the plane: each point may coincide with its own preimages, *or* it may coincide with all the preimages of the rank of the other point.

On the other hand it is important to notice that we may extract at most one point coinciding with itself, among the preimages of the rank 1 of any point of E; just a little consideration suffices to prove that any point of E has infinite preimages and so *every point of* E *is a limit point for the set of its own preimages* ([1]).

3) A very interesting property of the points of E is: *every point of* E *is the limit of the points of* E; that's to say, in an arbitrarily small neighbourhood of a point P of E, we get

$$[\zeta = \varphi_p(\zeta) \; ; \; | \varphi'_p(\zeta) | > 1]$$

there always is an infinity of points of E, roots of equations satisfying

$$z = \varphi_n(z) \quad \text{where} \quad | \varphi'_n(z) | > 1$$

In fact, whatever small is the area D, surrounding a point P of E, one can find in D a distinct point from P, preimage of P, of an enough high rank m.

Let P_{-m} be that point; then one can surround it with a small area area Δ_{-m}, which is inside D and not including P, so that the transformed of Δ_{-m} by $\varphi_m(z)$, which is an area Δ surrounding P (as small as desired: but this is irrelevant, it is important to assure that Δ surrounds P), is a simple and with only one layer.

The iteration of rank n of Δ is an iteration of rank $m + n$ of Δ_{-m}, and one can take a big index n to assure that the small finite area is completely inside a layer of Δ_n (this result induces us to consider that none of the preimages of a point of E cannot be one of two exceptional points occurring in the first and second case of our fundamental theorem).

Then the fraction $\varphi_{m+n}(z)$ maps the area Δ_{-m} into an area Δ_n, that can be mapped to itself like a Riemann surface so that Δ_{-m} is inside a layer of Δ_n; that's to say on the considered layer, Δ_{-m} and its contour include only interior points of Δ_n, not boundary points of Δ_n.

([1]) A point of E is never one of these two exceptional points, expected by the fundamental theorem.

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It is enough to assert ([1]) that the area Δ_{-m} includes a root of $z = \varphi_{m+n}(z)$ such that it satisfies $|\varphi'_{m+n}(z)| > 1$ and this root is distinct from P since Δ_{-m} does not include P.
It is proved that every arbitrarily small area D, surrounding a point P of E, includes a point of E, different from P; therefore it includes an infinite number of points of E: therefore every point of E is the limit of the points of E.

Moreover, it is proven that E surely includes at least one point, since *it always includes a countable infinity of points*. We can now speak about a set E′ *derived from* E.
E′ *includes* E: this is what is asserted by the 3rd corollary we have just demonstrated.

4) The previous demonstration is based upon principles proving that every preimage of a point of E belongs to E′ ; finally it is evident that E′ a *perfect set* because, on one hand, it includes its derived E" and, on the other hand, it includes E and every point of E′ since it is the limit of the points of E and the limit of the points E′ too.
Therefore E′ enjoys the same properties of E": then E′ *is perfect*.

Remarks. - The fact is very important: each point of E is the limit of points of E; we obtained it as consequence of the fundamental theorem by an extremely natural analysis.
A new direct demonstration follows: it is based onto the Picard's theorem and it has been improved by Landau; the enunciate is the following:

Let α *and* β be *two arbitrary complex numbers* (β ≠ 0), *then there is a number* R(α,β), *not depending either on* α *and on* β, *so that each holomorphic function*

$$f(z) = \alpha + \beta z + \dots ,$$

assumes the value 0 *or the value* 1 *in a circle* $|z| < R$.

Let us suppose that the origin is a point of E and that, to simplify the explanation, there is a root of φ(*z*) satisfying $|\varphi'(z)| > 1$ so that the generality of the prove is not restricted, as we have already said several times before.
φ(*z*) develops around the origin under the form

$$z_1 = \varphi(z) = a_1 z + a_2 z^2 + \dots , \quad (|a_1| > 1).$$

The iteration of rank *n* of φ has an development which can be easily calculated: it is

$$z_n = \varphi_n(z) = a_1^n z + a_2 a_1^{n-1} (a_1^n - 1) z^2 + \dots .$$

Now

$$\frac{\varphi_n(z)}{z} = a_1^n + a_2 a_1^{n-1} (a_1^n - 1) z + \dots \qquad \left(|a_1^n| > 1 \right)$$

It is assured that in a circle of radius $R\left[a_1^n, a_2 a_1^{n-1} (a_1^n - 1)\right]$, $\frac{\varphi_n(z)}{z}$ assumes the value *zero* or the value 1, if it is uniform in the circle and in a distinct point of O.

([1]) Read *Preliminaries*, § 5.

Hurwitz showed (*read* PICARD, *Traité d'Analyse*, 2e edition, t. 3, p. 377) that

$$R(\alpha,\beta) \le 16\frac{1}{|\beta|}\sqrt[3]{|\alpha|^2}\sqrt{|\alpha-1|} \quad .$$

One may give more precise expressions of R but this one is enough for us.

If this one calculates the value R for $\frac{\varphi_n(z)}{z}$, then one finds an expression of order of

$$\frac{|a_1^n|^{\frac{2}{3}+\frac{1}{2}}}{|a_1|^{2n-1}} = |a_1|^{-\frac{5n}{6}+1} \quad ,$$

that's to say *a value converging to zero as n grows indefinitely*.

The conclusion is that in an circle, centered at O and with radius at an order of $|a_1|^{-\frac{5n}{6}+1}$ (a small radius if n is big enough), $\frac{\varphi_n(z)}{z}$ has a pole or a zero, or it assumes the value 1 elsewhere but at the origin.

1. If $\frac{\varphi_n}{z}$ assumes the value 1, then the theorem is proven because in the point z one gets $\varphi_n = z$. The only embarrassing cases occur when, whatever n starts from a certain rank, $\frac{\varphi_n}{z}$ assumes the values 0 and ∞ but never the value 1.

2. If the value is zero, then the conclusion is that O is the limit point of its own preimages and the question is easily solved by the method of the third corollary: the *arbitrarily small* circle C includes a preimage P_{-n} of the origin which can be surrounded by a small enough area, included in C, so that its iteration Δ (surrounding O) by $\varphi_n(z)$ is arbitrarily small.

The local study of the neighbourhood of O suffices, when C is arbitrarily small, to prove that a certain iteration Δ_p of Δ includes C in its interior and so Δ_{-n} too; then, one concludes that both Δ_{-n} and so C includes a point of E, distinct from O, for which:

$$z = \varphi_{n-p}(z) \ , \qquad |\varphi'_{n-p}(z)| > 1 \quad .$$

3. Finally if $\frac{\varphi_n(z)}{z}$ has always a pole in the circle C, then it means that O is a limit point for the set consisting of the preimages of the point at infinity.

If such point of the plane exists, so that it is distinct from O and whose preimages do not admit O as limit point, then a proper homographic mapping can show that both the question of the point to infinity and the previous difficulty may be solved. It would not be impossible to conjecture that O is a limit point of preimages of any point of the plane. But since the origin has preimages distinct from itself, we will then refer to the 2nd case and the theorem has been proven again.

After all, the second demonstration is not different from the first one, especially if one thinks that Montel deducted the Picard-Landau theorem from the notions (he introduced) of normal families of functions.

One will notice the narrow link between these two following facts:

1. Every point of E is a limit point of the points of E;
2. Every point of E is a limit point of its own preimages.

13. *The set E′, derived from E.* – We saw that E′ is perfect.
When every point P of E′ is a limit of the points of E and whatever is the arbitrarily small area D surrounding P, there is always a point of E in D, so there are at most two points of the plane (in conformity with the first and second case of our fundamental theorem) that are outside all iterations D_i of D.
This is a characteristic property of the points of E′.

THEOREM. - Given the following property: *however small is the area surrounding* P, *at the most two points of the extended plane can be exterior of all iterations* D_i (*this case refers to the* 1° *and* 2° *of our fundamental theorem*); *then the necessary and sufficient condition for a point* P *of the extended plane to enjoy this fundamental property is that* D *is a point of* E′.

If P belongs to E′, the condition is verified: so it is sufficient.
It is evident that it will be necessary too: in fact, whatever is D (surrounding a point P), the iterations D_i of D include any point of the extended plane, except at most two points; so it is evident that P is a limit point for the set of the preimages of *any* point of the plane (except two points of the plane if the system of two points is the same as the set of all its preimages and of all its images; this brings us back to the 1st and 2nd case of our fundamental theorem).
Evidently every point of E lies among these points whose preimages have P as one of their limit points (it is been proven that a point of E can be never one of the two exceptional points).
Therefore P belongs to E′, when it is the limit point of the preimages of a point of E, since, on one hand, every preimage of a point of E belongs to E′ and, on the other hand, E′ is a perfect set. Since:

1. P is a point of E′.
2. D is an arbitrarily small area surrounding P.
3. Δ is any area of the extended plane so that Δ does not include one or two determined points in its interior and on its boundary.

then, when φ(z) is one of the fractions expected by the first or the second case of the fundamental theorem and since Δ may coincide with the extended plane even if φ(z) is generic, there is an index i so that all D_i, iterations of D, whose index $\geq i$, completely cover Δ ([1]).

14. *Corollaries*. - 1) Every point of E′ is a limit point for the set of the preimages of any point of the extended plane, except at most two points so that each one of them coincides with its own preimages or with all the preimages of rank 1 of the other point (1[st] and 2[nd] case of fundamental theorem, respectively).
In other words, there are at the most two complex values, finite or infinite, which cannot be assumed by anyone of the following functions

$$\varphi(z)\ ,\ \varphi_2(z),\ \ldots.,\quad \varphi_n(z),\ \ldots$$

in a given and arbitrarily small neighbourhood D of P.

2) All the preimages of a point of E′ and all its images belong to E′ ; both of them enjoy, like the considered point of E′, the characteristic property of the theorem above ([2]). One can say that E′ is a perfect invariant set for a simply rational mapping of the plane $z_1 = \varphi(z)$ and for the k branches of the inverse transformation.

3) The preimages of any point P of E′ generate an everywhere dense set in E′ ([3]), that's to say when any area, including points of E′, includes the preimages of the considered point P too.

([1]) More precisely, every point of E′, since it cannot be one of two exceptional points, expected by the previous theorem, is *inside an area* D_i, and since the set E′ is a perfect set (and bounded if it is placed onto the Riemann sphere), one can find a finite number of D_i, or simply a D_{i0}, with an enough high index i_0 and including *in its interior the whole set* E′, such as all D_i with an index $i > i_0$ .Whatever small is the area D of E′ and surrounding P, it includes a small circular area surrounding a point of E; and for the iterations of this small area, being completely in the *interior* of the images of D with the same rank, the previous property, already viewed, still holds.
([2]) The result of the previous notice, while taking into account the second corollary, is that if we consider an arbitrarily small area D (for example, a circle, centered at O and surrounding a point P of E′) and the part ξ′ of E′, *inside the area* D, then, since the image of D_{i0} of D includes in its interior the whole set E′ :
a) Every iteration of rank i_0 of a point of ξ′ is a point of E′ and inside D_{i0};
b) Every point of E′, inside D_{i0}, has an interior preimage with rank i_0 of D: this preimage is a point of ξ′; the conclusion is …
c) Every point of E′ is included in a set coming out from an iteration with rank i_0 (and in all iterations with rank $i > i_0$) of ξ′.
One can say that, starting from an arbitrarily small part ξ′ of E′, one may generate the whole E′ with the help of a finite number of iterations of ξ′; this is useful for us for discussing about the structure of E′, because one may say, as a consequence of the present note, that: *the structure of the whole* E′ *is the same as the structure of any part* ξ′ *of* E′, *consisting of interior points of* E′ *in an arbitrarily small area* (*of the plane*) *surrounding an arbitrarily point of* E′.
([3]) It is not the same as the images of any point of E′, since one knows *a priori* that every point of E or every preimage of a point of E has a finite number of images.

Since any point P of E′ cannot be one of the two exceptional points expected by the previous theorem, it is evident that each point of E′ is a limit point for the preimages of the considered point P.

4) *In a point* P *of* E′, *it is impossible that the family* $\varphi(z)$, $\varphi_2(z)$, …, $\varphi_n(z)$, … *or that any family, generated by an infinity of* φ_i, *is normal*. However small the circle C is, centered at P, there is in fact an index i so that all the areas C_j, iterations of C and with an index $j \geq i$, cover every previously fixed area Δ so that they leave at the most two determined points of the plane on its *exterior*.
If the considered family is normal in P, or if any infinite sequence, extracted from the sequence φ_i is normal in P, then one could find a sequence

$$\varphi_n(z),\ \varphi_{n2}(z),\ \ldots,\ \varphi_{np}(z),\ \ldots$$

which, supposed that C is small enough, converges uniformly to an analytic function $f(z)$, that can be a constant. But suppose that C is enough small; then, starting from a certain rank, the areas C_{np}, C_{np+1}, …, iterations of C, will be near to Γ, the transformed area of C by $f(z)$.
This contradicts that C_j covers Δ, for any index j, bigger than or the same as i.

5) In every arbitrarily small area D, surrounding a point P of E′, there are at most two complex values (finite or infinite) which cannot be assumed by any of the functions $\varphi'(z), \varphi_2'(z), ..., \varphi_n'(z)$, or, it is the same, they cannot be assumed by any of the functions of an infinite sequence extracted from the previous one:

$$\varphi'_{n1}(z), \varphi'_{n2}(z), ..., \varphi'_{ni}(z),$$

because, if there are three different complex values which are not assumed by any of the following functions

$$\varphi'_{n1}(z), \varphi'_{n2}(z), ..., \varphi'_{ni}(z),$$

in D, then this sequence is *normal in* D; so one deduces that

$$\varphi_{n1}(z), \varphi_{n2}(z), ..., \varphi_{ni}(z),$$

is also normal, in contradiction with to the 4th corollary.

15. *The structure of E′.* – Some simple examples show immediately that E′ can be:

a) a perfect discontinuous set , or …
b) a continuous line perfect set ([1]), but …
c) nothing excludes *a priori* the possibility for E′ to include continuous areas .

At this purpose the following proposition is interesting:

([1]) *A priori*, E′ can be at the same time discontinuous in certain regions and linearly continuous in other regions. We'll see further whether it may be or not.

THEOREM. – *The set* E′ *cannot include some areas without including all the extended plane*; *in other terms*, *if there is a point of the extended plane that does not belong to* E′, *then* E′ *can be only a continuous line set or a perfect discontinuous set; but it does not certainly include any area of the plane*. *In other words*, E′ *does not include any interior point in this case*.

In fact, if A is a point of the plane, not belonging to E′, then A may be surrounded by a small area Δ, whose no point belongs to E′ ; none of the preimages of Δ includes points of E′ ; then, a point P of E′ cannot be a limit point for the preimages of an *arbitrarily chosen point in* Δ ([1]) so that, in every circle centered at P, there are no points belonging to E′.

Therefore it excludes the possibility for P to be an «*interior point*» of E′, that's to say the centre of a certain small circle whose all points belong to E′.

As we will see it further, the local study about the neighbourhoods of remarkable points of the plane, noteworthy by Kœnigs, allows to define some regions that do not include any points of E′.
A priori E′ can therefore:

1. Be a perfect set, discontinuous or linearly continuous ([2]) ;
2. Be the same as extended plane.

([1]) It discards the objection against the current reasoning which could give birth to the hypothesis so that all preimages of A sum are counted by a finite number; this hypothesis corresponds to the first and second case of the fundamental theorem, since the point A is supposed to be one of those exceptional points, mentioned above.
Then we will see further that those two exceptional points, in the first and second case, never belong to E′, and that one may surround each one of those two points with an area which does not include any point of E′: in fact they generate a periodic cycle of rank 2, so that each point is a limit point with uniform convergence.
([2]) Here is an immediate consequence of the note ([1]) in the page 42 of this Memoir: if in a arbitrarily small region R, belonging to the plane, the points of E′ generate an everywhere discontinuous set, that's to say «*poorly connected between two any of its points*», then the whole E′ is *everywhere discontinuous*.
In fact, every point of E′ can be surrounded by a small region D, so that an iteration D_i includes R in its interior (because a point of E′ is never one of the two exceptional points that can escape from all the D_i).
So there is a preimage R_{-i}, with rank *i* in R, which is completely included in D.
In D, the points of E′ generate a poorly connected set between an arbitrary couple of its points, because if this partial set would be well connected between an arbitrary couple of its points, then its iteration with rank *i* is well connected too; or, that iteration with rank *i* includes the part of E′, included in R (since D includes R_{-i}), which, by hypothesis, is everywhere discontinuous in R.
In the same way, if in an arbitrarily small region R (of the plane) the points of E′ generate an linearly *continuous set*, that's to say well connected between any couple of the points of R, then the entire E′ is well connected between any couple of points of E′: E′ is still a linearly continuous.
Therefore E′ cannot be everywhere discontinuous in a region and continuous in another one.
It cannot consist of two continuous parts without any common points; but nothing stops *a priori* E′ to include points of E′ in the whole region of the plane so that E′ is well connected and to include other points so that E′ is poorly connected; that's to say, nothing may prevent that an entire region of the plane (whatever small this region is) includes points of E′ so that it includes both continuous and discontinuous portions of E′.

16.

1. Here some simple examples of the first case:
E′ is a perfect discontinuous set: it suffices to examine the rational fractions noticed by Fatou in its Note of the *Comptes Rendus* of October 15th 1906: $z_1 = \frac{z^k}{z^k + 2}$, for which it is shown that any point of the extended plane, not belonging to a perfect discontinuous set E′, has image points converging to the origin. [By the transformation $z = \frac{1}{Z}$, these fractions are reduced to polynomials

$$Z_1 = 2Z^k + 1$$

and the point of convergence in all the plane, except E′, is the point at infinity.]
E′ is obtained by Fatou in a very simple manner: he determines a circle C_0, centered at the origin, whose radius ρ is calculated according to Kœnigs formulas so that if $|z| < \rho$, then one gets

$$|z_1| < K|z| \qquad (0 < K < 1)$$

K is found by knowing the value $\frac{dz_1}{dz}$ vanishing at the origin. C_0, in the current case, includes the only two critical points of the algebraic function $z(z_1)$, inverse of $z_1 = \varphi(z)$, being $z_1 = 1$ and $z_1 = 0$.
Then if one takes the preimages of C_0, one gets k curves, then k^2 curves, ..., k^q curves; the portion of the plane, bounded by C_0 and including O, has a portion of the plane as preimage, including the previous portion and bounded by k preimage curves of C_0; this last portion has the portion of the plane as preimage, including O and bounded by k^2 preimages of rank 2 belonging to C_0 , etc.
Thus each one of the determined preimage regions includes its image; and each one of its limit curves defines a region (not including O) where there are k limit curves of the preimage regions of the considered region.
During this process one sees that the preimage curves of C_0 indefinitely shrink in all their dimensions.
The set of points, not belonging to any region among the examined previously regions (which are the preimages of the area of the circle C_0), *is a everywhere discontinuous perfect set*.
Each point of this set is a limit point for the preimages of any point in the plane, except the origin (this is an exceptional point of the same type referring to the second case of our fundamental theorem); it agrees evidently with the same definition of this set; one recognizes as well as this set is the set E′, related to the fraction $z_1 = \frac{z^k}{z^k + 2}$.

2. E′ is a continuous line: it suffices to take the example $z_1 = z^2$ entering into the first case of the fundamental theorem.
Here the iteration of rank n is

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$$z_n = \varphi_n(z) = z^{2n}$$

The module of the roots ζ of $z = \varphi_n(z)$ is $= 1$, except $z = 0$.

There are roots of

$$z^{2n-1} - 1 = 0$$

In each of these roots, one gets

$$\varphi'(\zeta) = 2\,\zeta$$

therefore

$$|\,\varphi'(\zeta)\,| = 2$$

and then,

$$|\,\varphi'_n(\zeta)\,| > 1$$

All roots of

$$z^{2n-1} - 1 = 0 \qquad\qquad (n = 1, 2, \ldots, \infty),$$

belong to E.
They are everywhere dense on the circle $|\,z\,| = 1$.
E′ *consists of the whole circumference* $|\,z\,| = 1$.

Moreover, the equation $z = \varphi(z)$ admits the roots 0 and ∞ [they are the two exceptional points of the fundamental theorem (first case)], where $|\,\varphi'(z)\,|$ vanishes.

These are respectively the limit points of the images of each point of the region of the plane, bounded by the circle $|\,z\,| = 1$.

3. We will see further what complexity can be offered by the set E′ in cases where it is a linear set.

We will see in particular that it can split the plane in an *infinity number of regions*; that's to say it is possible to give simple fractions $\varphi(z)$ related to an infinity of different regions in the plane so that each one of them is bounded by a part of E′.

17. The question is to know if E′ can effectively include the entire extended plane and to reply to this question, both negatively and positively, by giving an example of rational function $\varphi(z)$ so that each point of the plane belongs to E′.

We will see further that such a function is not either a polynomial or a fraction related to the first case of our fundamental theorem. Now let us show how, by analogy with a known and simple question, one may think that E′ may include the whole plane.

E′ is defined by its characteristic property: e*ach point of* E′ *is a limit point for the preimages of any point of the extended plane*, *except at most for two points*.

Now, consider the preimages of a point z of the plane; then solve the equation

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$\varphi(z_{-1}) = z$, giving k preimages z_{-1} of rank 1.
$\varphi(z_{-2}) = z$, giving k preimages z_{-2} of rank 2.
…………………………………………

This problem is a special case of the following: as $f(z,z_1) = 0$ is an algebraic relation between z and z_1 (f is a polynomial in z and z_1), then z is called the *preimage* of z_1; so that z_{-1} is defined the *preimage* of z by $f(z_{-1},z) = 0$, etc.

By these means, the preimages of z are defined and then the question is to examine the derived set of the set consisting of that preimages.

Now it is easy to see that, for some choices of f, the derived set consists of the entire plane, whatever the examined point z is. In fact, if

$$z' = S_1(z), \qquad z' = S_2(z), \quad \ldots, \quad z' = S_n(z)$$

are the fundamental substitutions of an *automorphism group*, *improperly discontinuous in the entire plane* (a Picard's group, for example, where these substitutions are

$$z' = z + 1 \quad , \quad z' = z + \mathrm{I} \quad , \quad z' = -\frac{1}{z}, \qquad z' = -\frac{iz}{i}$$

assuming

$$f(z, z_1) = [z_1 - S_1(z)]\,[z_2 - S_2(z)] \ldots [z_1 - S_n(z)] = 0$$

the preimages of any point z_1 of the plane won't evidently be something else but the homologous points of z_1 in the considered group.

Since this group is improperly discontinuous whatever z_1 is in the whole plane, these homologous points are everywhere dense in the plane; the derived set of the set of these homologous points consists of *the whole extended plane*, whatever the point z_1 is where one starts from.

Now, we are interested in this question: the determination of E′ by a relation

$$z_1 - \varphi(z) = 0$$

where φ is rational in z; and the determination of the points in the plane where an automorphism group stops being properly discontinuous.

These are two special cases of a more general following question: the determination of the derived set by the set of preimages of any point z_1 any in the plane, where the preimages are progressively defined by the algebraic relation:

$$f(z,z_1) = 0$$

Since the second special case (automorphic groups) easily gives some examples where E′ coincides with the extended plane, then one may think (at least) that there's no *a priori* reason so that the question we are interested in this Memoir (iterations of the rational functions) does not involve such examples; and the analogy would be rather a supposition in favour of the affirmative reply ([1]).

I exploit this occasion for pointing out of other analogies between the iterations and automorphic groups.

18. If one consider $S_1(z)$, …, $S_n(z)$ as the substitutions of a fuchsian group in the fundamental circle, then it is clear that, for any point of the plane, the derived set of the set of the preimages consists of all points of the fundamental circle: E′ consists of the whole circumference preserved by that group.

One found an analogous result in the second example, notified before ($z_1 = z^2$).

If $S_i(z)$ are the fundamental substitutions of a kleinian group of that type preserving the interior of a continuous curve, tangency provided in an infinity of points and lacking in the obsculator circle in these points (curves notified by Poincaré in its Memoir about the kleinian groups), then E′ consists of all the points of this curve.

Later we will give examples about those rational functions so that E′ consists of all points (no tangent provided) of a Jordan curve.

Finally if $S_i(z)$ are, for example, the fundamental substitutions of a fuchsian group so that the fundamental polygon has sides of the second kind, then we're induced to consider fuchsian functions defined *in all the plane*, except into the points of a perfect discontinuous set E′, lying on the fundamental circle; it is the analogous case of the first notified example $z_1 = \frac{z^k}{z^k + 2}$, where the set is perfect and discontinuous.

([1]) Since the depositing of this Memoir, Lattes showed in a Note of *Comptes Rendus* (January, 7th, 1918) an interesting example of this case. Assuming

$$z = \rho(u) \qquad (g_2 = 4)$$
$$z_1 = \rho(2u) \qquad (g_3 = 0)$$

one gets

$$z_1 = \frac{(z^2+1)^2}{4z(z^2-1)}$$

All points $u = 2\omega(v + iw)$, for which v and w have *simply periodic developments* in the system of base 2 with n digits in the period, involves that $2^n u = u$ has one period, which is near and corresponding to the roots of $z = z_n = \varphi_n(z)$ satisfying $\varphi'(z) = 2^n$. These are points of E; since the precedent u are dense in the periodic parallelogram, E is dense in all the plane; E′ is the same as the extended plane.

One will easily find an example of such $S_i(z)$ by:

1. taking three exterior circles (two by two) C_1, C_2, C_3;
2. considering the exterior area of each circle;
3. reflecting this last area on a circle (C_1 for example);
4. considering the set of the primitive area and its image as the fundamental domain of a fuchsian group;
5. corresponding C_2 and its image C'_2, C_3 and C'_3 in the new obtained fundamental domain.

If $S_i(z)$ are the fundamental substitutions of a kleinian group, given kleinian functions defined in the entire plane except at the points of a perfect discontinuous set (Schottky group, for example), then one will get again a great number of examples so that E′ is perfect and discontinuous but it includes no points on the fundamental circle.

I insisted on the comparison between the set E′, related to the iteration of a rational fraction, and the set ξ′ of those points where an automorphism group stops being properly discontinuous: my opinion is that each one of two questions about these two sets (iteration of a rational fraction and the study of an automorphism group) should clarify the other question because both of them are special cases of a more general question of the algebraic function $z(z_1)$, defined by $f(z,z_1) = 0$.

In our two special cases, every point of E′ (or of ξ′) is a limit point of the set of pre-images (or their homologous points) for *any point of the plane*: this distinctive property of E′ (like ξ′) is so much important that we will light up the narrow links between the two questions above.

But I stop this digression here since I do not want to escape from the proper subject of this Memoir: the study of the iteration of a rational fraction.

19. Without dwelling upon the question to effectively know if E′ may consist of the entire plane for some rational fractions $\varphi(z)$, I will pursuit the studies of the case where E′ is not an area.

More often, it is useful for this purpose to make a local study of neighbourhoods of both a limit point [$z = \varphi(z)$, $|\varphi'(z)| < 1$] and a periodic cycle.

We will show that if one recognizes, for the fraction $z_1 = \varphi(z)$, the existence of a limit point with uniform convergence, that's to say a point A, root of

$$z = \varphi(z) \quad \text{where} \quad |\varphi'(z)| < 1$$

then one may fix a neighbourhood of the point, wherein none of the equations

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$$z = \varphi_n(z) \qquad (n = 1, 2, \ldots \infty)$$

have distinct roots from A (1) .

One says, in fact, that if ζ is the affixe of A, then one may find a number ρ such that if

$$| z - \zeta | < \rho$$

then one gets

$$| z_1 - \zeta | < H\rho \ ,$$

where H ranges between 0 and 1.

Then

$$| z_n - \zeta | < H^n\rho$$

as z describes the circle C, centered at A and with radius ρ, z_1 describes a curve C_1, included inside C and without any common points with C.

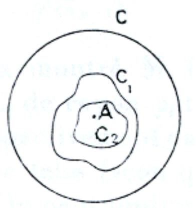

Fig. 12

If z describes the area D inside C then z_1 describes the area D_1 inside C_1.

If $\varphi'(\zeta) \neq 0$ then one may suppose that ρ is small enough so that, when the area D is simple and with one layer, D_1 is described by z_1 and it has one layer too; it is clear that each one of the iterations of D is enclosed in the previous one so that they converge to A in all their dimensions.

If $\varphi'(\zeta) = 0$ then the area described by z_i has two layers, whatever is i, and it admits A as a branch point; but this is not important: what matters is just that, *in all cases*, as z describes the circle C in the positive direction around A, then $z_1, z_2, \ldots, z_n, \ldots$ describe all the *completely interior curves of* C.

Then, as I indicated in the *Preliminaries*, the variation of the argument of $z - z_n$, when z describes C, is equal to 2π like the variation of the argument of $z - \zeta$; it follows that, in the area D, for the equations

$$z - \varphi_n(z) = 0 \qquad \text{for} \qquad n = 1, 2, \ldots, \infty$$

ζ is the only root.

The set E consists of root points of such equations; it can be seen that no point of E, neither of E′, appears in the area D.

(1) Every polynomial admits the point at infinity as a limit point with uniform convergence; it may be tested by an arbitrarily homographic mapping taking back the point at infinity to a finite distance. E′ cannot be an area for any polynomial.

The demonstration, for a limit group $\zeta, \zeta_1, \ldots, \zeta_{p-1}$ comes from the roots of

Then E′ is not surely an area.

$$z = \varphi_p(z), \qquad \zeta_1 = \varphi(\zeta), \qquad \ldots, \qquad \zeta_{p-1} = \varphi(\zeta_{p-2}), \quad \zeta = \varphi(\zeta_{p-1})$$
$$\varphi'_p(\zeta_i) = \varphi'(\zeta)\varphi'(\zeta_1) \ldots \varphi'(\zeta_{p-1}).$$

is just different ([1]).

As Kœnigs showed, each of ζ_i can be surrounded by a circle C_i with radius ρ_i so that if z is inside a C_i, then its iterations fall respectively in the circles C; z_p comes back falling into C_i, so that $| z - \zeta_i | < \rho$, determining $|z - \zeta_i| < H\rho_i$ for $H < 1$.

One may always suppose that the areas C_i are, two by two, exterior and small enough; then, if z describes C, then $z_1, z_2, \ldots, z_{p-1}$ describe some curves in the exterior of C and respectively near to $\zeta_1, \ldots, \zeta_p$; then the variation of argument of $z - z_i$ ($i = 1, 2, \ldots, p, \ldots$) along C vanishes.

Immediately, one sees that the variation will be the same, whatever i is, except for those value of i, multiple of p; the variation of $\arg(z - z_i)$ will always vanish along C, if i is not a multiple of p; it will equals to 2π, if i is multiple of p.

One concludes that, in the area D bounded by C, the following equations

$$z - \varphi_n(z) = 0 \qquad\qquad (n = 1, 2, \ldots, \infty)$$

have no roots, if n is not a multiple of p and one root (which is ζ) if n is multiple of p. The same reasoning is worth for the areas $D_1, D_2, \ldots, D_{p-1}$, bounded by the circles $C_1, C_2, \ldots, C_{p-1}$ ([2]).

As we compare what we have just said about the third corollary of our fundamental theorem, finally one may affirm: «In the set of the roots of the equations

$$z - \varphi_n(z) = 0 \qquad\qquad (n = 1, 2, \ldots, \infty)$$

a) if a root ζ retrieves $| \varphi'_n(\zeta) | < 1$, then it is isolated in the set ;
b) if a root ζ retrieves $| \varphi'_n(\zeta) | > 1$, then ζ is the limit of the points of the set E and the third corollary asserts that ζ is the limit for the roots of the equations $z = \varphi_p(z)$ satisfying $| \varphi'_n(\zeta) | > 1$. »

([1]) One may always suppose that all the points of the group have finite distance, without restricting the generality.

([2]) For any fraction of the type corresponding to the first case of our fundamental theorem, that's to say when it may be reduced, by an homographic mapping, to

$$Z_1 - a = \frac{1}{(Z-a)^k} \ ,$$

then the point a and the point at infinity (that's to say the two exceptional points of the theorem) generate a periodic cycle of rank 2.

Then it is very easy to differ the points of E from limit points with uniform convergence or from periodic cycles.

20. It may occur that some special properties of the considered fraction $\varphi(z)$ allow to enunciate the properties of the set E′ and then, in particular, to recognize when the set is not an area.

This is the case of fractions preserving the interior and the circumference of the fundamental circle; this case has been notified by Fatou in a note of *Comptes Rendus* (May 21^{th} 1917).

It may be always supposed that the circle C is the circle $|z| \leq 1$; then, for $|z| < 1$, one has $|\varphi(z)| < 1$, and for $|z| = 1$, $|\varphi(z)| = 1$ is retrieved.
[Fatou has given the needed form of $\varphi(z)$ enjoying this property by taking back the fundamental circle to the analytic upper half-plane with an homographic mapping.]

$\varphi(z)$ is supposed to be of degree k.

One sees that two symmetric points z and z' in respect of the circle C are mapped to two symmetric points $\varphi(z)$ and $\varphi(z')$ in respect of the circle.

In general the inverse algebraic function of $\varphi(z)$ has k - 1 critical points inside C and other (k - 1) critical points, symmetric to the first ones in respect of C.

If one considers the Riemann surface R, related to this algebraic function, then the circumference $|z| = 1$ intersects, in its *k layers*, a simply connected area S including O; S has k layers and it is bounded by the circumference $|z| = 1$, whatever k is.

The algebraic function $z = \psi(z_1)$, inverse of $\varphi(z)$, maps conformally the interior of the surface S with k layers on the interior of the circle C; by the same function, the portion of the Riemann surface R, exterior from $|z| = 1$, is conformally mapped on the exterior of the circle $|z| = 1$.

$z = \psi(z_1)$ maps conformally in the complex plane the Riemann surface R, which is closed, with *genus zero* and completed by the k points to infinity of the k layers.

If z intersects C only once in the positive sense, then $z_1 = \varphi(z)$ *always intersects* C *in the positive sense for k times*: so there are at least k - 1 roots of $z = \varphi(z)$ in C; there are at most k - 1 or k +1 roots, that's to say all roots lie in C.

If there are k +1 roots, then all the roots belonging to the equations $z = \varphi_n(z)$ (for an index $n = 1, 2, \ldots, \infty$) lie in C.

In fact, z_2 describes C in the positive sense k^2 times, z_3 in k^3 times. Each one of the equations $z = \varphi_n(z)$ has at least $k^n - 1$ roots in C. The equation has at most $k^n + 1$.

Two roots cannot exist outside C: if each one of them is root of $z = \varphi(z)$, then it is impossible, since $z = \varphi(z)$ has its $k + 1$ roots in C, or if the two considered roots generate a circular group for $z = \varphi_2(z)$; but, one sees immediately that those two roots are symmetric in respect of C.

Then it is impossible to accept the current hypothesis since the transformed $\varphi(z)$ of the interior root of C would be ζ (outside C); our hypotheses assure that it is impossible.

Conclusion. – All equations $z = \varphi_n(z)$, $(n = 1, 2, \ldots, \infty)$ have all their roots in C; the equation $z = \varphi(z)$ has two symmetric ([1]) roots in respect of C: they do not lie in C.
All its other roots are in C like all the other roots of $z = \varphi_n(z)$ $(n = 1, 2, \ldots, \infty)$ are in C.

This information of the fundamental circle allows to assert immediately that E′ is not an area.

Fatou enunciated in his Note that if all roots are in the circle C, then it may happen that all of them generate a set whose derived set (involving E′ evidently) is perfect discontinuous or consisting of the whole C.

([1]) It may happen that two roots are indistinct on C; so they satisfy $z - \varphi(z) = 0$ and $\varphi'(z) = 1$, involving a new double root of $z - \varphi(z) = 0$; but this is a special case.

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21. If $z = \varphi(z)$ admits two symmetric roots in respect of C, not lying onto C, then one may suppose, given auxiliary homographic mapping, that those roots are 0 and ∞.

Then $\varphi(z)$ behaves:

$$\varphi(0) = 0 ;$$
$$|\varphi(z)| < 1 \text{ if } |z| < 1 ; \quad |\varphi(z)| = 1 \text{ if } |z| = 1 ; \quad |\varphi(z)| > 1 \text{ if } |z| > 1 ;$$
$$\varphi(\infty) \equiv \infty ;$$

Then the Schwarz's Lemma, cited in the *Preliminaries* (§ 5) proves, as $\varphi(z)$ is not linear, that one gets for $|z| < 1$

$$|\varphi(z)| < |z| \quad (^1)$$

without any possible relation of equality; and it proves that one has $|\varphi'(0)| < 1$ at the origin with any possible relation of equality since a small circle γ, centered at O, is mapped into a curve inside γ by $\varphi(z)$ (read *Preliminaries*).

The origin is a limit point with uniform convergence; one sees that it is not the same as a point at infinity.

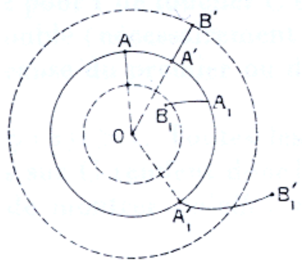

Fig. 13

It is easy to prove that, by the current hypothesis, *in every point of* C *one gets*

$$|\varphi'(z)| > 1 \ (^2).$$

In fact, for $|z| = 1$, then

$$|\varphi(z)| = 1,$$

and for $|z| < 1$,

$$|\varphi(z)| < |z|.$$

Each segment AB, normal at the circle $|z| = 1$, is mapped into an arc of the curve A_1B_1 by linking A_1 (on the circle $|z| = 1$) to B_1 (inside the circle with radius OB).

(1) One gets, for $|z| > 1$, $|\varphi(z)| > |z|$.
(2) This preposition seems to be a very useful complement to the Note of Fatou, notified further.

Therefore, the length of the arc A_1B_1 is always *bigger* than AB without any possible relation of equality ([1]).
This involves that $| \varphi'(z) | \geq 1$ in A ([2]).

Moreover, since $\Gamma : | \varphi'(z) | = 1$ is and algebraic curve, it can intersect C only in a finite number of points; one will easily see that in these points Γ would not intersect or, at least, be tangent to C in a point A, or, finally, it would have a multiple point in A without any contradictions arising from one of two following facts:

1) in the neighbourhoods of a common point A of C and Γ, except A itself, there exists a segment AB, satisfying $| \varphi'(z) | < 1$, on the radius OA or on its continuation; this segment AB correspond to an arc A_1B_1, longer than AB itself;

2) on C, in the neighbourhoods of A, some points satisfying $| \varphi'(z) | < 1$.

The figures *a* and *b* show the impossibility for Γ to intersect C in a simple point A; the region, near to A, has been dashed where $| \varphi'(z) | < 1$.

The figure *c* shows the impossibility for Γ to touch C and the figure *d* shows the impossibility for Γ to have no double point in a point of Γ, according to the first or second condition.

Therefore one has $| \varphi'(z) | > 1$ in each point of C.

All roots of $z = \varphi_n(z)$ ($n = 1, 2, \ldots, \infty$), lying on C, retrieve $| \varphi_n'(z) | > 1$.

All of them belong to the set E.

It is easy to prove *that these points are everywhere dense in* C. In fact, on C one gets

$$| \varphi'(z) | > M > 1,$$

so that M is never equal to 1.

If z describes any arc ab of C, then the iterated z_1 describes an arc a_1b_1, in the same sense as ab, which will be $> M \times (\text{ arc } ab)$; again z_2 will describe an arc

$$a^2b^2 > M^2(\text{ arc } ab) \ldots.$$

([1]) An arc $A'_1B'_1$ is always $> A'B'$ which is normal and exterior from C.
([2]) In fact, one gets

$$arcA_1B_1 = \int_A^B | \varphi'(z) \| dz| \quad ,$$

and if, in A, $|\varphi'(z)| < 1$, then $|\varphi'(z)| < 1$ over a certain segment AB of OA and one will get

$$arcA_1B_1 < \overline{AB} \quad .$$

The arcs $a_i b_i$ grow indefinitely. There is an iteration z_n describing an arc $a_n b_n$ that includes as many circumferences as one will see in the positive sense, if z describes *ab* in the positive sense:

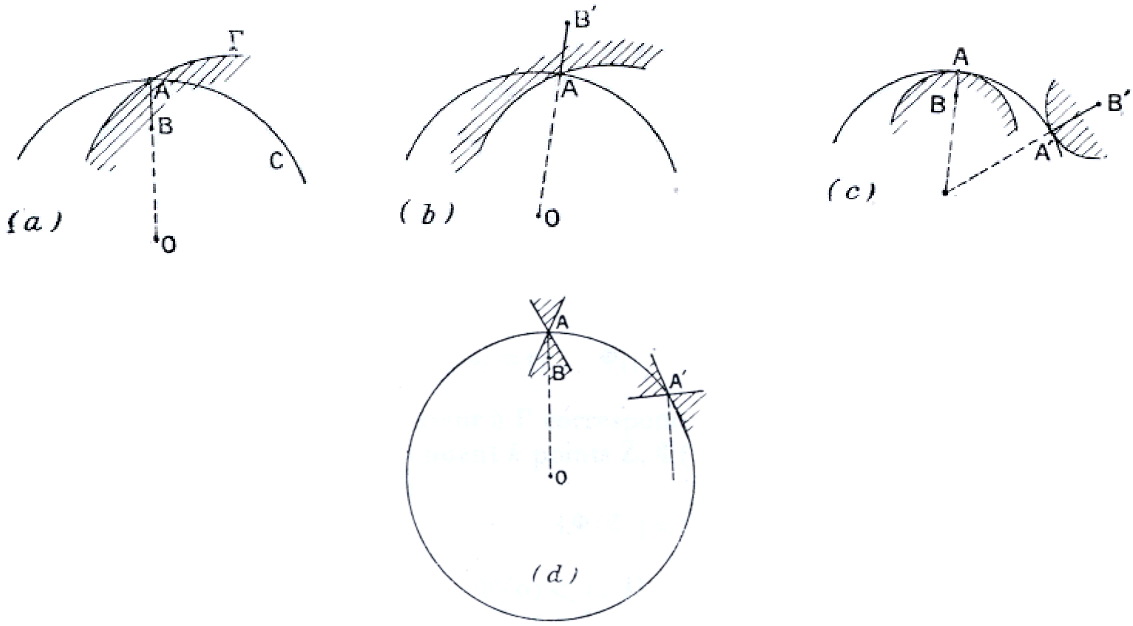

Fig. 14

therefore it happen evidently that z_n coincides with z for a certain position of z; that's to say: $z = \varphi_n(z)$ has a root in the arc *ab*. So *ab* may be any arc: that's to say, the roots of the equations $z - \varphi_n(z) = 0$ are everywhere dense in the circle C; E′ *is the same as the whole circle* C.

22. *Remark* ([1]). – If one considers, instead of a circle C, a simple analytic curve Γ, separating two regions so that each one of them is mapped into itself for k times ([2]) by $\varphi(z)$ like the area bounded by C is mapped into itself for k times, then one will get the same conclusions as before.

We will call this curve *the fundamental curve* Γ.

All the roots of $z = \varphi_n(z)$ (for $n = 1, 2, \ldots, \infty$) are on Γ; so only two of them, the roots of $z = \varphi(z)$, are outside Γ.

In the second case, if one maps conformally the interior ([3]) of Γ in the interior of the

([1]) Before this Memoir, Fatou has demonstrated in a Note of *Comptes Rendus* (February, 4th , 1918), that Γ cannot be *anything else but a circle*, because it consists of *a regular arc* of *an analytic curve*. Therefore this remark is no more valid and the reader is suggested to not take it in account further.

([2]) That's to say, an interior z in the curve corresponds to a point z_1, inside the curve and with z_1, in general to k distinct points; if z is over the curve, then z_1 is on the curve. Since the curve Γ is invariant for an infinity of rational substitutions $z_i = \varphi_i(z)$, then it shall belong to *genus zero* or *one*, if it is belongs to the algebraic kind, according to a theorem of Painlevé.

([3]) We call *interior of* Γ any of the two regions in which Γ splits the extended plane and, for fixing the ideas, the region including one root of $z=\varphi(z)$. However, we will see further that, in this case, each one of the two regions includes one of those roots in its interior and that, for those roots, we will get $|\varphi'(z)| < 1$.

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circle C(| Z | ≤ 1) by the function $z = f(Z)$, which is analytic in Γ and on Γ (Γ is in fact an analytic simple curve without singularities), then the relation $z_1 = \varphi(z)$, implying that z_1 is mapped to z in Γ, becomes a relation $Z_1 = \Phi(Z)$ making Z_1 mapped to Z in C.

So Φ is analytic in C and on C; if Z is on C, then Z_1 is on C_1.

If a root of $z = \varphi(z)$ (inside Γ) is mapped to Z = 0, then one gets

$$\Phi(0) = 0\ ,\ \Phi|(Z)| \leq 1 \qquad \text{if} \qquad |Z| \leq 1$$

k points z are mapped to a point z_1 inside Γ.

Therefore k points Z are mapped to a point Z_1 inside C; Φ(Z) is not linear, therefore

$$\text{for} \qquad |Z| < 1 \qquad \text{then,} \qquad |\Phi(Z)| < |Z|.$$

Therefore, at the origin, $\Phi'(0) < 1$.

On the following, the points $z = \varphi(z)$, out of Γ, retrieve $|\varphi'(z)| < 1$.

In fact in those points:

$$\frac{d\varphi}{dz} = \frac{dz_1}{dz}\frac{dZ_1}{dZ}\frac{dZ}{dz}$$

and like $z = z_1$ and $Z = Z_1$ at the considered points, one gets, in those points,

$$\frac{dz_1}{dZ_1} = \frac{dz}{dZ}, \qquad \text{therefore} \qquad \frac{dz_1}{dz} = \frac{dZ_1}{dZ} = \Phi'(0)\ .$$

The inverse function of $\varphi(z)$ has both $k - 1$ critical points (in general) inside Γ and $k - 1$ exterior points from Γ; on the k layers of its Riemann surface, Γ intersects a simply connected area with k layers; the area is inside Γ and it is bounded by the curve Γ; finally it is intersected k times ([1]).

In the same way, the inverse function of Φ(Z) has k - 1 critical points inside C; on the k layers of its Riemann surface, C intersects an area S with k interior layers of C which includes the origin and it is simply connected and bounded by C, which is intersected k times.

The function $Z = \psi(Z_1)$, inverse of ψ(Z), is therefore the conformal map on the simple area | Z | ≤ 1 of the area S ([2]).

Since the boundaries of the two corresponding areas are analytic, this conformal map can be analytically extended. Let us complete S by adding its symmetric S′ in respect of C and joining S and S′ along C, which is intersected for k times.

([1]) This area is described by z_1 as z describes the interior of Γ.
([2]) It is said that an homographic substitution preserving the area | Z | ≤ 1 is determined to the function $\psi(Z_1)$ achieving this application.

The set S + S′ generates a closed Riemann surface R of *genus zero* with *k* layers.

With the help of the Schwarz′s principle, Z and Z′, two symmetric points in respect of C, are mapped to the Z_1 and Z_1', still symmetric in respect of C; with the help of both the function $\psi(Z_1)$ and its previous extension, one realizes the conformal application of R on the extended plane.

But a function, applying a closed surface of *genus zero* with *k* layers (Z_1) on the extended plane of Z, is an algebraic function $Z = \psi(Z_1)$, the inverse function of the rational fraction (1) with degree *k*.

Therefore one sees that $Z_1 = \Phi(Z)$ is necessarily a rational function of Z and it is one of some the rational functions in the fundamental circle preserving the origin (as those we have considered above).

This area is that one being defined by z_1, since z defines the interior of Γ.
Some roots of $z = \varphi_n(z)$, since they are everywhere dense on Γ, are mapped by a conformal map $z = f(Z)$ to the roots of $Z = \Phi_n(Z)$ ($n = 1, 2, \ldots, \infty$), which are everywhere dense (2) on C: in this case two roots of $z = \varphi(z)$ are outside Γ and one sees that one root is *interior* and the other is *exterior* to Γ. Again E′ is the same as the curve Γ.

A priori it could be odd that the function $z = f(Z)$, which maps conformally the interior of Γ on the interior of C ($|Z| \leq 1$) and which is not linear (if Γ is not a circle), gives a relation $Z_1 = \Phi(Z)$ (which is the transformed function of $z_1 = \varphi(z)$), rational and with the same degree *k* as $z_1 = \varphi(z)$.

In fact, the relation $z = f(Z)$ maps not only the curve Γ (and its interior) to the interior of C, but also it maps an entire small strip, bordering with Γ (exterior to Γ), to a small strip, bordering with C (exterior from C), since, when the curve Γ is analytic, the analytic extension of $z = f(Z)$, beyond C and Γ, is made by the method of Schwarz's symmetries.

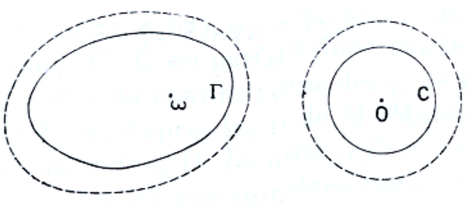

Fig. 15

(1) One may say that Z_1 is a analytic function of Z, so that it includes in the extended plane anything else but the poles; therefore it is a rational function.
(2) It is easily demonstrated that, if Z and *z* are mapped, so that

$$Z = \Phi_n(Z) \;, \qquad z = \varphi_n(z) \;,$$

then one gets in Z and *z*

$$\Phi'_n(Z) = \varphi'_n(z)$$

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Then one may think that $z = f(Z)$ maps the whole z plane to the plane Z ([1]); *surely this is not true*, if Γ is not a circle because $f(Z)$ cannot be linear; the function $f(Z)$ has a domain of existence which is bigger than C *but surely it does not extend on the entire plane*; since Γ not a circle, it is assured that the extension of $f(Z)$ *is not indefinitely possible* beyond C.
There is no more paradox so that the rational relation $z_1 = \varphi(z)$ would turn into a rational relation $Z_1 = \Phi(Z)$ with the help of a non-linear auxiliary substitution

$$z = f(Z),\ [z_1 = f(Z_1)],$$

and when $f(Z)$ exists only in a part of the plane Z.
If the function $z = f(Z)$, turning $z_1 = \varphi(z)$ into $Z_1 = \Phi(Z)$, is defined in the whole Z plane, then it shall be *linear*.
But if it exists only in a part of the Z plane, beyond that part where it is no more analytically extendable, then it is not necessary for that relation to be linear.

23. *Study of the region of the plane, not including any point of* E′.
Let D be any area of the plane, not including any point of E′.
It is clear that any of the iterations of D won't include points belonging to E′ because the opposite hypothesis, (when E′ coincides with all its preimages) would induce us to admit that D includes some points of E′. Immediately it implies that, in the area D, any of the functions of the family $\varphi_1(z), \varphi_2(z), \varphi_3(z), \ldots, \varphi_n(z), \ldots$ does not assume an affixe value of a point of E′.

E′ is a perfect set; therefore it has got an infinity of values (since it has the power of continuum) that none of $\varphi_i(z)$ may assume in D.
By the work of Montel, it follows that *the family of* $\varphi_i(z)$ *is normal in* D.

From every infinite sub-sequence that one may extract from the sequence of φ_i, *one may extract again a sequence converging uniformly in the interior of* D *to a limit function which is meromorphic in* D (*like the functions* φ_n) *and it may be a finite or infinite constant*.

This a fundamental result of the study of iteration and, in particular, in the study of the derived set (of the set of images of a point in the plane). If z is any interior point of D, then $z_1 = \varphi(z)$, $z_2 = \varphi_2(z), \ldots, z_n = \varphi_n(z)$ are the images. Let

$$\varphi_n(z),\ \varphi_{n2}(z),\ \ldots,\ \varphi_{np}(z),\ \ldots$$

be an infinite sequence, extracted from φ_i, which converges in every interior area of D to a meromorphic limit function $f(z)$ in D and it may be a constant.

([1]) The relation $Z_1 = \Phi(Z)$, the transformed relation of $z_1 = \varphi(z)$, is defined in the entire plane Z.

Then, since z is an any point inside D, its images

$$z_{n1} = \varphi_n(z),\ \ldots,\ z_{np} = \varphi_{np}(z),\ \ldots$$

converge to a limit point ζ, where ζ is the value in z of the limit function $f(z)$ in D of the sequence of $\varphi_{np}(z)$.
When z varies in every interior area of D, then ζ *could be fixed* and it may assume an *infinite value*.
If ζ does not *depend analytically from* z, then $\zeta(z)$ will be meromorphic in D ([1]).

This is a simple and very interesting result.

Then, immediately, if one knows that a special sequence $\varphi_n(z)$, $\varphi_{n2}(z)$, …, $\varphi_{np}(z)$, … converges uniformly in the whole interior area of D to a meromorphic function in D or to a infinite or finite constant value, then one may know the *behaviour of all sequence* $\varphi(z)$, $\varphi_2(z)$, …, $\varphi_n(z)$, … *in* D.

Evidently, if $\varphi_n(z)$, $\varphi_{n2}(z)$, …, $\varphi_{np}(z)$, … converges to $f(z)$ in D, then the sequence

$$\varphi_{n1+1}(z)\ ,\ \varphi_{n2+1}(z)\ ,\ \ldots\ ,\ \varphi_{np+1}(z)\ ,\ \ldots$$

converges uniformly to $\varphi[f(z)]$ in D, and generally

$$\varphi_{n1+i}(z)\ ,\ \varphi_{n2+i}(z)\ ,\ \ldots\ ,\ \varphi_{np+i}(z)\ ,\ \ldots$$

converges uniformly to $\varphi_i[f(z)]$ in the whole area inside D, whatever i is.

Evidently, every function $\varphi_k(z)$, with an index $k > n$, figures in one of the sequence of the precedent rows, for a certain value of i.

Resuming, therefore one may say that, when z is varying along D, the limit points of its image vary analytically with z.

Moreover, one sees that, in any point of E′ and so *in any area including a point of* E′, *any infinite sequence*, *extracted from the sequence of* φ_i, *cannot be a normal sequence*; therefore it follows that any infinite sequence

$$\varphi_n(z)\ ,\ \varphi_{n2}(z)\ ,\ \ldots,\ \varphi_{np}(z),\ \ldots$$

cannot converge to an analytic function in an area including a point of E′.

([1]) If one knows only that, since z is a determined point inside D (so that z does not belong to E′), ζ is a limit point of a sequence z_{n1}, z_{n2}, …, z_{np}, … of the images of z and of the sequence φ_{n1}, φ_{n2}, …, φ_{np}, …, normal in D, one may extract a sequence

$$\varphi_{n1},\ \varphi_{n2},\ \ldots,\ \varphi_{np},\ \ldots,$$

converging to a meromorphic limit function in the entire D and the value of that function, at the point z, won't be different from ζ; this shows that *each limit point* ζ, for the set of images of z, depends analytically on z in D.

Therefore, one sees that, while z varies in such an area, *no limit point of the set of the images can depend analytically on z in the whole area*.

The points of E′ look like *essential singular* points of the *limit functions* belonging to the sequence $\varphi_i(z)$.

The characteristic property of the points of E′ is the same as the property, explained by Picard's theorem about *essential singular points*.

That's why we call all points of E′ *singular points* of the iteration.

24. The importance of the points of E′ is fundamental *for defining the regions of the extended plane*, *where the behaviour of the function* $\varphi_i(z)$ *is always the same*.

Here is the necessary to intend: let us consider a region of the plane *consisting of one piece* and *having as boundary points only those belonging to* E′.

Starting from each point A of the plane, not belonging to E′, a region may be defined by the method of the analytic continuation: the region consists of one piece and it is bounded only by the points of E′, so that it could be defined as *the set of points that may be joint to* A *by simple line whose no point belongs to* E′.

We define R this region: any interior point does not belong to E′.

This region may be simply connected in some cases, if E′ is the same as a simple closed line (for example, the interior of the circle $| z | = 1$, in respect of the iteration of $z_1 = z^2$, is a region R); or it may be multiply connected even with an infinite order of connection ([1]), if E′ is a perfect discontinuous set in the entire plane.

As in the two previous examples, it could be a finite number of regions R in the entire plane; *there may be also an infinity of regions*, and we will see further some simple examples about it.

It is assured that the sequence of φ_i is *normal* in any interior area D of R.

Therefore, for any reason, if one has recognized that a partial sequence

$$\varphi_n(z),\ \varphi_{n2}(z),\ \ldots,\ \varphi_{np}(z),\ \ldots$$

([1]) From this point of view, one can make a interesting remark.
If one recognizes that *in a small area of the plane* the points of E′ generate a well connected set between any couple of its points, then one says that the whole E′ is a continuous line.
Then, every region R of the plane, consisting of one piece and whose boundary is generated by E′ or by a part of E′, is necessarily *simply connected*, because the opposite hypothesis would imply the existence of a simple closed line L, whose *all points are inside* R, so that each one of the two regions, in which L splits the extended plane, includes some points of E′ in its interior.
Therefore E′ won't be a set consisting of one piece, in contrast with hypothesis, since a point of E′, inside L, and a point of E′, exterior from L, will never be connected in E′, and since all points of L are at the distance to E′ $\geq \varepsilon$, where ε is a fixed positive number and different from zero.

converges, whatever z is in the interior of an arbitrarily small area D of R, to a limit function $f(z)$ which meromorphic in D, then one may affirm that, whatever is the interior area Δ of an examined region R, the sequence

$$\varphi_n(z),\ \varphi_{n2}(z),\ \ldots,\ \varphi_{np}(z),\ \ldots$$

converges uniformly to a limit function $f(z)$, which is meromorphic in each Δ.

To persuade oneself, it suffices to consider some areas so that each one may fit in the other including all areas D in their interior so that they have R as the limit area.

Each area Δ, inside R, can be enclosed in an area Δ′, inside R an including D in its interior. Now it is immediately clear that, on one hand, the sequence of

$$\varphi_n(z)\ ,\ \varphi_{n2}(z)\ ,\ \ldots\ ,\ \varphi_{np}(z)\ ,\ \ldots$$

is *normal* in Δ′ ; on the other hand, as this sequence converges uniformly to a meromorphic function $f(z)$ (to a infinite or finite constant value) in an area D of Δ′, then *this sequence converges uniformly to a meromorphic function* $f(z)$ in each Δ′ ([1]).

If the behaviour of the series $\varphi_n(z)$, $\varphi_{n2}(z)$, …, $\varphi_n(z)$, … is the same as in the whole region R, then an extracted partial sequence of the entire sequence converges in any area of R or *it does not converge in anyone of these areas*.

By this way, one may say that *the perfect set* E′ *delimits the different regions of convergence of the plane*: *in any region*, *consisting of one piece and bounding* E′, *the characteristics of convergence of the different infinite sequences*, *extracted from the sequence* $\varphi(z)$, $\varphi_2(z)$, …, $\varphi_n(z)$, … *are the same*.

25. The previous theorem acts like a real bridge from the general study of the iteration in the entire plane to the local study.
For example, the local study teaches us that if one finds a point ζ, root of $z = \varphi(z)$ and satisfying $|\ \varphi(z)\ | < 1$, then one may determine a circle C, centered at ζ, wherein:

1. There are no points of E′ ;
2. Whatever z is, its images z_1, z_2, …, z_n, … admit ζ and only ζ as limit point.

([1]) Here, I insist with the following theorem of Montel [*Sur les familles de functions analytiques* (*Annales de l' Ecole Normal*, 1912, p.531)]:
Let $f_i(x)$ (i = 1,2, …,∞) *be an infinite sequence of holomorphic functions in* D *and belonging to a normal family in* D.

1) *If the sequence converges to an infinity of points*, *inside their set* D, *then it converges in all the set* D.
2) *If the sequence converges in* D, then *the convergence is uniform in the interior of* D.

There is only to suppose that one has sent a point P of E' to infinity so that all poles of φ_i, since they are preimages of P, belong to E'; that's to say all the φ_i are holomorphic in R.

The previous theorem leads us to more general results:

1) ζ is inside a connected region R, uniquely determined by points of E′.

2) C is inside this region R and the sequence $\varphi(z)$, $\varphi_2(z)$, …, $\varphi_n(z)$ converges to the constant ζ in the interior of C.

3) It follows that, whatever z is in R, then the sequence $\varphi(z)$, $\varphi_2(z)$, …, $\varphi_n(z)$ converges to ζ; the images of any point z of R have ζ and only ζ as their limit point.

So we will say that R is the *immediate domain of convergence* of the limit point ζ.
Immediate means that R consists of one piece (i.e.: it is connected) with ζ.
The boundary of this set consists uniquely of points of E′ ; each one of these boundary points has only images belonging to E′ : so that they do not admit ζ as their limit point.

Analogous considerations may be made in respect of a point ζ, root of $z = \varphi_n(z)$ and satisfying $|\ \varphi'_n(z)\ | < 1$, and for points ζ_1, …, ζ_{n-1}, generating a periodic cycle with ζ itself.

26. Here some easy and fast applications, of this important theorem, follow:

1. If E′ is an *everywhere discontinuous set* ([1]), then there is only a region R consisting of the entire plane but E′ ; therefore, there is only a limit point with uniform convergence

$$z = \varphi(z)\ ,\ \text{satisfying}\ \ |\ \varphi'(z)\ | < 1$$

but there are never two distinct limit points with uniform convergence, neither a periodic cycle ([2]). The limit point with uniform convergence has the whole plane but E′ as its domain of convergence.
This applies, in particular, to rational fractions in the fundamental circle, if E′ is everywhere discontinuous.

2. In the fundamental circle C, for those fractions admitting two roots $z = \varphi(z)$, symmetric in respect of C, it's seen that each one of the roots satisfies $|\ \varphi'(\zeta)\ | < 1$; these roots are limit points with uniform convergence: E′ is the same as the circle C ;

~~([1]) In general, if E′ does not split the plane in many regions so that any path, going from an interior point~~ of a region to an interior point of another region, includes necessarily a point of E′.

([2]) In fact, if there are two distinct limit points with uniform convergence ζ_1 and ζ_2, then there are two areas D_1 and D_2, inside R and surrounding respectively ζ_1 and ζ_2, so that in D_1 the sequence of $\varphi_i(z)$ converges uniformly to the constant ζ_1 et, in D_2, the same sequence converges to ζ_2, contradicting our theorem.
If one has a periodic cycle $\zeta, \zeta_1, \dots, \zeta_{n-1}$, where

$$\zeta = \varphi_n(\zeta)\ ,\ |\ \varphi'_n(\zeta)\ | < 1$$

then one may surround the points ζ and ζ_1, which are distinguished by the small areas D and D_1, respectively so that the following sequence

$$\varphi_n(z), \varphi_{2n}(z), \dots, \varphi_{kn}(z), \dots$$

converges uniformly to the constant ζ in D and to the different ζ_1 in D_1: our theorem will be contradicted again.

the domain of convergence for each one of the two limit points consists of the region bounded by C and including that point.
The results (1) and (2) agree with what has been published by Fatou in his Note on May 21th 1917, in *Comptes Rendus de l'Académie des Sciences*.

3. If *a priori* one recognizes the simultaneous existence of two distinct limit points with uniform convergence or of a periodic cycle, then one may affirm that E′ encloses *a continuous linear set splitting the plane in many regions* ; since E′ cannot include the entire plane, neither it can be a perfect discontinuous set in any area of the plane, then the immediate domains of convergence to each one of two distinct limit points (for example) cannot share any common point, since one is separated from each other by E′, so that there are no paths connecting these two domains without including at least one point of E′.